Degrees of functionals

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In this paper we will discuss some problems of degree-theoretic nature in connection with recursion in normal objects of higher types.

Harrington [2] and Loewenthal [6] have proved some results concerning Post's problem and the Minimal Pair Problem, using recursion modulo subindividuals. Our degrees will be those obtained from Kleene-recursion modulo individuals. To solve our problems we then have to put some extra strength to ZFC. We will first assume $V = L$, and then we restrict ourselves to the situation of a recursive well-ordering and Martin's axiom.

We assume familiarity with recursion theory in higher types as presented in Kleene [3]. Further background is found in Harrington [2], Moldestad [9] and Normann [11]. We will survey the parts of these papers that we need.

In section 1 we give the general background for the arguments used later. In section 2 we prove some lemmas assuming $V = L$. In section 3, assuming $V = L$ we solve Post's problem and another problem using the finite injury method. We will thereby describe some of the methods needed for the more complex priority argument of section 4 where we give a solution to the minimal pair problem for extended r.e. degrees of functionals.

In section 5 we will see that if Martin's Axiom holds and we have a minimal well-ordering of $tp(1)$ recursive in ${}^3E$, we may use the same sort of arguments as in parts 3 and 4.

1 Preliminaries

1A Notation. For some fixed $k \geq 1$, let I be the set of functionals of type $\leq k$. We let $S \subseteq I$ be the set of functionals of type $< k$. The elements of S are called subindiviauals, they are denoted by i, j etc. n, m will be used for natural numbers, e mostly for indices. The elements a, b, c of I are called individuals.

$f : I \to \omega$ is called a function. We identity subsets of I with their characteristic functions.

F : functions $\to \omega$ is called a functional. A functional F is called normal if ${}^{k+2}E$ is recursive in F, where

$$ {}^{k+2}E(f) = \begin{cases} 0 & \text{if } \exists a \in I \quad f(a) = 0 \\ 1 & \text{if } \forall a \in I \quad f(a) \neq 0 \end{cases} $$

We will always assume f to be total.

By k+1-sc (F,a) we mean those subsets of I recursive in F and a.

By k+1-en (F,a) we mean those subsets of I semi-recursive in F and a.

By extended recursion, we mean recursion modulo an arbitrary individual.

1B Companion Theory

In Normann [11] a companion theory for recursion in a normal type $k+2$ object was developed and studied. The spectrum of a functional F was defined as follows:

Let $<$ be a partial ordering on I. Let $a \simeq b$ if $a < b$ and $b < a$. Let x be a set.

We say that $<$ is a code for x if ${}^{<}/_{\simeq}$ is isomorphic to

$(\in \cup =) \upharpoonright TC\{x\}$ (TC is the transitive closure).

Let $x \in M_a(F)$ if there is a code for x recursive in a and F.

$\langle M_a(F)\rangle_{a\in I}$ is called the spectrum of F and is denoted $\mathrm{Spec}(F)$.

Theorem 1.1 (Normann [11]) (For $F = {}^{k+2}E$ also MacQueen [7])

When F is a normal functional, $\mathrm{Spec}(F)$ is the least family $\langle M_a\rangle_{a\in I}$ satisfying:

i Each M_a is rudimentary closed in F.

ii If φ is a Δ_0-formula, $\vec{x}$ parameters from M_a, and if $\forall b \in I \; \exists x \in M_{\langle a,b\rangle}\varphi(x,\vec{x},F)$ then $\exists h \in M_a$ (h is a function and $\forall b \in I \quad \varphi(h(b),\vec{x},F)$).

This principle is called Σ^*-collection.

Remark Since $h \in M_{\langle a,b\rangle}$ and $b \in M_{\langle a,b\rangle}$, $h(b) \in M_{\langle a,b\rangle}$.

Definition Following Sacks [13] we say:

Let $A \subseteq V$ be a set. A is locally of type $k+1$ if

$$\forall x \in V \; (x \in A \iff x \text{ has a code in } A)$$

By the definition of the spectrum, it is clear that each $M_a(F)$ is locally of type $k+1$.

We will also have that each $M_a(F)$ is uniformly projectable to ω.

A subset $A \subseteq I$ is Σ^*_a-definable if there is a Δ_0-formula φ with parameters from M_a such that

$$b \in A \iff \exists x \in M_{\langle a,b\rangle}\varphi(x,b).$$

It is essentially proved both in Harrington [2] and in Normann [11] that

$$\Sigma^*_a(F) = k{+}1\text{-en}\,(F, {}^{k+2}E, a).$$

MacQueen [7] proved a selection principle for subindividuals and Harrington [2] used this to obtain the following:

<u>Theorem 1.2</u> (Harrington [2], Simple and further reflection)

Let $a \in I$, F a normal type $k+2$ functional.

Let $\langle M_a \rangle_{a \in I} = \mathrm{Spec}(F)$ and let $\mathcal{M}_a = \bigcup_{i \in S} M_{\langle a,i \rangle}$.

<u>a</u> $TC(M_a) <_{\Sigma_1} TC(\mathcal{M}_a)$

<u>b</u> Let $C \subseteq S$ be complete Σ_a^* among $\Sigma_a^* \restriction S$.
Identifying C with it's characteristic function we have $C \in I$ and obtain

$$M_a <_{\Sigma_1} M_{a,C}.$$

<u>1 C Another approach to recursion in normal functionals</u>

The construction of Spec(F) is relevant when we investigate $k+1 - en(F)$ and $k+1 - sc(F)$.
The definition of a Σ^*-subset of I was simple, but if we are interested in other 'semi-recursive' sets, the situation is more complicated. The following definition may be viewed as a generalization of recursion in a general E and a relation. In sections 2, 3 and 4 we will use it as a technical tool for making some notions precise and handy. In section 5 we use this theory to 'compare' theories on different domains.

<u>Definition of E(R)</u>

Let $R \subseteq V$ be a relation. We define the functions recursive relative to R with indices by the following schemes:

<u>i</u> $f(x_1,\ldots,x_n) = x_i$ $\quad e = \langle 1,n,i \rangle$

<u>ii</u> $f(x_1,\ldots,x_n) = x_i - x_j$ $\quad e = \langle 2,n,i,j \rangle$

<u>iii</u> $f(x_1,\ldots,x_n) = \{x_i,x_j\}$ $\quad e = \langle 3,n,i,j \rangle$

iv $f(x_1,\ldots,x_n) \simeq \bigcup_{y\in x_1} h(y,x_2,\ldots,x_n)$ $e = \langle 4,n,e'\rangle$ where e' is an index for h.

v $f(x_1,\ldots,x_n) \simeq h(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n))$
$e = \langle 5,n,m,e',e_1,\ldots,e_m\rangle$

vi $f(x_1,\ldots,x_n) \simeq x_i \cap R$ $e = \langle 6,n,i\rangle$

vii $f(e_1,x_1,\ldots,x_n,y_1,y_m) \simeq \{e_1\}^R(x_1,\ldots,x_n)$ $e = \langle 7,n,m\rangle$.

The functions defined by these schemes are called E-recursive relative to R, and the functions are denoted $\{e\}^R$.

Remark: All functions rudimentary in R will be E-recursive relative to R. Since for each $n \in \omega$ the constant function n is rudimentary, these functions will be E-recursive. Combining schemes i and v we may commute the arguments in the functions. E-recursion is nothing but the rudimentary function schemes augmented with diagonalization.

The schematic definition gives us canonical concepts of

i length of a computation $\|\ \|$

ii subcomputation

iii computation tree

By standard proofs we will obtain the recursion theorems, giving recursive fix-points and least recursive fix-points.

The connection between E-recursion and recursion in a normal functional is given by the following lemma, which also justifies the term E-recursion:

Lemma 1.3 In E-recursion theory there is an index e such that for arbitrary $R, x, e_1, \vec{x}$

$$\{e\}^R(x,e_1,\vec{x}) \simeq \begin{cases} 0 & \text{if } \forall y \in x \quad \{e_1\}^R(y,\vec{x}) \simeq 0 \\ 1 & \text{if } \forall y \in x \quad \{e_1\}^R(y,\vec{x})\downarrow \text{ and} \\ & \quad \exists y \in x \quad \{e_1\}^R(y,\vec{x}) \neq 0 \end{cases}$$

where $\downarrow$ means 'has a value'

Proof: There is a rudimentary function φ such that $\varphi(0) = 0$ and $\varphi(x) = 1$ for all $x \neq 0$. So we may assume that $\{e_1\}$ takes values $0 = \emptyset$ and $1 = \{\emptyset\}$ only.

Let $\{e\}^R(x,e_1,\vec{x}) = \bigcup_{y \in x} \{e_1\}(y,\vec{x})$.

Corollary 1.4 (Stage comparison)

There is an E-recursive function p such that $p(\sigma_1,\sigma_2)\downarrow$ if σ_1 or σ_2 is a computation and then

$$p(\sigma_1,\sigma_2) = \begin{cases} 0 & \text{if } \|\sigma_1\| \leq \|\sigma_2\| \\ 1 & \text{if } \|\sigma_1\| > \|\sigma_2\| \end{cases}$$

Proof: By Lemma 1.3 and the recursion theorem, the standard proof in higher type function theory is valid.

Remark: Corollary 1.4 may uniformly be relativized to an arbitrary R.

Corollary 1.5 In E-recursion we may uniformly select an element in a semirecursive nonempty subset of ω (Gandy Selection).

Proof: By Grilliot [1] this is a consequence of 1.4.

Definition Let $R \subseteq V$, $\vec{y} \in V^m$. Let φ be a partial map from V^n to V. We say that φ is recursive in $\vec{y}$ relative to R if there is an index e in E-recursion such that
$\forall \vec{x} \in V^n(\varphi(\vec{x}) \simeq \{e\}^R(\vec{x},\vec{y}))$.

We then obtain the natural definitions of recursive in $\vec{y}$ relative to R and semirecursive in $\vec{y}$ relative to R.

<u>Remark</u>: If P is E-recursive in R and φ is partially P-recursive, then φ is R-recursive.

<u>Definition</u> Let $A \subseteq V$, $R \subseteq V$. Let the <u>E-recursive closure of A relative to R</u> be

$$\mathfrak{R}(A;R) = \{\{e\}^R(\vec{x});\ e \in \omega,\ \vec{x} \in A^n,\ n \in \omega\}$$

A is <u>E-recursively closed relative to R</u> if

$$\mathfrak{R}(A;R) = A.$$

If A is E-recursively closed relative to R, we may split A up as follows

$$\langle \mathfrak{R}(\{x\};R)\rangle_{x\in A}$$

This structure will satisfy the following version of Σ^*-collection: If for some $z,\vec{x}$ in $\mathfrak{R}(\{x\};R)$ and some Δ_0-formula φ

$$\forall y \in z \exists r \in \mathfrak{R}(\{y,x\};R)\varphi(r,\vec{x},y,R) \quad \text{then}$$
$$\exists f \in \mathfrak{R}(\{x\};R)(\text{func}(f)\ \&\ \text{dom}(f) = z$$
$$\&\quad y \in z\varphi(f(y),\vec{x},y,R))$$

Also $\mathfrak{R}(\{x\};R)$ will be rudimentary closed in R.

On the other hand, if $\langle A_x\rangle_{x\in A}$ is any decomposition of A satisfying $\Sigma^*(R)$-collection and rudimentary closure relative to R, it follows by induction on the length of computations that each A_x will be E-recursively closed relative to R and that the computation tree of a computation in x will be in A_x. Combining these two observations we see that when F is a normal functional,

$$\text{Spec}(F) = \langle \mathfrak{R}(\{a,I\};F)\rangle_{a\in I}$$

1 D Semirecursion and relative recursion

Definition Let $R \subseteq V$ be a relation. Let

$$\mathrm{Spec}(R) = \langle \mathcal{R}(\{a,I\};R)\rangle_{a \in I}$$

We will first pose three problems which we have not answered, but believe to have negative solutions:

Problems

Let $R \subseteq V$, $\langle M_a \rangle_{a \in I} = \mathrm{Spec}(R)$, $M = \bigcup_{a \in I} M_a$. As proved in Normann [11], each M_a will be locally of type $k+2$.

1. If $x \in M$, is there an $a \in I$ such that

$$M_a = \mathcal{R}(\{I,x\};R)\ ?$$

2. If $x \in M$, is $\mathcal{R}(\{I,x\};R)$ locally of type $k+1$?

3. Is M E-semirecursive in I relative to R?

We omit some of the difficulties induced by these problems by restricting ourselves to ${}^1M$:

Definition When $\langle M_a \rangle_{a \in I}$ is a spectrum, let ${}^1M = \{\langle a,y \rangle ; y \in M_a\}$.

${}^1M$ is E-semirecursive in I relative to R and problems 1 and 2 are trivially correct for $x \in {}^1M$.

From now on, let R be fixed, $\langle M_a \rangle_{a \in I} = \mathrm{Spec}(R)$, $M = \bigcup_{a \in I} M_a$. Let, for $x \in M$, $M_x = \mathcal{R}(\{x,I\};R)$.

Definition Let $Q \subseteq M$, $a \in I$.

i We say that Q is $\Sigma_a^*(R)$-definable if there is a Δ_0-formula φ with parameters from M_a such that

$$x \in Q \iff \exists y \in M_{\{a,x\}} \varphi(x,y,R)$$

ii Q is weakly $\Sigma_a^*(R)$-definable ($w-\Sigma_a^*(R)$) if there is a Δ_o-formula φ such that

$$x \in Q \iff \forall b \in I(x \in M_{\langle a,b\rangle} \Rightarrow \exists y \in M_{\langle a,b\rangle}\varphi(x,y,R))$$

Remark: If Q is $\Sigma_a^*(R)$, Q will be $w-\Sigma_a^*(R)$.

If $Q \subseteq {}^1M$, or if Problem 1 has a positive solution, the two concepts coincide.

iii Q is $\Delta_a^*(R)$ ($w-\Delta_a^*(R)$) if both Q and $M \setminus Q$ are $\Sigma_a^*(R)$ ($w-\Sigma_a^*(R)$).

Definition Let $Q \subseteq I \times M$ $A_Q = \{\langle a,f\rangle;$ f is the characteristic function of a code for a set x and $\langle a,x\rangle \in Q\}$.

F_Q is the characteristic function of A_Q. By some effective identification of $tp(k) \times tp(k+1)$ with $tp(k+1)$, F_Q will be of type $k+2$.

