Models for Recursion Theory

by

Johan Moldestad and Dag Normann

Several results in the theory of recursion in higher types indicate that the effect of a higher type functional on the lower types does not reflect the high type, i.e. the same effect could be obtained by functionals of relatively low type. The two main results here are :

<u>Plus - 1 - theorem.</u> (G. Sacks [6] for $k = 1$, [7] for $k > 1$).
Let H be a normal functional of type $\geq k + 1$. Then there exists a normal functional F of type $k + 1$ such that $k\text{-}sc(F) = k\text{-}sc(H)$, i.e. the same subsets of $tp(k-1)$ are recursive in F and H.

<u>Plus - 2 - theorem.</u> (L. Harrington [1].
Let H be a normal functional of type $\geq k + 2$. Then there exists a normal functional F of type $k + 2$ such that $k\text{-}en(H) = k\text{-}en(F)$, i.e. the same subsets of $tp(k-1)$ are semirecursive in F and H.

The results in this paper also indicate that higher types cannot have too much influence on lower types. The key is the Skolem-Löwenheim theorem. Among the results we mention :

1. Let $n < m$. $A \subseteq tp(n) \times tp(m)$ be Kleene-semicomputable. Let $x \in B \iff \forall y \in tp(m)\ \langle x,y\rangle \in A$. Then B is Π^n_1. This result may be relativized to a functional of type $n + 1$.

2. Let ${}^k0$ be the type-k-functional that is constant zero. Let F be a functional of type $< k$. Then, for $i \leq k-2$
$i\text{-}sc(F,{}^k0) = i\text{-}sc(F)$, $\quad i\text{-}en(F,{}^k0) = \forall^{tp(i)}(i\text{-}en F)$

3. Let $n,m \geq 1$. Then there is a functional F of type $n + 2$ such that for $k \leq n$, $k + 1 - en(F) = \Pi^n_m(tp(k))$.

4. Π^n_1-positive inductive definitions over $tp(n)$ have Π^n_1 - least fixedpoints.

All these results have relativized versions.

This paper includes results from Moldestad & Normann [5]. There we proved a relativized version of 4 for $n = 1$, and derived 2 for $k = 3$. The proof of 3 from 2 follows the same ideas as in [5]. Also the discussions in § 8 of this paper are from [5]. These results and ideas are jointly due to both authors. The notion of recursion structures and theorem 1 are due to the second author.

2. Notation.

We will work with Kleene-recursion on objects of finite type, and we assume familiarity with the contents of Kleene [3]. We define the types as

$$tp(0) = \omega, \qquad tp(i+1) = {}^{tp(i)}\omega.$$

Let $X \subseteq tp(0)^{k_0} \times \cdots \times tp(n)^{k_n}$.

We say that X is Δ^0_0 if X is Kleene-recursive.

Assume Δ^n_0 is defined. Let $\Pi^n_0 = \Sigma^n_0 = \Delta^n_0$.

$$\Pi^n_{k+1} = \forall^{tp(n)}(\Sigma^n_k), \qquad \Sigma^n_{k+1} = \exists^{tp(n)}(\Pi^n_k)$$

$$\Delta^n_k = \Pi^n_k \cap \Sigma^n_k , \qquad \Delta^{n+1}_0 = \bigcup_{k\in\omega} \Delta^n_k$$

$\underline{F}$, $\underline{G}$, $\underline{H}$, $\underline{U}$ will denote formal symbols for functionals. To each symbol there is assigned a number indicating the type of the interpretations, F, G, H, U will denote standard interpretations of the symbols, $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{U}$ other interpretations.

3. Recursion structures.

Let $\vec{F}$ be a list of functional symbols, σ the associated list of type-indicators.

Let $i,j \in \omega$, $i < j$. By a type i,j,σ-recursion structure we mean a structure

$$\mathfrak{A} = \langle A_0,\dots,A_i,\dots,A_j,\vec{\mathcal{F}},E\rangle \text{ such that}$$

i $k \leq i \to A_k = tp(k)$

$i < k \leq j \to A_k \subseteq {}^{A_{k-1}}\omega$

ii $\vec{\mathcal{F}}$ is an interpretation of $\vec{F}$ such that if F is a symbol of type k, then $\mathcal{F}$ is in A_k.

iii Each A_k is closed under primitive recursive operations.

iv AC is satisfied in $\mathfrak{A}$

v E is the evaluation-relation on $\bigcup_{k \leq j} A_k$.

$E(x,y,n) \Longleftrightarrow x(y) \simeq n$

We will explain iv a bit :

Let φ be a formula in $L(\mathfrak{A})$ (the 1.order language with constants for all elements in $A_0,\dots,A_j$.) Let $k_1 \leq j$, $k_2 \leq j-1$. Assume

$$\mathfrak{A} \models \forall\alpha \in A_{k_2} \; \exists\beta \in A_{k_1} \varphi(\alpha,\beta)$$

Let $k = \max(k_1, k_2+1)$. Then

$$\mathfrak{A} \models \exists\beta \in A_k \forall\alpha \in A_k \; \varphi(\alpha, \lambda x^{k_1-1} \beta(\alpha,x)).$$

We assume here the existence of some standard coding of lower functionals to higher functionals.

