Actions of p-tori on projective spaces

TOR SKJELBRED

A p-torus of rank k is a direct product $T = (\underline{Z}_p)^k$ of the group $\underline{Z}_p$ of order p where p is a prime. The fixed point set of an action of a p-torus on a projective space has been studied extensively by P.A.Smith, A.Borel, G.E.Bredon, W.Y. Hsiang and J.C.Su [2,3,5,6,7,11]. The results for $p=2$ and for p an odd prime are usually quite different. For p odd there is the following conjecture of Bredon. Let $\underline{F}_p$ denote the field of order p.

Conjecture. For an action of $\underline{Z}_p$ on the quaternionic projective space HP^n, there is at most one component of the fixed point set which has the $\underline{F}_p$ cohomology groups of some HP^m with $m > 0$.

Bredon has shown [3] that this conjecture holds for $n \geq p - 2$. Hsiang and Su [7] have shown that it holds for all n in case a p-torus T of rank at least two is acting effectively. In the first part of this paper, we will show that the conjecture does not hold for $T=\underline{Z}_p$ if one allows HP^n to be replaced by a space X with $\underline{F}_p$ cohomology ring isomorphic to that of HP^n. We give examples of this kind for $n \leq (p+1)/2$. In case $n=(p+1)/2$, we show that the Steenrod operation P^1 is non-trivial in $H^*(X;\underline{F}_p)$. We also give an upper bound for the sum of the dimensions of the components of the fixed point set which have the $\underline{F}_p$ cohomology of a quaternionic projective space. A consequence of the counterexamples is that the invariants of $\underline{F}_p$ cohomology will not suffice to prove Bredon's conjecture.

For actions on the complex projective space $\underline{C}P^n$, the $\underline{F}_p$ cohomology of the fixed point set is completely known, with theory and examples in perfect correspondence. However, relatively little is known about actions without fixed points. A.Borel [1] has shown that $(\underline{Z}_p)^2$ acts on $\underline{C}P^{np-1}$ without fixed points. We prove that when a p-torus T acts on $\underline{C}P^{m-1}$, then all maximal isotropy groups K have the same order and that the index [T:K] divides m . For p=2, we must assume that T acts trivially on $H^*(X;\underline{Z}/4\underline{Z})$. There are actions by projective transformations of $T = (\underline{Z}_p)^{2a}$ on $\underline{C}P^{p^a-1}$ with maximal isotropy groups $K=(\underline{Z}_p)^a$, hence [T:K] equals m in this case.

We will assume that the T actions we consider are such that the localization theorem for the equivariant cohomology ring $H^*_T(X;\underline{F}_p)$ is valid. This is the case if X is compact or if the $\underline{F}_p$ cohomology dimension of X is finite. We assume that X is paracompact, in any case.

1. Actions of p-tori on quaternionic projective space

Let H be the division ring of quaternions and let $G=Sp(1)$ be the group of quaternions of modulus one. Letting H^{n+m+2} be a right vectorspace over H, the linear transformations

$$g(q_o,\dots,q_{n+m+1})=(q_o,\dots,q_n,gq_{n+1},\dots,gq_{n+m+1}),g \in G$$

define an action of G on HP^{n+m+1} with fixed point set $F=F^1+F^2$ where $F^1=HP^n$ and $F^2= RP^m$. We shall modify this action outside a tubular neighbourhood of F^1. In the process of doing this we replace HP^{n+m+1} by a cell complex X which has even dimensional cells only and with $X\sim_p HP^{n+m+1}$ for sufficiently big primes p. Here $X\sim_p Y$ means that the cohomology rings $H^*(X;\underline{F}_p)$ and $H^*(Y;\underline{F}_p)$ are isomorphic. The fixed point set in X has two components F^1 and F^2 where $F^1 = HP^n$ and $F^2\sim_p HP^m$ for p sufficiently large. We notice that the normal bundle of $HP^n\subset HP^{n+m+1}$ is $m+1$ times the quaternionic Hopf bundle η whose unit sphere bundle is a principal G bundle. We will view η as an orthogonal bundle where G is a group of bundle automorphisms. Then G acts semifreely on the discbundle $D(\eta)$ with fixed point set $HP^n=F^1$.

For any space X where a topological group G is acting continuously, we set $X_G=(E_G\times X)/G$ where E_G is a universal free G space and G acts diagonally on $E_G\times X$. We set $H^*_G(X;A) = H^*(X_G;A)$. Given a fibre bundle η over X with structural group S, such that G acts as a group of bundle automorphisms of η covering the action on X, the resulting bundle η_G over X_G has the same fibre and structural group as η. In case $G=Sp(1)$ is acting trivially on HP^n and η is the quaternicric Hopf bundle, η_G is a four dimensional orthogonal vectorbundle over $(HP^n)_G = HP^n \times B_G$, where $B_G = E_G/G$. The Euler class of η is a generator $x \in H^4(HP^n;\underline{Z})$, and the Euler class of η_G is $x + u$ where $H^*(B_G;\underline{Z}) = \underline{Z}[u]$, deg $u=4$. The discbundle $N = D((m+1)\eta)$ is the equivariant normal discbundle of $HP^n \subset HP^{n+m+1}$, and the Euler class of $N_G \to (HP^n)_G$ is $(x+u)^{m+1}$. Since $x+u$ is not a zerodivisor in $H^*(HP^n \times B_G;\underline{Z}) = H^*(HP^n;\underline{Z})[u]$, the Gysin sequence of the bundle

$$(1.1) \quad S^{4m+3} \to (\partial N)_G \xrightarrow{\pi} (HP^n)_G$$

is short exact.

$$(1.2) \quad 0 \to H^*_G(HP^n;\underline{Z}) \xrightarrow{(x+u)^{m+1}} H^*_G(HP^n;\underline{Z}) \xrightarrow{\pi^*} H^*_G(\partial N;\underline{Z}) \to 0$$

