Introduction. A locally compact group G is said to be an $[FC]^-$ group if all of its conjugacy classes $\{xgx^{-1}:x\in G\}$ are precompact. The class $[FIA]^-$ consists of all locally compact groups G possessing precompact inner automorphism group $I(G)$, where $I(G)$ is equipped with the relative topology from $Aut(G)$, the group of all topological automorphisms with the usual Birkhoff topology. The class $[FIA]^-$ is contained in $[FC]^-$. G belongs to the class $[SIN]$ if there is a fundamental system of neighborhoods of the identity e, invariant under inner automorphisms. One has $[FIA]^- = [FC]^- \cap [SIN]$. See [4].

We shall assume the reader is familar with Mackey's theory of induced representations as outlined in [1]. We refer to [16] and [17] for notation undefined in the present paper. Good references to the theory of the classes of group discussed here are [4], [11], and [14]. Structure theory and other basic information will be found there.

The paper is organized in the following way. In section 1 we prove that the dual space $\hat{G}$ of all equivalence classes of unitary continuous irreducible Hilbert space representations of an $[FC]^-$ group G has a finite number of connected components iff the subgroup of all periodic elements in G is finite ($x \in G$ is periodic if the closed subgroup generated by x is compact). $\hat{G}$ is endowed with the Hull - kernel topology.
Our proof depends on an analysis of the orbits in the dual space $\hat{H}$ of a closed normal subgroup H of G under the action of G by inner automorphisms:

$$x\cdot\rho(h) = \rho(x^{-1}hx)\ ,\ \text{all}\ \ x \in G\ ,\ h \in H\ ,\ \rho \in \hat{H}\ .$$

Under suitable conditions on the group H and the orbit $G\cdot\rho$ we show that $G\cdot\rho$ is finite (1.4).

This result also turns out to be useful in section 2 where we study square-integrable irreducible representations and prove that among the $[FC]^-$ groups only those of type I may possess such representations.
This gives that $\pi \in \hat{G}$ is square-integrable iff $\{\pi\}$ is open in $\hat{G}$.

We hope to study this question for a larger class of groups at a later occasion.

In [17] we studied the connection between the toplogy of the dual and the group structure for $[FC]^-$ groups. The main results of the present article extend Proposition (2.1), (3), and Theorem (2.4) of [17] , where type I and $[FIA]^-$ assumptions were posed on the groups.

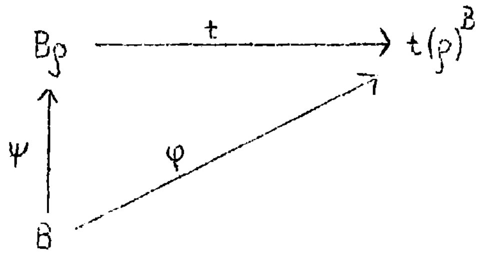
1. In this section the groups will not be assumed separable. Let G be a locally compact group and B a subgroup of the automorphism group $Aut(G)$. G is an $[FIA]_B^-$ group if B has compact closure in $Aut(G)$. The set $\mathfrak{X}^B(G)$ of B-characters consists of the nonzero extreme points of the convex set of continuous positive definite B-invariant functions φ on G with $\varphi(e) \leq 1$. $^B(G)$ is given the topology of uniform convergence on compacta.
If $G \in [FIA]_B^-$ and $B \supset I(G)$ there is an open and continuous surjection $t : \hat{G} \to \mathfrak{X}^B(G)$ given by $t(\pi)(x) = \int_{\bar{B}} < \pi \circ \beta(x)v, v > d\beta$, where $v \in H_\pi$ with $\|v\| = 1$, $x \in G$, and $d\beta$ denotes normalized Haar measure on the compact group $\bar{B}$, [14] Lemma 5.1.

<u>(1.1) Lemma</u>. Let $G \in [FIA]_B^-$ where B is a subgroup of $Aut(G)$ containing $I(G)$, and fix $\rho \in \hat{G}$. Put

$$(\beta\rho)(g) = \rho(\beta^{-1}(g)) , \text{ all } \beta \in B , g \in G .$$

Then the map $\psi : \beta \to \beta\rho$, $B \to \hat{G}$, is continuous.

<u>Proof</u>. If $f : G \to \mathbb{C}$, put $f^{\beta}(g) = f(\beta^{-1}(g))$, all $\beta \in B$, $g \in G$. Let $t : \hat{G} \to \mathfrak{X}^{B}(G)$ be the open and continuous surjection defined in [14] Lemma 5.1 . It is easily seen that $t(\beta\rho) = t(\rho)^{\beta}$, all $\beta \in B$, $\rho \in \hat{G}$; in other words the following diagram commutes:



where $\varphi : \beta \to t(\rho)^{\beta}$ is continuous by [14] Lemma 5.6. Since t is open and surjective it follows that ψ is continuous.