We say that a set $\underline{A \subseteq A_k \text{ is in } \mathfrak{A}}$ if its characteristic function is an element of A_{k+1}.

$\mathfrak{A}$ is <u>absolute</u> if well-foundedness is absolute with respect to $\mathfrak{A}$.

Now we are going to code some i,j,σ-structures as elements of $tp(i+1)$:

Let $q = \langle x_{i+1},\dots,x_j \rangle$ be a sequence of type $(i+1)$-elements. Let $\vec{f}$ be a list of elements from $tp(i)$ of the same length as $\vec{\underline{F}}$. Define

$$\alpha^z_{i+1} = \lambda y \; x_{i+1}(\langle y,z \rangle)$$

Inductively define $\alpha^z_{k+1}(\alpha^y_k) = x_{k+1}(\langle y,z \rangle)$.

<u>Remark:</u> There is a possibility that α^z_k may be a many-valued function. However,

'Each α^z_k is single-valued' is given by a Δ^0_{i+1}-statement in q. From now on we will, given q, assume that the α^z_k's are well-defined.

Now, given q, let $A_k = tp(k)$ for $k \le i$

$$A_k = \{\alpha^z_k \; ; \quad z \in tp(i)\} \quad \text{for} \quad i < k \le j$$

Define $\mathcal{F}_n$ to be $\alpha^{f_n}_k$ where k is the type-number of $\underline{F}_n$. Let $\mathfrak{A} = \langle A_1,\dots,A_j,\vec{\mathcal{F}},E \rangle$. We say that $q,\vec{f}$ code $\mathfrak{A}$.

For the sake of simplicity we denote 'i,j,σ-recursion structure' by 'structure' when no ambiguity may arise.

<u>Lemma 1.</u>

<u>a</u> '$q,\vec{f}$ code a structure' is Δ^{i+1}_1 .

<u>b</u> If $i > 0$, then '$q,\vec{f}$ code an absolute structure' is Δ^{i+1}_1 .

If $i = 0$, then this is Π^1_1 .

Proof. The language of $\mathfrak{A}$ is arithmetizable over $\mathfrak{A}$, and thus 'truth in $\mathfrak{A}$' is $\Delta^1_1(\mathfrak{A})$-expressible. A set quantifier over $\mathfrak{A}$ is nothing more than a type i+1-quantifier, and 1.order quantifiers in $\mathfrak{A}$ are tp(i)-quantifiers. Thus by standard coding:

'$q,\vec{f}$ code an $\mathfrak{A}$ such that $\mathfrak{A} \models AC$' is $\Delta^{i+1}_1(q,\vec{f})$, uniformly in q,f. The rest of the properties of a structure are arithmetic over $\mathfrak{A}$, and thus Δ^{n+1}_0 in $\langle q,f\rangle$.

This proves a. To prove b note that $\mathfrak{A}$ is absolute if

$\forall T \in |\mathfrak{A}|(T \in (wf)_{\mathfrak{A}} \Rightarrow T \in wf)$, where wf denotes the wellfounded relations. wf is Π^1_1 for $i = 0$, but Δ^{i+1}_0 else.

Remark: For $i > 0$, we always assume that a structure is absolute, since this does not affect the definability. Moreover, in our proofs, we do not need the full axiom of choice. We may give an upper estimate of the complexity of the formulas we need AC to hold for. In that case '$\langle q,\vec{f}\rangle$ codes a structure' will be Δ^i_n for some n, irrespectively of whether $i = 0$ or $i > 0$.

4. Recursion in the structures.

The purpose with these structures is to simulate recursion in a list of objects $\vec{F}$ over tp(i+1). We know from Kleene [3] that when the maximal type in $\vec{F}$ is j, then no tp(j-1)-functionals, except those in $\vec{F}$, will occur in any subcomputation. When δ is a computation in $\vec{F}$, we let δ^- be the list obtained from δ by replacing F_n by n in δ for each F_n in $\vec{F}$. By the computation tree of a computation in $\vec{F}$, we mean :

$\{\langle\delta^-_1,\sigma^-_2\rangle$; δ_1 is an immediate subcomputation of δ_2, which again either is a subcomputation of the given computation or is the given computation}. The computation tree will then be a subset of tp(j-2), and it will be well-founded. Moreover, there is a Π_1^{j-1}-formula ψ such that

$$T \text{ is a computation tree} \iff T \text{ is well-founded } \& \ \psi(T,\vec{F}).$$

Given $\mathfrak{A}$, $\psi_{\mathfrak{A}}(T,\vec{\mathcal{F}})$ has a natural interpretation, and we have two possibilities in defining recursion in higher types over $\mathfrak{A}$:

1. We use Kleene's inductive definition of recursion in higher types, i.e. an outside definition. The only new there is schema 8:
   Let $x_1 \in A_k$ and let e be the index of schema 8.

$$\{e\}(x_1,\dots,x_k) \simeq n \iff x_1(\lambda y\{e'\}(y,x_1,\dots,x_k)) \simeq n \quad .$$