Lemma 1.6 Let $Q \subseteq M$.

a $Q, A_Q \cap M$ and $F_Q \cap M$ are $w-\Delta^*(R)$ in each other.

b If $Q \in w-\Delta_a^*(R)$, F_Q is weakly Kleene-recursive in $F_R, {}^{k+2}E, a$ (For definition, see Moldestad [9] or the proof.)

c If $Q \in \Sigma_a^*(R)$ and $Q \subseteq {}^1M$, then A_Q is Kleene-semirecursive in $F_R, {}^{k+2}E, a$.

Proof: a is trivial since each M_b is locally of type $k+1$.
To prove b we must find an index e such that in Kleene-recursion

$$F_Q(\lambda b\{e'\}(F_R, {}^{k+2}E, a, b, \vec{c})) \simeq \{e\}(F_R, {}^{k+2}E, a, \vec{c}, e')$$

for arbitary $e' \in \omega$, $\vec{c} \in I^n$.

Δ_o-formulas over $I^n \times \mathcal{P}(I)^m$ are computable in ${}^{k+2}E$ and the unbounded quantifiers over $M_{a,\vec{c}}$ needed in the $w-\Delta_a^*$-definition of F_Q from R may be replaced by unbounded quantifiers over $k+1\text{-sc}(F_R, {}^{k+2}E, a, \vec{c})$ (see Normann [11] for details). We may then perform the wanted computation using Gandy's selection operator.

We use similar arguments to prove <u>c</u>.

<u>1 F</u> <u>Some more notations</u>

Let R be a relation, $M_x(R) = \mathcal{R}(\{x,I\};R)$

$K_o^x(R) = \text{Sup}(\text{On} \cap M_x(R))$

$K_n^x(R) = \text{Sup}\{K_o^{\{x,i\}}(R);\ i \in tp(n)\} = \text{Sup}(\text{On} \cap \bigcup_{i \in tp(n)} M_{\{x,i\}}(R))$

$\lambda_o^x(R)$ = Least ordinal not in $M_x(R)$

= Ordertype of $\{\alpha;$ there is an $E(R)$- computation with arguments x, I of length $\alpha\}$

$\lambda_n^x(R)$ = Least ordinal not in $\bigcup_{i \in tp(n)} M_{\{x,i\}}(R)$

= Ordertype of $\{\alpha;$ there is an $E(R)$-computation with arguments x, I and some $i \in tp(n)$ of length $\alpha\}$.

The equalities in the definition is fairly easy to show.

<u>Definition</u> Let $\sigma \in \text{On}$, $R \subseteq V$.

$M_a^\sigma(R) = \{x; \exists e(\{e\}^R(a,I) \simeq x \ \&\ \|\langle e,a,I,x\rangle\|_R \leq \sigma\}$.

$M^\sigma = \bigcup_{a \in I} M_a^\sigma$. ${}^1M^\sigma = \{\langle a,y\rangle;\ y \in M_a^\sigma\}$.

Now assume that we have a Gödel-enumeration $\{\varphi_n\}_{n \in \omega}$ of the Δ_o-formulas. Identify $\omega \times S$ with S, so that $i_o \in \omega$, $i_1 \in S$.

Our next problem will be to find recursive, monotone approximations of the w-$\Sigma^*_{i,a}(R)$-subsets of M:

Let $i = \langle i_o, i_1 \rangle$ and let

$x \in I^\sigma_{i,a}(R)$ if for some $b \in I$, $\sigma_1 \leq \sigma$ $x \in M^{\sigma_1}_{i,a,b}(R)$ and $\forall b \in I\ (x \in M^{\sigma_1}_{i,a,b}(R) \Rightarrow \exists y \in M^\sigma_{i,a,b} \exists z \in y\ \varphi_{i_o}(i_1,a,z,x,R))$

Let $I_{i,a}(R) = \bigcup_{\sigma \in On} I^\sigma_{i,a}(R)$.

<u>Lemma 1.7</u>

<u>a</u> If $\sigma_1 > \sigma_2$ then $I^{\sigma_2}_{i,a}(R) \subseteq I^{\sigma_1}_{i,a}(R)$

<u>b</u> $\{I_{i,a}(R);\ i \in S\} = \{w-\Sigma^*_{j,a}(R);\ j \in S\}$

<u>Proof</u>: The monotonity is immediate from the definition. To prove <u>b</u> we first prove:

<u>Claim</u>: The following are equivalent:

<u>i</u> $x \in I_{i,a}$

<u>ii</u> $\forall b \in I\ (x \in M_{i,a,b}(R) \Rightarrow \exists y \in M_{i,a,b} \exists z \in y\ \varphi_{i_o}(i_1,a,z,x,R))$.

<u>Proof</u>: <u>ii</u> $\Rightarrow$ <u>i</u>. Pick σ_1 s.t. for some b $x \in M_{i,a,b}$. We may then find a suitable σ by Σ^*-collection over $\{b;\ x \in M^{\sigma_1}_b\}$.

<u>i</u> $\Rightarrow$ <u>ii</u>. Let σ_1 and σ be as in the definition, and choose σ_1 minimal, and then σ minimal. Let b be such that $x \in M_{i,a,b}$. If $x \in M^{\sigma_1}_{i,a,b}$ there is no problem. If not, let σ_2 be minimal such that $x \in M^{\sigma_2}_{i,a,b}$. Then also $\sigma_1 \in M_{i,a,b}$ and $\sigma \in M_{i,a,b}$ by definitions.

Choose $y = M^\sigma \in M_{i,a,b}$.

The claim proves that $I_{i,a} \in w-\Sigma^*_{i,a}$.

Obviously any $w - \Sigma^*_{j,a}$ - set can be defined on the form of ii in the claim. This proves b.

Definition Let $a \in I$, $i = \langle i_1, i_2 \rangle \in S$.

By $J^\sigma_{i,a}(R)$ we mean the partial set

$$x \in J^\sigma_{i,a}(R) \Leftrightarrow x \in I^\sigma_{i_1,a}(R)$$

$$x \notin J^\sigma_{i,a}(R) \Leftrightarrow x \in I^\sigma_{i_2,a}(R)$$

whenever this is consistent.

When $J_{i,a}(R)$ is total and well defined, it will be a general $\Delta^*_{i,a}(R)$ - subsets of M.

2 V = L and the structure of the spectrum

In this section we will develop some machinery. So, let I, S be as in section 1 and let $<$ be a ${}^{k+2}E$-recursive well-ordering of I of length $\aleph_k$.

Each initial segment of $<$ can be put in a 1 - 1 correspondence with a subset of S.

If $a \in I$, let $a_i(j) = a(\langle i,j \rangle)$ and

$$S_a = \{a_i;\ i \in S\}$$

$\{b;\ S_b = \{c;\ c \leq a\}\}$ is uniformly recursive in a (and ${}^{k+2}E$ which we will always mean when nothing else is said), and by the recursive well-ordering we may pick the least.

This gives us:

Lemma 2.1

If $a < b$, there is a subindividual i such that a is recursive in b and i.

Now, let $\langle M_a \rangle_{a \in I} = \mathrm{Spec}({}^{k+2}E)$.

Let $\mathcal{M}_a = \bigcup_{i \in S} M_{\langle a,i \rangle}$.

Lemma 2.1 then gives

$$a < b \Rightarrow \mathcal{M}_a \subseteq \mathcal{M}_b$$

By simple reflection; $TC(M_a) <_{\Sigma_1} TC(\mathcal{M}_a)$, and using the recursive well ordering:

$$M_a <_{\Sigma_1} TC(M_a)$$

so

$$M_a <_{\Sigma_1} \mathcal{M}_a \text{ .}$$

This gives the following variant of Dependent Choice:

<u>Lemma 2.2</u>

Let $a \in I$ and let φ be a Δ_0-formula with parameters $\vec{x} \in M_a$. Assume $\forall c \forall x \in M_{c,a} \exists y \in \mathcal{M}_{c,a}\ \varphi(x,y,\vec{x})$. Then there is a sequence $\langle x_c \rangle_{c \in I}$ in M_a such that

$$\forall c\ \varphi(\langle x_d \rangle_{d<c}, x_c, \vec{x})$$

For the proof we use the reflection described above in combination with Gandy's selection and Σ^*-collection.

<u>Definition</u>

<u>a</u> $a \in I$ is called <u>minimal</u> if for no $b < a$, $a \in \mathcal{M}_b$.

<u>b</u> a' (read: a-jump) is the least b such that $b \notin \mathcal{M}_a$.

Let $\| \ \|$ be the norm induced by $<$.

<u>Lemma 2.3</u>

<u>a</u> $\|a'\| = \lambda^a_{k-1}$ = least ordinal not in $\mathcal{M}_a$

<u>b</u> $\mathcal{M}_a \in M_{a'}$.

Proof:

a By induction on the ordinals $\alpha < \aleph_k$ it follows that $\forall b(\alpha \in M_b \iff \exists c \in M_b(\|c\| = \alpha))$. The lemma follows trivially.

b By a and the equivalent definitions of λ^a_{k-1} we have

$$\forall b < a' \; \exists \sigma \in M_{a,b,a'} \; (\|b\| = \text{ordertype of } \{\alpha; \; \alpha \text{ is the length of a computation in } a \text{ and a subindividual } i \in S\})$$

Using Σ^*-collection over $\{b; \; b < a'\}$ we see that $K^a_{k-1} \in M_{a,a'}$ uniform in a .

Now $\{b < a'; \; a' = b'\} = \{b < a'; \; \forall c(b \leq c < a' \Rightarrow c \in \mathcal{M}_b)\}$ is $\Sigma^*_{a'}$-definable.

By Grillict-selection (MacQueen [7]) we pick a recursive subset of $\{b < a'; \; a' = b'\}$ and for each b in that set we find $K^b_{k-1} = K^a_{k-1}$ uniform in b, a' . But then $K^a_{k-1} \in M_{a'}$ by Σ^*-collection, and $\mathcal{M}_a = \bigcup_{b<a'} M_b^{K^a_{k-1}} \in M_{a'}$. □

Now, if c is the characteristic function of a complete Σ^*_a-subset of S , then $c \notin \mathcal{M}_a$, so $a' \leq c$. On the other hand, $c \in \mathcal{M}_{a'}$ since $a, \mathcal{M}_a \in \mathcal{M}_{a'}$. Thus $\mathcal{M}_{a'} = \mathcal{M}_c$, and $\mathcal{M}_a <_{\Sigma_1} \mathcal{M}_{a'}$ by further reflection.

Definition Let a be minimal. We say that <u>a is bad</u> if $\operatorname{Sup}\{K^b_{k-1}; \; b < a\} = K^a_{k-1}$.

We have not been able to decide upon the existence of bad points, but we are inclined to believe that they exist. By lemma 2.3 a jump is not bad, and it can for instance be proved that when a is bad, the order type of the minimal b's < a is $\|a\|$ itself.

We will now define two well orderings that will be useful in later proofs:

1. From standard definability theory we know that there is a well ordering of $M^\alpha \setminus \bigcup_{\beta<\alpha} M^\beta$ of ordertype $\aleph_k$, uniformly recursive in α. Let $\alpha(x)$ be the least α such that $x \in M^\alpha$.

Now, let $\underline{x <^1 y}$ if $\alpha(x) < \alpha(y)$ or $\alpha(x) = \alpha(y) = \alpha$ and x is less than y in the ordering on $M^\alpha \setminus \bigcup_{\beta<\alpha} M^\beta$.

Let $\| \ \|^1$ be the associated norm.

<u>Remark</u>: To define $<^1$ we do not need $V = L$, only a recursive in ${}^{k+2}E$ well ordering of I.

$<^1$ is recursive in the following sense:
Given γ, we may uniformly pick x such that $\|x\|^1 = \gamma$. Unfortunately the converse, i.e. compute $\|x\|^1$ from x may not be possible if we do not know for which a, $x \in M_a$. Thus

$\{\langle x, \|x\|^1 \rangle\}$ is $w - \Delta^*$ but probably not Δ^*.

Let $\nu < \aleph_k$. We say that <u>y is in row ν</u> if for some β, $\|y\|^1 = \aleph_k \cdot \beta + \nu$.

2. On each $\mathcal{M}_a$ there is a canonical well ordering $<_{\mathcal{M}_a}$ of length λ^a_{k-1} defined by

$x <_{\mathcal{M}_a} y$ if x is computed from a, I and some $i \in S$ before y is computed from a, I and some $j \in S$, or if they are computed by computations of the same length, but the index $\langle e,i\rangle$ of the computation of x is less that that of y.

$$\underline{x <^2 y} \iff \mu a(x \in \mathcal{M}_a) < \mu b(y \in \mathcal{M}_b)$$

$$\text{or } (\mu a(x \in \mathcal{M}_a) = \mu b(y \in \mathcal{M}_b) = c \ \& \ x <_{\mathcal{M}_c} y .$$

This well ordering has length $\aleph_k$, but is in no sense recursive. To be able to use it, we have to use recursive approximations:

$x <_\sigma y$ if we restrict the definition of $<^2$ to $\langle \mathcal{M}_a^\sigma \rangle_{a\in I}$.

Let $\| \ \|^2$ and $\| \ \|_\sigma$ be the associated norms.

For $x \in M^\sigma$, let $<_\sigma \restriction x = <_\sigma \restriction \{y;\ y \leq_\sigma x\}$.

To justify the term approximation we prove:

<u>Lemma 2.4</u>

<u>a</u> For any x ; $\{<_\sigma \restriction x;\ \sigma \in On\}$ has at most cardinality $\aleph_{k-1}$.

<u>b</u> If $x \in \mathcal{M}_a$, then $\forall\, \sigma \geq K^a_{k-1}$ $(<_\sigma \restriction x = <_{K^a_{k-1}} \restriction x)$

<u>c</u> For any x , $\{\|x\|_\sigma;\ \sigma \in On\}$ is finite.

<u>Remark</u>: We will not use <u>c</u> in this paper.

<u>Proof</u>: <u>a</u> Let σ be the least ordinal such that $x \in M^\sigma$ and let a be the least individual such that $x \in \mathcal{M}_a^\sigma$. If for some ordinal $\delta > \sigma$ $<_\delta \restriction x \neq \lim_{\delta_0 \to \delta} <_{\delta_0} \restriction x$, this is because we for some $b < a$ have $\mathcal{M}_b^\delta \setminus \bigcup_{\delta_0 < \delta} \mathcal{M}_b^{\delta_0} \neq \emptyset$. This only happens when $\delta \in \mathcal{M}_b \subseteq \mathcal{M}_a$. Since $\bar{\bar{\mathcal{M}}}_a = \aleph_{k-1}$ the lemma follows.