Since G acts freely on ∂N, the projection $(\partial N)_G \to \partial N/G$ is a homotopy equivalence. Setting $Y = \partial N/G$, the map π defines a unique homotopy class of maps $\pi^1 : Y \to (HP^n)_G$ such that π^{1*} is surjective in cohomology. Dividing by the action of G in the bundle (1.1), we obtain a bundle

$$(1.3) \quad HP^m \to Y \xrightarrow{\pi^o} HP^n$$

For any fibre bundle $F_1 \to B_1 \to X_1$ where fibre and base are cell complexes, the total space B_1 has a natural cell decomposition with the cells corresponding to the cells of $X_1 \times F_1$, but with possibly non-cellular attaching maps. This cell decomposition can be used to construct maps from B_1. We thus have, by (1.3),

$$Y = (HP^m \underset{\neg}{\vee} S^4) \cup \text{ cells of dimension } 4k,\ 2 \leq k \leq n+m,$$

where $\pi^o(S^4) = HP^1 \subset HP^n = F^1$, and HP^m is a fibre of (1.3). Suppose now that we have a map $g: Y \to HP^m$. Then G acts semifreely on the space $X = N \cup_g HP^m$ obtained by attaching ∂N to HP^m by $\partial N \to \partial N/G = Y \to HP^m$, and the fixed point set is $HP^n + HP^m$. We will construct a map like g so as to obtain the right cohomology ring for X . Thus we will prove,

<u>Theorem (1.4)</u> For any two integers $n,m \geq 1$, there is a cell complex X with cells of even dimensions only and a semifree action of $Sp(1)$ on X such that the fixed point set has two components F^1 and F^2 with $F^1 = HP^n$ and F^2 is a subcomplex of X . For primes p with $(p+1)/2 \geq n + m + 1$, we have $X \sim_p HP^{n+m+1}$ and $F^2 \sim_p HP^m$. If $(p+1)/2 = n + m + 1$, then the Steenrod operation P^1 is nontrivial in $H^*(X;\underline{F}_p)$.

<u>Corollary (1.5)</u> For any two integero $n,m \geq 1$, there is a cell complex X with cells of even dimensions only and a semifree circle action on X such that the fixed point set has two components F^1 and F^2 with $F^1 = HP^n$, and, writing $X \sim_o Y$ when the rational cohomology rings $H^*(X;\underline{Q})$ and $H^*(Y;\underline{Q})$ are isomorphic, $F^2 \sim_o HP^m$, and $X \sim_o HP^{n+m+1}$.

Proof: To construct g_o, we begin with $g_o : HP^m \vee S^4 \to HP^m$ which is the identity on HP^m and maps S^4 such that there is a generator $y \in H^4(HP^m;\underline{Z})$ with $g_o^*(y) = \pi^{1*}(x+u) \mid (HP^m \vee S^4)$

<u>Lemma (1.6)</u> The obstructions to extending g_o to Y lie in finite groups without p - primary components for primes p with $(p+1)/2 \geq n + m + 1$.

Proof: Those obstructions lie in the groups $\pi_{4k-1}(HP^m)$ for $2 \leq k \leq n + m$. There is the exact homotopy sequence

$0 \to \pi_i(S^{4m+3}) \to \pi_i(HP^m) \to \pi_{i-1}(S^3) \to 0$

For q odd and p an odd prime, $\pi_i(S^q)$ has trivial p-primary component for $i \leq q + 2p-4$, [10,p.517] , hence we find that $\pi_i(HP^m)$ has trivial p-primary component for $i \leq 2p$. Thus for $4(n+m) - 1 \leq 2p$, i.e. $n+m+1 \leq (p+1)/2$, the obstructions to extending g_o lie in groups with trivial p-primary component. The obstruction group $\pi_{4m+3}(HP^m)$ contains an infinite cyclic group, but the obstruction elements lie in the torsion subgroup. To see this, we may pass to the rational homotopy category, letting $f_{\underline{Q}} : X_{\underline{Q}} \to Y_{\underline{Q}}$ be the rational form of a map $f = X \to Y$ of simply connected cell complexes [12] . When $Y^{(k)}$ is the k-skeleton of Y , there is a unique extension of g_o, $g_{\underline{Q}} : (Y^{(4m)})_{\underline{Q}} \to (HP^m)_{\underline{Q}}$. An homotopy element $\iota = S^{4m+3} \to HP^m$ is torsion if and onlv y^{m+1} vanishes in $H^*(HP^m \cup_\iota e^{4m+4};\underline{Q})$. Hence, the obstruction to extending g_Q to $Y^{(4m+4)}$ is trivial if and only if $g_{\underline{Q}}^*(y)^{m+1}=0$ in $H^*(Y_Q)$. Here $g_{\underline{Q}}^*(y)$ is a class in $H^4(Y_Q)$. Restricted to $H^4(HP^m \vee S^4;\underline{Q})$, we have $g_{\underline{Q}}^*(y) = g_o^*(y)_{\underline{Q}} = \pi^{1*}(x+u)_{\underline{Q}}$, by definition of g_o.
When the homotopy
equivalence $(\partial N)_G \to Y$ is understood, we have $g_{\underline{Q}}^*(y)^{m+1} = \pi^{1*}(x+u)_{\underline{Q}}^{m+1} = \pi^*((x+u)^{m+1})_{\underline{Q}} = 0$, by (1.2) . This completes the proof of Lemma (1.6) .

Let $HP^m = K_o \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_3 \longrightarrow \cdots$ be the sequence of Lemma (1.8). The mapping telescope of this sequence is the space HP^m with all primes q with $(q+1)/2 < n+m+1$ made invertible. Replacing g_o by the composite map $g_i = fg_o$, $f=f_i \cdots f_2 f_1$, $g_i : HP^m \vee S^4 \to HP^m \to K_i$, it is clear that the obstruction to extending g_o will be mapped to zero in $\pi_*(K_i)$ for big i .