We then say

$$\{e\}_{\mathfrak{A}}(x_1,\dots,x_k) \simeq n \iff x_1(\lambda y \in A_{k-1}\{e'\}_{\mathfrak{A}}(y,x_1,\dots,x_k)) \simeq n.$$

2. We define

$$\{e\}_{\mathfrak{A}}(x_1,\dots,x_k) \simeq n \iff \mathfrak{A} \models \exists T \ (T \text{ is a computation tree for } \langle e,x_1,\dots,x_k,n\rangle^{-}) \ .$$

<u>Lemma 2.</u>

Let $\mathfrak{A}$ be a structure. Then 1. is equivalent to

3. $$\{e\}_{\mathfrak{A}}(x_1,\dots,x_k) \simeq n \iff (\exists T \in |\mathfrak{A}|) \ (T \text{ is well founded } \& \ \mathfrak{A} \models T \text{ is a computation tree for } \langle e_1,x_1,\dots,x_k,n\rangle^{-}).$$

<u>Proof.</u>

3 ⇒ 1 follows by induction on rank (T) .

1 ⇒ 3 We prove this by induction on the length of the computation. The only nontrivial case is case 8.

Let $x_1, \ldots, x_n$ be given, each $x_i \in A_{k_i}$ for some $k_i < j-1$. Let $x = x_1, \ldots, x_n$ or let x be from $\vec{\mathcal{F}}$. Assume that $x \in A_k$. Let e be the index such that

$$\{e\}_{\mathfrak{A}}(x_1, \ldots, x_n, \vec{\mathcal{F}}) = x(\lambda y \in A_{k-2}\{e'\}_{\mathfrak{A}}(x_1, \ldots, x_n, \vec{\mathcal{F}})).$$

Assume that $\lambda y \in A_{k-2}\{e'\}_{\mathfrak{A}}(x_1, \ldots, x_k, \vec{\mathcal{F}})$ is total over A_{k-2}. By induction hypothesis

$$* \quad \forall y \in A_{k-2}\ \exists T \in A_{j-1}\ \exists n\ (T \text{ is wellfounded } \& \ \mathfrak{A} \models \psi(T, \vec{\mathcal{F}})$$
$$\& \ \langle e', x_1, \ldots, x_n, n\rangle \text{ is the top of } T).$$

Moreover, ψ is such that T is unique (no standard computation has nonstandard subcomputations), and if $\langle e', x_1, \ldots, x_k, n\rangle$ and $\langle e', x_1, \ldots, x_k, m\rangle$ are two computations where at least one is standard, then $n = m$. This is proved by induction on the length of the standard computation. Then we may apply AC on * and obtain a well-founded computation tree for $\langle e, x_1, \ldots, x_k, \vec{\mathcal{F}}, n'\rangle$. This ends the proof of lemma 2.

<u>Remark.</u> By this lemma, 1.and 2. are equivalent for absolute structures, and 1. defines a stronger concept thant 2. i.e. all computations by 1. will be computations by 2. When nothing else is stated, we use 2. as our definition.

6. $\vec{F}$-structures and weak $\vec{F}$-structures.

<u>Definition.</u> Let $\vec{F}$ be a list of functionals. Let $\mathfrak{A}$ be a structure.

<u>1</u> We say that <u>$\mathfrak{A}$ is an $\vec{F}$-structure</u> if whenever $x_1, \ldots, x_k \in A_{i+1}$ we have

$$\{e\}_{\mathfrak{A}}(x_1, \ldots, x_k \vec{\mathcal{F}}) \simeq n \Rightarrow \{e\}_K(x_1, \ldots, x_k, \vec{F}) \simeq n .$$

Remark. $\{e\}_K$ denotes the Kleene-computable function with index e. $x_1,\ldots,x_k$ are all elements of type (i+1), so this is meaningful.

ii $\mathfrak{A}$ is a weak $\vec{F}$-structure if whenever $x_1,\ldots,x_k \in A_{i+1}$ we have $\{e\}_K(x_1,\ldots,x_k,\vec{F}) \simeq n \Rightarrow \{e\}_{\mathfrak{A}}(x_1,\ldots,x_k,\vec{\mathcal{F}}) \simeq n$.

Remark. '$q,\vec{f}$ code an $\vec{F}$-structure' will be semicomputable in $\vec{F}$ and ${}^{i+2}E$.

Lemma 3. Assume $j \leq i+3$. If $\mathfrak{A}$ is an $\vec{F}$-structure, then $\mathfrak{A}$ is a weak $\vec{F}$-structure.