<u>b</u> Immediate from the definition and the considerations in the proof of <u>a</u>.

<u>c</u> Since we do not need the result, we will not give the details in the proof:

Let a be σ-minimal if $\langle \mathcal{M}_a^\sigma \rangle_{a\in I} \models a$ is minimal.

<u>Claim</u> Let a be σ-minimal. If $\| <_\sigma \restriction \bigcup_{b<a} \mathcal{M}_b^\sigma \| \neq \|a\|$, a is not $\sigma{+}1$-minimal.

<u>Proof</u>: If $\neq$ is $>$, a will be definable from element no $\|a\|$ in $\|<_\sigma \restriction \bigcup_{b<a} \mathcal{M}_b^\sigma\|$. If $\neq$ is $<$, $\bigcup_{b<a} \mathcal{M}_b^\sigma$ will be definable from σ and the $b<a$ such that $\|b\| = \|<_\sigma \restriction \bigcup_{b<a} \mathcal{M}_b^\sigma\|$.

Now, let x be given and let a_σ be the least a such that $x \in \mathcal{M}_a^\sigma$. $\{a_\sigma;\ \sigma \in On\}$ is finite since $\sigma_1 > \sigma \Rightarrow a_{\sigma_1} \leq a_\sigma$. In general we prove from the claim that if $\|x\|_\sigma \neq \lim_{\sigma_1 \to \sigma} \|x\|_{\sigma_1}$ we have $a_\sigma \neq a_{\sigma+1}$ or σ is a successor and $a_{\sigma-1} \neq a_\sigma$. □

Now we will prove a few results about order types of partial orderings on I.

Let $\prec$ be a partial ordering on I. Let A, B, C be subsets of field $(\prec)$. Let

$$\varphi(A,B,C) \iff \forall a,b,c(a \in A \ \&\ b \in B \ \&\ c \in C \implies a \prec b \wedge \neg(c \prec a) \wedge \neg(b \prec c)).$$

$\prec$ satisfies * if for all $A,B,C \subseteq$ field $(\prec)$ of cardinality $< \bar{\bar{I}}$.

$\varphi(A,B,C) \implies$ there is a $d \in$ field $(\prec)$ such that $A \prec d \prec B$ and for all $c \in C$, c and d are $\prec$-incomparable.

<u>Lemma 2.5</u> Let $\prec_1$ and $\prec_2$ be two partial orderings on I satisfying *. Then $\prec_1$ and $\prec_2$ are isomorphic.

For proof, see e.g. Sacks [12], Theorem 16.3. This is almost the same as proving that countable dense linear orderings are isomorphic.

<u>Remark</u>: $V = L$ is not required in Lemma 2.5.

<u>Lemma 2.6</u> <u>a</u> If GCH holds, all partial orderings $\prec^1$ on I

can be imbedded in a partial ordering $\prec^2$ on I satisfying $*$.

b If V = L holds, there is a partial ordering on I, recursive in ${}^{k+2}E$, satisfying $*$.

<u>Remark</u>: Both GCH and V = L are too strong assumptions for the respective statements.

<u>Proof</u>: To prove a it is sufficient to find one partial ordering satisfying $*$, by the proof of Lemma 2.5 we may imbed any partial ordering in one satisfying $*$. We prove b, which will just be an effective version of a.

Let $<$ be the minimal well-ordering of I recursive in ${}^{k+2}E$.
For $\nu < \aleph_k$, let a_ν be element no ν in $<$.
Let $\langle\, ,\,\rangle : I^2 \to I$ be onto and recursive such that
$\forall a,b$, $\|\langle a,b\rangle\| \geq \max\{\|a\|, \|b\|\}$.
We will define $\{<_\nu;\ \nu < \aleph_k\}$ to be an increasing sequence of partial orderings, uniformly recursive in a_ν, such that cardinality (field $(<_\nu)$) $\leq \aleph_{k-1}$. We may then for each ν find a b uniformly recursive in a_ν such that field $(<_\nu) = S_b$. Since $({}^{S}2)^3$ may be regarded as a subset of I, there is a well-ordering of this set recursive in b. This is used for the following:

The tripples A, B, C of subsets of field $(<_\nu)$ may be indexed uniformly recursive in a_ν in the following way:

$$\langle A_{\langle a_\nu,c\rangle},\ B_{\langle a_\nu,c\rangle},\ C_{\langle a_\nu,c\rangle}\rangle_{c\in I}$$

When $<_\nu$ is constructed, we automatically perform the indexing described above.

We now describe the construction:

$$<_0 = \emptyset$$

If λ is a limit, let $<_\lambda = \bigcup_{\nu<\lambda} <_\nu$.

Assume $<_\nu$ is constructed. Pick tripple $\langle A_{a_\nu}, B_{a_\nu}, C_{a_\nu}\rangle$ of subsets of field $(<_\nu)$.

Let φ be as in the definition of * . If $\varphi(A_{a_\nu}, B_{a_\nu}, C_{a_\nu})$, add a_ν to field $(<_\nu)$, and let

$$A_{a_\nu} <_{\nu+1} a_\nu <_{\nu+1} B_{a_\nu} \quad \text{and for each} \quad c \in C ,$$

let a_ν and c be incomparable, and extend $<_{\nu+1}$ to a transitive relation. (We will not add new relations between elements of field $(<_\nu)$.)

If $\neg\varphi(A_{a_\nu}, B_{a_\nu}, C_{a_\nu})$, let $<_{\nu+1} = <_\nu$.

Since φ is first order over I , this construction is recursive. Let $<^* = \bigcup_{\nu<\aleph_k} <_\nu$.

By construction $<^*$ satisfies * , and $<^*$ is recursive in ^{k+2}E .

3 V = L and the finite priority method

In this section we will give a solution to Post's problem and a problem requiring a similar proof for extended recursion in functionals. We will assume V = L .

In the proof we also give terminology and methods required for the more complex priority argument in section 4.

Recall the notions in section 2. Let $I = tp(k)$. Let $\langle M_a\rangle_{a\in I} = Spec(^{k+2}E)$, $^1M = \{\langle a,x\rangle ;\ x \in M_a\}$. By reasons of convenience, let 'card($\aleph_{-1}$)' mean 'finite'.

<u>Theorem 3.1</u> (V = L)

There is Σ^*-definable subset $Q \subseteq {}^1M \times I$ such that when $\langle N_a \rangle_{a \in I} = \mathrm{Spec}(Q)$ we have

<u>i</u> a is minimal and not bad $\Rightarrow \mathcal{N}_a = \mathcal{M}_a$.

<u>ii</u> a is minimal but bad $\Rightarrow \mathcal{N}_a \subseteq \mathcal{M}_{a'}$.

Let $Q_b = \{x; \langle x,b \rangle \in Q\}$

$Q_{-b} = \{\langle x,a \rangle; \langle x,a \rangle \in Q \ \& \ b \neq a\}$

<u>iii</u> $\forall a,b \quad Q_b \notin \Delta^*_a(Q_{-b})$ over $\mathrm{Spec}(Q_{-b})$. □

<u>Remark</u>: Since $Q_b \subseteq {}^1M$, Δ^*_a and $w - \Delta^*_a$ will be the same. Using results in section 1 we obtain

<u>Corollary 3.2</u> (V = L)

There is a subset A of $tp(k+1) \times I$ semirecursive in ${}^{k+2}E$ such that

<u>i</u> If a is minimal and not bad:

$$k+1 - sc(A, {}^{k+2}E, a) = k+1 - sc({}^{k+2}E, a)$$

<u>ii</u> If a is minimal but bad:

$$k+1 - sc(A, {}^{k+2}E, a) \subseteq k+1 - sc({}^{k+2}E, a')$$

<u>iii</u> $\forall a,b \quad A_b$ is not weakly recursive in $A_{-b}, {}^{k+2}E, a$. □

To obtain a solution to Post's problem, let $a \neq b$ be two recursive elements of I . Then for all $c \in I$:

A_a is not weakly recursive in $A_b, c, b, {}^{k+2}E$, since A_b is recursive in $A_{-a}, b, {}^{k+2}E$. So $A_a \not<_w A_b, {}^{k+2}E, c$ where $<_w$ means 'weakly recursive in'. The opposite will hold by symmetry.

By lemmas 2.5, 2.6 and corollary 3.2 we may obtain

Corollary 3.3 $(V = L)$

Let $\prec$ be a partial ordering on I. Then there are subsets $\{B_a\}_{a \in \text{field}(\prec)}$ of $tp(k+1) \times I$ such that

i Each B_a is semirecursive in ${}^{k+2}E$ and some individual.

ii $a \preceq b \Rightarrow B_a$ is recursive in $B_b, {}^{k+2}E$ and some individual.

iii $\neg(a \preceq b) \Rightarrow B_a$ is not weakly recursive in $B_b, {}^{k+2}E$ and any individual.

Proof: By lemmas 2.5 and 2.6 we may assume that $\prec$ is recursive in ${}^{k+2}E$. Let A be as in corollary 3.2. Let for $a \in \text{field}(\prec)$:

$$B_a = \{\langle f,b\rangle ;\ \langle f,b\rangle \in A \ \&\ b \preceq a\} .$$

Then, if $a \preceq b$, B_a is recursive in B_b and a, while if $\neg(a \preceq b)$, B_a is recursive in A_{-b}, a and A_b is recursive in B_b and b. So, if $B_b <_w B_a, c, {}^{k+2}E$ we would have

$$A_b <_w A_{-b}, a, b, c, {}^{k+2}E \ , \quad \text{impossible by corollary 3.2.}$$

□

The rest of this section is devoted to the proof of theorem 3.1.

If b is recursive in a via subindividual i and natural number e, we write $b = [e,i]^a$. We code $\langle e,i\rangle$ to one $j \in S$ and write $b = [j]^a$.

There are two kinds of conditions we want to meet:

1.i.j.a : $M \sim Q_{[i]^a} \neq I_{j,a}(Q_{-[i]^a})$

2.i.a : Protect the statement
$\exists x \in \mathcal{M}_a(Q)\varphi_i(x,a,Q)$
where $i = \langle e,j\rangle$ and in some Gödel-enumeration
$\varphi_i(x,a,Q) = \varphi_e(x,a,j,Q)$.

Each condition is coded as a pair $\langle i,a\rangle \in S\times I$, and by the recursive well-orderings on S and I, we order the conditions in the antilexicographical ordering. The order type will be $\aleph_k$. We will let ν denote both a condition and its place in the ordering.

If $\nu = \langle i,a\rangle$ we call ν an <u>a-condition</u>

If $\nu = 1,i.j.a$ we call ν a <u>1-condition</u>

If $\nu = 2.i.a$ we call ν a <u>2-condition</u>

We construct Q^ξ by induction on $\xi = \langle\nu,\sigma\rangle$ in the antilexicographical ordering, where ν is a condition, $\sigma \in M$ is called the stage and ξ the position. During the construction we will create requirements for a condition ν, and if we are able to keep the requirement disjoint from Q, ν will be met. If we at some position ξ add something in a requirement to $Q^{\xi+1}$, we <u>injure</u> the requirement. A requirement z is <u>active</u> at position ξ if $z \cap Q^\xi = \emptyset$. Otherwise it is <u>inactive</u>.

To meet the 1-condition $\nu = 1.i.j.a$ we will appoint <u>candidates</u> $\langle r,[i]^a\rangle$ for ν, where $r = \langle b,r_1\rangle$ for some r_1 in $\mathrm{row}(\nu)$, $b \in I$ such that $r_1 \in M_b$.
We will <u>reject</u> the candidate if we create a requirement for a condition $\nu_1 < \nu$. A candidate will always be a new element on the construction. Since we only add unrejected candidates to Q, the priority problem is taken care of this way. When we put a candidate into Q, we <u>realize</u> it.

We will try to meet the a-conditions inside $\mathcal{M}_a$. To keep control over the construction it is essential that no injury of an a-condition takes place outside $\mathcal{M}_a$. Thus we will refuse to do anything with a 1.a-condition outside $\mathcal{M}_a$.

We will now describe the construction:

Let $Q^{o} = \emptyset$

If ξ is a limit-position, let $Q^{\xi} = \bigcup_{\xi_1<\xi} Q^{\xi_1}$

Let Q^{ξ}_{b} and Q^{ξ}_{-b} be as defined in Theorem 3.1.

Let $\xi = \langle \nu,\sigma \rangle$

Case 1 $\nu = 1$ i.j.a Do nothing unless there is an E-computation in I, a and some subindividual of length σ. (Proceed to the next position)

Ask: Is there an active requirement for ν at position ξ?

If yes, let $Q^{\xi+1} = Q^{\xi}$ and proceed to the next position. If no, let r_1 be element no. $\langle \nu,\sigma+1 \rangle$ in $<^1$. If $\mathcal{M}^{\sigma}_{a} \models [i]^{a}$ is defined (=b), let c be the least individual $\geq$ a such that $r_1 \in M_c$ and let $\langle\langle c, r_1\rangle,b\rangle$ be a candidate for ν.

Ask: $\exists r \in \mathcal{M}^{\sigma}_{a}[($ r is a candidate for ν that is not rejected)

$\& \mathcal{M}^{\sigma}_{a} \models [i]^{a}$ is defined $(=b)$ & $r = \langle r_1,b \rangle$

& $r_1 \in I^{\sigma}_{j,a}(Q^{\xi}_{-b})]$?

If yes, choose the first appointed such r and let $(M^{\sigma} \times I)_{-b} - Q^{\xi}_{-b}$ be a requirement for ν.

Reject all unrealized candidates for conditions $\nu_1 > \nu$.

For $\nu_1 > \nu$, let $Q^{\langle \nu_1,\sigma \rangle} = Q^{\xi} \cup \{r\}$ and proceed to the next stage.

If no, let $Q^{\xi+1} = Q^{\xi}$ and proceed to the next position.

Case 2 $\nu = 2$.i.a

Let $Q^{\xi+1} = Q^{\xi}$.

<u>Ask</u>: Is there an active requirement for ν ? If yes, proceed to the next position. If no

<u>Ask</u>: $x \in \mathcal{M}_a^\sigma(Q^\xi)[\varphi_i(x,a,Q^\xi)]$?