Hence g_i extends to $g: Y \to K_i$ for some i. We set $F^2 = K_i$, $g : Y \to F^2$, and we define $X = N \cup_g F^2$. Then G acts semifreely on X with fixed point set $F = F^1 + F^2$ where $F^1 = HP^n$ and $F^2 \sim_p HP^m$ for primes p with $n+m+1 \leq (p+1)/2$. As a cell complex, N is obtained from ∂N by attaching a cell of dimension $4k$ for $m+1 \leq k \leq n+m+1$ and F^2 has a cell of dimension $4k$ for $0 \leq k \leq m$. Thus, X has a cell of dimension $4k$ for $0 \leq k \leq n+m+1$ and those are all the cells of X. There is an exact sequence.

$$(1.7) \quad 0 \to H^{4q}(N, \partial N; \underline{Z}) \xrightarrow{\alpha^*} H^{4q}(X;\underline{Z}) \xrightarrow{\varphi_2^*} H^{4q}(F^2;\underline{Z}) \to 0,$$

where $\varphi_i : F^i \subset X$ and $\alpha : X \to X/F^2 = N/\partial N$. Let $z \in H^4(X;\underline{Z})$ be a generator. We must show that z^a is a generator of $H^{4a}(X;\underline{F}_p)$ for all primes p with $(p+1)/2 \geq n+m+1$ and all a. Since φ_2^* is an isomorphism for $q \leq m$, z^a is a generator for $a \leq m$. To show that z^{m+1} is a generator, we will consider the skeleton $X^{(4m+4)} = F^2 \cup e^{4m+4}$, by construction. The cell of dimension $4m+4$ is attached by the map

$S^{4m+3} \subset \partial N \to \partial N/G \approx Y \xrightarrow{g} F^2$, where S^{4m+3} is the fibre of $\partial N \to F^1 = HP^n$. This map fits into a commutativediagram

$$\begin{array}{ccccc} \partial N & \longrightarrow & Y & & \\ \cup & & \cup & \searrow^{g} & \\ S^{4m+3} & \longrightarrow & HP^m & \xrightarrow{f} & F^2 . \end{array}$$

Thus f defines a map of mapping cones, $f' : HP^{m+1} \to X^{(4m+4)}$. Since $f^* : H^4(F^2; \underline{F}_p) \to H^4(HP^m;\underline{F}_p)$ is an isomorphism, it follows that $f'^* : H^*(X^{(4m+4)};\underline{F}_p) \to H^*(HP^{m+1};\underline{F}_p)$ is an isomorphism, hence z^{m+1} is a genarator. Since $H^{4m+4}(N,\partial N;\underline{Z})$

is generated by the Thom class, it is not difficult to see, using (1.7), that z generates the algebra $H^*(X;\underline{F}_p)$ if $\varphi_1^* : H^4(X;\underline{F}_p) \to H^4(F^1;\underline{F}_p)$ is nontrivial. We have seen that the Euler class of the bundle $N_G \to F^1{}_G$ is $(x+u)^{m+1}$. Clearly this Euler class is $\varphi_1^*\alpha^*$ (U) where U is the Thom class in $H_G^{4m+4}(N,\partial N;\underline{Z})$. The ring $H_G^*(X;\underline{F}_p)$ is generated in degrees $\leq 4m+4$ by u and w where w is a lift of z over $H_G^4(X;\underline{Z}) \to H^4(X;\underline{Z})$. Hence there is a homogeneous polynomial P with coefficients in $\underline{F}_p$ such that $\alpha^*(U) = P(w,u)$. Setting $\varphi_1^*(w) = a_1x+a_2u \in H_G^4(F^1;\underline{Z})$, we obtain,

$$(x+u)^{m+1} = \varphi_1^*\alpha^*(U) = \varphi_1^*P(w,u) = P(a_1x+a_2u,u) ,$$

in $H_G^*(F^1;\underline{F}_p)$. Since $m+1 < p$, it follows that $a_1 \neq 0$ (mod p) , hence the composite map $H_G^4(X;\underline{F}_p) \to H^4(X;\underline{F}_p) \to H^4(F^1;\underline{F}_p)$ nontrivial.

Lemma (1.8). Lett K be a cell complex without cells of dimension one and let S be a set of primes. Then the localized homotopy type $S^{-1}K$ can be obtained as the mapping telescope (or homotopy direct limit) of a sequence

$$K = K_0 \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \to \dots.$$

where each K_i has the same number of cells in each dimension as K, and each f_i induces an isomorphism of cellular chain groups $S^{-1}C_*(K_{i-1}) \cong S^{-1}C_*(K_i)$.

Proof: This is an immediate consequence of D . Sullivan's construction of $S^{-1}K$ by attaching cones on S-local spheres [12].

It follows from Theorem (1.9) below that the condition $n+m+1 \leq (p+1)/2$ cannot be relaxed. In fact for every odd prime q with $(q+1)/2 < n+m+1$, the ring $H^*(X;\underline{F}_q)$ is not generated by one element when X is the space of Theorem (1.4). In case $(p+1)/2 = n+m+1$, if the Steenrod operation P^1 were trivial in $H^*(X;\underline{F}_p)$, it follows from Theorem (1.9) that $n+m \leq (p-3)/2$. This contradiction shows that P^1 is nontrivial for this p.

Theorem (1.9). Assume that $T = \underline{Z}_p$ acts on a space $X \sim_p HP^k$ where p is an odd prime and let $F^1, \ldots, F^s$ be the components of the fixed point set. Assume that $F^i \sim_p HP^{k_i}$ with $k_i > 0$ for $i \leq q$. If $q \geq 2$, then

$$k_1 + k_2 + \ldots + k_q \leq (p-1)/2 .$$

If the Stenrod operation P^1 is trivial in $H^*(X;\underline{F}_p)$, then $k_1 + k_2 + \ldots + k_q \leq (p-3)/2$.