Proof. We prove that $\{e\}_K(x_1,\ldots,x_n,\vec{F}) \simeq n \Rightarrow \{e\}_{\mathfrak{A}}(x_1,\ldots,x_n,\vec{\mathcal{F}}) \simeq n$ by induction on the length of the Kleene-computation. The induction will be trivial except in case 8, and then only recursion in some F of type k from $\vec{F}$ is interesting. So assume

$$\{e\}_K(x_1,\ldots,x_n,\vec{F}) = F(\lambda y\{e'\}(y,x_1,\ldots,x_n,\vec{F})) \simeq n .$$

By induction hypothesis $\lambda y \in A_{k-2}\{e'\}_K(y,x_1,\ldots,x_n,\vec{F}) =$
$= \lambda y \in A_{k-2}\{e'\}_{\mathfrak{A}}(y,x_1,\ldots,x_n,\vec{\mathcal{F}}) \in A_{k-1}$. (Here we use that $j \leq i+3$).
Then $\mathcal{F}(\lambda y \in A_{k-2}\{e'\}_{\mathfrak{A}}(y,x_1,\ldots,x_n,\vec{\mathcal{F}}) = m$ for some $m \in \omega$, and

$$\{e\}_{\mathfrak{A}}(x_1,\ldots,x_n\vec{\mathcal{F}}) \simeq m .$$

Since $\mathfrak{A}$ is an $\vec{F}$-structure, $\{e\}_K(x_1,\ldots,x_n,\vec{F}) \simeq m$. Then $n = m$.

Remark. The condition that $j \leq i+3$ is essential for this lemma. Assume the lemma holds for $j = i+4$. Let F be a normal type j+4-functional. Then j+2-en(F) is closed under type (j+1)-existential quantifiers.(MacQueen [4] or Harrington & MacQueen [2]).

By the lemma, however, and lemma 4, the following holds

δ is not a computation in F on type j+1

$\Longleftrightarrow \exists(q,f)$ (q,f code an i,i+4-F-structure & δ is not a computation in $\mathcal{F}$).

By MacQueen's theorem this would be semicomputable in F. Also note that lemmas 3 and 4 give a new proof of MacQueen's result that for a functional F of tp k+2, k+1-en F is not closed under tp(k)-existensial quantifier. Our proof works for all functionals in which ${}^{k+1}E$ is recursive. MacQueen's proof is by a delicate analysis of the subcomputation-relation and works for all functionals.

<u>Lemma 4.</u> Let $\vec{F}$ be a list of functionals of type σ, $\alpha_1,\ldots,\alpha_n \in tp(i+1)$, where $i \leq j$, $j \geq \max \sigma$. Then there are $q,\vec{f}$ from tp(i+1) coding an i,j,σ-$\vec{F}$-structure $\mathfrak{A}$ such that $\alpha_1,\ldots,\alpha_n \in A_{i+1}$. Moreover $\mathfrak{A}$ will be absolute.

<u>Proof.</u> Regard the structure

$$\mathfrak{A}_0 = \langle x, tp(i+1),\ldots,tp(j),\vec{F},\alpha_1,\ldots,\alpha_n,E,=\rangle_{x \in tp(l), l \leq i}$$

By Skolem-Löwenheim theorem, let $\mathfrak{A}'$ be a substructure of $\mathfrak{A}_0$ such that $\mathfrak{A}'$ and $\mathfrak{A}_0$ are elementarily equivalent and $\mathfrak{A}'$ has the same cardinality as tp(i). Let $\mathfrak{A}$ be the transitive structure obtained from $\mathfrak{A}'$. Since $\mathfrak{A}_0$ is an $\vec{F}$-structure and $\mathfrak{A}$ and $\mathfrak{A}_0$ are elementary equivalent, $\mathfrak{A}$ will be an $\vec{F}$-structure.

Now assume that $T \in A_k$ is wellfounded in $\mathfrak{A}$. Then T comes from a T' by Mostowskis isomorphism, and by elementary equivalence, T' will be well-founded in the real world. A descending chain in T will be mapped on a descending chain in T' by the inverse Mostowski's isomorphism. This proves the lemma.

Lemma 5. Let $\mathfrak{A}$ be a structure. Then there exists a list of functionals $\vec{F}$ such that $\mathfrak{A}$ is a weak $\vec{F}$-structure. Moreover if $\underline{S}$ is a type $i+2$-symbol in $\vec{\underline{F}}$ and for some S, $S \restriction A_{i+1} = \zeta$ (the interpretation of $\underline{S}$ in $\mathfrak{A}$), then we may interprete $\underline{S}$ by S.

Remark. For technical reasons we cannot prove the theorem for more than one S, for instance if $\zeta_1 = \zeta_2$ while $S_1 \neq S_2$ the proof won't work.

Proof.. We will define a function $\varphi\colon |\mathfrak{A}| \to V$ such that $x \in A_k \Rightarrow \varphi(x) \in tp(k)$, and such that

$$\{e\}_K(\varphi(x_1),\ldots,\varphi(x_n)) \simeq m \Rightarrow \{e\}_{\mathfrak{A}}(x_1,\ldots,x_n) \simeq m.$$

The converse of this implication will not hold in general. We define φ by induction on the type k.

$$k \leq i+1, \quad \text{let } \varphi \text{ be the identity.}$$

Else, assume that φ is defined for all elements in A_k. Let $x \in A_{k+1}$, $y \in tp(k)$.

If $y \restriction \varphi''(A_{k-1})$ (= φ-image of A_{k-1}) $= \varphi(z) \restriction \varphi''(A_{k-1})$ for some $z \in A_k$, then let $\varphi(x)(y) = x(z)$.

(If $x = \zeta$ and $y \in tp(i+1)$, then we are in this case if $y \in A_{i+1}$. Thus $\varphi(\zeta)(y) = \zeta(y) = S(y)$ here).

