Introduction

Let G be a compact Lie group acting as a transformation group in Euclidean space $\mathbb{R}^n$. It is a conjecture that the orbit space $\mathbb{R}^n/G$ is acyclic for integral cohomology. Conner and Floyd have proven that this conjecture holds if G is finite and if G is abelian, [2, 6]. In [1], they also proved the conjecture to hold when G is a closed subgroup of $SO(5)$. For any G, the rational cohomology group $H^*(\mathbb{R}^n/G; Q)$ is acyclic, this is a result due to Borel, see [1]. Since $H^*(\mathbb{R}^n/G; \mathbb{Z})$ is a torsion group, it suffices to shows that $H^*(\mathbb{R}^n/G; \mathbb{F}_p)$ is acyclic for each prime p, where $\mathbb{F}_p$ is the field of order p. We therefore consider the conjecture that for each G space X with $H^*(X; \mathbb{F}_p)$ acyclic, the cohomology of the orbit space, $H^*(X/G; \mathbb{F}_p)$ is also acyclic. By a technique which we will call the Borel induction step (1.1), [1], this conjecture depends on the existence, for each simple connected nonabelian G, of a G space X without fixed points such that $H^*(X; \mathbb{F}_p)$ and $H^*(X/G; \mathbb{F}_p)$ are acyclic. We will construct a number of such G spaces, with the exceptions listed in the following theorem.

<u>Theorem 1.</u> <u>Let</u> G <u>be a compact connected simple nonabelian Lie group, and let</u> p <u>be a prime number. With the only possible exceptions listed below, there is a compact, finite-dimensional space</u> X <u>where</u> G <u>is acting without fixed points and such that</u> $H^*(X;\mathbb{F}_p)$ <u>and</u> $H^*(X/G;\mathbb{F}_p)$ <u>are both acyclic. The possible exceptions are,</u>

$p = 2$, $G = G_2$, $\mathbb{E}_7$, $\mathbb{E}_8$, $SO(2^a)$, $SU(2^a)$, $a \geq 3$, $Sp(2^a)$, $a \geq 2$,

$p = 3$, $G = \mathbb{F}_4$, $\mathbb{E}_6$, $\mathbb{E}_8$,

$p = 5$, $G = \mathbb{E}_8$, *and*

$p \geq 3$, $G = SU(p^a)$, $a \geq 1$.

For $p = 2$, none of the groups listed in theorem 1 are contained in $SU(6)$.

Theorem 2. Let X *be a space where a group* G *is acting, and such that* $H^*(X; \mathbb{F}_2)$ *is acyclic. If* G *is a closed subgroup of* $SU(6)$, *then the cohomology group* $H^*(X/G; \mathbb{F}_2)$ *is also acyclic.*

For $p \geq 7$, the possible exceptions listed in theorem 1 are of the form $SU(n)$, $n = p^a$. This is surprising, since the unitary groups are usually the best-behaved groups when cohomology is concerned. In question 4 at the end of the paper, we make precise the kind of result that needs to be proven for $SU(p^a)$.

Let X be a G space and let $f : X \to X$ be an equivariant map such that the orbit map f' preserves the components of X/G. We then want to ask what are the eigenvalues of f^* in $H^*(X; \mathbb{k})$, and if they are related to the eigenvalues of f'^* in $H^*(X/G; \mathbb{k})$. By the substitution $Z \to f^*$ or $Z \to f'^*$, we can consider $H^*(X; \mathbb{k})$ and $H^*(X/G; \mathbb{k})$ as $\mathbb{k}[Z]$ modules, and the eigenvalues of f^* or f'^* then correspond to the simple $\mathbb{k}[Z]$ submodules of $H^*(X; \mathbb{k})$ or of $H^*(X/G; \mathbb{k})$. There is a close relationship between the acyclicity of $\mathbb{R}^n/G$ and a property of the eigenvalues of f^* and f'^*.

Theorem 2. Let G *be a compact Lie group acting on a space* X, *and let* $f : X \to X$ *be an equivariant map. Assume that the orbit* *space* X/G *is connected. Then each simple* $Q[Z]$ *submodule of* $H^i(X/G;Q)$ *for each* $i>0$ *is isomorphic to a* $Q[Z]$ *submodule of* $H^j(X;Q)$ *for some* $j>0$ *. Here the module structures are given by* $Z \to f'^*$ *and* $Z \to f^*$ *, respectively.*

The above theorem includes the statement that $H^*(X/G;Q)$ is acyclic when $H^*(X;Q)$ is ; this is seen by taking f to be the identity map. Let $A(X)$ be the ring gnerated (as a free Abelian group) by the monoid of equivariant maps $X \to X$, which preserve the components of X/G . Then $H^*(X)$ and $H^*(X/G)$ are $A(X)$ modules. Theorem 3 is valid for the $A(X)$ module structures, in the sense that for each $i>0$, each simple $A(X)$ subquotient of $H^i(X/G;Q)$ is isomorphic to an $A(X)$ subquotient of $H^j(X;Q)$ for some $j>0$. This reformulation of the question if (and for rational cohomology, the theorem that) $\mathbb{R}^n/G$ is $\mathbb{k}$-acyclic for all actions of G on $\mathbb{R}^n$, makes this kind of question more interesting and may be a help in settling the question itself.

Montgomery has conjectured that $\mathbb{R}^n/G$ is contratible when a compact Lie group G is acting on $\mathbb{R}^n$. It is well known that $\mathbb{R}^n/G$ is simply connected. A main result of Conner's paper [1] is that if $H^*(\mathbb{R}^m/K;\mathbb{Z})$ is acyclic for all actions of a closed subgroup K of G on some $\mathbb{R}^m$, then $\mathbb{R}^m/K$ is an absolute neighbourhood retract, and hence is locally contractible and contractible. Conner proves, using Floyd's maps of degree zero (see below), that $\mathbb{R}^n/SO(k)$ is contractible for $k \leq 5$.

Throughout this paper we make some basic topological assumptions on the G spaces, namely that the action of G has a finite number of conjugacy classes of isotropy groups, and that X is regular, paracompact and of finite cohomology dimension over $\mathbb{Z}$.

It suffices however to prove the results of this paper when X is compact, since the extension to the more general case is an easy technical trick made possible by the fact that the Hsiang test spaces defined below are inverse limits of compact differentable G manifolds. The technical problem thus avoided is the question of the validity of certain Leray spactral sequences. To avoid such technicalities, we will by and large confine our attention to the case where X is compact.

I want to express my gratitude to Professor Wu-yi Hsiang for pointing out to me the conjetures conserning $\mathbb{R}^n/G$, and for explaining some points of [7] as well as showing me how to construct equivariant self-maps of the 26 dimensional representation of the exceptional group $\mathbb{F}_4$.

1. Cohomology eigenvalues of equivariant maps.

In this chapter we will show that the conjecture on acyclic orbitspaces is equivalent to a conjecture on cohomology eigenvalues of equivariant maps. The more general eigenvalue statement sheds light on the Borel induction step, and enables us to construct a large number of $\mathbb{F}_p$ acyclic G spaces without fixed points and with $\mathbb{F}_p$ acyclic orbitspaces. We will briefly describe Borel's induction step, and extensions of this technique. A mapping torus construction will give the new G spaces just mentioned.

Let $\mathbb{k}$ be one of the prime fields $\mathbb{F}_p$ and $\mathbb{Q}$, and let G be a compact Lie group. There are three interrelated conjectures on the cohomology of orbit spaces of G spaces.

$\mathbb{k}$-acyclicity conjecture. *When X is a G space with $H^*(X;\mathbb{k})$ acyclic, then $H^*(X/G;\mathbb{k})$ is also acyclic.*

Let V be a module over an algebra A over the field $\mathbb{k}$. By a *decomposition factor* of V we will understand a simple A module of the form V_2/V_1 where $V_1 \subset V_2 \subset V$ are submodules of V. The module V_2/V_1 will also be called a simple subquotient of V. We do not insist that V be of finite dimension, and hence there may be an infinite number of nonisomorphic decomposition factors of V. We will consider $A(X)$ decomposition factors of $H^+(X;\mathbb{k})$ and of $H^+(X/G;\mathbb{k})$ where $H^+ = \Sigma_{i>0} H^i$. By a *constant* $A(X)$ module V we will understand a module such that $(\Sigma a_i f_i)x = \Sigma a_i x$ for all $(\Sigma a_i f_i) \in A(X)$ and all $x \in V$.

$\mathbb{k}$ - *eigenvalue conjecture. For a* G *space* X *such that* X/G *is connected, each decomposition factor of the* $A(X)$ *module* $H^{+}(X/G;\mathbb{k})$ *is also a decomposition factor of the* $A(X)$ *module* $H^{+}(X;\mathbb{k})$.