If no, proceed to the next position. If yes

<u>Ask</u>: Is this verifiable from negative information about Q contained in some active requirement of higher priority? If yes, let the active requirement of highest priority containing such information be a requirement for ν and proceed to the next position (we do not reject candidates unnecessarily). If no, let $M^\sigma - Q^\xi$ be a requirement for ν and reject all unrealized candidates for conditions $\nu_1 > \nu$. Then proceed to the next position.

This ends the construction, now it just remains to prove that it works.

First note that we sometimes proceed to the next stage, sometimes proceed to the next position. There are technical reasons for not wanting to add more than one element to Q at each stage while we do not hesitate too much in dealing with the 2-conditions.

If we at stage σ ask the questions about ν given above, we say that we <u>pay attention</u> to ν at stage σ.

By construction, Q^ξ is uniformly recursive in ξ. Moreover, Q^ξ is a subset of ${}^1M \times I$. To prove that Q is Σ^*, we must prove that when $r = \langle\langle c,r_1\rangle,b\rangle$ is put into Q^ξ, $\xi \in \mathcal{M}_r$. If r_1 is in row ν, ν will be recursive in c and some subindividual, by choice of c. But the stage σ at which we realize r is recursive in ν and some subindividual, so $\xi = \langle\nu,\sigma\rangle \in \mathcal{M}_c \subseteq \mathcal{M}_r$.

We make a change on a condition ν at position ξ_1 if we realize or reject a candidate for ν, create or injure a requirement for ν at position ξ_1.

Claim 1 Let ν be a condition
$\{\xi_1$; we make a change on ν at $\xi_1\}$ has at most cardinality $\aleph_{k-1}$.

Proof: We cannot make a change on a condition ν more that once without making a change on a condition $< \nu$. Then the proof is by standard induction on ν.

Corollary $\forall \nu \exists \xi$ (After ξ we do not make a change on ν)

Proof: This follows by claim 1, since the cofinality of our construction is $\aleph_k$.

Remark: The argument used in claim 1 will be refered to as 'the priority argument'.

Claim 2 Let a be minimal and not bad. Let ν be an a-condition. There is a stage $\sigma \in \mathcal{M}_a$ after which we will always pay attention to ν. In particular, after stage σ, no injury of a ν-requirement will take place.

Proof: After $\sigma_o = \mathrm{Sup}\{K^b_{k-1}; b \leq a\}$ we will only realize candidates for c-conditions where $c \geq a$. There are at most $\aleph_{k-2}$ such conditions of higher priority that ν, and for each such conditions there will by the priority argument be realized at most $\aleph_{k-2}$ candidates after σ_o. Since the only reason not to pay attention to a condition is that we at the same stage realize a candidate for a condition of higher priority, and since K^a_{k-1} has cofinality $\aleph_{k-1}$, the claim follows by the standard argument.

<u>Claim 3</u> $\mathcal{M}_a$ is rudimentary closed relative to Q.

<u>Proof</u>: Let $x \in \mathcal{M}_a$. Let b be the least individual such that $x \in TC(\mathcal{M}_b)$, and let $y \in \mathcal{M}_b$ be transitive such that $x \in y$, $x \subseteq y$. By definition, b is minimal and not bad. In E-recursion there is an index e such that $y = \{e\}(b,I,i)$ for some subindividual i, so the formula

$$\forall u \in y \, (u \in Q \vee u \notin Q)$$

is protected by some 2.b-condition ν.
By claim 2 there will be a $\sigma \in \mathcal{M}_b$ after which we always pay attention to ν. Thus, at the first $\sigma_1 > \sigma$ such that $\|\langle e,a,I,i,y\rangle\| \leq \sigma_1$ (as a E-computation) there will either be a requirement for ν or we will create one. This requirement will never be injured. Thus, for some $\xi \in \mathcal{M}_b$, $y \cap Q^\xi = y \cap Q \in \mathcal{M}_b$. Since $b \leq a$, $Q^\xi \in \mathcal{M}_a$, and $x \cap Q^\xi = x \cap Q \in \mathcal{M}_a$.

<u>Definition</u>. Let $x \in M$. We say that '$x \in \mathcal{M}_a(Q)$' is <u>finally protected</u> at stage σ if for some $e \in \omega$, $i \in S$, the statement $\{e\}^Q(i,a,I) = x$ is protected by a requirement active at stage σ that is never injured.

<u>Claim 4</u> Let $a,c \in I$. Let $\delta \in \mathcal{M}_{a,c}$ be an ordinal and let '$\vec{x} \in \mathcal{M}_a(Q)$' be finally protected at stage δ. Assume that in E-recursion $\{e\}^Q(\vec{x}) \simeq x$. Then there is a $\sigma > \delta$, $\sigma \in \mathcal{M}_{a,c}$ such that

$$x \in \mathcal{M}_a^\sigma(Q^{\langle o,\sigma\rangle})(x \simeq \{e\}^{Q^{\langle o,\sigma\rangle}}(\vec{x}))$$

<u>Remark</u>: In the application, $\vec{x}$ will come from $I \cup \{I\}$, in which case the assumption is trivially true. The assumption on $\vec{x}$ seems essential to make the inductive proof work.

Proof: We prove this by induction on the length of the computation $\{e\}^Q(\vec{x}) \simeq x$. We give the cases where schemes v or iv is used. The methods used here cover vii as well. i, ii, iii are trivial and vi is covered by claim 3.

Case v Composition

$$\{e\}(\vec{x}) = \{e_o\}(\{e_1\}(\vec{x}),\ldots,\{e_n\}(\vec{x}),\vec{x})$$

Let $\delta_o = K^{a,c}_{k-1}$. By the induction hypothesis there will be stages $\delta_1,\ldots,\delta_n$ in $\mathcal{M}_{\langle a,c\rangle'}$ such that for $1 \leq m \leq n$

$$\exists\, y_m \in \mathcal{M}_a^{\delta_m}(Q^{\langle o,\delta_m\rangle})(y_m \simeq \{e_m\}^{Q^{\langle o,\delta_m\rangle}}(\vec{x}))\,.$$

The associated conditions will be a-conditions, so they will be paid attention to and never injured after $\delta_{n+1} = \mathrm{Max}\{\delta_m; 1\leq m\leq n\} \geq K^{a,c}_{k-1}$. Thus at stage δ_{n+1}, all '$y_m \in \mathcal{M}_a(Q)$' is finally protected. By the induction hypothesis again, there is a $\delta_{n+2} \geq \delta_{n+1}$ in $\mathcal{M}_{\langle a,c\rangle'}$ such that

$$\exists\, x \in \mathcal{M}_a^{\delta_{n+2}}(Q^{\langle o,\delta_{n+2}\rangle})(x \simeq \{e_o\}^{Q^{\langle o,\delta_{n+2}\rangle}}(y_1,\ldots,y_n,\vec{x}))$$

Since $\mathcal{M}_{\langle a,c\rangle} <_{\Sigma_1} \mathcal{M}_{\langle a,c\rangle'}$, we find a σ in $\mathcal{M}_{\langle a,c\rangle}$ having the same property as δ_{n+2} above.

Case iv $\quad \{e\}(x_1,\ldots,x_n) \simeq \bigcup_{y\in x_1} \{e_1\}(y,x_2,\ldots,x_n)$

where $x_1 \in \mathcal{M}_a(Q),\ldots,x_n \in \mathcal{M}_a(Q)$ are all finally protected at stage δ.

First note that when x_1 is computed from a and I, there will be a 1 - 1 map f from an initial segment of $\langle I,<\rangle$ onto x_1 uniformly recursive in the computation of x_1. We regard the case when f is defined on the whole of I. The other case is simpler.

For each $y = f(b) \in x_1$, '$y \in \mathcal{M}_{a,b}(Q)$' will be finally protected at stage δ.

We want to find a stage where all computations $\{e_1\}^Q(f(b),x_2,\ldots,x_n)$ for $b \in I$ are convergent, and first we do this for all initial segments of I.

<u>Subclaim</u>

$$\forall c \forall \gamma \in \mathcal{M}_{a,c} \forall b \in I \exists \sigma_b \in \mathcal{M}_{\langle c,a,b\rangle}(\sigma_b > \gamma \ \&$$

$$\forall d \leq b \exists x_d \in \mathcal{M}^{\sigma_b}_{a,d}(Q^{\langle o,\sigma_b\rangle})(\{e_1\}^{Q^{\langle o,\sigma_b\rangle}}(f(d),x_2,\ldots,x_n) = x_d))$$

<u>Proof</u>: After $K^{\langle c,a,b\rangle}_{k-1}$ none of the associated conditions will be injured for $d \leq b$. By the induction hypothesis there will for all $d \leq b$ be an ordinal after $K^{\langle c,a,b\rangle}_{k-1}$ at which the computation

$$\{e_1\}(f(d),x_2,\ldots,x_n)$$

is protected, and the associated requirement is never injured. Thus, using Σ^*-collection we find a candidate for σ_b in $\mathcal{M}_{\langle c,a,b\rangle}$, and by reflection we find it in $\mathcal{M}_{\langle c,a,b\rangle}$. This proves the subclaim.

Now we use the DC described in section 2.
Let $\delta_o = K^{\langle a,c\rangle}_{k-1}$. By the subclaim, find a sequence $\langle\delta_b\rangle_{b\in I} \in \mathcal{M}_{\langle a,c\rangle}$, such that $b_1 < b_2 \Rightarrow \delta_{b_1} < \delta_{b_2}$ and

$$\forall b \forall d \leq b \exists x_d \in \mathcal{M}^{\delta_b}_{d,a}(Q^{\langle o,\delta_b\rangle}(\{e_1\}^{Q^{\langle o,\delta_b\rangle}}(f(d),x_2,\ldots,x_n) \simeq x_d).$$

Let $\sigma_o = \operatorname{Sup}\{\delta_b ; b \in I\}$.
$\sigma_o \in \mathcal{M}_{\langle c,a\rangle}$, and $\sigma_o > \delta$.
Since the cofinality of σ_o is $\aleph_k$, we may use the priority argument on the construction below σ_o, i.e. for each condition ν there is some stage $\sigma_\nu < \sigma_o$ such that between σ_ν and σ_o we do

not change on ν. The sequence $\{\delta_b\}_{b\in I_\xi}$ is constructed so that for each $d \in I$, the statement $\{e_1\}^{Q^\xi}(f(d),x_2,\ldots,x_n)\downarrow$ will be protected cofinally many times below σ_o. Thus for all d, this computation converges at stage $\langle o,\sigma_o\rangle$.

But then $\exists\, x \in \mathcal{M}^{\sigma_o}_{\langle a,c\rangle}(Q^{\langle o,\sigma_o\rangle})(x = \bigcup_{d\in I}\{e_i\}^{Q^{\langle o,\sigma_o\rangle}}(f(d),x_2,\ldots,x_n))$.

Using reflection we find σ in $\mathcal{M}_{\langle a,c\rangle}$ gaving the same property.

$\square$

Claim 5

If a is minimal and not bad

$$\mathcal{M}_a(Q) = \mathcal{M}_a .$$

If a is minimal and bad

$$\mathcal{M}_a(Q) \subseteq \mathcal{M}_{a'} .$$

Proof: Let a be minimal but not bad.

If $x \in \mathcal{M}_a(Q)$ there is an index e and subindividual i such that $x = \{e\}^Q(a,i,I)$.

a, I and i are all finally protected as elements of $\mathcal{M}_a$ from the very beginning. There will be an a-condition ν associated with the statement

$$\exists\, x \in \mathcal{M}_a(Q)\ \{e\}^Q(a,i,I) \simeq x$$

By claim 2 there is a $\sigma \in \mathcal{M}_a$ such that after σ we always pay attention to ν. By claim 4 there is a $\sigma_1 > \sigma$ in $\mathcal{M}_a$ such that

$$\exists x \in \mathcal{M}_a^{\sigma_1}\ \{e\}^{Q^{\langle o,\sigma_1\rangle}}(a,i,I) \simeq x$$

Since we pay attention to ν at stage σ_1, $Q^{\langle o,\sigma_1\rangle} = Q^{\langle \nu,\sigma_1\rangle}$.

If there is no active requierment for ν at stage σ_1, we will create one, and this requirement will never be injured.

Then $x = \{e\}^{Q^{\langle\nu,\sigma_1\rangle}}(a,i,I) = \{e\}^{Q}(a,i,I)$ by the same computation. Since $\sigma_1 \in \mathcal{M}_a$ and $\nu \in \mathcal{M}_a$, $\{e\}^{Q^{\langle\nu,\sigma_1\rangle}}(a,i,I) \in \mathcal{M}_a$. This was what we wanted to prove.

If a is bad, we use claim 4 again, noting that after K^a_{k-1}, ν will always be paid attention to. □

Remarks

1. We have now verified parts i and ii in the theorem.
2. If a is bad and $\mathcal{M}_a \neq \mathcal{M}_a(Q)$, then K^a_{k-1} will be in $\mathcal{M}_a(Q)$, $a' \in \mathcal{M}_a(Q)$ so $\mathcal{M}_a(Q) = \mathcal{M}_{a'}$.
3. By Gandy's selection operator, the general statement '$\exists x \in \mathcal{M}_a(Q)\varphi_i(x,a,I)$' is equivalent to the convergence of a certain computation. Thus we have 'met' all 2-conditions by claim 5.

Claim 6 If a is minimal, not bad and not the jump of a bad, and if ν is a 1.a-condition, there is a $\sigma \in \mathcal{M}_a$ after which

i We will always pay attention to ν

ii No candidate for ν is rejected.

Proof: i is known from claim 2.

To prove ii we prove the following:

Subclaim Let ν_1 be another condition. We reject a candidate for ν due to ν_1 if we create a requirement for ν_1 while we reject the candidate.

If we at a stage after $K = \mathrm{Sup}\{K^b_{k-1}; b < a\}$ reject a candidate for ν due to a condition ν_1, ν_1 is an a-condition.

Proof of subclaim:

Assume that the subclaim is false, let $\sigma > K, \nu_1$ constitute a counterexample. Since we are not dealing with 1.b-conditions for $b < a$ after K, ν_1 is a 2.b-condition for some $b < a$.

Assume that $\exists x \in \mathcal{M}_b^\sigma(Q^{\langle \nu_1, \sigma \rangle})\varphi_j(x,b,I)$ where $\nu_1 = 2.j,b$. Let b_o be minimal such that $b \in \mathcal{M}_{b_o}$. Then there is an $i \in S$ such that

$$\forall x[\varphi_j(x,b,I) \Longleftrightarrow \varphi_i(x,b_o,I)]$$

Let ν_2 be the condition protecting

$$\exists x \in \mathcal{M}_{b_o}(Q)\varphi_i(x,b_o,I).$$

By claim 5, this will be met in $\mathcal{M}_{b_o}$ if b_o is not bad, and in $\mathcal{M}_{b_o'}$ if b_o is bad.