Else, let $\varphi(x)(y)$ be anything you want, for example $\varphi(x)(y) = 0$ (If $x = \zeta$, we may choose $\varphi(\zeta)(y) = S(y)$ in this case.

We must verify that there is no ambiguity here. We prove that for $x \in A_k$, $y \in A_k$

$$\varphi(x) \restriction \varphi''(A_{k-1}) = \varphi(y) \restriction \varphi''(A_{k-1}) \Rightarrow x = y$$

Uniqueness on type $k+1$ will then follow.

Let $x \in A_k$, $y \in A_k$, $x \neq y$. Then there is a $z \in A_{k-1}$ such that $x(z) \neq y(z)$. But

$$\varphi(x)(\varphi(z)) = x(z) \quad \text{and} \quad \varphi(y)(\varphi(z)) = y(z).$$

Thus

$$\varphi(x) \upharpoonright \varphi''(A_{k-1}) \neq \varphi(y) \upharpoonright \varphi''(A_{k-1}) .$$

Here we have used uniqueness of φ on A_{k-1} and A_k.

Now we prove by induction on the length of the computation that $\{e\}_K(\varphi(x_1),\cdots,\varphi(x_k)) \simeq n \Rightarrow \{e\}_{\mathfrak{A}}(x_1,\cdots,x_k) \simeq n$.

Since $\varphi(x_i)(\varphi(x_j)) = x_i(x_j)$ by definition, all cases except case 8 are trivial. Assume $x_i \in A_k$ and

$$\{e\}_K(\varphi(x_1),\cdots,\varphi(x_k)) = \varphi(x_i)(\lambda y\{e'\}_K(y,\varphi(x_1),\cdots,\varphi(x_k)) .$$

By the induction hypothesis we have for all $y \in A_{k-2}$

$$\{e'\}_K(\varphi(y),\varphi(x_1),\cdots,\varphi(x_k)) = \{e'\}_{\mathfrak{A}}(y,x_1,\cdots,x_n) .$$

Let $x = \lambda y \in A_{k-2} \ \{e'\}_{\mathfrak{A}}(y,x_1,\cdots,x_k)$. Then

$$\varphi(x) \upharpoonright \varphi''A_{k-2} = \lambda y\{e\}_K(y,\varphi(x_1),\cdots,\varphi(x_k)) \upharpoonright \varphi''A_{k-2}$$

and

$$\varphi(x_i)(\lambda y\{e\}_K(y,\varphi(x_1),\cdots,\varphi(x_k))) = x_i(x) = \{e\}_{\mathfrak{A}}(x_1,\cdots,x_k).$$

This proves the claim.

By letting $\vec{F} = \varphi(\vec{\mathfrak{F}})$, the lemma also follows.

6. Applications on large quantifiers.

Theorem 1.

Let $0 < i < j$, S a functional of type $\leq i+1$.

Let $A \subseteq tp(i) \times tp(j)$ be semicomputable in S. Define B by

$$x \in B \iff \forall y \in tp(j) \langle x,y \rangle \in A.$$

Then B is $\Pi^1_1(S)$ in the following sense :

$\Delta^0_0(S)$ = recursive in S. The rest is as in the definition of Π^n_m.

<u>Proof.</u> Let e be a Kleene-index such that

$<x,U> \in A \Leftrightarrow \{e\}(x,U,S) \simeq 0$. Thus

$x \in B \Leftrightarrow \forall U\{e\}(x,U,S) \simeq 0$.

<u>Claim.</u> $\forall U\{e\}(x,U,S) \simeq 0$

$\Leftrightarrow \forall (q,u,s) \in tp(i)$ (q,u,s codes $<A_0,\cdots,A_j,\mathcal{U},\zeta,F>$

$\& \ \zeta = S \cap A_i \ \& \ x \in A_{i+1} \Rightarrow \{e\}_{\mathfrak{A}}(x,\mathcal{U},\zeta) \simeq 0$).

<u>Proof of claim.</u>

$\Rightarrow$ Let q,u,s be given satisfying the premise. By lemma 5 we find U such that $\mathfrak{A}$ is a weak U,S-structure. By assumption $\{e\}(x,U,S) \simeq 0$, and since $\mathfrak{A}$ is a weak U,S-structure, $\{e\}_{\mathfrak{A}}(x,\mathcal{U},\zeta) \simeq 0$.

$\Leftarrow$ Let U be given. By lemma 4, let $\mathfrak{A}$ be a U,S-structure containing x. Obviously $S \cap A_{i+1} = \zeta$. By assumption

$\{e\}_{\mathfrak{A}}(x,\mathcal{U},\zeta) \simeq 0$, and since $\mathfrak{A}$ is a U,S-structure,

$\{e\}_K(x,U,S) \simeq 0$.

By the claim, the theorem follows, since what is inside the paranthesis is $\Delta^1_0(S)$.

<u>Corollary 1.</u> If $A \subseteq tp(i) \times tp(j)$ is Kleene semi-computable and B is defined by

$x \in B \Leftrightarrow \forall U \in tp(j) <x,U> \in A$,

then B is Π^1_1.