It is clear that if the $\mathbb{k}$-eigenvalue conjecture holds, then the $\mathbb{k}$-acyclicity conjecture will also hold.

$\mathbb{k}$ - *test space conjecture. There is a* G *space* Z *without fixed points such that both* $H^*(Z;\mathbb{k})$ *and* $H^*(Z/G;\mathbb{k})$ *are acyclic.*

Lemma (1.1) (Borel's induction step)
If the $\mathbb{k}$*-acyclicity conjecture holds for all proper closed subgroups of* G *and if the* $\mathbb{k}$*-test space conjecture holds for* G *, then the* $\mathbb{k}$*-acyclicity conjecture holds for* G .

Proof [1]. Let X be a $\mathbb{k}$-acyclic G space and let Z be the test space. Let $Z \times X$ be a G space with the diagonal action and set $Z \times_G X = (Z \times X)/G$. There are natural maps induced by the two projections,

$$\begin{array}{ccccc} Z/G_x & \to & Z \times_G X & \to & X/G \\ X/G_z & \to & Z \times_G X & \to & Z/G \end{array} \qquad (1.2)$$

where the fibres are $Z \times_G G(x) = Z/G_x$ and $G(z) \times_G X = X/G_z$. Since $G_z \neq G$ for $z \in Z$, the assumptions imply that $H^*(X/G_z;\mathbb{k})$ and $H^*(Z/G_x;\mathbb{k})$ are acyclic for all $x \in X$ and $z \in Z$. The Vietoris-Begle theorem implies that $H^*(X/G;\mathbb{k}) \cong H^*(Z \times_G X;\mathbb{k}) \cong$ $\cong H^*(Z/G;\mathbb{k})$, and hence is acyclic. Hence the $\mathbb{k}$-acyclicity conjecture holds for G .

<u>Theorem (1.3)</u> <u>If the identity component</u> G^o <u>of</u> G <u>is abelian, then the</u> $\mathbb{k}$<u>-eigenvalue conjecture holds for</u> G <u>and all fields</u> $\mathbb{k}$.

<u>Proof.</u> If the theorem holds for a closed normal subgroup $N \subset G$, and if it holds for the quotient group G/N, then it holds for G since $X/G = (X/N)/(G/N)$. Hence we may assume that G is either a finite group or a circle group. In case G is a circle group, let N be a finite subgroup containing all finite isotropy groups. Then G/N is acting semifreely on X/N, hence we need to prove the theorem for semifree circle actions only. If G is finite, let $H \subset G$ be a p-Sylow subgroup when $\mathbb{k} = \mathbb{F}_p$, and let $H = 1$ when $\mathbb{k} = \mathbb{Q}$. Then $H^*(X/G;\mathbb{k}) \to H^*(X/H;\mathbb{k})$ is injective. Since H is solvable, we may assume $G = \mathbb{Z}_p$ in case G is finite.
Let $G = \mathbb{Z}_p$, $\mathbb{k} = \mathbb{F}_p$, and let X be a G space .
Set $F = F(\mathbb{Z}_p,X)$, and consider the diagram of long exact sequences of $\mathbb{k}$ cohomology induced by the projection $(X_G,F_G) \to (X/G,F)$, where $X_G = E_G \times_G X$ with E_G a contractible free G space , and the projection $X_G \to X/G$ is given by $G(e,x) \to G(x)$.

$$\begin{tikzcd}
\to H^*_G(X,F) \arrow[r] & H^*_G(X) \arrow[r] & H^*_G(F) \arrow[r, "\delta"] & H^*_G(X,F) \to \\
\to H^*(X/G,F) \arrow[r] \arrow[u, "\cong"'] & H^*(X/G) \arrow[r] \arrow[u] & H^*(F) \arrow[r, "\delta"] \arrow[u] & H^*(X/G,F) \to \arrow[u, "\cong"']
\end{tikzcd}$$

Since G is acting semifreely, there are two isomorphisms in the diagram. From this we obtain a long exact Mayer-Vietoris sequence,

$$\to H^*(X/G) \to H^*(F) \oplus H^*_G(X) \to H^*_G(F) \xrightarrow{\delta} \qquad (1.4)$$

This is a sequence of $A(X)$ modules, <u>via</u> the restriction map $A(X) \to A(F)$. It suffices to prove the theorem when X/G is connected. If F is not empty, then X is also connected. Let SX be the unreduced suspension of X, and let v be a vertex of SX .

We then have isomorphisms of $A(X)$ modules, $H^+(X;\mathbb{k}) \cong H^*(SX,v;\mathbb{k})$ and $H^+(X/G;\mathbb{k}) \cong H^*(SX/G,v;\mathbb{k})$. Let E_r, $r \geq 1$, be the spectral sequence converging to $H^*_G(SX,v;\mathbb{k})$ with

$$E_1 = C^*(B_G; H^*(SX,v;\mathbb{k})) .$$

Then E_r is a spectral sequence of $A(X)$ modules and each decomposition factor of E_1 is a decomposition factor of $H^+(X;\mathbb{k})$. Hence each decomposition factor of E_r, $1 \leq r \leq \infty$, and of $H^*_G(SX,v;\mathbb{k})$ is a decomposition factor of $H^+(X;\mathbb{k})$. The localization theorem for $\mathbb{Z}_p$ actions now implies that each decomposition factor of $H^*_G(SF,v;\mathbb{k})$ and of $H^*(SF,v;\mathbb{k})$ is a decomposition factor of $H^+(X;\mathbb{k})$. Using a relative Mayer-Vietoris sequence of the form (1.4) ,

$$\to H^*(SX/G,v) \to H^*(SF,v) \oplus H^*_G(SX,v) \to H^*_G(SF,v) \to ,$$

we see that every decomposition factor of $H^*(SX/G,v;\mathbb{k}) \cong H^+(X/G;\mathbb{k})$ is a decomposition factor of $H^+(X;\mathbb{k})$. This completes the proof when F is not empty. When F is empty, we have $H^*(X/G;\mathbb{k}) \cong H^*_G(X;\mathbb{k})$. Let E_r, $r \geq 1$, be the spectral sequence converging to $H^*_G(X;\mathbb{k})$ with $E_1 = C^*(B_G;H^*(X;\mathbb{k}))$. Here each decomposition factor of E_r^{st} for $t > 0$ is a decomposition factor of $H^+(X;\mathbb{k})$. If X is connected, $E_2^{so} \cong H^s(B_G;\mathbb{k})$ is a constant $A(X)$ module. If $H^+(X;\mathbb{k})$ has no constant decomposition factor, then no differential could map nontrivially into E_r^{so}, $r \geq 2$, and hence $H^*(B_G;\mathbb{k}) \subset H^*_G(X;\mathbb{k})$. But this is not the case since the fixed point set is empty. It follows that $H^+(X;\mathbb{k})$ has a constant decomposition factor. Hence each decomposition factor of E_r, $r \geq 1$, and of $H^*(X/G;\mathbb{k}) \cong H^*_G(X;\mathbb{k})$ is a decomposition factor of $H^+(X;\mathbb{k})$.

If X is not connected, it must have p components which are being permuted transitively by G . Hence, $E_2^{*0} = H^*(\mathbb{Z}_p, \mathbb{k}\mathbb{Z}_p)$, the group cohomology of $\mathbb{Z}_p$ with coefficients in the left regular representation $\mathbb{k}\mathbb{Z}_p$, and it follows that $E_2^{s0} = 0$ for $s > 0$. Thus for $(s,t) \neq (0,0)$, each decomposition factor of E_r^{st} , $r \geq 2$, is a decomposition factor of $H^+(X; \mathbb{k})$. Thus the same holds for $H_G^+(X; \mathbb{k}) \cong H^+(X/G; \mathbb{k})$, and this concludes the proof for G finite.

If G is a circle group acting semifreely, we may assume that X is connected, and a proof quite similar to the one given above, will apply for any coefficient field.

It follows from the above theorem and its proof, that if we want to prove the $\mathbb{k}$-eigenvalue conjecture for all G , it suffices to do so in case G is simple, connected, and of dimension > 1 .

Remark. In case the fixed point set is not empty, the above proof uses the suspension of X , which need not be paracompact unless X is compact. Avoiding the suspensions, we can use the absolute Mayer-Vietoris sequence (1.4) and a more careful algebraic argument.

In order to discuss the relationship between the eigenvalue conjecture and the test space conjecture, we need the cohomological lemma (1.5). Here the Hsiang test spaces are of importance. Assume that X is a compact connected G space and let $f: X \to X$ be an equivariant map. Let $X_\infty = \varprojlim_f X$ be the inverse limit of the system

$$\cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots \xrightarrow{f} X .$$

If X has no fixed points, then X_∞ has no fixed points. Because G is compact, we have $X_\infty/G = \varprojlim_{f'} X/G$.