In any case, since a is neither bad nor the jump of a bad, there is some $\sigma_o < K$ such that at stage σ_o, ν_2 is finally met with a requirement.

Moreover, for some $\sigma_1 < K$, $b \in \mathcal{M}_{b_o}^{\sigma_1}$.

Thus after $\mathrm{Max}(\sigma_o,\sigma_1)$, if we pay attention to ν_1, all information we need is contained in the still active requierment for ν_2. But then we would not reject anything. This proves the subclaim.

To end the proof of the claim, note that between K and K^a_{k-1} the set of conditions due to which we reject a candidate for ν has cardinality $\leq \aleph_{k-2}$, and we may apply the priority argument. $\square$

We are now ready to end the proof of the theorem, i.e. prove iii. To obtain a contradiction, assume that for some a, b, j_o, $M \smallsetminus Q_b = I_{j_o,a}(Q_{-b})$.

Let c be minimal, not bad and not the jump of a bad such that $a,b \in \mathcal{M}_c$. Then for some $i,j \in S$, $b = [i]^c$ and $I_{j_o,a}(Q_{-b}) = I_{j,a}(Q_{-[i]^c})$.

Let ν be $\langle 1,i,j,c \rangle$. By claim 6, let $\sigma \in \mathcal{M}_c$ be such that after σ, no requirement for ν will be injured, we will always pay attention to ν and no candidate for ν will be rejected.

If we at some stage $\sigma_1 > \sigma$ realize a candidate $r = \langle r_1,b \rangle$ for ν, r_1 will be a counterexample to $M \setminus Q_b = I_{j_o,a}(Q_{-b})$, since $r_1 \in Q_b$, $r \in I_{j_o,a}(Q_{-b})$.

So let $r = \langle r_1,b \rangle$ be a candidate that is neither rejected nor realized. Then $r_1 \notin Q_b$, so $r_1 \in I_{j,c}(Q_{-b})$. Using claim 5 we find $\sigma_1 > \sigma$ such that

$r_1 \in I^{\sigma_1}_{j,c}(Q^{\langle \nu,\sigma_1 \rangle})$ and we pay attention to ν at stage σ_1. But then we would add something to Q_b at stage σ_1, or there would exist an active requierment for ν at stage σ_1. In both cases we obtain a contradiction.

This ends the proof of theorem 3.1.

4. V = L and the minimal pair problem for extended degrees of functionals

Let $\langle M_a \rangle_{a \in I} = \mathrm{Spec}(^{k+2}E)$. Recall from section 1 the definition of ^1M, $I^\sigma_{i,a}$ and the partial set $J^\sigma_{\langle i,j \rangle,a}$, and the definitions of row, $<^1$, $<_\sigma$ etc. from section 2.

Our aim in this section will be to give a solution to the minimal pair problem, in the style of section 3. The main theorem will be the following:

<u>Theorem 4.1</u> (V = L)

There exist two disjoint subsets A and B of ${}^1M$ (both recursive in $A \cup B$) both Σ^*-definable such that

<u>i</u> $\forall a \in I$, neither A nor B are Δ^*-definable.

<u>ii</u> If a is a jump, then $\mathcal{M}_a(A \cup B) = \mathcal{M}_a$.
If a is a limit of jumps, $\mathcal{M}_a(A \cup B) \subseteq \mathcal{M}_{a'}$.

<u>iii</u> If C is $w - \Delta_a^*(A)$-definable over Spec(A) and $w - \Delta_b^*(B)$-definable over Spec(B) for some $a,b \in I$, there is a $c \in I$ such that C is $w - \Delta_c^*$-definable over $\mathrm{Spec}({}^{k+2}E)$.

<u>Corollary 4.2</u> (V = L)

There exist two subsets A_1 and B_1 of tp(k+1), both semi-recursive in ${}^{k+2}E$ such that neither A_1 nor B_1 is recursive in ${}^{k+2}E$ and any individual, and whenever a type k+2 functional F is weakly recursive both in A_1 and an individual and in B_1 and an individual, then F is weakly recursive in ${}^{k+2}E$ and an individual.

The rest of this section will be devoted to the proof of theorem 4.1. The proof is based on the ω-case (Lachlan [4], Yates [5]) as presented in Shoenfield [14], with inspiration from Lerman-Sacks [5]. It will be an advantage to have the proof in Shoenfield [14] in mind.

We are led to the following conditions

<u>1.A.i.a</u> $A \neq J_{i,a}$ <u>1.B.i.a</u> $B \neq J_{i,a}$

<u>2.i.a</u> Protection of the statement

$$\exists x \in \mathcal{M}_a(A \cup B)\varphi_i(x,a,A \cup B)$$

3.$i_1, i_2, \langle a_1, a_2 \rangle$: If $J_{i_1,a_1}(A) = J_{i_2,a_2}(B)$ and both are total, then this set is weakly Δ_a^*-definable over $\langle M_b \rangle_{b \in I}$ for some a.

As in section 3. we use the notions a-conditions, 1-conditions, 2-conditions, 3-conditions and in addition A-conditions and B-conditions. The meaning of these notions should be clear.

Throughout the construction we will concentrate on the A-cases. If nothing else is mentioned, there will be an analogue B-case.

As in section 3 we index the conditions by pairs $\langle i,a \rangle$ ordered in the antilexicographical ordering. We identify a condition with its place ν in this ordering. Define position and stage as in section 3.

To satisfy the 3-conditions we need infinitely many requirements, and the problem of priority will be more difficult than in section 3. Before we begin on the formal construction we will give a brief idea of what will happen:

For each position $\xi = \langle \nu, \sigma \rangle$ we define subsets A^ξ and B^ξ of ${}^1M$, uniformly recursive in ν, σ. We let $A = \bigcup_{\xi \in Pos.} A^\xi$ and $B = \bigcup_{\xi \in Pos.} B^\xi$.

It will follow from the construction that if $r \in A$ there is a $\xi \in M_r$ such that $r \in A^\xi$. Thus A will be Σ^*-definable. The same will hold for B.

We only put elements into A to meet the 1.A-conditions, and for each condition, we put at most one element into A. At certain points in the construction we will appoint candidates $\langle a,r \rangle$ for a 1-condition ν, where r will be in row ν. These may be realized or rejected. For reasons of convenience, we say that a candidate $\langle a,r \rangle$ is <u>from</u> row ν if r is <u>in</u> row ν.

To meet the 2-conditions we act like we did in section 3.

To meet a 3-condition we need M-infinitely many requirements. Given $y \in M$, we may want to protect $y \in J_{i,a}(A)$ or $y \notin J_{i,a}(A)$ by a requirement z for A with argument y and value 'yes' or 'No' according to which statement we protect.

We use active and inactive as in section 3. If ν is a 3-condition and if z is a ν-requirement for A active at position ξ, we call z effective if there is no ν-requirement z_1 for B active at position ξ with the same argument and value as z. Otherwise z is called ineffective. A requirement is called essential if it is effective at position ξ for all sufficiently large ξ. Otherwise it is called inessential.

We use realize and reject for candidates as in section 3. Through the rejecting of candidates we take care of the priority problem and some other technical problems.

We will now state some important properties about candidates and requirements, and thereby prove a claim:

1. A candidate r for A can only be realized if it is not rejected, and we realize at most one candidate from each row.

2. When we appoint r at some position ξ, r will not be in any requirement created at some position $\xi_1 \leq \xi$.

3. When a requirement for a 2-condition ν is created, all unrealized candidates from rows $> \nu$ will be rejected (we will also reject some candidates when we create a 3-requirement, see the construction).

4. If we realize a candidate from row ν, we reject all unrealized candidates from rows $\geq \nu$.

From 1, 2 and 3 it follows that an unrealized candidate for ν will never be in a 2-requirement for a condition $< \nu$. Adding 4 we obtain

<u>Claim 1</u> Let z_1 and z_2 be two requirements active at position ξ, and assume that they are injured at stages σ_1 and σ_2 by r_1 and r_2 resp. Assume $\sigma_1 < \sigma_2$, r_1 is from ν_1 and r_2 is from ν_2. Then $\nu_2 < \nu_1$.

<u>Proof</u>: Both r_1 and r_2 are appointed before position ξ by 2. If $\nu_1 < \nu_2$, r_2 would have been rejected when r_1 was realized, by 4, and by 1 would not have been realized itself. By the other part of 1, $\nu_1 \neq \nu_2$ and the claim follows.

<u>Definition of P and Q</u>

For each condition ν_o and set $y \in M$, we define sets $P^A_\xi(\nu_o,y)$ and $Q^A_\xi(\nu_o)$ by induction on $\xi = \langle \nu,\sigma \rangle$ as follows:

$r \in P^A_\xi(\nu_o,y)$ if $r \in Q^A_{\xi_1}(\nu_o)$ for all $\xi_1 = \langle \nu_1,\sigma_1 \rangle$

such that $\|y\|^1 \leq \xi_1 < \xi$ and r is from ν_1.

$r \in Q^A_\xi(\nu_o)$ if there is a $\nu_1 \leq \nu_o$ such that for some y there is a ν_1-requirement z for A with argument y effective at position ξ, and $r \notin P^A_\xi(\nu_1,y)$ and $r \in z$.

<u>Remarks</u>

1. We have the following monotonity properties:

<u>a</u> $\nu_1 < \nu_2 \Rightarrow Q^A_\xi(\nu_1) \subseteq Q^A_\xi(\nu_2)$

<u>b</u> $\xi_1 < \xi_2 \Rightarrow P^A_{\xi_1}(\nu,y) \supseteq P^A_{\xi_2}(\nu,y)$

2. When $\xi \leq \|y\|^1$, P^A_ξ is the entire universe. However, we will

only deal with $P^A_\xi(\nu,y)$ in the case when it is an element of M.

Now recall from section 2 the definition of $<^2$ and $<_\sigma$.

If z is a requirement for a 3-condition ν with argument y, $r \in z$ is called a <u>key-element</u> of z at stage σ if $r \in {}^1M$ is from row ν_1 and $\nu_1 \geq \nu + \|y\|_\sigma$.

If z is created at stage σ, we will reject all unrealized candidates from rows $\geq \nu_1 + \|y\|_\sigma$. This takes care of some of the priority problems for 3-conditions. In addition elements from row ν in $Q^A_\xi(\nu)$ will not be put into A at position ξ.

We are now ready to describe the construction:

<u>Case 1.A.i,a = ν</u> $\quad \xi = \langle \nu,\sigma \rangle$

<u>Ask</u> Is there an element in A^ξ from row ν?
If yes, let $A^{\xi+1} = A^\xi$, $B^{\xi+1} = B^\xi$ and proceed to the next position. If no, let r be element no. $\langle \nu,\sigma+1 \rangle$ in $<^1$.
Let b be the least individual $\geq a$ such that $r \in M^{\sigma+1}_b$, and let $r_1 = \langle b,r \rangle$ be a candidate for ν.
Reject all candidates r_2 from row ν not satisfying
$\forall c$ (if for some $i \in S$ σ is the length of a computation in c,i,I then $r_2 \in \mathcal{M}^\sigma_c$).
(We reject candidates not being recursive in the stage.) Then

<u>Ask</u> Is there an unrejected candidate $r = \langle b,r_1 \rangle$ for ν such that $r \notin J^\sigma_{i,a}$, $r \notin Q^A_\xi(\nu)$, but $\xi \in M^\sigma_b$. If there is such candidate from row ν, let r be the first appointed one.
Let $A^{\xi+\nu_0} = A^\xi \cup \{r\}$, $B^{\xi+\nu_0} = B^\xi$ for all ν_0 such that $1 \leq \nu_0 < \aleph_k$. Reject all unrealized candidates for conditions $\geq \nu$, and proceed to the next stage.

If there is no such candidate, let $A^{\xi+1} = A^{\xi}$, $B^{\xi+1} = B^{\xi}$ and proceed to the next position.

Case 1.B.i,a = ν $\xi = \langle \nu, \sigma \rangle$

This is like the case above, with A and B interchanged.

Remark If we in one of these cases put something into A from row ν, it is clear that we meet condition ν.

Case 2.i.a = ν $\xi = \langle \nu, \sigma \rangle$

Let $A^{\xi+1} = A^{\xi}$, $B^{\xi+1} = B^{\xi}$.

Ask: Is there a ν-requirement active at position ξ?

If yes, proceed to the next position. If no

Ask: $\exists x \in \mathcal{M}_a^{\sigma}(A^{\xi} \cup B^{\xi})[\varphi_i(x,a,A^{\xi} \cup B^{\xi})]$?

If no, proceed to the next position. If yes, create a requirement for ν consisting of $M^{\sigma} - A^{\xi}$ and reject all unrealized candidates for conditions $> \nu$. Then proceed to the next position.

Remark If we at stage σ appoint a candidate r, $r \in M^{\sigma+1} \setminus M^{\sigma}$, and will thus be outside the requirement created here.

Case 3. $i_1.i_2.\langle a_1, a_2 \rangle = \nu$ $\xi = \langle \nu, \sigma \rangle$

This case is divided in an A-part and a B-part. We describe the A-part. The B-part is symmetric to the A-part. Let $A^{\xi+1} = A^{\xi}$, $B^{\xi+1} = B^{\xi}$.

Let $y \in \bigcup_{\delta<\sigma} M^{\delta}$ be the $<_{\sigma}$-least element such that there is no active ν-requirement for A with argument y, if such y exist. If not, proceed to the next position. Do nothing unless no $y_1 <_{\sigma} y$

is the argument of an effective ν-requirement for A and

$$y \in \bigcup_{\delta<\sigma} J^{\delta}_{i_1,a_1}(A^{\xi}) \quad \text{or} \quad y \notin \bigcup_{\delta<\sigma} J^{\delta}_{i_1,a_1}(A^{\xi})$$

(recall that $J^{\alpha}_{i_1,a_1}(A^{\xi})$ is a partial set).