Corollary 2. Let $i < j-1$. Let S be a type $i+1$ functional. ${}^{j}0$ the constant zero type j-functional.

Then $i+1\text{-en}(S,{}^{j}0) \subseteq \Pi^{i}_{1}(S)$.

Moreover, if S is normal, then $i+1\text{-en}(S,{}^{j}0) = \Pi^{i}_{1}(S)$.

Proof. ${}^{j}0$ is uniformly computable in any type-j-functional U, i.e. there is a primitive recursive function f such that

$$\{e\}(x,S,{}^{j}0) \simeq n \iff \forall U \in tp(j)\ (\{f(e)\}(x,S,U) \simeq n).$$

To obtain the first part, use theorem 1. To obtain the second part note that when S is normal, $\Delta^{i}_{0}(S) \subseteq i+1\text{-sc}(S)$, and $i+1\text{-en}(S,{}^{j}0)$ is closed under $\forall^{tp(j-2)}$.

Corollary 3. Let $n,m > 0$, $i \leq n$. Then there exists a functional F such that $i+1\text{-en}(F) = \Pi^{n}_{m}$ over type (i).

Proof. For $n = m = 1$, this is well known, let $F = {}^{2}E$. For $m = 1$, $n > 1$, let $F = \langle {}^{n+2}0, {}^{n+1}E \rangle$. By corollary 2,

$$n+1\text{-en}({}^{n+2}0, {}^{n+1}E) = \Pi^{n}_{1}({}^{n+1}E).$$

However, $n+1\text{-sc}({}^{n+1}E) \subseteq \Delta^{n}_{1}$ for $n > 1$. The corollary then follows in this case.

For $n \geq 1$, $m > 1$, let S be the characteristic function of a complete Σ^{n}_{m-1}-set. Again S is normal and since $n+1\text{-sc}(S) \subseteq \Delta^{n}_{m}$, $n+1\text{-en}({}^{n+2}0,S) = \Pi^{n}_{1}(S) = \Pi^{n}_{m}$.

Remark. It is not known whether some Σ^{n}_{m}-sets are envelopes of functionals. However, we have

Σ^{n}_{m} is never the $n+1$-envelope of any functional of type $\geq n+2$, and if $n+1\text{-en}\ F \subseteq \Sigma^{n}_{m}$ for a functional of type $n+2$, then $n+1\text{-en}\ F \subseteq \Delta^{n}_{m}$. This is seen by the result of MacQueen [4] that says :

There exists a set A which is semicomputable in F such that $\langle e,\sigma,k\rangle$ is not a F-computation $\Leftrightarrow \exists x(x,e,\sigma,k) \in A$, where σ is a tuple in tp(n) and x varies over tp(n).

7. Skolem-Löwenheim and inductive definitions.

In Moldestad & Normann [5] we proved a result on relativized Π^1_1-inductive definitions as a key to recursion in $^3 0$. For $n = 1$, corollary 3 was proved. The proof in [5] may be generalized to higher types. We prove the theorem here, although we have no applications of it.

Before we are able to formulate our result, we need a definition of Π^n_1 relative to a set of higher type objects.

Let S be a functional of tp(n+1), $A \subseteq$ tp(n), $\vec{x}$ objects of type $\leq n$. $R(\vec{x},A,S)$ is simple if R is defined using the connectives $\vee$ and $\neg$, evaluation in types, the $\in$-relation in $tp(n) \times \mathcal{P}(tp(n))$ and function symbols for all primitive recursive operations on tp(k) for $k \leq n$.

We say that <u>A occurs positively in R</u> if all subformulas $t \in A$ occurs positively in R, where t is a term. We define $\Pi^n_1(S)$ as before, when we replace $\Delta^0_0(S)$ by simple relative to S.

Let $\Gamma: \mathcal{P}(tp(n)) \to \mathcal{P}(tp(n))$

We say that <u>Γ is a positive $\Pi^n_1(S)$-operator</u> if '$x \in \Gamma(A)$' is $\Pi^n_1(S)$ such that A occurs positively.

<u>Remark:</u> $\Pi^n_1(S)$ has the usual closure properties, i.e. closure under $\vee,\wedge$ and quantifiers of lower type.

<u>Theorem 2.</u> If Γ is a positive $\Pi^n_1(S)$-operator over tp(n), then Γ^∞ is $\Pi^n_1(S)$.

Proof. Let $x \in \Gamma(A) \Longleftrightarrow \forall y \varphi(x,y,A,S)$, where φ is Δ_0^n and A is positive in φ.

Let $B \subseteq tp(n)$. Define

$$x \in \Gamma_B(A) \Longleftrightarrow \forall y \in B \varphi(x,y,A) \quad \& \quad x \in B.$$

Note that Γ_B will be a monotone operator.

We say that B is sufficiently closed if

B includes all tp(n-1)-elements and is closed under primitive recursion.

Note that if $x,y \in B$ and B is sufficiently closed, then

$$\varphi(x,y,A,S) \Longleftrightarrow \varphi(x,y,A \cap B,S) .$$

Now let $\alpha \in tp(n)$. Define $B_\alpha = \{\lambda z(\alpha(z,y)) \; ; \; y \in tp(n-1)\}$.