Hence $H^*(X_\infty; \mathbb{k}) = \varinjlim_{f^*} H^*(X; \mathbb{k})$

and $H^*(X_\infty/G; \mathbb{k}) = \varinjlim_{f'^*} H^*(X/G; \mathbb{k})$.

Thus if f^* is nilpotent in $H^+(X; \mathbb{k})$ and if f'^* is nilpotent in $H^+(X/G; \mathbb{k})$, we see that the $\mathbb{k}$-test space conjecture holds for G, by setting $Z = X_\infty$. Floyd has proven that for $G = SO(2n+1)$, there is an action of G on a sphere X without fixed points, and an equivariant map $f : X \to X$ of degree 0, and hence that $H^*(X_\infty; \mathbb{Z})$ is acyclic. Moreover, the quotient space $X/SO(2n+1)$ is contractible ([1] p.352, for a proof, see [7]). Hsiang and Hsiang [7] show that for any connected nonabelian G, there is a sphere X which is a G space without fixed points and an equivariant map $f : X \to X$ of degree 0. It is generally not known if f'^* is nilpotent in $H^+(X/G; \mathbb{Z})$. We will call X_∞ the <u>Hsiang test space for</u> G and note that it is an inverse limit of differentiable G manifolds.

<u>Lemma</u> 1.5). <u>Let</u> Z <u>be a compact connected</u> G <u>space without fixed points which is either a differentiable</u> G <u>manifold or a directed inverse limit of differentiable</u> G <u>manifolds. Let</u> X <u>be a connected</u> G <u>space, and let</u> $pr_1 : Z \times_G X \to Z/G$ <u>be the projection. Assume that the</u> $\mathbb{k}$-<u>eigenvalue conjecture holds for all</u> G_z, $z \in Z$. <u>Then in the exact sequence</u>

$$0 \to A^* \to H^*(Z/G; \mathbb{k}) \xrightarrow{pr_1^*} H^*(Z \times_G X; \mathbb{k}) \to B^* \to 0 ,$$

<u>each</u> $A(X)$ <u>decomposition factor of</u> A^* <u>or</u> B^* <u>is a decomposition factor of</u> $H^+(X; \mathbb{k})$. <u>Here</u> A^* <u>is a constant</u> $A(X)$ <u>module.</u>

<u>Proof.</u> Assume first that Z is differentiable. Then Z/G has a triangulation [11] such that the singularity strata are subcomplexes.

For each simplex $\sigma \in Z/G$, choose a point $z \in \sigma^o$ and set $G_\sigma = G_z$. The spectral sequence converging to $H^*(Z \times_G X; \mathbb{k})$ associated to the skeletal filtration of Z/G, has the form E_r, $r \geq 1$, where $E_1 = \Pi_\sigma H^*(\sigma, \partial\sigma) \otimes H^*(X/G_\sigma; \mathbb{k})$.

Here $H^s(Z/G; \mathbb{k}) \cong E_2^{so}$ and each decomposition factor of E_r^{st}, $r \geq 1$, is a decomposition factor of $H^t(X/G_\sigma; \mathbb{k})$ for some σ, and for $t > 0$ it is hence a decomposition factor of $H^+(X; \mathbb{k})$, by assumption. Thus each simple decomposition factor of B^* is a decomposition factor of $H^+(X; \mathbb{k})$. If $A^* \neq 0$, then some differential $d_r : E_r^{*r-1} \to E_r^{*o}$, $r \geq 2$, must be nontrivial, and hence $H^+(X;\mathbb{k})$ must have a constant decomposition factor for the same reason as before.

If $Z = \varprojlim Z_i$ is an inverse limit of differentiable manifolds Z_i, a cofinal system of Z_i will be without fixed points, since Z is. We obtain exact sequences,

$$0 \to A_i^* \to H^*(Z_i/G; \mathbb{k}) \xrightarrow{pr_1^*} H^*(Z_i \times_G X; \mathbb{k}) \to B_i^* \to 0,$$

and the direct limit of those sequences is the sequence of the lemma. Therefore it suffices to remark that each decomposition factor of $\varinjlim B_i^*$ is a decomposition factor of some B_i^* and hence of $H^+(X;\mathbb{k})$.

<u>Proposition (1.6).</u> <u>Let</u> G <u>be a compact Lie group and let</u> $\mathbb{k}$ <u>be a field. Then the</u> $\mathbb{k}$<u>-eigenvalue conjecture holds for all closed subgroups of</u> G <u>if and only if the</u> $\mathbb{k}$<u>-test space conjecture holds for all connected nonabelian simple subgroups of</u> G.

<u>Proof.</u> Assume that the $\mathbb{k}$-test space conjecture holds for the nonabelian connected simple subgroups of G, and let $K \subset G$ be a closed subgroup. By induction we may assume that the $\mathbb{k}$-eigenvalue

conjecture holds for all proper closed subgroups of K. If K is not connected, we see that the $\mathbb{k}$-eigenvalue conjecture must hold for K by applying theorem (1.3) to the finite group K/K^0. If K is connected but not simple, then K is covered by $K_1 \times K_2$ where K_1 and K_2 are proper subgroups of K and hence the $\mathbb{k}$-eigenvalue conjecture must hold for K. If K is simple connected, we use theorem (1.3) if K is a circle. If K is nonabelian, let Z be the Hsiang test space. Then $H^*(Z/K; \mathbb{k})$ is acyclic by assumption and by lemma (1.1). Let X be a compact K space. Using the mapping

$$Z/K_x \to Z \times_K X \to X/K$$

we see that $H^*(Z \times_K X; \mathbb{k}) \cong H^*(X/K; \mathbb{k})$ because $H^*(Z/H; \mathbb{k})$ is acyclic for all closed subgroups of K. Since Z is an inverse limit of differentiable K manifolds, we can use lemma (1.5) to obtain an exact sequence

$$0 \to \mathbb{k} \to H^*(X/K; \mathbb{k}) \to B^* \to 0$$

where $\mathbb{k} \cong H^*(Z/K; \mathbb{k})$ and $B^* \cong H^+(X/K; \mathbb{k})$. It follows from lemma (1.5) that every decomposition factor of B^* is a decomposition factor of $H^+(X; \mathbb{k})$.

Conversely, assume that the $\mathbb{k}$-eigenvalue conjecture holds for all $K \subset G$, and let Z be the Hsiang test space for K. Since $H^+(Z; \mathbb{k}) = 0$, it follows that $H^+(Z/K; \mathbb{k}) = 0$, so that the $\mathbb{k}$-test space conjecture holds for all connected nonabelian simple $K \subset G$.

It is well known that the $\mathbb{Q}$-test space conjecture holds for every G whose identity component G^0 is nonabelian. Letting T be a maximal torus of G and $N(T)$ its normalizer, the homogeneous space $Z = G/N(T)$ is $\mathbb{Q}$-acyclic, and Z/G is a point [1]. Proposition (1.6) then implies,

<u>Theorem (1.7).</u> <u>Every</u> $A(X)$ <u>decomposition factor of</u> $H^+(X/G;Q)$ <u>is an</u> $A(X)$ <u>decomposition factor of</u> $H^+(X;Q)$.

We will now introduce an equivariant mapping torus construction which we will use to verify the $\mathbb{k}$-test space conjecture for a number of simple groups. Let $f,g : X \to Y$ be two continous maps and let the <u>mapping torus</u> $T(f,g)$ be the identification space obtained from $X \times [0,1] \cup Y$ by identifying $(x,0)$ with $g(x) \in Y$ and $(x,1)$ with $f(x) \in Y$. Let

$$i : X \times [0,1] \cup Y \to T(f,g)$$

be the resulting identifiaction map. Let $f',g' : X/G \to Y/G$ be the maps of orbit-spaces. There is a natural homeomorphism

$$T(f,g)/G = T(f',g') .$$

There is a natural map $j : T(f,g) \to S^1$ where $S^1 = [0,1]/\{0,1\}$, given by $ji(x,t) = t$. To compute the cohomology of $T(f,g)$, we use the Mayer-Vietoris sequence of the subspaces

$$A = j^{-1}[1/3,2/3] \quad \text{and} \quad B = T(f,g) - j^{-1}(1/3,2/3) .$$

This sequence has the form

$$\xrightarrow{\delta} H^*(T(f,g)) \to H^*(X) \oplus H^*(Y) \xrightarrow{L} H^*(X) \oplus H^*(X) \xrightarrow{\delta}$$

where L has the matrix form

$$L = \begin{pmatrix} 1 & g^* \\ 1 & f^* \end{pmatrix} .$$

Dividing by the exact subsequence

$$0 \to H^*(X) \oplus 0 \xrightarrow{L} \operatorname{diag}(H^*(X) \oplus H^*(X)) \to 0 ,$$

the result is a long exact sequence,

$$(1.8) \quad \xrightarrow{\delta} H^*(T(f,g)) \to H^*(Y) \xrightarrow{g^*-f^*} H^*(X) \xrightarrow{\delta} .$$

This sequence is naturally isomorphic to the long exact cohomology sequence of the pair $(T(f,g),Y)$.