We will summarize Hsiang's result [5] on the structure of the equivariant cohomology ring $H^*_T(X;\underline{F}_p)$ when a p-torus T acts on $X \sim_p HP^k$. As an algebra over $H^*(B_T;\underline{F}_p)$, we have

$$H^*_T(X;\underline{F}_p) = H^*(B_T;\underline{F}_p)[Y]/(H(Y))$$

where $Y \to y \in H^4_T(X;\underline{F}_p)$ and the restriction of y in $H^4(X;\underline{F}_p)$ is nontrivial. The polynomial $H(Y)$ has the form $H(Y) = \prod_{i=1}^{s} (Y - A_i)^{k_i+1}$ where $k_i + 1 = \dim H^*(F^i;\underline{F}_p)$ and $A_i = p_i^*(y)$ where p_i^* is the homomorphism $H^*_T(X) \to H^*_T(p_i) = H^*(B_T)$ defined by a point $p_i \in F^i$. Setting $\varphi_i : F^i \subset X$, we have $\operatorname{Ker} \varphi_i^* = (y-A_i)^{k_i+1} \cdot H^*_T(X;\underline{F}_p)$. When $F^i \sim_p HP^{k_i}$ with $k_i > 0$, we have $\varphi_i^*(y) = y_i + A_i$ where y_i is a generator of $H^4(F^i;\underline{F}_p)$. The differences $A_i - A_j \neq 0$ lie in the ring generated by $\beta H^1(B_T;\underline{F}_p)$ where β is the Bockstein operator.

Proof of Theorem (1.9): In case $T = \underline{Z}_p$, we have $H^*(B_T;\underline{F}_p) = \underline{F}_p[t]\langle u\rangle$ where $\deg u = 1$, $t = \beta(u)$, and $u^2 = 0$. Setting $r = (p+1)/2$, there is a polynonvial $f(Y) = C_oY^r + C_1Y^{r-1}+\dots+ C_r$ with $C_i \in H^*(B_T;\underline{F}_p)$ and $P^1(y) = f(y)$. We have $A_i = a_it^2$ and $C_i = c_it^{2i}$ where $a_i, c_i \in \underline{F}_p$. Hence P^1 is trivial in $H^*(X;\underline{F}_p)$ if and only if $C_o = 0$ if and only if $\deg f(Y) < r = (p+1)/2$. As $\varphi_i^*(y) = y_i+a_it^2$ for $i \leq q$, we get

$$\varphi_i^*(f(y)-2a_it^{p+1}) = \varphi_i^*(P^1(y) - P^1(a_it^2)) = P^1(y_i) = b_i\ y_i^r$$

for some $b_i \in \underline{F}_p$. It follows that the polynominal $f(Y) - 2a_it^{p+1}$ is divisible by $(Y - a_it^2)^{r_i}$ where $r_i=\min(r,k_i+1)$; hence the derivative $f'(Y)$ is divisible by $(Y-a_it^2)^{r_i-1}$, and $\sum_{i=1}^{q}(r_i-1) \leq \deg f(Y) - 1 \leq r - 1$. Since $r_i \geq \min(2,2)=2$ we must have $r_i - 1 < r - 1$ when $q \geq 2$, and so, $r_i=k_i+1$. This gives the inequality $\sum_{i=1}^{q} k_i \leq \deg f(Y)-1 \leq (p-1)/2$, completing the proof.

Proposition (1.10) Let $T = \underline{Z}_p$ act on a space $X\sim_p HP^k$ such that the fixed pointset has more than $(p+1)/2$ components. Then the fixed point set consists of $k+1$ acyclic components, $k < p$, and the Steenrod operation P^1 is trivial in $H^*(X;\underline{F}_p)$. For each odd prime p and each $k < p$, there is a space $X\sim_p HP^k$ with an action of $\underline{Z}_p$ with $k+1$ isolated fixed points.

Proof: We keep the notation from the proof of the above theorem, setting $F(Y) = f(Y) - 2Yt^{p-1}$. Thus we have $P^1(y)=F(y)+2yt^{p-1}$.

There is a relation of Steenrod operations [4], $2P^2 = (P^1)^2$, hence we obtain

$$2P^2(y)=P^1(F(y)+2yt^{p-1})=P^1(y)F'(y)+P^1(t)\partial F(y)/\partial t+P^1(y)2t^{p-1}-2yt^{2p-2},$$

keeping in mind that P^1 is a derivation. Since $F(Y)$ is a weighted homogeneous polynomial in t and Y, we have

$t\partial F(Y)/\partial t = F(Y)-2F'(Y)$.

This gives,

$$2y^p=2P^2(y)=F(y)(F'(y)+3t^{p-1})+2yt^{2p-2} ,$$

$$(1.11) \quad 2y(y^{p-1}-t^{2p-2})=F(y)(F'(y)+3t^{p-1})$$

Asimilar equation is used by Bredon [3].

For each component F^i of the fixed point set, $F(a_i t^2)=p_i^* F(y)= p_i^*(P^1(y)-2yt^{p-1})=P^1(a_i t^2)-2a_i t^{p+1}=0$. This shows that the $a_i t^2$, $1 \leq i \leq s$, are roots of $F(Y)$.

If there are more than $(p+1)/2$ components, the number of roots exceeds the degree of $F(Y)$, hence $F(Y)=0$. From (1.11) it follows that $2y(y^{p-1}-t^{2p-2})=0$, hence $H(Y)$ must divide $Y(Y^{p-1}-t^{2p-2})=\Pi(Y-ct^2),(c \in \underline{F}_p)$. Since $H(Y)$ is square-free, we have $k_i+1 = \dim H^*(F^i;\underline{F}_p)=1$ for all i , meaning that each F^i is $\underline{F}_p$ acyclic. Further, $P^1(y)=F(y)+2yt^{p-1}=2yt^{p-1}$ implies that P^1 is trivial in $H^*(X;\underline{F}_p)$. It also follows that $k+1=\deg H(Y) \leq p$.