Q. e. d.

If H is a closed normal subgroup of the locally compact G , put $I(H,G)$ = the set of all inner automorphisms of G restricted to H . Thus $I(H,G)$ is a subgroup of $Aut(H)$.
If $\rho \in \hat{H}$ then G acts on ρ by inner automorphisms : $x \cdot \rho(h) = \rho(x^{-1}hx)$, all $x \in G$, $h \in H$.

<u>(1.2) Corollary</u>. Let $G \in [FIA]^{-}$, H a closed normal subgroup of G , and $\rho \in \hat{H}$. Assume that each point $x.\rho$ of the orbit $G.\rho$ is open in $B\rho$, where $B\rho$ is given the topology induced from $\hat{H}$. Then $G.\rho$ is finite.

<u>Proof</u>. We have $H \in [FIA]^{-}_{G}$, i.e. the closure B of $I(H,G)$ in $Aut(H)$ is compact. By lemma (1.1) B operates continuously

on the orbit $B\rho$. Hence $B\rho$ is compact and this implies $G\cdot\rho$ is finite, since each point of $G\cdot\rho$ is open in $B\rho$.

Q. e. d.

We got the idea of the next two results from the arguments on page 283 of [12] .

(1.3) Lemma. Let H be a compactly generated closed normal subgroup of the locally compact group G . Suppose that for each $h \in H$, $h \neq e$, there is a continuous G-invariant function $\varphi = \varphi_h : G \to \mathbb{C}$ such that $\varphi(h) \neq 0$ and $\varphi(e) = 0$. Then H possesses a fundamental system of G-invariant neighborhoods of e , i.e. $H \in [SIN]_G$.

Proof.. Arguing as in [3] 17.3.7 we see that the following condition is satisfied:

(*) If C is a compact subset of H such that $e \notin C$ then there is a neighborhood V of e in H such that $V \cap C = \emptyset$ and V is G-invariant.

The Lemma then follows as [3] 17.3.8.

Q. e. d.

(1.4) Lemma. Let $G \in [FC]^-$, H a compactly generated closed normal subgroup of G such that each G-orbit $G\cdot\dot{x} = \{y(xH)y^{-1} : y \in G\}$ in G/H is finite. Assume $\rho \in \hat{H}$ is such that each point $x\cdot\rho$ of $G\cdot\rho$ is open in $\hat{H}$. Then $G\cdot\rho$ consists of only a finite number of points.

Proof. Assume each point of the orbit $G\cdot\rho$ is open in $\hat{H}$. Put $\sigma = \bigoplus_{\tau \in G\cdot\rho} \tau$ and $N = \{x \in G : \sigma(x) = I\}$. N is a closed normal subgroup of G (it is normal because of the definition of σ) and σ may be lifted to a representation of the factor group G/N .

Replacing G by G/N , without change of notation, we may assume that σ is injective.

Fix $h_0 \in H$ with $h_0 \neq e$. Since $\sigma(h_0) \neq I$ we may choose a vector $v \in H_\sigma$ with $||v|| = 1$ and $\sigma(h_0)v \neq v$. Put

$$\varphi(h) = 1 - \langle\sigma(h)v,v\rangle \quad \text{all} \quad h \in H .$$

Then φ is continuous, G-invariant , $\varphi(e) = 0$, and $\varphi(h_0) \neq 0$ so that $H \in [SIN]_G$ by Lemma (1.3).

Now we consider the topological group $\mathcal{G}$ obtained as follows: $\mathcal{G}$ and G are to be equal as sets, however, $\mathcal{G}$ is equipped with the topology in which the induced topology on H is unchanged, and which makes H open in $\mathcal{G}$. Then $\mathcal{G} \in [SIN]$ since $H \in [SIN]_{\mathcal{G}}$. By hypothesis $\mathcal{G}/H$ has finite $\mathcal{G}$-orbits and it follows that $\mathcal{G} \in [FC]^-$. Hence $\mathcal{G} \in [FIA]^- = [FC]^- \cap [SIN]$, and we may apply Corollary (1.2) to see that the $\mathcal{G}$-orbit of ρ is finite. Clearly, $G\cdot\rho$ and $\mathcal{G}\cdot\rho$ are identical, and the proof is complete.