When $X = Y$, and $g = 1_X$, we set $T(f) = T(1_X,f)$. Let f^m be the m^{th} power of the map f . There is a map d^m such that the diagram commutes,

$$\begin{array}{ccc} T(f^m) & \xrightarrow{d^m} & T(f) \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{z^m} & S^1 . \end{array}$$

Here z^m denotes the m^{th} power homomorphism of the circle group. Note that $T(f) = i(X \times [0,1])$ where $(x,1)$ is identified with $(f(x),0)$. The map d^m is defined as follows,

$$d^m i\,(x,a) = i\,(f^q(x)\,,\,ma-q) \quad \text{for} \quad q/m \leq a \leq (q+1)/m \;,$$

$$0 \leq q \leq m-1 \,, \quad \text{where} \quad q \quad \text{is an integer.}$$

d^m is multiplicative in the sense that

$$d^m d^n = d^{mn} : \; T(f^{mn}) \to T(f^m) \to T(f) \;.$$

<u>Lemma (1.9). Let</u> p <u>be a prime and let</u> $f : X \to X$ <u>be a self map on a compact space</u> X <u>such that</u> $H^*(X;\mathbb{F}_p)$ <u>is finitely generated. Let</u> $T(f^\infty)$ <u>be the inverse limit of the system</u>

$$\cdots \to T(f^{p^{n+1}}) \xrightarrow{d^p} T(f^{p^n}) \xrightarrow{d^p} \cdots \to T(f^p) \xrightarrow{d^p} T(f) \;.$$

<u>Then</u> $H^*(T(f^\infty;\mathbb{F}_p) \cong H^*(X;\mathbb{F}_p)^{(1)}$ <u>where</u> $H^*(X;\mathbb{F}_p)^{(1)}$ <u>is the largest subspace of</u> $H^*(X;\mathbb{F}_p)$ <u>on which</u> $I - f^*$ <u>is nilpotent.</u>

Proof. Define $i_o : X \to T(f^m)$ by $i_o(x) = i(x,0)$. Then $i_o d^m = i_o$, and hence there is a limit map $i_\infty : X \to T(f^\infty)$. The maps d^m are maps of pairs, $d^m : (T(f^m), i_o(X)) \to (T(f), i_o(X))$, and hence define homomorphisms of sequences of the form (1.8). The coefficient group is $\mathbb{F}_p$.

$$\begin{array}{ccccccccc}
\longrightarrow H^*(X) & \xrightarrow{\delta} & H^*(T(f^{p^n})) & \xrightarrow{i_o^*} & H^*(X) & \xrightarrow{I-f^{*p^n}} & H^*(X) & \xrightarrow{\delta} \\
\downarrow t^{p^n} & & \downarrow d^{p*} & & \downarrow I & & \downarrow t^{p^n} & \\
\longrightarrow H^*(X) & \xrightarrow{\delta} & H^*(Tf^{p^{n+1}})) & \xrightarrow{i_o^*} & H^*(X) & \xrightarrow{I-f^{*p^{n+1}}} & H^*(X) & \xrightarrow{\delta}
\end{array}$$

Here $t = I + f^* + \cdots + f^{*(p-1)}$. For sufficiently big n, the kernel of $(I-f^{*p^n}) = (I-f^*)^{p^n}$ is independent of n. Then $\delta \circ t^{p^n} = 0$, and hence,

$$\mathrm{Im}(d^{p*}) \cong \mathrm{Im}(i_o^* d^{p*}) = \mathrm{Im}(i_o^*) = \mathrm{Ker}\ (I-f^*)^{p^n} = H^*(X; \mathbb{F}_p)^{(1)} .$$

It follows that

$$H^*(T(f^\infty); \mathbb{F}_p) \cong \varinjlim H^*(T(f^{p^n}); \mathbb{F}_p) \cong H^*(X; \mathbb{F}_p)^{(1)}$$

Theorem (1.10). <u>Assume that there exists a compact connected</u> G <u>space</u> X <u>without fixed points with an equivariant map</u> $f : X \to X$ <u>such that</u> $I - f^*$ <u>is an automorphism of</u> $H^+(X; \mathbb{F}_p)$ <u>and</u> $I - f'^*$ <u>is an automorphism of</u> $H^+(X/G; \mathbb{F}_p)$. <u>Then the</u> $\mathbb{F}_p$<u>-test space conjecture holds for</u> G.

Proof. Setting $Z = T(f^\infty)$ as defined above, we have $Z/G = T(f'^\infty)$. Lemma (1.9) implies that $H^*(Z; \mathbb{F}_p) \cong H^*(Z/G; \mathbb{F}_p) \cong \mathbb{F}_p$. Since X is without fixed points, Z cannot have fixed points. Hence Z satisfies the conditions of the $\mathbb{F}_p$-test space conjecture.

Remark (1.11). There is a version of (1.10) which permits a mapping torus of the form $T(f,g)$ to be used. Let X and Y be connected G spaces without fixed points and let $f,g : X \to Y$ be equivariant maps such that $g^* - f^*$ is an isomorphism $H^+(Y;\mathbb{k}) \to H^+(X;\mathbb{k})$ and such that $g'^* - f'^*$ is an isomorphism $H^+(Y/G;\mathbb{k}) \to H^+(X/G;\mathbb{k})$. Then if the $\mathbb{k}$-acyclicity conjecture holds for all proper subgroups of G, it also holds for G. We will only give a sketchy proof. Set $T = T(f,g)$; it then follows from (1.8) that
$j^* : H^*(S^1;\mathbb{k}) \to H^*(T;\mathbb{k})$ is an isomorphism, and that
$j'^* : H^*(S^1;\mathbb{k}) \to H^*(T/G;\mathbb{k})$ is an isomorphism. Letting C_j be the mapping cone of $j : T \to S^1$, it follows that $H^*(C_j;\mathbb{k})$ is acyclic, and hence by assumption that $H^*(C_j/K;\mathbb{k})$ is acyclic for all proper subgroups K of G. It follows that
$j^* : H^*(S^1;\mathbb{k}) \to H^*(T/K;\mathbb{k})$ is an isomorphism for all subgroups K of G. Let Z be an arbitrary G space with $H^*(Z;\mathbb{k})$ acyclic, let $z \in Z$, $t \in T$, and consider the diagrams.

$$Z/G_t \longrightarrow T \times_G Z \longrightarrow T/G$$

$$\begin{array}{ccccc} T/G_z & \longrightarrow & T \times_G Z & \longrightarrow & Z/G \\ {\scriptstyle j}\downarrow & & {\scriptstyle j\times 1}\downarrow & & \downarrow{\scriptstyle =} \\ S^1 & \longrightarrow & S^1 \times Z/G & \longrightarrow & Z/G \end{array}$$

There are isomorphisms,

$$\begin{aligned} H^*(S^1;\mathbb{k}) &\cong H^*(T/G;\mathbb{k}) \cong H^*(T \times_G Z;\mathbb{k}) \\ &\cong H^*(S^1 \times Z/G;\mathbb{k}), \end{aligned}$$

and hence $H^*(Z/G;\mathbb{k})$ is acyclic.

Assuming that the $\mathbb{k}$-acyclicity conjecture holds for all proper subgroups of G, the main question considered in this paper is how to compute $H^*(Z/G; \mathbb{k})$ when Z is a G space without fixed points with $H^*(Z; \mathbb{k})$ acyclic. In this situation there is a Vietoris-Begle isomorphism, for every G space X without fixed points,

$$pr_2^* : H^*(X/G; \mathbb{k}) \cong H^*(Z \times_G X; \mathbb{k}) .$$

Thus there is a homomorphism

$$(pr_2^*)^{-1} pr_1^* : H^*(Z/G; \mathbb{k}) \to H^*(Z \times_G X ; \mathbb{k}) \cong H^*(X/G; \mathbb{k}) .$$

and this homomorphism is natural for equivariant maps $f : X \to Y$ of G spaces without fixed points.

<u>Lemma (1.12).</u> <u>Assume that the $\mathbb{k}$-acyclicity conjecture holds for all proper subgroups of</u> G, <u>and let</u> X <u>be a connected</u> G <u>space without fixed points such that</u> $H^+(X; \mathbb{k})$ <u>has no constant</u> $A(X)$ <u>decomposition factor. Then there is an exact sequence</u>

$$0 \to H^*(Z/G; \mathbb{k}) \to H^*(X/G; \mathbb{k}) \to B^* \to 0$$

<u>where</u> B^* <u>has no constant</u> $A(X)$ <u>decomposition factor.</u>

<u>Proof.</u> This follows directly from lemma (1.5).