To construct examples, let X be the $4k$ skeleton of the loopspace ΩS^5 in the unique cell structure with even dimensional cells only. Then, $X^k \sim_p HP^k$ for $k < p$ as $H^*(\Omega S^5;\underline{Z})$ is a ring of divided powers of one variable [10,p.514] .

We now use James' construction [8] of $\Omega\Sigma A$ for a pointed space A, where ΣA is the reduced suspension of A. There are closed subspaces $X^0 \subset X^1 \subset \ldots$ with $\cup X^k = \Omega\Sigma A$ and $X^k/X^{k-1} = A \wedge \ldots \wedge A$ (k fold smash product). This construction is functorial for maps leaving the base point fixed. We take $A = S^4$ with a $\underline{Z}_p$ action with two fixed points. Then $\underline{Z}_p$ acts on $X^k/X^{k-1} = S^{4k}$ with two fixed points. Hence $\underline{Z}_p$ has k+1 isolated fixed points in X^k.

2. Actions of p-tori on projective spaces without fixed points

Let $p \geq 2$ be a prime and let T be a p-torus. We want to consider actions of T on a space X such that $H^*(X;\underline{F}_p)$ is a ring generated by one element, and the fixed point set is empty. For example, there is a principal S^1 bundle $\underline{R}P^{2n-1} \to \underline{C}P^{n-1}$, showing that $\underline{Z}_2$ acts on the real projective space $\underline{R}P^{2n-1}$ without fixed points.

Lemma (2.1) When p is odd, every action of $\underline{Z}_p$ on X has fixed points if $H^*(X;\underline{F}_p)$ is generated by one element of even degree. When $p=2$, every action of $\underline{Z}_2$ on X has a fixed point if $H^*(X;\underline{F}_2)$ is generated by an element of even degree, and $\underline{Z}_2$ acts trivially on $H^*(X;\underline{Z}/4\underline{Z})$.

We remark that since the mod 2 Steenrod algebra is generated by P^{2^i}, $i \geq 0$, it follows that if $H^*(X;\underline{F}_2)$ is generated by one element, and $\dim H^*(X;\underline{F}_2) \geq 3$, then the generator has degree 2^i, $i \geq 0$. The next lemma follows from Bredon [3].

Lemma (2.2) Under the conditions of Lemma (2.1), each component F^i of the fixed point set is either acyclic over $\underline{F}_p$ or $H^*(F^i;\underline{F}_p)$ is generated by an element of even degree.

Proof of Lemma (2.1): Consider the spectral sequences $E(2)$ and $E(1)$ of the bundle $X \to X_{\underline{Z}_p} \to B_{\underline{Z}_p}$ with coefficients in $\underline{Z}/p^2\underline{Z}$ and in $\underline{Z}/p\underline{Z}=\underline{F}_p$. For each prime $p \geq 2$, we have that $E_2^{o*}(2) \to E_2^{o*}(1)$ is onto and $H^{odd}(B_{\underline{Z}_p};\ \underline{Z}/p^2Z) \to H^{odd}(B_{\underline{Z}_p};\underline{F}_p)$ is trivial where both maps are reduction mod p. It follows that the spectral sequence $E(1)$ has trivial differentials, because the generator of $H^*(X;\underline{F}_p)$ is of even degree, hence the fixed point set is not empty.

Lemma (2.3) A 2-torus T of rank ≥ 3 cannot act freely on a space $X \sim_2 RP^n$. If $\underline{Z}_2 \times \underline{Z}_2$ acts freely on X, then the action on $H^n(X;\underline{Z}/4\underline{Z})$ is trivial and $n+1=0 (\mathrm{mod}\ 4)$.

Proof: Assume that $T=(\underline{Z}_2)^3$ acts freely on X. Let (E_r, d_r) be the spectral sequence for $\underline{F}_2$ cohomology of the bundle $X \to X_T \xrightarrow{\pi} B_T$. Let $0 \neq x \in H^1(X;\underline{F}_2)$, and notice that the differentials $d_r(x^{r-1})$, when defined, are elements of the ring $H^*(B_T;\underline{F}_2)=\underline{F}_2[t_1\ t_2,t_3]$ modulo the images of d_s for $s < r$. A subgroup $K \subset T$ has fixed points in X if and only if $d_2(x)|B_K=0$, in other words if and only if $d_2(x) \in PK$ where PK is the prime ideal which is the kernel of $H^*(B_T) \to H^*(B_K)$. We have

$$d_2(x) = a_1t_1^2+a_2t_2^2+a_3t_3^2+b_1t_2t_3+b_2t_3t_1+b_3t_1t_2$$

where $a_i, b_i \in \underline{F}_2$. When $K \subset T$ is the subgroup with $PK=(t_1,t_2)$, we have $d_2(x)=a_3t_3^2$ mod PK, hence $a_i=1$ when the action is free, $i = 1,2,3$. Choosing K with $PK=(t_1+t_2,t_3)$, we have $d_2(x)=b_3t_1^2$ mod PK, hence $b_i=1$ for $i=1,2,3$. Choosing K with $PK = (t_1+t_2,t_2+t_3)$, we obtain $d_2(x)=0$ mod PK, hence the action cannot be free.