Q. e. d.

Recall that the set of all periodic elements of an $[FC]^-$ group G forms a closed characteristic subgroup P of G , [4] , called the periodic subgroup.

<u>(1.5) Proposition</u>. Let $G \in [FC]^-$. Then the dual space $\hat{G}$ has only a finite number of connected components iff the periodic subgroup P is finite.

<u>Proof</u>. Suppose $\hat{G}$ has only a finite number of connected components. We may fix a compact open G-invariant subgroup K of P such that $G/K \simeq \mathbb{R}^h \times D$ where D is a discrete group with a finite number of G-orbits , [11] Proposition 2.1 . Since K is compact $\widehat{G/K}$ is embedded in $\hat{G}$ as an open and closed subspace in the natural way. By hypothesis each connected component $\mathcal{C}_\pi$ of $\hat{G}$

is open and closed. Let $\alpha \in \widehat{G/K}$, the above implies that the connected component of α in $\hat{G}$ is actually contained in $\widehat{G/K}$. Hence the number of connected components in $\widehat{G/K}$ is finite.

We shall see next that P/K is finite. Now G/K is an $[FIA]^-$ group , hence we may use the continuous and open surjection $t : \widehat{G/K} \to \mathfrak{X}(G/K)$, ([14] 5.1 and 5.2) . It follows that $\mathfrak{X}(G/K)$ has only a finite number of connected components. By [16] Proposition (2.10) the periodic subgroup of G/K , which equals P/K , is finite.

It remains to show that K is finite. Since K is compact each $\pi \in \hat{G}$ lies over some G-orbit $G\cdot\rho$ in $\hat{K}$. We shall denote by $\hat{G}_{\rho,K}$ the set of all $\pi \in \hat{G}$ which lies over the orbit of $\rho \in \hat{K}$. Each $\hat{G}_{\rho,K}$ is an open and closed subspace of $\hat{G}$ ([17] Lemma (1.2)) and hence the connected component $\mathcal{C}_\pi$ of each $\pi \in \hat{G}_{\rho,K}$ is contained in $\hat{G}_{\rho,K}$. By hypothesis the number of $\hat{G}_{\rho,K}$'s must then be finite. It is well known that the $\hat{G}_{\rho,K}$'s are in bijective correspondence with the G-orbits in $\hat{K}$, so that the number of orbits is finite. By Lemma (1.4) each orbit is finite, and hence $\hat{K}$ is finite. Then a well known result says that K is finite, [2] .

Conversely, assume the periodic group P is finite. By [11] Proposition 2.7 $G = \mathbb{R}^n \times H$ where H is a discrete $[FC]^-$ group , hence $G \in [FIA]^-$ and it follows from [6] p. 79 (remark d) that $\hat{G}$ has only finitely many connected components.

Q. e. d.

2. This section is devoted to the study of square-integrable irreducible representations of $[FC]^-$ groups. J. Dixmier has asked if such representations are necessarily open as points in the reduced dual $\hat{G}_r$ of separable locally compact (unimodular) type I groups G . ([3] 18.9.1) . In [15] M. Rieffel studied this

question using Hilbert algebra tecniques, concluding that more group theoretic considerations would be necessary. R. Lipsman showed that the answer to Dixmier's question is affirmative for split-rank one semisimple groups, [10]. The author settled the question for type I $[FC]^-$ groups in [17] Theorem (2.4). In this connection it is interesting to note that noncompact [SIN]-groups have no square-integrable irreducible representations, which may be seen as follows: If $\pi \in \hat{G}$ is square-integrable we may assume π is a subrepresentation of the left regular representation of G on $L^2(G)$, hence $\pi(f)$ is a compact operator for all $f \in L^1(G)$;

$$\pi(f)h(x) = \int_G k(x,y)h(y)dy \quad , \quad \text{all} \quad x \in G, h \in L^2(G) \ ;$$

where the kernel $k(x,y) = f(xy^{-1})$. Now we choose f equal to the characteristic function χ_V of a compact invariant neighborhood V of e . Then χ_V is a central function on G and we have

$$\pi(\chi_V)\pi(\varphi) = \pi(\chi_V * \varphi) = \pi(\varphi * \chi_V) = \pi(\varphi)\pi(\chi_V) \ ,$$

all $\varphi \in L^1(G)$. In other words, $\pi(\chi_V)$ commutes with the irreducible π , and hence $\pi(\chi_V) = c(V)I$ where $c(V) \in \mathbb{C}$. Letting $h \in H_\pi$ with $\|h\| = 1$ we have

$$c(V) = \langle\pi(\chi_V)h,h\rangle = \int_V \langle\pi(x)h,h\rangle dx \ ,$$

and since π is continuous and $G \in [SIN]$ there is a compact neighborhood V of e , invariant under inner automorphisms, such that $c(V) \neq 0$. Thus the identity operator on H_π is compact and this forces $\dim H_\pi < \infty$. By a result of A. Weil ([18] p. 70) G is compact.