<u>Theorem (1.13).</u> <u>With the assumptions of lemma (1.12), let</u> Y <u>be a connected</u> G <u>space without fixed points such that</u> $H^+(Y; \mathbb{k})$ <u>has no constant</u> $A(Y)$ <u>decomposition factor. Let</u> $X * Y$ <u>be the join of</u> X <u>and</u> Y <u>with the natural inclusion</u> $i : X \subset X * Y$. <u>Then the image of</u>

$$i^* : H^*((X * Y)/G; \mathbb{k}) \to H^*(X/G; \mathbb{k})$$

<u>is isomorphic to</u> $H^*(Z/G; \mathbb{k})$ <u>where</u> Z <u>is a</u> G <u>space without fixed points and with</u> $H^*(Z; \mathbb{k})$ <u>acyclic. We assume that</u> X <u>and</u> Y <u>are compact.</u>

Proof. Because $H^+(X*Y;\mathbb{k}) \cong H^+(X;\mathbb{k}) \otimes H^+(Y;\mathbb{k})$, $H^+(X*Y;\mathbb{k})$ has no constant $A(Y)$ decomposition factor, where $A(Y) \subset A(X*Y)$. It follows from lemma (1.12) that there is a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^*(Z/G;\mathbb{k}) & \longrightarrow & H^*(X/G;\mathbb{k}) & \longrightarrow & B^* & \longrightarrow & 0 \\
 & & \uparrow{=} & & \uparrow{i^*} & & \uparrow{i^*} & & \\
0 & \longrightarrow & H^*(Z/G;\mathbb{k}) & \longrightarrow & H^*((X*Y)/G;\mathbb{k}) & \longrightarrow & C^* & \longrightarrow & 0 ,
\end{array}$$

where C^* has no constant $A(Y)$ decomposition factor. Hence $i^* : C^* \to B^*$ is trivial, and the proof is completed by diagram chasing.

Let X be the sphere of Floyd-Hsiang that has a G action without fixed points and admits an equivariant self map of degree zero. If the $\mathbb{k}$-acyclicity conjecture holds for all proper subgroups of G, then $H^*(X_\infty/G;\mathbb{k})$ is isomorphic to the image of

$$i^* : H^*((X*X)/G;\mathbb{k}) \to H^*(X/G;\mathbb{k}) .$$

Thus, the group $H^*(X_\infty/G\,;\mathbb{k})$ is independent of the choice of equivariant map of degree 0, and we need only compute the effect of i in cohomology, rather than the effect of the complicated self maps of degree zero. We hope that this observation will turn out to be of use.

2. Degrees of some equivariant maps of spheres without fixed points.

In this chapter, we will establish the following list of triples G, V, d, where G is a simple compact connected Lie group, V is a real linear representation space of G, and d is the degree of some equivariant self map of the unit sphere S_1V. Moreover, V will have no direct summand of dimension one, and the quotient space S_1V/G will be contractible. Wu-yi Hsiang has constructed a self-map (2.12) of degree 4 when $G = \mathbb{F}_4$, and $\dim V = 26$. Also, there is a construction in the thesis of Robert A. Oliver [9] which produces a finite cell complex Z which is a $\mathbb{F}_4$ space without fixed points such that $Z/\mathbb{F}_4$ is contractible and $H^*(Z;\mathbb{F}_2)$ is acyclic. Hence the $\mathbb{F}_2$-test space conjecture holds for $\mathbb{F}_4$, using either result.

The List (2.0).

G	V	d	condition, remark
$SO(2n+1)$	$S_0^2\,\mathbb{R}^{2n+1}$	$1-\binom{n}{k}$	$0 \leq k \leq n$
$SO(2n)$	$S_0^2\,\mathbb{R}^{2n}$	$1-\binom{n}{k}$	$1 \leq k \leq n-1$
$SO(n)$	$so(n)$	$1-2^k$	$n = 2k+1,\ 2k+2$
$SU(n)$	$su(n)$	$1-\binom{n}{k}$	$1 \leq k \leq n-1$
$Sp(n)$	$sp(n)$	$1-2^k\binom{n}{k}$	$1 \leq k \leq n$
$Sp(n)$	$\Lambda_0^2\,\mathbb{H}$	$1-\binom{n}{k}$	$1 \leq k \leq n-1$
$\mathbb{G}_2$	$\dim V = 7$	-1	(2.11)
$\mathbb{F}_4$	$\dim V = 26$	4	(2.12)

G	V	d	condition, remark
$\mathbb{E}_6$	e_6	$1-27$	$SO(10)SO(2) \subset \mathbb{E}_6$
$\mathbb{E}_7$	e_7	-1	$\dim e_7$ is odd
$\mathbb{E}_8$	e_8	$1-5\cdot16\cdot27$	$SO(14)SO(2) \subset \mathbb{E}_8$
G	$\mathfrak{g}$	$1-\lvert W\rvert$	W = Weyl group

We shall need the two theorems below to compute the degrees of equivariant maps of differentiable G manifolds. More complete results, and proofs, will appear elsewhere [10]. For the sake of simplicity, we will assume that M is a compact connected differentiable G manifold without boundary, and that M and the interior of M/G are orientable. Let $f : M \to M$ be a continuous equivariant map. Then in the orbit space, we obtain $f' : (M/G, \partial(M/G)) \to (M/G, \partial(M/G))$, and hence f' has a degree. Let G be connected, and let H be the principal isotropic group.

Theorem (2.1). The degrees of f and of the orbit map f' are equal if the adjoint action of the normalizer $N(H)$ of H preserves the orientation of H^o. If H is finite, this always holds.

Lemma (2.2). Let G be a torus or a finite p-group acting on a sphere M. Let $f : M \to M$ be an equivariant map, and let $F \subset M$ be the fixed point set. Then the degrees of f and of $f|F$ are equal if G is a torus, and they are equal mod p if G is a p-group. If there are no fixed points in M, the degree of f is 1 if G is a torus, and it is $\equiv 1 \pmod p$ if G is a p-group.

Let Ad_G be the adjoint representation of the group G on its Lie algebra g. Let $h \subset g$ be the Cartan subalgebra corresponding to a maximal torus T. The singular hyperplanes through the origin of h cut out the Weyl chambers in h and determine a simplicial structure on S_1h. Each Ad_G orbit in S_1g intersects each simplex of S_1h in exactly one point, and the Weyl group $W = N(T)/T$ acts simply transitively on the set of top dimensional simplexes of S_1h.

Theorem (2.3). The unit sphere S_1g of the adjoint representation of G admits an Ad_G equivariant self map of degree $1-|W|$.

Proof. Let σ be a top dimensional simplex of S_1h as described above. Given any map $f: \sigma \to S_1h$ with $C_x \subset C_{f(x)}$ for $x \in \sigma$ where C_x is the centralizer of x in G, it extends uniquely to an equivariant map $f: S_1g \to S_1g$. We let f be a homeomorphism

$$f: \sigma \to S_1h - \mathring{\sigma}$$

which is the identity on the boundary of σ. Then σ is mapped with degree -1 over $|W|-1$ simplexes. Since $W\sigma = S_1h$, it is clear that $\deg(f|S_1h) = 1-|W|$. Using lemma (2.2), we conclude that $\deg f = 1-|W|$ because $h = g^T$.

To construct more self maps of S_1g we will first consider W equivariant self maps of the unit sphere of h. We will consider the more general case of Coxeter groups. Let h be a Euclidean space with a set of hyperplanes $\{a_i\}$ and let W be the group generated by orthogonal reflections in the a_i, such that,

(1) W permutes the hyperplanes $\{a_i\}$,

(2) the hyperplanes a_i determine a triangulation of S_1h,

(3) each top dimensional simplex σ of the triangulation (2) intersects each W orbit in a single point,

(4) the isotropy group of $x \in h$, W_x, is generated by the reflections in the hyperplanes a_i which contain x.

(5) W_x acts simply transitively on the set of top dimensional simplexes containing x.

Of course, (1) - (5) are interdependent. Let α and β be opposite faces of σ, such that $\sigma = \alpha * \beta$, the join of α and β. We then define a map $f_{\alpha\beta}: \sigma \to S_1h$ by

$$f(x) = x \quad \text{for} \quad x \in \beta, \quad f(y) = -y \quad \text{for} \quad y \in \alpha,$$

and f maps the shortest geodesic form x to y onto the shortest geodesic form x to $-y$, with constant speed, when $x \in \beta$ and $y \in \alpha$.