Assume that $T=\underline{Z}_2 \times \underline{Z}_2$ acts freely. Then n must be odd, and we will show that $n \neq 4m+1$. Assuming $n=4m+1$, we will calculate explicitly the spectral sequence E_r. We set $R=H^*(B_T;\underline{F}_2)=\underline{F}_2[t_1,t_2]$ and notice that E_r is a bigraded R algebra with 1. We must have $d_2(x)=t_1^2+t_1t_2+t_2^2$ because every other second degree form contains a linear form defining a subgroup with fixed points. Let β be the Bockstein operator and notice that

$\beta\ \pi^*(d_2(x))=\pi^*(\beta(t_1^2+t_1t_2+t_2^2))=\pi^*(t_1t_2(t_1+t_2))=0.$

It follows that $d_3(x^2) = t_1t_2(t_1+t_2) \pmod{d_2(x)}$. Clearly, $E_3^{*q} = 0$ for q odd and $E_3^{*2q} \cong R/(t_1^2+t_1t_2+t_2^2)$ for $0 \leq q \leq 2m$. E_3 is generated as an R algebra by $x^2=y$, and $d_3(y^{2m+1}) = y^{2m}d_3(y) = y^{2m}t_1t_2(t_1+t_2) \neq 0$. This is impossible as $y^{2m+1} = 0$. Thus we must have $n+1=0 \pmod 4$, say $n=4m-1$, when T is acting freely.

If T acts nontrivially on $H^n(X;\underline{Z}/4\underline{Z})$, there is a subgroup K of order 2 which also acts nontrivially. Then, dim $H_K^*(X;\underline{F}_2)=2m$, and a direct computation of the spectral requence converging to $H_K^*(X;\underline{Z})$ shows that $H_K^q(X;\underline{Z}) \neq 0$ for infinitely many q, this is a contradiction. (We will not make this ad hoc argument explicit because more general arguments are available, using the G-Euler characteristic for free actions. If we cannot use the universal coefficient theorem for $H_K^*(X;\underline{Z})$, then it should be replaced by $\lim_a H_K^*(X;\underline{Z}/2^a\underline{Z})$, cfr. [9])

<u>Lemma (2.4)</u> Let K be a 2-torus acting om a space X such that $H^*(X;\underline{F}_2)$ is generated by an element of degree $n > 0$, and assume that the fixed point set $F(K)$ is nonempty with components F^i, $1 \leq i \leq s$. Set $d_i = \dim H^*(F^i;\underline{F}_2)$ and let $n_i \geq 0$ be the degree of the generator of $H^*(F^i;\underline{F}_2)$. Then

$$|K| \geq \sum_{d_i \geq 3} n/n_i + \#\{i \mid d_i = 2\}$$

Proof. If $d = \dim H^*(X;\underline{F}_2) \leq 2$, the inequality is easily verified, so we will assume $d \geq 3$. When the group $\underline{Z}_2$ acts on X, Theorem (4.1) of Bredon [3] says that either $F(\underline{Z}_2, X)$ is connected with a cohomology generator of degree $n/2$, or $F(\underline{Z}_2, X)$ has two components F_1 and F_2 such that $H^*(X;\underline{F}_2) \to H^*(F_i;\underline{F}_2)$ is surjective for $i=1,2$. We will prove the Lemma by induction on rank K. When $K=\underline{Z}_2$, the inequality follows directly from Bredon's result. If rank $K \geq 2$, choose a subgroup $\underline{Z}_2 \subset K$ and consider the fixed point set $F(\underline{Z}_2)$. If it has two components F_1 and F_2, we may assume $F^i \subset F_1$ for $i \leq a$ and $F^i \subset F_2$ for $i > a$. By induction, we have inequalities for the actions of $K/\underline{Z}_2$ on F_1 and F_2,

$$|K/\underline{Z}_2| \geq \sum_{i \leq a, d_i \geq 3} n/n_i + \# \{i \mid d_i = 2, i \leq a\}$$

$$|K/\underline{Z}_2| \geq \sum_{i > a, d_i \geq 3} n/n_i + \# \{i \mid d_i = 2, i > a\}$$

Those inequalities add up to the inequality of the Lemma. If $F(\underline{Z}_2)$ is connected, the ineqality for the action of $K/\underline{Z}_2$ on $F(\underline{Z}_2)$ is

$$|K/\underline{Z}_2| \geq \sum_{d_i \geq 3} n/2n_i + \# \{i \mid d_i = 2\},$$

and the Lemma follows.

<u>Remark</u> If $d_i \geq 3$ for all i and the action is cohomology effective, that is, $H^*(X, F(\underline{Z}_2);\underline{F}_2) \neq 0$ for all $\underline{Z}_2 \subset K$, then the inequality (2.4) is an equation.

<u>Theorem (2.5)</u> Let T be a p-torus acting on a space X such that the ring $H^*(X;\underline{F}_p)$ is generated by one element of even

degree and such that T has no fixed points in X. Then for each maximal isotropy group K, the index $[T:K]$ divides dim $H^*(X;\underline{F}_p)$ and for p odd, $|K|^2 \geq |T|$, whereas $2|K|^2 \geq |T|$ for $p=2$. When p is odd, all maximal isotropy groups have the same order, and if K and L are subgroups with $F(K) \neq \emptyset$ and $F(L) \neq \emptyset$, and $|K|>|L|$, there is a subgroup $A \subset K$ with $|AL|=|K|$ and $F(AL) \neq \emptyset$.