We shall prove below that the type I hypothesis in [17] Theorem (2.4) gives no loss of generality. In fact we shall demonstrate the following: If $G \in [FC]^-$ is separable and $\pi \in \hat{G}$ is square-integrable then G is type I . Along the road we generalize to multiplier representations the well known result stating that for infinite discrete groups the left regular representation fails to have irreducible subrepresentations. We shall therefore need some facts concerning multipliers on locally compact groups.

Let ω be a normalized multiplier on the locally compact unimodular group G , i.e., $\omega : G \times G \to S^1$ has the properties

(i) $\omega(x,e) = \omega(e,x) = 1$

(ii) $\omega(x,y)\omega(xy,z) = \omega(x,yz)\omega(y,z)$

(iii) $\omega(x,x^{-1}) = 1$,

all $x,y,z \in G$, and

(iv) ω is a measurable function of $G\times G$ into S^1 .

Here S^1 denotes the circle group.

We let

$$f *_{\omega} h(x) = \int_G \omega(x^{-1},y)f(y)h(y^{-1}x)dy ,$$

all $f,h \in C_c(G)$, $x \in G$, and

$$f^*(x) = \overline{f(x^{-1})} , \qquad \text{all } f \in C_c(G) , x \in G .$$

The set $C_c(G)$ of all complex valued continuous functions on G with compact support becomes a Hilbert algebra with the multiplication and involution defined above and the L^2- inner product. We shall denote this Hilbert algebra by $A(G,\omega)$.

Let π be a unitary continuous irreducible ω-representation of G on the Hilbert space H_π (*) . We say that π is <u>square-integrable</u> if all the coordinate functions $x \to \langle\pi(x)v,v\rangle$

(*) I.e. $\pi(x)\pi(y) = \omega(x,y)\pi(xy)$, all $x,y \in G$.

are L^2 - fuctions of G , or what amounts to the same, if π is unitary equivalent to an (irreducible) direct summand of the left regular ω-representation L^ω of G on the Hilbert space $L^2(G)$, where

$$L_y^\omega f(x) = \omega(x^{-1},y)f(y^{-1}x) \text{ , all } f \in L^2(G),x,y \in G \text{ .}$$

Given an ω-representation ρ of G we may form a multiplicative*-representation (also denoted by ρ) of the algebra $A(G,\omega)$ in the usual way:

$$\rho(h)v = \int_G h(x)\rho(x)v dx \text{ , all } v \in H_\rho \text{ , } h \in A(G,\omega) \text{ .}$$

The algebra representation corresponding to L^ω is given by left convolution,

$$L_h^\omega f = h *_\omega f \text{ , all } h \in A(G,\omega) \text{ , } f \in L^2(G) \text{ .}$$

It is easy to verify that the irreducible ω-representation π of G is square-integrale iff the linear map $h \mapsto \pi(h)$ is continuous, i.e., there is a constant C such that $||\pi(h)|| \leq C||h||_2$, all $h \in A(G,\omega)$.

In what follows we shall have the opportunity to apply the theory of square-integrable representations of Hilbert algebras as developed in [15] . The following result is analogous to Corollary 5.12 in [15] where it is proved for ordinary unitary representations.

(2.1) Lemma. Let G be an infinite discrete group. Then G has no irreducible square-integrable multiplier representations.

Proof. Let ω be a normalized multiplier on G . Since $A(G,\omega)$ has an identity we see from [15] Corollary 5.11 that every square-integrable irreducible representation of $A(G,\omega)$ has finite dimension. Hence the same holds for the irreducible square-inte-

grable ω-representations of G. Suppose π is such a finite dimensional ω-representation of G, and consider the central group extension defined by $\omega : (e) \to S^1 \to G(\omega) \to G \to (e)$.
Let $\chi_0 : t \mapsto t$ be the generating character of the circle group S^1, and χ_0 an extension of χ_0 to an $\bar{\omega}$-representation of $G(\omega)$ (for example, we may let $\tilde{\chi}_0(s,x) = \bar{s}$, all $s \in S^1$, $x \in G$). Then the representation $\tau = \tilde{\chi}_0 \otimes \pi$ is an ordinary unitary representation of $G(\omega)$ and is of finite dimensjon.
If π were square-integrable, so were τ (since S^1 is compact). But A. Weil has proved that noncompact groups have no finite dimensional square-integrable representations, [17] p.70, and $G(\omega)$ is noncompact since G is infinite.