Clearly, if $m \in \sigma$ lies on a hyperplane a_i, then so does $f_{\alpha\beta}(m)$. This is clear if $m \in \alpha$ or $m \in \beta$. If $m \in a_i - \alpha - \beta$, then we note that $a_i \cap \sigma$ is a face (of some codimension) of σ, and hence $a_i \cap \sigma = \alpha_i * \beta_i$ where α_i, β_i are faces of α, β. The geodesic lying in σ going from β_i to α_i containing m is contained in a_i, hence $f_{\alpha\beta}(m) \in a_i$. It follows from (4) that $W_m \subset W_{m'}$ when $m' = f_{\alpha\beta}(m)$, and by (3), $f_{\alpha\beta}$ extends uniquely to an equivariant map $f_{\alpha\beta}: S_1h \to S_1h$.

<u>Lemma (2.4).</u> <u>Let</u> $|f_{\alpha\beta}(\sigma)|$ <u>be the number of top dimensional simplexes in</u> $f_{\alpha\beta}(\sigma)$. <u>Then the degree of</u> $f_{\alpha\beta}: S_1h \to S_1h$ <u>is</u> $(-1)^{a+1}|f_{\alpha\beta}(\sigma)|$, <u>where</u> $a = \dim \alpha$.

Proof. This is clear except for the sign. Let L_β be the smallest linear subspace of h containing β, and let R_β denote the orthogonal reflection in L_β. It is easily seen that $\deg(R_\beta f_{\alpha\beta}) > 0$. The codimension of L_β in h is $a+1$, hence $(-1)^{a+1}\deg f_{\alpha\beta} > 0$.

In the sequel, we shall only use maps $f_{\alpha\beta}$ where α is a vertex, $\dim \alpha = 0$. We also must compute the degree in the orbit space S_1h/W of $f'_{\alpha\beta}$. We begin with the case of the symmetric group. Let $\mathbb{R}^n$ be Euclidean space with coordinates $(x_1,\dots,x_n)$ and set $h = \{x \mid x_1+\dots+x_n = 0\}$. Let $S_1h = S^{n-2} = \{x \in h \mid x_1^2+\dots+x_n^2 = 1\}$, and choose $a_{ij} = \{x \mid x_i = x_j\}$ as the set of hyperplanes. Reflection in a_{ij} generates the symmetric group $W = S_n$ and $\sigma = \{x \in S^{n-2} \mid x_1 \geq x_2 \geq \dots \geq x_n\}$ is a simplex of dimension $n-2$.

Lemma (2.5). The S_n space S^{n-2} just defined admits maps of the form $f_{\alpha\beta}$ of degrees

$$1 - \binom{n}{k}, \quad 1 \leq k \leq n-1 .$$

It admits maps of the form $f_{\alpha\beta}$ whose degrees in the orbit space are

$$1 - \binom{m}{k}, \quad 0 \leq k \leq m \quad \text{when} \quad n = 2m+1 \geq 3 ,$$

$$1 - \binom{m}{k}, \quad 1 \leq k \leq m-1 \quad \text{when} \quad n = 2m \geq 4 .$$

Proof. The action of S_n on h is given by

$$(wx)_i = x_{w^{-1}i}, \quad w \in S_n ,$$

when S_n is considered as a permutation group of $\{1,\dots,n\}$. The $n-2$ dimensional simplexes of S^{n-2} are of the form $w\sigma$ where

$$w\sigma = \{x \in S^{n-2} \mid x_{w1} \geq x_{w2} \geq \dots \geq x_{wn}\} .$$

We choose an orientation of σ, and hence obtain an orientation of S^{n-2}. We orient the orbitspace by requiring that the homeomorphism $\sigma \to S^{n-2}/S_n$ be orientation preserving. Then the degree of the homeomorphism $w\sigma \to S^{n-2}/S_n$ (given by $x \to S_n(x)$) equals the degree of $w: \sigma \to w\sigma$, and this degree is $\det(w) = \pm 1$.

Let v_k be the vertex of σ which lies on the hyperplanes a_{ii+1} for $i \neq k$, $1 \leq k \leq n-1$. Let $\sigma = v_k * \beta_k$ and set $f_k = f_{v_k \beta_k}$.
Let

$$L_k = \{x \in S^{n-2} \mid x_i \geq x_{i+1} \text{ for } i \neq k\};$$

then σ is a neighbourhood of v_k in L_k and

$$L_k = \sigma \cup f_k(\sigma).$$

A simplex $w^{-1}\sigma$ lies in L_k if and only if $wi < w(i+1)$ for $i \neq k$. This follows from the above sets of inequalities describing $w\sigma$ and L_K. A permutation with the property

$$l_k: \ wi < w(i+1) \quad \text{for } i \neq k,$$

is determined by the set $w\{1,\ldots,k\}$, hence there are $\binom{n}{k}$ such permutations. It follows from lemma (2.4) that $\deg f_k = 1 - \binom{n}{k}$, $1 \leq k \leq n-1$.
We define

$$N(k,n) = \Sigma\{\deg(\tau \to S^{n-2}/S_n) \mid \tau = w^{-1}\sigma \subset L_k\}.$$

It is then clear that

$$\deg(f_k': S^{n-2}/S_n \to S^{n-2}/S_n) = 1 - N(k,n).$$

Also, $N(k,n) = \Sigma\{\det(w) \mid w \in l_k\}$. This number can be computed by induction on k and n. Recalling that

$$\det(w) = (-1)^c, \quad c = \mathrm{card}\{(i,j) \mid i < j, wi > wj\},$$

we set $l_k = l'_k \cup l''_k$ where $w \in l'_k$ when $w(1) = 1$, and $w \in l''_k$ when $w(k+1) = 1$. Accordingly,

$$N(k,n) = N(k-1,n-1) + (-1)^k N(k,n-1) .$$

Hence, $N(k,n) = N(k-2,n-2) + (-1)^{k-1}N(k-1,n-2) + (-1)^k N(k-1,n-2) + N(k,n-2) = N(k-2,n-2) + N(k,n-2)$.

Using those recursion formulas for $N(k,n)$, we find,

$$N(2q+1,2m) = 0 \quad \text{for all } q, m, \text{ and,}$$

$$N(2q+1,2m+1) = N(2q,2m+1) = N(2q,2m) = \binom{m}{q} .$$

If $n = 2m+1$, it follows that $\deg f'_k = 1 - N(k,2m+1)$, $1 \leq k \leq 2m$, takes the values $1 - \binom{m}{q}$, $0 \leq q \leq m$. In case $n = 2m$, $\deg f'_k = 1 - N(k,2m)$, $1 \leq k \leq 2m-1$, takes the values 1 and $1 - \binom{m}{q}$, $1 \leq q \leq m-1$. This completes the proof.

Lemma (2.6). Let W be a Coxeter group generated by reflections in the hyperplanes a_i of a Euclidean space h, such that (1) - (5) holds. Then for each vertex v of the simplicial complex S_1h, there is a W equivariant map $f_{v\beta} : S_1h \to S_1h$ of degree $1 - [W : W_v]$.

Proof. Let $\sigma = v * \beta$ be a top dimensional simplex containing the vertex v. Set $L_\sigma = \sigma \cup f_{v\beta}(\sigma)$. For each top dimensional simplex t of S_1h, there is a unique top dimensional simplex τ containing v such that $t \subset L_\tau$. By (5), those simplexes τ are permuted simply transitively by W_v. It follows that the number of simplexes of top dimension contained in L_σ is $[W : W_v]$. Lemma (2.4) then implies that $\deg f_{v\beta} = 1 - [W : W_v]$.

Example. With the notation of lemma (2.5) and its proof, the isotropy group of v_k is $S_k \times S_{n-k} \subset S_n$. Hence (2.5) and (2.6) give the same result, $\deg f_k = 1 - [S_n : S_k \times S_{n-k}] = 1 - \binom{n}{k}$.

Theorem (2.7). Let G be a simple connected compact Lie group of rank ≥ 2. Let K be the centralizer of a circle subgroup of G such that the identity component of the centre of K is the circle subgroup. Then the unit sphere $S_1 g$ of the adjoint representation of G admits an equivariant self map of degree $1 - [WG : WK]$ where WG, WK are the Weyl groups of G, K.

Proof. Let $T \subset K \subset G$ be a maximal torus of K and of G, and let $h \subset k \subset g$ be the corresponding Lie algebras. The centre of k contains an edge of a Weyl chamber in h with respect to G. Set $W = WG$, and let $v \in S_1 h \cap \text{centre}(k)$. Then $W_v = WK$. Let $v * \beta$ be a top dimensional simplex of $S_1 h$, with respect to W. Then $\deg f_{v\beta} = 1 - [W : W_v]$ by lemma (2.6). Each G orbit in $S_1 g$ intersects $v * \beta = \sigma$ in a single point, and the isotropy group G_x of $x \in h$ is the unique closed connected subgroup of G containing T whose Weyl group is the subgroup $W_x \subset W$. Hence the relation $G_x \subset G_y$ holds for $x, y \in h$ if and only if $W_x \subset W_y$. It follows that $f_{v\beta}|\sigma$ extends uniquely to an equivariant map $f : S_1 g \to S_1 g$. By lemma (2.2), $\deg f = \deg f_{v\beta} = 1 - [W : W_v] = 1 - [WG : WK]$.