Proof. Assume that p is odd and let K be a maximal isotropy group for the action of T. Let $F(K)^i$ be a component of $F(K)$. The group which keeps $F(K)^i$ invariant is equal to K, for if $K \subset L$ and $L/K = \underline{Z}_p$ and $F(K)^i$ is invariant under L, then $F(L, F(K)^i) \neq \emptyset$ by Lemma (2.1). Since K is a maximal isotropy group, this is a contradiction. It follows that T/K permutes the components of $F(K)$ freely. Since $F(K)$ has at most $|K|$ components, we obtain $|T/K| \leq |K|$ or $|K|^2 \geq |T|$. Let Y be a union of one component of $F(K)$ from each T/K orbit of components. Then, dim $H^*(X;\underline{F}_p)$=dim $H^*(F(K);\underline{F}_p)$= dim $H^*((T/K)Y;\underline{F}_p)=[T:K]$. dim $H^*(Y;\underline{F}_p)$. Let K and L be subgroups with $F(K) \neq \emptyset, F(L) \neq \emptyset$, and $|K|>|L|$. Both K and L must have fixed points in each component Y of $F(K \cap L)$, hence Y is invariant under KL. Thus the group $N=KL/K \cap L= (K/K \cap L) \times (L/K \cap L)$ is a transformationgroup on Y. Since $|L/K \cap L|^2 < |N|$, $L/K \cap L$ is not a maximal isotropy group for the action of N on Y. Hence there is an element $a \in K-L$ such that the group $\langle a \rangle L$ has fixed points in Y, and, consequently, $F(\langle a \rangle L, X) \neq \emptyset$ Replacing L by $\langle a \rangle L$ and repeating the above argument if $|\langle a \rangle L| < |K|$, we find a subgroup $A \subset K$ with $|AL|=|K|$ and $F(AL,X) \neq \emptyset$. If $|L|<|K|$, it follows that L and K cannot

both be maximal isotropy groups, that is, $|L|=|K|$ for maximal isotropy groups L and K.
When $p=2$, and $K \subset T$ is a maximal isotropy group, let $F(K)^i$ be a component of $F(K)$ and let $L \supset K$ be the stabilizing subgroup of $F(K)^i$; then L/K acts freely on $F(K)^i$, and $|L/K|=1,2$, or 4 by Lemmas (2.1) and (2.3). If $L=K$, then $[T:K]$ divides dim $H^*(TF(K)^i;\underline{F}_2)$. If $|L/K|=2$, then dim $H^*(F(K)^i, \underline{F}_2)$ is even, hence dim $H^*(TF(K)^i;\underline{F}_2)=[T:L]\dim H^*(F(K)^i;\underline{F}_2)$ is divisible by $[T:K]$. If $|L/K|=4$, then, by Lemmas (2.1) and (2.3), $F(K)^i \sim_2 RP^{4m-1}$ hence $[T:K]$ must divide dim $H^*(TF(K)^i;\underline{F}_2)$ in this case as well. It follows that dim $H^*(X;\underline{F}_2)=\dim H^*(F(K);\underline{F}_2)$ is divisible by $[T:K]$.
To prove the inequality $2|K|^2 \geq |T|$, we first assume $|L/K|=1$ or 2. Then $F(K)$ has at least $|T/K|/2 \leq |T/L|$ components, hence $|K| \geq |T/K|/2$ or $2|K|^2 \geq T$. If $|L/K|=4$, we have $F(K)^i \sim_2 RP^{4m-1}$ as above, $m \geq 1$. Since we are assuming that $H^*(X;\underline{F}_2)$ is generated by an element of degree $n=2^a \geq 2$, the inequality of Lemma (2.4) yeilds, counting only the components $T \cdot F(K)^i$ of $F(K)$,

$$|K| \geq |T/L| n \geq 2|T/L| = |T/K|/2$$

which shows that $2|K|^2 \geq |T|$. This concludes the proof of the theorem.

<u>Remark</u> If $p=2$ and $X \sim_2 CP^n$ and T acts trivially on $H^*(X;\underline{Z}/4\underline{Z})$, then, for maximal isotropy groups K and L, we have $|K|=|L|$ and $|K|^2 \geq |T|$.

3. Examples of actions of p-tori without fixed points

In this section we will give some examples of actions in order to show that the inequalities of Theorem (2.5) cannot be generally improved. For each integer $a \geq 1$, we will construct a group Γ^a of linear transformations of $\underline{C}^{p^a}$ as follows: Set $\underline{C}^{p^a} = \underline{C}[Z_1,\ldots,Z_a]/(Z_1^p-1,\ldots,Z_a^p-1)$ and let $\rho \in \underline{C}$ be a primitive p^{th} roof of 1. We define generators f_i and g_i $(1 \leq i \leq a)$ of Γ^a by

$$f_i(Z_1^{\alpha_1}\ldots Z_a^{\alpha_a}) = \rho^{\alpha_i} Z_1^{\alpha_1}\ldots Z_a^{\alpha_a}$$

$$g_i(Z_1^{\alpha_1}\ldots Z_a^{\alpha_a}) = Z_i\, Z_1^{\alpha_1}\ldots Z_a^{\alpha_a}$$

There are relations,

$$f_i g_j = g_j f_i\ ,\quad f_i f_j = f_j f_i, \quad \text{and} \quad g_i g_j = g_j g_i \quad \text{for } i \neq j\ ,$$

$$f_i^p = g_i^p = 1, \quad \text{and} \quad f_i g_i = \rho g_i f_i.$$

The centre C of Γ^a is generated by ρ, and Γ^a/C is abelian. The subgroup Γ_f generated by the f_i and the subgroup Γ_g generated by the g_i are both p-tori of rank a, and the natural map

$$\Gamma_f \times \Gamma_g \to \Gamma^a/C$$

is an isomorphism. The defining representation of Γ^a of degree p^a is irreducible. In fact, the p^a monomials $Z_1^{\alpha_1}\ldots Z_a^{\alpha_a}$ span linear subspaces invariant under Γ_f which is represented in those subspaces by p^a different characters. Also, Γ_g permutes those monomials transitively, hence the representation of Γ^a is irreducible. On the Grassmann variety G_{k,p^a} of k-dimensional subspaces of $\underline{C}^{p^a}$, the subgroup C

acts trivially and it follows that the p-rorus Γ^a/C of rank $2a$ acts on G_{k,p^a} without fixed points. In particular we obtain an action on $\underline{C}P^{p^a-1}$ without fixed points.

Remark . The group Γ^1 and the corresponding action on $\underline{C}P^{p-1}$ is defined by A. Borel [1]. He also shows that $(\underline{Z}_3)^3$ acts on the Cayley projective plane without fixed points.