Q. e. d.

Let ω be a multipler on the discrete group G. $x \in G$ is said to be <u>ω-regular</u> if $\omega(x,a) = \omega(a,x)$ whenever a commutes with x. If $x \in G$ is ω-regular then all conjugates yxy^{-1} of y are ω-regular ([7] Lemma 3). In order to illustrate that the above result has applications in the theory of multiplier representations we prove the following.

<u>(2.2) Corollary</u>. Let G be a discrete group, ω a normalized multiplier on G. Suppose the number of finite ω-regular conjugacy classes is finite. Then the left regular ω-representation L^ω is type I iff G is finite.

<u>Proof</u>. By [7] Theorem 3 L^ω is the direct sum of a finite number primary ω-representations. If L^ω is type I it follows that L^ω is a direct sum of irreducible ω-representations. Hence G is finite by Lemma (2.1). The converse is clear.

Q.e.d.

We turn next to the main result of this section. The structure theory of $[FC]^-$ groups, [4], [11], and Mackey's theory of induced representations, [1], are important ingredients of our proof. If H is a closed subgroup of G and ρ a unitary representation of H we denote by $\mathrm{Ind}_H^G(\rho)$ the unitary representation of G induced from ρ. If τ is a representation of a factor group of G we let τ' be the inflation of τ back to $G: \tau'(g) = \tau(\dot{g})$ where $g \mapsto \dot{g}$ denotes the quotient map.

(2.3) Theorem. Let $G \in [FC]^-$ be separable.

(i) Suppose there is an irreducible discrete summand of the regular representation of G on $L^2(G)$. Then G is of type I, and G satisfies an exact sequence of topological groups $(e) \to K \to G \to \mathbb{R}^n \to (e)$ where K is compact.

(ii) $\pi \in \hat{G}$ is square-integrable iff $\{\pi\}$ is open in $\hat{G}$.

Proof. By the structure theorem for $[FC]^-$ groups we may choose an open normal subgroup H of G on the form $(e) \to K \to H \to \mathbb{R}^n \to (e)$ where K is compact. Such groups H are well known to be (and easily seen to be) type I, [11]. Assume there is a square-integrable representation π in $\hat{G}$. Since H is open in G we may use the restriction of Haar measure on G to H as a Haar measure on H. Then it becomes clear that each vector $v \in H_\pi$ gives a square-integrable coordinate function $h \to \langle\pi(h)v,v\rangle$ for the restriction $\pi|_H$. Hence $\pi|_H$ splits into a direct sum of irreducible square-integrable representations of H (Kunze [9], Corollary to Thm.2), and it may be seen that $\pi|_H$ is concentrated on a single G-orbit in $\hat{H}$: $\pi|_H = m\cdot\bigoplus_{\tau\in G\cdot\rho}\tau$, for some $\rho \in \hat{H}$. Now H is type I and therefore [17] Theorem (2.4) implies $\{x\cdot\rho\}$ is open in $\hat{G}$, all $x \in G$.

Hence, by Lemma (1.4) the orbit $G \cdot \rho$ is finite, and the isotropy group $G(\rho) = \{x \in G : x \cdot \rho \simeq \rho\}$ has finite index in G.

We show next that $G(\rho)/H$ is finite. Now $\pi = \mathrm{Ind}_{G(\rho)}^{G}(\sigma)$ for some σ in $\widehat{G(\rho)}_{\rho}$, and σ is on the form $\tilde{\rho} \otimes \gamma'$ where γ is an irreducible multiplier representation of $G(\rho)/H$ (say ω-representation) and γ' its inflation back to $G(\rho)$, and $\tilde{\rho}$ denotes some extension of ρ to an $\bar{\omega}$-representation of $G(\rho)$, [1]. Since π is square-integrable γ is so ([8] Corollary 11.1). Thus $G(\rho)/H$ is finite by Lemma (2.1).

Hence $|G/H| < \infty$ and H is type I so that G must be type I ([5] Corollary 2.5). Hence $\{\pi\}$ is open in $\hat{G}$ by [17] Theorem (2.4).

Since H is in the form $(e) \to K \to H \to \mathbb{R}^n \to (e)$ G must be on the form $(e) \to K \to G \xrightarrow{p} \mathbb{R}^n \times F \to (e)$ where K is compact and F is finite. Replacing K with $p^{-1}(f)$ the theorem follows.

Q. e. d.

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