Examples (2.8). We will list a number of examples of a simple group G and a maximal rank subgroup K with centre of dimension one. Both G and K are compact connected Lie groups, and K is the centralizer of the central circle group.

$SO(14)SO(2) \subset \mathbb{E}_8$, $[WG : WK] = 5\cdot 16\cdot 27$,

$SO(10)SO(2) \subset \mathbb{E}_6$, $[WG : WK] = 27$,

$U(k)Sp(n-k) \subset Sp(n)$, $[WG : WK] = 2^k\binom{n}{k}$, $1 \leq k \leq n$,

$S(U(k)U(n-k)) \subset SU(n)$, $[WG : WK] = \binom{n}{k}$, $1 \leq k \leq n-1$,

$SO(n-2k)U(k) \subset SO(n)$, $[WG : WK] = 2^k\binom{m}{k}$, $1 \leq k \leq m-1$

when $n = 2m+1$, $1 \leq k \leq m-2$ when $n = 2m$.

$U(m) \subset SO(n)$, $[WG : WK] = 2^m$ when $n = 2m+1$,

$[WG : WK] = 2^{m-1}$ when $n = 2m$.

In addition to the adjoint representations, there are other representations admitting self maps based on the triangulation defined by certain Coxeter groups.

__Lemma (2.9).__ __Let__ $V = S_0^2\mathbb{R}^n$ __be the linear space of symmetric matrixes of trace__ 0 , __and let__ $g \in O(n)$ __act on this space by__ $g(v) = gv\,{}^tg$. __Then__ V __is an irreducible representation of__ $O(n)$, __and the unit sphere__ S_1V __admits equivariant self maps of degrees__

$$1 - \binom{m}{k} , \quad 0 \leq k \leq m \quad \text{when} \quad n = 2m+1 ,$$

__and of degrees__

$$1 - \binom{m}{k} , \quad 1 \leq k \leq m-1 \quad \text{when} \quad n = 2m \geq 4 .$$

__The orbit space__ $S_1V/SO(n)$ __is contractible__.

__Proof.__ Let $\mathbb{Z}_2^n \subset O(n)$ be the subgroup of diagonal matrixes. The fixed point set in V, $F(\mathbb{Z}_2^n,V)$ consists of all diagonal matrixes A in V, $A = \mathrm{diag}(x_1,\ldots,x_n)$. The isotropy group $O(n)_A$ consists of all $g \in O(n)$ such that $gA\,{}^tg = A$, that is, such that $gA = Ag$. If $x_i \neq x_j$ for $i \neq j$, it follows that $O(n)_A = \mathbb{Z}_2^n$.

If the different values of x_i occur $m_1,\ldots,m_s$ times, $m_1+\ldots+m_s = n$, $O(n)_A \cong O(m_1)\times\ldots\times O(m_s)$, the embedding in $O(n)$ being determined by the orthogonal splitting of $\mathbb{R}^n$ into eigenspaces of A of dimensions $m_1,\ldots,m_s$. Since every orbit contains a diagonal matrix, it follows that $\mathbb{Z}_2^n$ is the principal isotropy group of the action. The normalizer of $\mathbb{Z}_2^n$ is generated by the permutations of the x_i, and by $\mathbb{Z}_2^n$. Let $W = S_n$ be the group of permutations. It is clear that the action of S_n on $h = F(\mathbb{Z}_2^n,V)$ is the Coxeter group of (2.5). Each S_n orbit in h intersects the cone

$$C = \{A\in h \mid x_1\geq\ldots\geq x_n\}$$

in a single point. For $A\in h$, the isotropy group $O(n)_A$ determines the set of those hyperplanes

$$a_{ij} = \{H\in h \mid x_i = x_j\}$$

on which A lies, and conversely, this set determines the isotropy group $O(n)_A$. The same statement holds for the isotropy group $(S_n)_A$, hence $(S_n)_A$ determines $O(n)_A$, and $O(n)_A$ determines $(S_n)_A = O(n)_A\cap S_n$. For a matrix $B\in V$, the orbit $O(n)(B)$ intersects C in a single point obtained by arranging the eigenvalues of B in decreasing order, and hence, $C\cong V/O(n) = V/SO(n)$. It follows that $S_1V/SO(n)$ is contractible. Every equivariant map $f_o\colon S_1h\to S_1h$ with respect to S_n extends uniquely to an equivariant map $f\colon S_1V\to S_1V$. Since the principal isotropy group for the action of $SO(n)\subset O(n)$ is $\mathbb{Z}_2^{n-1}$ which is finite, theorem (2.1) implies that the degree of f equals the degree of $f'\colon S_1/SO(n)\to S_1V/SO(n)$ in the orbit space. To conclude the proof, we note that $S_1V/SO(n) = S_1h/S_n$, and use lemma (2.5).

Lemma (2.10). *Let* $V = \Lambda^2_0 \mathbb{H}^n$ *be a real irreducible summand of the natural representation of* $Sp(n)$ *in the space of skewsymmetric* $2n$ *by* $2n$ *complex matrixes. Then the unit sphere* S_1V *admits* $Sp(n)$ *equivariant self maps of degrees*

$$1 - \binom{n}{k} \ , \quad 1 \le k \le n-1 \ ,$$

and the orbit space $S_1V/Sp(n)$ *is contractible.*

Proof. When M_i are 2 by 2 complex matrixes, let $\mathrm{diag}(M_1,\ldots,M_n)$ be the $2n$ by $2n$ matrix constructed diagonally from those blocks. Set

$$j = \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \ , \text{ and } \ J = \mathrm{diag}(j,\ldots,j) \ ,$$

$$Sp(n,C) = \{g \in SL(2n,C) \mid gJ\,{}^tg = J\} \ ,$$

and set $Sp(n) = Sp(n,C) \cap U(2n)$.

Let A be a $2n$ by $2n$ matrix, and let $g \in Sp(n)$. Then, $\det(zI - gA\,{}^tg\,J) = \det(zI - gAJg^{-1}) = \det(zI - AJ)$, and hence the eigenvalues of AJ are invariant under the action $gA\,{}^tg$. It follows that $\mathrm{Tr}(AJ) = \mathrm{Tr}(gA\,{}^tg\,J)$. Now define $\varphi(A) = J\bar{A}J^{-1}$, and notice that $\varphi(AB) = \varphi(A)\varphi(B)$, $\varphi(\bar{A}) = \overline{\varphi(A)}$, and $\varphi({}^tA) = {}^t\varphi(A)$.

Define,

$$V = \{A \mid \varphi(A) = A \ , \ \ \mathrm{Tr}(AJ) = 0, \ A + {}^tA = 0\}$$

and, $V \supset h = \{\mathrm{diag}(x_1 j,\ldots,x_n j) \mid x_i \in \mathbb{R}, \ \Sigma x_i = 0\}$.

Then $Sp(1)^n \subset Sp(n)$ is acting trivially on h , and letting $\mathbb{Z}_4^n$ and T^n be subgroups such that T^n is a maximal torus of $Sp(1)^n$, and $J \in \mathbb{Z}_4^n \subset T^n \subset Sp(1)^n$, a direct computation shows that $h = F(\mathbb{Z}_4^n,V)$, and hence,

$$h = F(\mathbb{Z}_4^n,V) = F(T^n,V) = F(Sp(1)^n,V) \ .$$

Replacing the entries in a permutation n by n matrix according to the rule

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad 0 \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain an embedding $S_n \subset Sp(n)$, and S_n acts on h by permuting the x_i's. Thus h is isomorphic to the S_n space of lemma (2.5). The principal isotropy group of V is $Sp(1)^n$, according to [8], and hence every orbit intersects h. For a point $x \in h$ we have isotropy groups

$$(S_n)_x = S_{m_1} \times \dots \times S_{m_k}, Sp(n)_x = Sp(m_1) \times \dots \times Sp(m_k) .$$

Hence those groups determine each other, and every S_n equivariant map of h extends uniquely to a $Sp(n)$ equivariant map of V. The orbit space $S_1V/Sp(n) = S_1h/S_n$ is contractible. By lemma (2.5), there are equivariant self maps of degrees $1 - \binom{n}{k}$, $1 \leq k \leq n-1$, of S_1h, and since $h = F(T^n, V)$, it follows that the extension to S_1V have the same degrees by lemma (2.2).