Claim 1 'B_α is sufficiently closed' is Δ_1^n.

Proof. Let $\alpha_y = \lambda z \, \alpha \, (z,y)$. Primitive recursion in α_y is Δ_1^n-definable, and equility is Π_1^{n-1}-definable. Then observe

$$x \in \Gamma(A) \Rightarrow \forall \alpha(B_\alpha \text{ is sufficiently closed } \& \ x \in B_\alpha \Rightarrow x \in \Gamma_{B_\alpha}(A)).$$

We obtain

Claim 2. $x \in \Gamma^\gamma \Rightarrow \forall \alpha(B_\alpha$ is sufficiently closed and $x \in B_\alpha \Rightarrow x \in \Gamma_{B_\alpha}{}^\gamma)$.

Proof by induction on γ. For $\gamma = 0$ and for limit ordinals γ this is trivial.

Now let $x \in \Gamma(\Gamma^\gamma)$. Then

$\forall y \varphi(x,y,\Gamma^\gamma)$. Let $x \in B_\alpha$ and let $y \in B_\alpha$ be arbitrary. Since B_α is sufficiently closed

$\varphi(x,y,\Gamma^{\gamma} \wedge B_{\alpha})$ will hold.

By induction hypothesis, $\Gamma^{\gamma} \wedge B_{\alpha} \subseteq \Gamma_{B_{\alpha}}{}^{\gamma} \cap B_{\alpha} = \Gamma^{\gamma}_{B_{\alpha}}$.

Thus $\varphi(x,y,\Gamma_{B_{\alpha}}{}^{\gamma})$ holds by monotonicity, and

$$x \in \Gamma_{B_{\alpha}}(\Gamma_{B_{\alpha}}{}^{\gamma}) = \Gamma^{\gamma+1}_{B_{\alpha}}$$ and the claim is proved.

From the claim we derive :

$$x \in \Gamma^{\infty} \Rightarrow \forall\alpha(B_{\alpha} \text{ is sufficiently closed } \& \ x \in B_{\alpha} \Rightarrow x \in \Gamma^{\infty}_{B_{\sigma}}$$

By the next claim the converse will also hold.

Claim 3. If $x \notin \Gamma^{\infty}$, then there exists an α such that B_{α} is sufficiently closed, $x \in B_{\alpha}$ and $x \notin \Gamma^{\infty}_{B_{\alpha}}$.

Proof. Let $K > |\Gamma|$ and let $M = \langle V_K, tp(n-1), x, S, \in\rangle$. By Skolem-Löwenheim theorem M has an elementary submodel of the same cardinality as tp((n-1). Let $\mathcal{M}$ be transitive, of cardinality $\overline{\overline{tp(n-1)}}$ and isomorphic to the submodel of M. Let $B = \mathcal{M} \cap tp(n)$. Note that $S_{\mathcal{M}} = S \restriction B$, and that $\mathcal{M} \models x \notin \Gamma^{\infty}$ Now $(\Gamma^{\infty})_{\mathcal{M}} = \Gamma_B$ since all lower-type quantifiers are made absolute. Moreover B is sufficiently closed since tp(n) is in it. By cardinality there is an α such that $B = B_{\alpha}$.

Proof of the theorem. By claims 2 and 3 we obtain

$$x \in \Gamma^{\infty} \Leftrightarrow \forall\alpha(B_{\alpha} \text{ is sufficiently closed } \& \ x \in B_{\alpha} \Rightarrow x \in \Gamma_{B_{\alpha}}{}^{\infty}).$$

Since $\Gamma_{B_{\alpha}}$ is nothing more than an inductive definition over tp(n-1), $\Gamma_{B_{\alpha}}{}^{\infty}$ will be $\Pi^n_1(S)$ uniformly in α, by standard proof. Thus Γ^{∞} is $\Pi^n_1(S)$.

It is still an open problem whether Π^n_m-positive inductive definitions have Π^n_m-fixpoints (over tp(n)).

Remarks. In claim 3 we did not use the fact that Γ was positive or $\Pi^n_1(S)$. Thus this claim holds for all definable inductive definitions.

8. Some properties of $^j 0$.

Theorem 3. Let S be a functional of type $\leq n + 2$. If $n > 0$ assume ^{n+2}E is recursive in S. Then

i $n + 2 - sc(^{n+k}0, S) = n + 2 - sc(S)$,

ii $n + 2 - en(^{n+k}0, S) = \Pi^{n+1}_1(S)$ for $k > 3$.

Proof. ii is already verified, i follows by a reindexing f, i.e.

$$\{e\}^{S, {}^{n+k}0}(x) \simeq n \Rightarrow \{f(e)\}^S(x) \simeq m.$$

$^{n+k}0$ can 'only check totality', so we replace recursion in $^{n+k}0$ by the total zero function on indices. We omit the details here. See [5].

The reflections that follow are done for recursion over tp(1). Similar reflections may be done for recursion in higher types.

Corollary 4.

$2\text{-en}(^3 0) = 2\text{-en}(^3 0, {}^2E) = 2\text{-en}(^2E) = \Pi^1_1$.