Let δ be the least ordinal such that $y \in J^{\delta}_{i_1,a_1}(A^{\xi})$ or $y \notin J^{\delta}_{i_1,a_1}(A^{\xi})$. Then create a ν-requirement z for A with argument y and value i, i being the answer to the question $y \in I^{\delta}_{i_1,a_1}(A^{\xi})$? Let z consist of $(M^{\delta}(A^{\xi}) \smallsetminus A^{\xi}) \cap {}^1M^{\sigma}$. Reject all unrealized candidates that are from rows $\geq \nu + \|y\|_{\sigma}$. Then proceed to the next position.

When ξ is a limit, we let $A^{\xi} = \bigcup_{\xi_1<\xi} A^{\xi_1}$ and $B^{\xi} = \bigcup_{\xi_1<\xi} B^{\xi_1}$.

This ends the construction of A and B.

<u>Claim 2</u> Both A and B are Σ^*-definable.

<u>Proof</u>: $r \in A \Leftrightarrow \exists \xi \in \mathcal{M}_r\, (r \in A^{\xi})$ and A^{ξ} is uniformly recursive in ξ. The same will hold for B. □

In the M-finite injury method in section 3, we satisfied all 2-conditions ν by paying attention to them at all stages in $\mathcal{M}_{\nu'} \smallsetminus \mathcal{M}_{\nu}$. In the present situation we do not stop realizing candidates for ν at K^{ν}_{k-1}, so we have to prove that the methods from section 3 can be used.

<u>Claim 3</u> Let a be minimal, $c = a'$.

<u>a</u> We will pay attention to all a-conditions at all stages between K^{a}_{k-1} and K^{c}_{k-1}.

<u>b</u> If ν is an a-condition and z is a ν-requirement active at some position between K^{a}_{k-1} and K^{c}_{k-1}, then z is never injured.

<u>Proof</u>: Let $b \leq a$ and assume that we at some stage σ, $K^a_{k-1} \leq \sigma < K^c_{k-1}$ realize a candidate r for some b-condition ν_o. Then

$$\mathcal{M}_c \models \exists \xi \exists r \in A^\xi \ (r \text{ is from row } \nu_o)$$

Since $\nu_o \in \mathcal{M}_a$ we may use reflection, which gives

$$\mathcal{M}_a \models \exists \xi \exists r \in A^\xi \ (r \text{ is from row } \nu_o)$$

But if that is the case we would do nothing with ν_o at stage σ. This proves <u>a</u>.

To prove <u>b</u>, let $z \in \mathcal{M}_c$ be the requirement. If $r \in z$ is put into A, r would have been appointed as a candidate before the creation of z. Since r was not rejected when z was created, r is from a row $\nu_1 < \nu$. ν_1 will be recursive in a and a sub-individual. Assume such r exist for a condition $\nu_1 < \nu$. There are two possibilities:

1. $r \in A^{\xi_1} \cup B^{\xi_1}$ for a $\xi_1 = \langle \nu_1, \sigma_1 \rangle < K^c_{k-1}$.

   By <u>a</u>, there is a $\xi_2 < K^a_{k-1}$ such that $r \in A^{\xi_2} \cup B^{\xi_2}$.

   But this contradicts the assumption on r and z.

2. r is put into $A \cup B$ at some position $\xi_1 = \langle \nu_1, \sigma_1 \rangle \geq K^c_{k-1}$. Since ξ_1 shall be recursive in r, we cannot have $r \in \mathcal{M}_c$. But since r was appointed when z was created, there will be some ordinal $\sigma \in \mathcal{M}_c$ such that r is appointed at stage σ and such that there for some $i \in S$ is a computation in i,c,I of length σ. But then we would reject r at this stage, which leads to a contradiction.

<u>Claim 4</u> If $a = b'$ and ν is a 2.a-condition, then ν is injured at most $\aleph_{k-2}$ times between stage K^b_{k-1} and K^a_{k-1}, and from claim 3, not between K^a_{k-1} and $K^{a'}_{k-1}$.

Proof: By claim 3, if a 1-condition recursive in b and some $i \in S$ is met below K^a_{k-1}, it is met below K^b_{k-1}. In addition to such conditions there are at most $\aleph_{k-2}$ conditions which are allowed to injure ν, and each will do it at most once.

□

Combining claims 3 and 4 with the methods from the proof of theorem 3.1, we see that ii in the theorem must hold.

Remark We will obtain that $\mathcal{M}_a(A \cup B) = \mathcal{M}_a$ whenever a is minimal and

$$\bigcup_{b<a} \mathcal{M}_b <_{\Sigma_1} \mathcal{M}_a .$$

This is known not to hold for certain a, but definitely for more than just the jumps.

Claim 5 Let y, ν be given, ν a 3-condition. Then the set of ν-requirements with argument y has cardinality at most $\aleph_{k-1}$.

Proof: We can injure a requirement z with argument y only if we put into z an r not being a key-element of z at the stage when z was created, i.e. for some σ, r is from a row $< \nu + \|y\|_\sigma$.

By lemma 2.4.a there is an ordinal $\gamma < \aleph_k$ such that $\forall \sigma \|y\|_\sigma < \gamma$. Thus r will be from a row $< \nu + \gamma$. Since we never add more than one element from each row to $A \cup B$, the claim follows.

Claim 6 If z is an inessential requirement with argument y, then for some ξ_0, $\forall \xi > \xi_0$, z is ineffective at position ξ.

Proof: Let z be an inessential A-requirement for ν with argument y. Then by claim 5, the set of B-requirements for ν with argument y has cardinality at most $\aleph_{k-1}$. Assume that z is never injured. If all B-requirements for ν with argument y are injured, z is essential, so let z_1 be a B-requierment for ν with argument y that is never injured. Thus, when both z and z_1 are created, they must either both be effective or both ineffective. The latter must hold since z is inessential.

If z is injured, it is ineffective after that stage.

<u>Claim 7</u> Let ν be a 3-condition. The set of essential ν-requirements is M-finite, i.e. an element of M.

<u>Proof</u>: Let z be essential with argument y. (If no such z exist the claim is trivial.) Let σ_1 be such that after stage σ_1, z is effective. By lemma 2.4.a there is a $\sigma_2 \geq \sigma_1$ such that for all $\sigma \geq \sigma_2$, $<_\sigma \restriction y = <_{\sigma_2} \restriction y$.
By claim 5 there will throughout the entire construction be created at most $\aleph_{k-1}$ A - or B-requierments for ν with arguments $<_{\sigma_2} y$. Let these be created at a stage $\sigma_3 \geq \sigma_2$. Then, after stage σ_3 no new ν-requirements will be created (see the construction, part 3). Let X be the set of ν-requirements active at stage σ_3.

<u>Subclaim</u> There is a stage $\sigma_4 \geq \sigma_3$ such that for all $z \in X$, if z is ever injured, z will be injured before stage σ_4.

<u>Proof of subclaim</u>: It is sufficient to prove that we only injure elements of X at a finite number of stages after stage σ_3. Let $z_1, z_2 \in X$. Assume that at stages $\sigma_{11} < \sigma_{22}$, z_1 and z_2 are injured by r_1 and r_2 from rows ν_1 and ν_2 resp. By claim 1 $\nu_2 < \nu_1$. Thus an infinite sequence of injuries gives an infinite descending sequence of rows. This proves the subclaim.

Then all ν-requirements active at stage σ_4 will be active for ever, and a requirement is essential if and only it is effective at stage σ_4. This proves the claim.

□

<u>Claim 8</u> Let $\nu = i_1, i_2, \langle a_1, a_2\rangle$. Assume that $J_{i_1,a_1}(A) = J_{i_2,a_2}(B)$ and that both are total. Choose b such that $\langle a_1, a_2\rangle \in \mathcal{M}_b$ and let $y, \xi \in \mathcal{M}_b$. Then for some $\xi_1 \in \mathcal{M}_b$ ($\xi_1 > \xi$ and there are ineffective requirements for A and B with argument y, active at position ξ_1).

<u>Proof</u>: We find such ξ_1 in $\mathcal{M}_{b'}$, and then use reflection.

Let $\xi_2 = \sigma_2 = K^b_{k-1}$ ($\xi_2 = \langle o, \sigma_2\rangle$). By claim 2.4.b, $\tau \geq \sigma_2 \Longrightarrow <_\tau \restriction y = <_{\sigma_2} \restriction y$. Moreover there will be some 2.b'-condition ν_o protecting the following Σ_1-statement:

$$\forall y_1 \leq_{\sigma_2} y \; (y_1 \in J_{i_1,a_1}(A) \Leftrightarrow y_1 \in J_{i_2,a_2}(B))$$

and since <u>ii</u> of the theorem holds, there will at some stage $\sigma_3 \geq \sigma_2$, $\sigma_3 \in \mathcal{M}_{b'}$ be a <u>permanent</u> requirement protecting this fact (i.e. the requirement is never inured). By claim 3.a we will pay attention to ν at all stages between σ_3 and $K^{b'}_{k-1}$. Thus it follows by induction on $\|y_1\|_{\sigma_2}$ for $y_1 \leq_{\sigma_2} y$ that at position $\aleph_k \cdot \sigma_3 + \aleph_k \cdot \|y_1\|_{\sigma_2} + \nu$, there will be ineffective ν-requirements for A and B with argument y_1.
(See the relevant part of case 3 in the construction.)
Let $\xi_1 = \aleph_k \cdot (\sigma_3 + \|y\|_\sigma) + \nu$. $\xi_1 \in \mathcal{M}_{b'}$ and has the wanted property.

□

<u>Definition</u> $r \in Q^A(\nu)$ if $r \in {}^1M$ and for all sufficiently large $\xi_1 = \langle \nu_1, \sigma_1\rangle$, if r is from row ν_1, then $r \in Q^A_{\xi_1}(\nu)$.

<u>Remark</u> From the definition of $Q^A_{\xi_1}(\nu)$ and the construction of requirements in case 3 of the construction it follows ahat $Q^A_{\xi_1}(\nu) \subseteq {}^1M$.

In particular all requirements for 3-conditions are subsets of ${}^1M$.

<u>Claim 9</u> Let z be an essential ν-requirement with argument y. Then $z \subseteq Q^A(\nu)$.

<u>Proof</u>: Let σ_1 be such that after stage σ_1, z is effective and $\|y\|^1 < \langle o,\sigma_1\rangle$. Let $r \in z$ be from row η. Assume that for cofinally many $\sigma > \sigma_1$, $r \notin Q^A_{\langle\eta,\sigma\rangle}(\nu)$. We will obtain a contradiction:

Let $\sigma_2 > \sigma_1$ be arbitrary and let $\sigma_3 > \sigma_2$ be such that $r \notin Q^A{}_{\langle\eta,\sigma_3\rangle}(\nu)$. Using the definition of $Q^A_{\langle\eta,\sigma_3\rangle}(\nu)$ and the fact that z with argument y is effective at position $\langle\eta,\sigma_3\rangle$, we see that $r \in P^A_{\langle\eta,\sigma_3\rangle}(\nu,y)$. But then $r \in Q^A_{\xi_4}(\nu)$ for all $\xi_4 = \langle\eta,\sigma_4\rangle < \langle\eta,\sigma_3\rangle$ such that $\|y\|^1 \leq \xi_4$, by definition of $P^A_{\langle\eta,\sigma_3\rangle}(\nu,y)$. This is satisfied by $\xi_4 = \langle\eta,\sigma_2\rangle$, so $r \in Q^A_{\langle\eta,\sigma_2\rangle}(\nu)$. σ_2 was arbitrary chosen. This contradicts the assumption and $r \in Q^A(\nu)$. □

<u>Claim 10</u> Let $r \in Q^A(\nu)$. Then there is a $\nu_1 \leq \nu$ such that r is the element of an essential ν_1-requirement.

<u>Proof</u>: The 3-conditions will be of two types:
Let $\nu_1 = \langle i_1,i_2,\langle a_1,a_2\rangle\rangle$.

<u>Type 1</u> There is an essential ν_1-requirement, or for some y, y is not the argument of any permanent ν_1-requirement for A or B. (In this last case, either $J_{i_1,a_1}(A)$ or $J_{i_2,a_2}(B)$ is not total.)

<u>Type 2</u> There are permanent ineffective ν_1-requirements for A and B with argument y for all $y \in M$. (In this case $J_{i_1,a_1}(A) = J_{i_2,a_2}(B)$ and both are total.)

For conditions ν_1 of type 1, there will be a stage after which we neither create nor injure ν_1-requirements (see proof of claim 7). Since there are at most $\aleph_{k-1}$ conditions $\nu_1 \leq \nu$ we find a σ_o so large that

1. For $\sigma \geq \sigma_o$, $r \in Q^A_{\langle \eta,\sigma\rangle}(\nu)$ where r is from row η.
2. For $\nu_1 \leq \nu$ of type 1, no ν_1-requirements are created or injured after stage σ_o.

Now, let $\sigma \geq \sigma_o$. Let $\nu_1 \leq \nu$ be a 3-condition of type 2. Then there is a 2-condition protecting the following statement:

$$\forall y \in M^\sigma \; (y \in J_{i_1,a_1}(A) \iff y \in J_{i_2,a_2}(B))$$

and since ν_1 is of type 2, this will be met at some stage $\delta_1 \geq \sigma$. (Since <u>ii</u> in the theorem holds.) Let b be such that $M^\sigma \in \mathcal{M}_b$ and $\delta_1 \in \mathcal{M}_b$. Now, if $y \in M^\sigma_c$, there will, by claim 8, be a position $\langle \nu_1,\delta_2\rangle$ in $\mathcal{M}_{b,c}$ such that $\delta_2 \geq \delta_1$ and y is the argument of ineffective ν_1-requirements for A and B at position $\langle \nu_1,\delta_2\rangle$.

By choice of δ_1, $J^{\delta_1}_{i_1 a_1}(A^{\langle \nu_1,\delta_2\rangle}) = J^{\delta_1}_{i_1 a_1}(A^{\langle o,\delta_1+1\rangle})$ and the same will hold for B. Also '$y \in J^{\delta_1}_{i_1 a_1}(A^{\langle \nu_i,\delta_2\rangle})$' has a value. Then by the construction, part 3, the ν_1-requirements mentioned above will be subsets of $M^{\delta_1}(A^\xi \cup B^\xi)$.. When we at stage δ_1 created a permanent requirement for the 2-condition, we prevented new r's from M^{δ_1} to be added to $A \cup B$. Thus the ν_1-requirements will be permanent.

Using Σ^*-collection over M^σ we find $\delta_3 \in \mathcal{M}_b$ such that

$\forall y \in M^\sigma$ (y is the argument of permanent ν_1-requirements for A and B, ineffective at stage δ_3).