<u>Lemma (2.11).</u> <u>Let V be an irreducible representation of G of odd dimension ≥ 3. Then the unit sphere S_1V admits an equivariand self map of degree -1. The group $\mathbb{G}_2$ and the simple groups of odd rank all admit irreducible odd dimensional representations such that the orbsit space of S_1V is contractible.</u>

<u>Proof.</u> For $\mathbb{G}_2$, we take the 7 dimensional representation whose unit sphere is $\mathbb{G}_2/SU(3)$, and for G of odd rank, we take $V = g$, the Lie algebra of G, with the adjoint representation.

<u>Theorem (2.12) (W.Y. Hsiang).</u> <u>The irreducible representation of $\mathbb{F}_4$ of dimension 26 admits an equivariant self map of degree 4 on</u>

<u>the unit sphere.</u> <u>The orbit space ot the unit sphere is contractible.</u>

<u>Proof</u>. The weights of this representation are the short roots of $\mathbb{F}_4$ and two 0-weights. Using Hsiang's theory of isotropy groups [8], we let $T \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9) \subset \mathbb{F}_4$, where T is a maximal torus. Since the roots of $\mathrm{Spin}(8)$ are the long roots of $\mathbb{F}_4$, none of those are weights, and hence $F(T,V) = F(\mathrm{Spin}(8),V)$ which is a 2 dimensional subspace, h say, of the representation space V of dimension 26. Restricted to $\mathrm{Spin}(9)$, the representation V splits as a sum of three representations, (i) a trivial one-dimensional representation, (ii) the usual representation of $\mathrm{Spin}(9)$ in $\mathbb{R}^9$, and (iii) the isotropy representation of $\mathbb{F}_4/\mathrm{Spin}(9)$. Hence, $\dim F(\mathrm{Spin}(9),V) = 1$. Let N be the normalizer of $\mathrm{Spin}(8)$ in $\mathbb{F}_4$, then $N/\mathrm{Spin}(8) \cong S_3$, the symmetric group of order 6. Conjugation by N gives two conjugates $\mathrm{Spin}(9)'$ and $\mathrm{Spin}(9)''$ of $\mathrm{Spin}(9)$. Then $F(\mathrm{Spin}(9),V) \cap F(\mathrm{Spin}(9)',V) = F(\mathbb{F}_4,V) = 0$, and it follows that setting

$$l^i = F(\mathrm{Spin}(9)^{(i)},V)\,,\quad i=0,1,2\,,\quad l^i \subset h\,,$$

the l^i are three lines in h permuted transitively by S_3. Thus the representation of S_3 in h is effective, and it follows that the isotropy groups of points $x \in h$ are

$$\mathbb{F}_4\,,\quad \mathrm{Spin}(8)\,,\quad \text{and}\quad \mathrm{Spin}(9)^{(i)}\,.$$

Let $\omega \in \mathbb{C}$ be a primitive 6^{th} root of unity, and choose an isometry $\mathbb{C} \cong h$ such that the isotropy group of $\pm\omega^i$ is $\mathrm{Spin}(9)^{(i)}$, $i = 0,1,2$. It follows from Hsiang [8] that the isotropy groups are all of maximal rank, hence every orbit in V intersects h. We then find that every orbit in S_1V intersects the arc

$$\sigma = \{\exp(\pi i t/3) \mid 0 \leq t \leq 1\} \subset \mathbb{C} \cong h$$

in a single point, and hence that

$$S_1V/\mathbb{F}_4 \cong \sigma$$

is contractible. Also, the map $f: \sigma \to S_1h$ given by

$$f(\exp(\pi i t/3)) = \exp(4\pi i t/3)$$

extends uniquely to an equivariant map $S_1V \to S_1V$. By lemma (2.2), this map has degree 4 because $h = F(T,V)$ and $f|S_1h$ has degree 4.

Remark. It follows that the extension of T by the 3-Sylow subgroup of the Weyl group of $\mathbb{F}_4$ has no fixed points in S_1V. Using lemma (2.2), this implies that every equivariant self map of S_1V has degree $\equiv 1 (\mathrm{mod}\ 3)$. In fact, looking closely at h, we see that those maps are all of degrees $6k+4$, $k \in \mathbb{Z}$.

Here is a proof of theorem 1 of the Introduction. According to theorem (1.10), if a group G has an irreducible linear representation V such that the unit sphere S_1V admits a self-map of degree $\neq 1 (\mathrm{mod}\ p)$, and such that the orbit space S_1V/G is contractible, then the $\mathbb{F}_p$-test space conjecture holds for G. Therefore, it suffices to show that, with the exceptions mentioned in theorem 1, such representations V are proveded by the list (2.0). The validity of the list itself is a consequence of (2.3), (2.7), (2.8), (2.9), (2.10), (2.11), and (2.12).

As examples of how to use the list, let us consider the cases $SO(n)$, $SU(n)$, and $\mathbb{E}_8$. For $SO(n)$, $n \geq 5$, there is a map of degree $1-2^k$ for some $k \geq 1$. As $1-2^k \neq 1 (\mathrm{mod}\ p)$ for all odd primes p, only the prime 2 causes a problem. The Floyd map of degree 0 in case $G = SO(2n+1)$ settles those groups for all primes. If $G = SO(2n)$, $n \geq 3$, there are maps of degrees $1-\binom{n}{k}$, $1 \leq k \leq n-1$, in the list (2.0). If $1-\binom{n}{k} \equiv 1 (\mathrm{mod}\ 2)$ for all those k, then n is a power of 2, and hence $G = SO(2^a)$. This leaves $G = SO(2^a)$ with the prime 2, and they appear in the list of exceptions of theorem 1.

For $G = SU(n)$, there are self-maps of degrees $1-\binom{n}{k}$, $1 \leq k \leq 1$. If $1-\binom{n}{k} \equiv 1 (\mathrm{mod}\ p)$ for $1 \leq k \leq n-1$, then $n = p^a$. The local isomorphisms $SU(2) \sim SO(3)$ and $SU(4) \sim SO(6)$ settles the problem for those two groups, with $p = 2$. The other pairs $(p, SU(p^a))$ appear in the list of exceptions of theorem 1.

For $G = \mathbb{E}_8$, we see that there is a self-map of degree $1-16 \cdot 27 \cdot 5$, and hence the primes 2, 3, and 5 remain unsettled in this case.

For a given prime p, the smallest simple groups (by the relation of local inclusion) in the list of exceptions of theorem 1 are, for $p = 2$, $G = \mathbb{G}_2$ and $Sp(4)$, and for $p \geq 3$, $G = SU(p)$.

Since $\mathbb{G}_2$ has no nonconstant complex representation of dimension < 7, and $Sp(4)$ has no such representation of dimension < 8, it follows that $SU(6)$ contains no simple group appearing in the list of exceptions of theorem 1, with $p = 2$. Thus the $\mathbb{F}_2$-test space conjecture holds for all simple subgroups of $SU(6)$. This proves theorem 2 of the Introduction.

Theorem 3 of the Introduction follows from theorem (1.7).

Further Questions.

1. Let $T = (\mathbb{Z}_p)^k$ act on X such that X/T is connected and let $f: X \to X$ be an equivariant map. Let $M(u) \in \mathbb{F}_p[u]$ be the polynomial of smallest degree such that $M(f^*)$ vanishes in $H^i(X; \mathbb{F}_p)$ for all $i > 0$. It then follows from theorem (1.3) that $M((f|F)^*)$ is nilpotent in $H^i(F; \mathbb{F}_p)$ for all $i > 0$ where F is the fixed point set. Is $M((f|F)^*)$ actually trivial for $i > 0$, or is there an example showing that it need not be trivial?

2. Conversely, let $M(u)$ be a polynomial such that $M((f|F)^*)$ is trivial in $H^+(F; \mathbb{F}_p)$. Can one conclude that in $H^+(X; \mathbb{F}_p)$, $\operatorname{Ker} M(f^*) = \operatorname{Ker} M(f^*)^2$?

3. Let p be a prime. Is it true that for each simple group G, there is an integer N such that for each $\mathbb{F}_p$-acyclic G space X, the cohomology group $H^*(X/G; \mathbb{F}_p)$ is generated by at most N elements lying in degrees $\leq N$? This is an approximation to the $\mathbb{F}_p$-acyclicity conjecture.

4. For each prime p, is there a compact connected $SU(p^a)$ space X without fixed points such that neither $H^+(X; \mathbb{F}_p)$ nor $H^+(X/SU(p^a); \mathbb{F}_p)$ has constant $A(X)$ decomposition factors, where $A(X)$ is the ring generated by the monoid of equivariant maps $X \to X$. - There are such spaces when $p^a = 2$ or 4.

5. Can the irreducible representations of $\mathbb{E}_6$ and $\mathbb{E}_7$ with principal isotropy group $Spin(8)$ provide contributions to the list (2.0)? See Hsiang [8] for the maximal weights of those representations.

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