$2\text{-sc}(^3 0)$ = the recursive sets .

$2\text{-sc}(^3 0, {}^2E) = 2\text{-sc}(^2E) = \Delta^1_1$.

For any functional U, let Th(U) denote the Kleene theory of U over tp(1) with associated length function. We see that

$Th({}^2E)$ and $Th({}^2E, {}^3O)$ have the same 2-envelopes and the same 2-sections. In both theories we have arbitrarily long countable computations. However, if S is arbitrary of $tp \leq 2$ we shall see that in $Th(S, {}^3O)$ it is a 'quick' operation to check that a tuple is a computation.

The set of computations in $Th(S, {}^3O)$ is given by a $\Pi^1_1(S)$-expression.

$$\sigma \text{ is a computation} \Longleftrightarrow \forall\alpha\ \exists n\ \varphi(\alpha,n,\sigma,S)$$

where φ is simple. Given α, n and σ we may effectively in α, σ, S decide whether $\varphi(\alpha,n,\sigma,S)$ holds or not. Thus there is a S-recursive function f such that $f(\alpha,\sigma)\downarrow \Longleftrightarrow \exists n\ \varphi(\alpha,n,\sigma,S)$, and when $f(\alpha,\sigma)\downarrow$, the computation will be finite.

Let $g(\sigma) = W(\lambda\alpha f(\alpha,\sigma))$. If $g(\sigma)\downarrow$ the length of the computation will be at most ω.

<u>Corollary 5.</u> $Th({}^3O, S)$ is not p-normal, i.e. we cannot compare lengths.

<u>Proof.</u> It is not hard to construct a computation in ${}^3O$ with length greater than ω, and which has a natural number as argument. p-normality and the observation above would yield that the set of computations were computable.

Thus $Th({}^2E)$ and $Th({}^2E, {}^3O)$ are different, although they have the same envelopes and the same sections. This contrasts that in the normal case, equality between evelopes gives equivalence between theories.

Observe that $2\text{-en}(S, {}^3O)$ will always be closed under $\exists^\omega$.

However,

<u>Conjecture.</u> Let S be an arbitrary type-2-functional. In general, the functional

$$\varphi(A,a) \simeq 0 \iff \exists n \in \omega(\langle n,a\rangle \in A)$$

will not be ${}^3 0$,S-computable in the sense of Moschovakis i.e. there is no index e such that

$$\{e\}^{W,S}(e',\vec{x}) \simeq 0 \iff \exists n\{e\}^{W,S}(n,\vec{x}) \simeq 0$$

and

$$\|\langle e,e'x\rangle\|_{W,S} > \inf\{\|\langle e',n,\vec{x},0\rangle\|_{W,S} \; ; \; n \in \omega\} .$$

The conclusion is false when $S = {}^2E$.

P. Aczel proved that the partial functional

$$\varphi(f) \simeq 0 \iff \exists n f(n) \downarrow$$

is not computable in any total functional.

Let $\Pi^1_1(S)\text{-ind} = \{\Gamma^\infty \; ; \quad \Gamma \text{ is a positive } \Pi^1_1(S)\text{-operator}\}$,

$|\Pi^1_1(S)|_{df} = \mathrm{Sup}\}|\Gamma| \quad \Gamma \text{ is a positive } \Pi^1_1(S)\text{-operator}\}$.

<u>Problem.</u> Let $|S,{}^3 0|$ denote the supremum of the lengths of computations in $Th(S,{}^3 0)$. Will $|S,{}^3 0| = |\Pi^1_1(S)|$?

<u>Remark.</u> If the conjecture above is disproved for arbitrary S, we have a positive solution to the problem, by the first recursion theorem. We will always have $|S,{}^3 0| \leq |\Pi^1_1(S)|$, since the set of computations is given by a $\Pi^1_1(S)$-inductive definition.

We end this note by the following observation :

<u>Observation.</u> Let $U \in tp(n+2)$. Then the following statements are equivalent

$\underline{a}$ U is normal

$\underline{b}$ $Th(U)$ over $tp(n)$ is p-normal.

<u>Proof.</u> $\underline{a} \Rightarrow \underline{b}$ is well known. To prove $\underline{b} \Rightarrow \underline{a}$, regard the following way of computing ${}^{n+2}E$.

$${}^{n+2}E(\lambda x\{e'\}(x,\vec{y},U)) :$$

Compute $0 \circ U(\lambda x\{e'\}(x,\vec{y},U))$. This converges if and only if $\lambda x\{e'\}(x,\vec{y},U)$ is total.

To check if $\forall x\{e'\}(x,\vec{y},U) \simeq 0$, we may use $0 \circ U$ on the functional $\varphi(\{e'\}(x,\vec{y},U))$ where $\varphi(x) = \begin{cases} 0 & \text{if } x = 0 \\ \text{undefined} & \text{o.w.} \end{cases}$

When this computation converges, it will have shorter length than the computation of $0 \circ 0 \circ 0 \circ U(\lambda x\{e'\}(x,\vec{y},U))$, which converges. By p-normality then, we may decide whether $\varphi(\{e'\}(x,\vec{y},U))$ converges or not, i.e. compute ${}^{n+2}E$.

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