Since there are at most $\aleph_{k-1}$ conditions $\nu_1 \leq \nu$, we may find a $\delta_4 > \sigma$ such that

* $\forall \nu_1 < \nu$ (ν_1 of type 2 $\Longrightarrow$ $y \in M^\sigma$ (y is the argument of permanent ν_1-requirements for A and B, ineffective at stage δ_4))

Let $\{\sigma_n\}_{n\in\omega}$ be an increasing sequence starting with the given σ_0 such that the relation between σ_{n+1} and σ_n is as * between δ_4 and σ.

This is not constructive, so we use full ordinary DC.

Let $\sigma = \text{Sup}\{\sigma_n\}$.

By choice of σ_0, $r \in Q^A_{\langle\eta,\sigma\rangle}(\nu)$, and by definition of $Q^A_{\langle\eta,\sigma\rangle}$ there is a $\nu_1 \leq \nu$ such that r is the element of an effective ν_1-requirement with some argument y at position $\langle\eta,\sigma\rangle$.

If ν_1 is of type 1, we are safe since then after stage σ_0, effective and essential ν_1-requirements are the same.

We will prove that ν_1 is not of type 2.

Assume it is. Then $y \in \bigcup_{\delta<\sigma} M^\delta$ since these are the only arguments considered up to and at stage σ. But then for some σ_n, $y \in M^{\sigma_n}$, and after stage σ_{n+1}, y is argument of permanent, ineffective ν_1-requirements. This contradicts that y is the argument of an effective ν_1-requirement at position $\langle\eta,\sigma\rangle$.

□

<u>Remark</u> In the proof of this lemma we did not use the properties of P and several of the properties of Q. The construction of the sequence σ_n is, however, not valid in the ω-case, so the analogous point in that proof is, in idea, more complicated.

From claims 7, 9 and 10 we obtain

<u>Claim 11</u> For each ν, $Q^A(\nu) \in M$.

<u>Proof</u>: $Q^A(\nu) = \{r ; \exists z \exists \nu_1 \leq \nu \, (r \in z$ and z is an essential ν_1-requirement$)\}$,

by claims 9 and 10. By claim 7, for each $\nu_1 \leq \nu$

$\{r ; \exists z \, (r \in z$ and z is an essential ν_1-requirement$)\}$

is in M, and M is closed under subsets of cardinality at most $\aleph_{k-1}$. □

<u>Observation</u>

A candidate from row ν can be rejected for four reasons:

1. We realize a candidate for a condition $\leq \nu$. Since we realize at most one candidate for each condition, this way of rejecting candidates takes an end.

2. We create a ν_1-requirement for a 2-condition $\nu_1 < \nu$. By the priority argument this happens at most $\aleph_{k-1}$ times, and takes an end.

3. We create a ν_1-requirement with argument y, where ν_1 is a 3-condition, and $\nu \geq \nu_1 + \|y\|_\sigma$. For each ν_1 there are at most $\aleph_{k-1}$ arguments y that will satisfy the inequality, by lemma 2.4.a, and for each y there is by claim 5 at most $\aleph_{k-1}$ such requirements. Thus this rejecting also comes to an end.

4. r is rejected when we appoint a candidate less complex than r.

<u>Claim 12</u> Let $r, y, \nu \in \mathcal{M}_b$ and assume that $r \notin Q^A(\nu)$.

<u>a</u> There is a $\xi \in \mathcal{M}_b$ such that $r \notin P^A_\xi(\nu, y)$.

<u>b</u> For any $\xi \in \mathcal{M}_b$ there is a $\xi_1 \in \mathcal{M}_b$, $\xi_1 > \xi$, such that

$r \notin Q^A_{\xi_1}(\nu)$, where $\xi_1 = \langle \nu_1, \sigma_1 \rangle$ and r is from row ν_1 (if $r \notin {}^1M$, $r \notin Q^A_{\xi_1}(\nu)$ for all ξ_1, ν).

<u>Proof</u>: <u>a</u> follows from <u>b</u> by chosing $\xi = \|y\|^1$: If for $\xi_1 = \langle \nu_1, \sigma_1 \rangle$ we have that $r \notin Q^A_{\xi_1}(\nu)$, we have that $r \notin P_{\xi_1+1}(\nu, y)$ by definition of $P_{\xi_1+1}(\nu, y)$.

<u>Proof of b</u> We seek ξ_1 in $\mathcal{M}_{b'}$, and then use reflection.

<u>Subclaim</u> There is an increasing sequence $\langle \delta_a \rangle_{a \in I}$ in $\mathcal{M}_{b'}$ such that $\delta_o = K^b_{k-1}$ and

$\forall a \forall \nu_2 \leq \nu$ (ν_2 is a 3-condition $\Longrightarrow \forall z_1, y_1$ (If z_1 is an effective ν_2-requirement for A with argument y_1 at stage δ_a, and $r \in z_1$, then z_1 will be ineffective at some stage between δ_a and δ_{a+1})).

<u>Proof</u>: We will use DC over I, so let $\delta_a \in \mathcal{M}_{a,b'}$ be given, and assume $\delta_a \geq K^b_{k-1}$.

For some $\nu_2 \leq \nu$, let z_1 be a ν_2-requirement with argument y_1, arbitrarily chosen such that $y_1 \in M^{\delta_a}$ and $r \in z_1$. Let c be such that $y_1 \in \mathcal{M}^{\delta_a}_c$. Since $r \in z_1$ and $r \notin Q_A(\nu)$, z_1 will not be essential by claim 9. Thus z_1 will either be injured or there will be some permanent ν_2-requirement for B with argument y_1 and the same value as z_1. In the first case the injury will, by claim 3 take place before stage $K^{a,b',c}_{k-1}$. In the second case, when we have a permanent ν_2-requirement for B, this will be created before stage $K^{\langle a,b',c \rangle'}_{k-1}$, and by reflection there will be a ν_2-requirement for B with the same value and answer as z_1, active at some position in $\mathcal{M}_{a,b',c}$. In both cases there is a position $\xi_2 \in \mathcal{M}_{a,b',c}$ such that $\xi_2 > \langle o, \delta_a \rangle$ and z_1 is ineffective at position ξ_2.

Now we may use Σ^*-collection on M^{δ_a} and find δ_{a+1} as required. This ends the proof of the subclaim.

Now, let $\sigma_3 = \mathrm{Sup}\{\delta_a;\ a \in I\}$
Assume $r \in Q^A_{\langle\nu_1,\sigma_3\rangle}(\nu)$. From this we will prove a contradiction: By assumption r will at position $\langle\nu_1,\sigma_3\rangle$ be the element of some effective ν_2-requirement z_1 with argument y_1 for some $\nu_2 \leq \nu$ and $y_1 \in \bigcup_{\delta<\sigma_3} M^\delta$.

There are three possibilities:

a z_1 is effective at position $\langle o,\sigma_3\rangle$

b z_1 is active but ineffective at position $\langle o,\sigma_3\rangle$

c z_1 is created between position $\langle o,\sigma_3\rangle$ and $\langle\nu_1,\sigma_3\rangle$

Impossibility of a: If z_1 is effective at position $\langle o,\sigma_3\rangle$, z_1 would be effective at all stages below σ_3 except on a proper initial segment. This is impossible by choice of σ_3.

Impossibility of b: If b holds, there will be an injury of a ν_2-requirement for B with value y_1 somewhere between positions $\langle o,\sigma_3\rangle$ and $\langle\nu_1,\sigma_3\rangle$. Assume that some r_3 from some row ν_3 is put into B before position $\langle\nu_1,\sigma_3\rangle$ at stage σ_3. $\nu_3 < \nu$, so by reflection this would have been done before $K^b_{k-1} < \sigma_3$.

Impossibility of c: Let z_1 be a ν_2-requirement, $\nu_2 = i_1, i_2\langle a_1, a_2\rangle$. Since we at stage σ_3 create z_1 with argument y_1 there will for all $y_2 <_\sigma y_1$ in $\bigcup_{\delta<\sigma} M^\delta$ be ineffective ν_2-requirements for A and B with argument y_2 at stage σ_3. These will all be created at some stage $\sigma_4 < \sigma_3$, since σ_3 has cofinality $\aleph_k$. We may also assume that for $\sigma_4 \leq \sigma < \sigma_3$, $<_\sigma \upharpoonright y_1 = <_{\sigma_4} \upharpoonright y_1$.

Since we create z_1 with argument y_1 at stage σ_3, by case 3 in the construction (recall that J is partial):

$$y \in \bigcup_{\delta<\sigma} J^{\delta}_{i_1,a_1}(A^{\langle\nu_2,\sigma_3\rangle}) \vee y \notin \bigcup_{\delta<\sigma} J^{\delta}_{i_1,a_1}(A^{\langle\nu_2,\sigma_3\rangle})$$

Now $A^{\langle\nu_2,\sigma_3\rangle} = A^{\langle o,\sigma_3\rangle}$ by the proof of the impossibility of b.
Then for some $\sigma_5 < \sigma_3$

$$y \in J^{\sigma_5}_{i_1,a_1}(A^{\langle o,\sigma_3\rangle}) \vee y \notin J^{\sigma_5}_{i_1,a_1}(A^{\langle o,\sigma_3\rangle}).$$

Since σ_3 has cofinality $\aleph_k$ we may use the priority argument and some 2-condition to find a $\sigma_6 \geq \max\{\sigma_5,\sigma_4\}$ and $\sigma_6 < \sigma_3$ such that for $\sigma_6 \leq \sigma < \sigma_3$

$$J^{\sigma_5}_{i_1,a_1}(A^{\langle o,\sigma_3\rangle}) = J^{\sigma_5}_{i_1,a_1}(A^{\langle o,\sigma\rangle}).$$

But then a ν_2-requirement with argument y_1 would have been created first time we paid attention to ν_2 after stage σ_6. Since we are above K^b_{k-1}, this requirement cannot be injured, so we cannot have c.

These arguments show that $r \notin Q^A_{\langle\nu_1,\sigma_3\rangle}(\nu)$.
Let $\xi_1 = \langle\nu_1,\sigma_3\rangle$. $\xi_1 \in \mathcal{M}_{b'}$, but by reflection we find a similar one in $\mathcal{M}_b$.

We are now ready to prove i of the theorem:

Assume that $A = J_{i,a}$. We want to obtain a contradiction. Let $\nu = 1.A.i.a.$ If we ever put an r from row ν into A, we know that $A \neq J_{i,a}$ must hold, so there is no element in A from row ν.

There will be a stage σ in the construction such that

i After σ we do not reject candidates from row ν due to reasons 1 - 3 in the observation.

ii After σ we will always pay attention to ν

iii All elements in $Q^A(\nu)$ that are in A will be in $A^{\langle o,\sigma\rangle}$, and $Q^A(\nu) \in M^\sigma$.

We may assume that $\sigma = K^b_{k-1}$ for some b. Then we will appoint a candidate r for ν at stage σ. Since $r \notin A$, $r \notin J_{i,a}$ and there will be a $\sigma_1 \in \mathcal{M}_{b'}$ such that $\sigma_1 \geq \sigma$ and $r \notin J^{\sigma_1}_{i,a}$.
By claim 12.b there will be a position $\xi \in \mathcal{M}_{b'}$ such that $\xi = \langle \nu,\sigma_2\rangle$ for some σ_2, and $r \notin Q^A_\xi(\nu)$. (Recall that $r \notin Q^A(\nu)$.)
But ξ will be recursive in r, so at stage σ_2 we will put something into A from row ν (see case 1 of the construction). This gives the contradiction, since by choice of $\sigma = K^b_{k-1}$ we will not reject r due to reason 4. □

We will now end this proof by proving iii in the theorem.

Let $\nu = i_1,i_2,\langle a_1,a_2\rangle$ and assume that $J_{i_1a_1}(A) = J_{i_2,a_2}(B)$ and that both are total.
By claim 11, $Q^A(\nu)$ and $Q^B(\nu)$ are both elements of M.
Let σ_1 be so large that $A^{\langle o,\sigma_1\rangle} \cap Q^A(\nu) = A \cap Q^A(\nu)$ and $B^{\langle o,\sigma_1\rangle} \cap Q^B(\nu) = B \cap Q^B(\nu)$.
Let $\sigma_2 \geq \sigma_1$ be so large that all r's form rows $\leq \nu$ that ever go into $A \cup B$ will be there at stage σ_2. Let b be such that $\nu,\sigma_2,Q^A(\nu),Q^B(\nu)$ are all elements of $\mathcal{M}_b$. We will prove that $J_{i_1,a_1}(A)$ is $w-\Delta^*_b$.

Let $y \in \mathcal{M}_{b,c}$ be given. For some $\sigma \in \mathcal{M}_{b,c}$, $P^A_{\langle o,\sigma\rangle} \in \mathcal{M}_{b,c}$ (e.g. $\|y\|^1 < \langle o,\sigma\rangle$)
By claim 12 a:

$$\forall d \forall r \in [P^A_{\langle o,\sigma\rangle}(\nu,y) \setminus Q^A(\nu)] \cap \mathcal{M}_{b,c,d} \exists \xi \in \mathcal{M}_{b,c,d}\, r \notin P^A_\xi(\nu,y)$$

Since $P^A_\xi(\nu,y)$ is monotonously shrinking there will by Σ^*- collection be a $\sigma_3 \in \mathcal{M}_{b,c}$ such that $P^A_{\langle o,\sigma\rangle}(\nu,y) \subseteq Q^A(\nu)$ and $P^B_{\langle o,\sigma\rangle}(\nu,y) \subseteq Q^B(\nu)$.

By claim 8 there will be a stage $\sigma_4 \in \mathcal{M}_{c,b}$ such that $\sigma_4 \geq \sigma_3$ and at position $\langle \nu,\sigma_4\rangle$ there are ineffective ν-requirements with argument y. We will prove that the values of these requirements will be the values of

$$y \in J_{i_1,a_1}(A)\,? \quad \text{and} \quad y \in J_{i_2,a_2}(B)\,?$$

If that is correct, we may give the following $w-\Delta^*_b$-description of $J_{i_1,a_1}(A)$:

For $y \in M_{b,c}$

$$y \in J_{i_1,a_1}(A) \iff \exists \sigma \in \mathcal{M}_{b,c}\ (P^A_{\langle o,\sigma\rangle}(\nu,y) \subseteq Q^A(\nu)$$

and $P^B_{\langle o,\sigma\rangle}(\nu,y) \subseteq Q^B(\nu)$ and at stage σ there are ineffective ν-requirements for A and B with argument y and value 'yes')

$$y \notin J_{i_1,a_1}(A) \iff \exists \sigma \in \mathcal{M}_{b,c}\ (P^A_{\langle o,\sigma\rangle}(\nu,y) \subseteq Q^A(\nu)$$

and $P^B_{\langle o,\sigma\rangle}(\nu,y) \subseteq Q^B(\nu)$ and at stage σ there are ineffective ν-requirements for A and B with argument y and value 'no')

<u>Proof of the claim</u>: We know that there will be permanent ν-requirements for A and B with argument y giving the right value. That the requierments at stage σ_4 above have the right value will then follow from

<u>Subclaim</u> Let σ_4 be as above, i the value for the ν-requirements for A and B with argument y. At all positions after $\langle o,\sigma_4\rangle$ there will be at least one active ν-requirement with argu-

ment y and value i.