It is clar that $F(\Gamma_f,\underline{C}P^{p^a-1})$ has p^a isolated fixed points represented by the monomials in $Z_1,\dots,Z_a$, and that Γ_f is a maximal isotropy group. Thus the inequalities of Theorem (2.5) cannot by improved since we have $|K|^2=|T|=p^{2a}$ and $[T:K] = p^a=\dim H^*(X;\underline{F}_p)$ for the action on $X=\underline{C}P^{p^a-1}$.

Let E_r be the cohomology spectral sequence of $X \to X_T \to B_T$; $T = \Gamma_f \times \Gamma_g$. Let $x\in H^2(X;\underline{F}_p)$ be a generator, and let $d=d_3(x) \in H^3(B_T;\underline{F}_p)$. Then a subgroup $K \in T$ has fixed points in X if and only if $d|B_K = 0$. We have, for p odd,

$$H^*(B_T;\underline{F}_p) = \underline{F}_p[t_1,\dots,t_{2a}]\langle s_1,\dots,s_{2a}\rangle$$

where $\deg s_i=1$ and $\beta s_i = t_i$ where β is the Bockstein operator. Let C_i be the subgroup generated by f_i if $i\leq a$, and by g_{i-a} if $i>a$. Then we choose generators s_i such that $s_j \mid B_{C_i} = 0$ for $i \neq j$. We have

$$d = \sum_{i<j<k} c_{ijk}\, s_i s_j s_k + \sum_{i,j} c_{ij}\, s_i t_j \,.$$

Since x lifts to integral coefficients, d must be an integral class, and $\beta(d) = 0$. It follows that $c_{ijk} = 0$ and $c_{ij}+c_{ji}=0$, hence

$$d = \beta \sum_{i<j} c_{ji}\, s_i s_j .$$

To determine the c_{ji}, it suffices to restrict the action to the subgroup $C_j C_i \subset \Gamma_f \times \Gamma_g$. If $j \neq i+a$, C_j and C_i generate an abelian subgroup of Γ^a, hence $C_j C_i$ must have fixed points, hence $c_{ji} = 0$, and $d = \beta \sum_{i=1}^{a} c_i\, s_i\, s_{i+a}$. By permuting the subscripts of $Z_1, \ldots, Z_a \in \underline{C}^p$, we obtain automorphisms of the representation of Γ^a, inducing an automorphism of Γ^a which permutes the sets $\{f_1, \ldots, f_a\}$, $\{g_1 \ldots, g_a\}$, $\{s_1, \ldots, s_a\}$, and $\{s_{1+a}, \ldots, s_{2a}\}$ accordingly. It follows that the c_i are independent of i, hence,

$$d = c\,\beta \sum_{i=1}^{a} s_i\, s_{i+a} = \frac{1}{2} c\,\beta \sum_{i=1}^{a} (s_i\, s_{i+a} - s_{i+a} s_i)$$

Here $\Sigma(s_i\, s_{i+a} - s_{i+a} s_i)$ is a symplectic form on $H_1(B_T; \underline{F}_p)$, and the maximal subspaces where this form vanishes are all of dimension a. Thus we can verify the statements of Theorem (2.5) on isotropy groups by using properties of nondegenerate forms.

In the case $p=2$, the equation $\beta(d)=0$ implies $d = \sum_{i<j} c_{ij}\, t_i\, t_j\, (t_i + t_j)$ where $H^*(B_T; \underline{F}_2) = \underline{F}_2[t_1, \ldots, t_{2a}]$. Using permutation invariance and the fact that $C_i C_j$ has fixed points for $i<j \neq i+a$ we obtain, $d = \beta \sum_{i=1}^{a} t_i\, t_{i+a}$. Thus, for the action of $T = (\underline{Z}_2)^{2a}$, the statements of Theorem (2.5) on isotropy groups follow from properties of the quadratic function $\Sigma\, t_i t_{i+a}$. The representation of Γ^a in $\underline{C}^{2^a}$ is real, hence $\Gamma_f \times \Gamma_g$ acts on $RP^{2^a - 1}$ without fixed points. The differential d_2 of the generator of E_2^{o2} is $\sum_{i=1}^{a} t_i t_{i+a}$.

REFERENCES

[1] A. Borel, Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tohoku Math. J. 13(1961)216-240.

[2] A. Borel et al , Seminar on transformation groups. Annals of Math. Studies Number 46, Princeton University Press (1961).

[3] G.E. Bredon, The cohomology ring structure of a fixed point set. Annals of Math. 80(1964)524-537.

[4] D.B.A. Epstein and N.E.Steenrod, Cohomology operations. Annals of Math. Studies Number 50, Princeton University Press (1962).

[5] W.Y.Hsiang, On generalizations of a theorem of A. Borel and their applications in the study of topological actions. Topology of manifolds, Proc. Univ. of Georgia (1969)274-290. Editors Cantrell and Edwards, Markham Publ. Co., Chicago (1970).

[6] W.Y.Hsiang, On some fundamental theorems in cohomology theory of topological transormation groups. Taita J.Math. (Taiwan) 2(1970)66-87.

[7] W.Y. Hsiang and J.C.Su, On the geometric weight system of topological actions on cohomology quaternionic projective spaces. (mimeographed).

[8] I.M. James, Reduced product spaces. Ann. of Math. 62(1955) 170-197.

[9] T. Skjelbred, Integral global weights for torusactions on projective spaces. Math. Scand, 34(1974)249-253.

[10] E.H. Spanier, Algebraic topology. Mc Graw-Hill Book Company, New York (1966).

[11] J.C. Su, Transformation groups on cohomology projective spaces. Trans. Amer. Math. Soc. 106(1963)305-318.

[12] D.Sullivan, Geometric topology, part I. M.I.T. Cambridge, Massachusetts (1970).