<u>Proof of subclaim</u>: The proof is by induction on $\xi \geq \langle o,\sigma_4\rangle$. The successor step is like the proof in Shoenfield [14], while we use a trick borrowed from Lerman-Sacks [5] to pass the limits.

1. <u>Successor case $\xi + 1$</u>

<u>a</u> If there are active ν-requirements with argument y and value i for both A and B at position ξ, we cannot injure more than one of them, since we do not put elements into both A and B at the same position.

<u>b</u> Assume that there is a ν-requierment z for A with argument y and value i, but not for B. z is then effective. We will obtain a contradiction from the assumption that some $r \in z$ is put into A at position $\xi + 1$. Let r above be from row η. By case 1 of the construction, $r \notin Q^A_{\xi+1}(\nu)$, and by choice of σ_2, $\nu < \eta$. Since r is in z and z is effective, we have $r \in P_{\xi+1}(\nu,y)$, using the definition of $r \in Q^A_{\xi+1}(\nu)$. By choice of σ_3, $r \in Q^A(\nu)$. But this is impossible by choice of σ_1, and we obtain a contradiction.

2. <u>Limit case</u> To go through a limit position it is sufficient to prove that we will not injure ν-requirements with argument y for A and B alternately more than a finite number of times. This follows from the following considerations:

Assume that we between $\langle o,\sigma_4\rangle$ and ξ alternately have injured ν-requirements with argument y for A and B in an ω-sequence. By the successor case there will at all positions below ξ be at least one active ν-requirement with value y and argument i. Let z_1 be the requirement for A and z_2 the requirement for B active at $\rho_0 = \langle o,\sigma_4 \rangle$. By symmetry we may assume that we first injure z_1 by putting r_1 from row ν_1 into z_1 at position $\rho_1 > \rho_0$. When we then injure z_2 with r_2 from row ν_2 at position $\rho_2 > \rho_1$, there will be a ν-requirement z_3 for A with argument y active at position ρ_2. When we injure z_3 with r_3 from row ν_3 at position $\rho_3 > \rho_2$, there will be some ν-requirement z_4 for B with argument y active at position ρ_3 etc. We find a sequence of requirements injured by r_n from row ν_n at position ρ_n.

Since both z_n and z_{n+1} are created at position ρ_n, it follows from claim 1 that $\nu_{n+1} < \nu_n$. This is indeed a contradiction.

This proves the subclaim, the claim, and the proof of theorem 4.1 is completed.

5. Martins axiom and recursion in a normal type - 3 - object

In sections 3 and 4 we used V = L to perform certain priority arguments. The only properties we actually used was the Generalized Continuum Hypothesis (GCH) and the existence of a recursive well-ordering of minimal length! A natural problem is then: How can these assumptoins be weakened ?

In this section we will restrict ourselves to recursion in a normal type - 3 - object F. We will assume that there is a minimal well-ordering of I = tp(1) recursive in F. Instead of CH we will use Martins Axiom or the axiom $A_{\aleph}$ for $\aleph < 2^{\aleph_0}$ as described in Martin-Solovay [8].

The different lemmas and theorems will be marked with MA , $A_{\aleph}$

resp. $<$ when we assume Martins Axiom, $A_{\aleph}$ resp. existence of minimal recursive well-ordering. We will let F be a fixed normal functional of type-3.

Our aim is to establish sufficient machinery to use the proofs in section 3 and 4. This is done by proving that a recursive set of cardinality $< 2^{\aleph_0}$ share important properties with the subindividuals in the general theory. To do this we refer to a paper by Moldestad [9] on general recursion on two types, where he proves e.g. the reflection principles for recursion in functionals over the more general domains.

In Martin-Solovay [8] it is proved that if $\aleph_1 < 2^{\aleph_0}$, if $A_{\aleph_1}$ holds and if there is a $\underset{\sim}{\Pi}^1_1$-set of cardinality $\aleph_1$, then all sets of cardinality $\aleph_1$ are $\underset{\sim}{\Pi}^1_1$. In the following we are using methods from that proof only.

For $x \subseteq \omega$, $n,m \in \omega$ define f_x to be the characteristic function of x, $\bar{f}_x(m) = \langle f_x(o),\ldots,f_x(m)\rangle$ and $S_{x,n} = \{\bar{f}_x(m)\,;\ m$ is a power of the $n+1$'st prime number$\}$

For $B \subseteq \mathcal{P}(\omega)$, $t \subseteq \omega$ let

$$B * t = \{a\,;\exists b \in B\ (n \in a \iff t \cap S_{b,n} \text{ is finite})\}$$

<u>Theorem 5.1</u> ($A_{\aleph}$, Martin-Solovay [8])
Let $B \subseteq \mathcal{P}(\omega)$ be of cardinality $\aleph$ and let $A \subseteq \mathcal{P}(\omega)$ be of cardinality at most $\aleph$.

Then there is a set $t \subseteq \omega$ such that $A = B * t$.

<u>Remark</u> A will be $\Sigma_1(B,t)$ uniformly in B and t, and thus recursive in B,t and ${}^3E$.

Corollary 5.2 $(A_{\aleph})$

By Ext - 2 - sc (F) we mean $\bigcup_{a\in I} 2\text{-}sc(F,a)$

The following are equivalent

i $\exists B \subseteq I\ (\bar{\bar{B}} = \aleph \wedge B \in \text{Ext-2-sc}(F))$

ii $\forall B \subseteq I\ (\bar{\bar{B}} = \aleph \Rightarrow B \in \text{Ext-2-sc}(F))$

Proof: Since $\mathcal{P}(\omega)$ and ${}^{\omega}\omega$ are essentially the same modulo F, this follows directly from theorem 5.1.

Corollary 5.2 $(A_{\aleph})$

Assume there is a $B \in \text{Ext-2-sc}(F)$ such that $\bar{\bar{B}} = \aleph$. Let $\langle M_a \rangle_{a\in I} = \text{Spec}(F)$, $M = \bigcup_{a\in I} M_a$. Then

a M is closed under subsets of cardinality $\aleph$.

b $cf(K_1^F) > \aleph$.

Proof: b follows from a. To prove a, let $x \subseteq M$ be of cardinality $\aleph$. For each $y \in x$ pick one pair e_y, a_y such that in E-recursion $\{e_y\}(a_y, I) = y$. Let $A = \{\langle e_y, a_y \rangle ; y \in x\}$. Then $A \in M$ by corollary 5.2. Using Σ^*-collection over A we see that $x \in M$.

□

By MA we may prove that $2^{\aleph_0}$ is regular. We will for instance obtain this from Theorem 5.1. Also Theorem 5.1 gives $\aleph_0 \leq \aleph < 2^{\aleph_0} \Rightarrow 2^{\aleph} = 2^{\aleph_0}$. This is sufficient to find a partial ordering $\prec$ on I satisfying * from section 2. Adding the well-ordering we obtain:

Lemma 5.4 (MA , <)

There exists a partial ordering $\prec$ on I recursive in F such that $\prec$ satisfies $*$ of section 2.

Proof: The only part of the proof of lemma 2.6 which we cannot do immediately here, is the effective indexing of triples of subsets of field $(<_\nu) = D$. But for $t = \langle t_1, t_2, t_3 \rangle$ let $A_t, B_t, C_t = \langle D*t_1, D*t_2, D*t_3 \rangle$. We order these triples by the given ordering on the t's, and the effective indexing is given.

To simplify arguments we will now assume that functionals act upon subsets of the domain instead of on functions on the domain. What normal functionals concerns, this is no restriction or addition to the theory. In particular, F acts on subsets of ω.

Let $A \subseteq I$ be recursive in F, a. Let $\bar{A}$ be the closure of A under primitive recursive operations. When A is infinite, A and $\bar{A}$ will have the same cardinality, and $\bar{A}$ is recursive in F, a. We assume $A = \bar{A}$, e.g. $\omega \subseteq A$ and A is closed under pairing.

Now, let $I_0 = \mathcal{P}(\omega)$, $B = \mathcal{P}(A)$. I_0 and I are essentially the same, and so are B and ${}^{A}\omega$. Following Moldestad [9] we let $B(\supseteq A)$ be a comain for recursion on two types. When

$$ {}^{B}E(X) = \begin{cases} 0 & \text{when } X = \emptyset \\ 1 & \text{when } X \neq \emptyset \end{cases} \qquad \text{for } X \subseteq B, $$

we will have $\mathcal{P}(\omega) \subseteq B$ as a set recursive in ${}^{B}E$.

Let $F_1(X) = F(X \cap \mathcal{P}(\omega))$. We will prove that the theory θ_1 in F_1 over B is 'equivalent' to the theory θ_2 in F over I_0.

Definition Let $X \subseteq A$. We say that t codes X if $A * t = X$.

Lemma 5.5 The set of codes is recursive in F and A.

Proof: '$a \in A * t$' is a recursive relation, and

$$ t \text{ is a code} \iff \forall a (a \in A*t \Rightarrow a \in A) $$

<u>Lemma 5.6</u> $(A_{\aleph})$

Let $A = \bar{A}$ be recursive in F, a, and assume $\bar{\bar{A}} = \aleph$

<u>a</u> In E-recursion there is an index e such that $B = \{e\}^{F}(a, I)$

<u>b</u> F_1 is E-recursive in a, I relative to F

<u>c</u> F is E-recursive in a, I relative to F_1.

<u>Proof</u>:

<u>a</u> $B = \{A * t ; t$ is a code$\}$. We use Σ^*-collection over the set of codes.

<u>b</u> and <u>c</u> are even more trivial.

This lemma leads to the following result.

<u>Theorem 5.7</u> $(A_{\aleph})$

Let $A \in 2\text{-sc}(F, a)$ and assume $A = \bar{A}$ and $\bar{\bar{A}} = \aleph$.

Let $B = \mathcal{P}(A)$. For arbitrary $x \in V$,

$$\mathcal{R}(\{x, a, I\}; F) = \mathcal{R}(\{x, a, B\}; F_1)$$

<u>Corollary 5.8</u> $(A_{\aleph})$

Let A, a be as in theorem 5.7. Then

$$TC(M_a) <_{\Sigma_1} TC(\bigcup_{b \in A} M_{\langle a, b \rangle})$$

<u>Proof</u>: By theorem 5.7 this is nothing more than simple reflection in Moldestad's theory on two types for F_1.

<u>Corollary 5.9</u> $(A_{\aleph})$

Let A, a be as in theorem 5.7.

Define $M_x = \mathcal{R}(\{x, I\}; F)$.

If $x \subseteq A$ is complete Σ^*_a-definable, then $M_a <_{\Sigma_1} M_{x, a}$.

Proof: By theorem 5.7 this reduces to further reflection in the theory on two types, verified by Moldestad in [9].

The program is now to fix notation such that the proofs of theorems 3.1 and 4.1 can be repeated with as few modifiactions as possible.

Definition $(<)$

a For $a \in I$, let $\mathcal{M}_a = \bigcup_{b \leq a} M_{b,a}$

b a is minimal if $a = \mu b(a \in \mathcal{M}_b)$

c $a' = \mu b(b \notin \mathcal{M}_a)$

d $K^a_{<a} = \mathrm{Sup}(On \cap \mathcal{M}_a)$

e $\lambda^a_{<a}$ = Least ordinal not in $\mathcal{M}_a$ = ordertype of $\{\alpha;\ \alpha$ is the length of a computation relative to F in I, a and some $b \leq a\}$.

It is clear that $b < a \Rightarrow \mathcal{M}_b \subseteq \mathcal{M}_a$, and by corollary 5.8, $TC(M_a) <_{\Sigma_1} TC(\mathcal{M}_a)$.

By the recursive wellordering we then obtain

$$M_a <_{\Sigma_1} TC(\mathcal{M}_a)$$

Lemma 5.10 $(MA, <)$

$\forall a(\mathcal{M}_a \in M_{a'})$

Proof: We may use the proof of lemma 2.3.

Lemma 5.11 $(MA, <)$

$\mathcal{M}_a <_{\Sigma_1} \mathcal{M}_{a'}$

Proof: By corollary 5.9 we must prove that a' is E-recursively equivalent to a complete Σ^*_a - subset of $\bar{A}$ modulo a, where

$A = \{c ; c \leq a\}$.

Since $\mathcal{M}_a \in M_{a'}$ and $a < a'$, a complete Σ^*_a-definable subset of $\bar{A}$ is clearly definable from a'. Now let $x \subset \bar{A}$ be the set of pairs $\{\langle e,c\rangle ; c \in \bar{A}\ \&\ \{e\}^F(c,I)\downarrow\}$. Then by Σ^*-collection $\mathcal{M}_a \in M_x$. But then $\{b ; b \notin \mathcal{M}_a\} \in M_x$ and $\mu b(b \notin \mathcal{M}_a) \in M_x$. But this b is a', so $a' \in M_x$.

□

Definition (MA, <)

Let $a \in I$. Card(a) $= \mu b(\exists t (\{c; c \leq a\} = \{c; c \leq b\} * t))$.

Card(a) will be the least b such that the initial segments has the same cardinality.

a is called a cardinal if Card(a) $= a$. Then $\|a\|$ will be a cardinal in the ordinary sense.

Lemma 5.12 (MA, <)

There is a recursive minimal well-ordering $<^o$ such that if a is an infinite cardinal in $<^o$, then $\{c; c <_o a\}$ is closed under primitive recursion.

Proof: By induction on the cardinals $\aleph < 2^{\aleph_0}$ we define $<^o_\aleph$ uniformly recursive in the $a_\aleph$ such that $\|a_\aleph\| = \aleph$, and if necessary extend $<^o$ to I.

On limit cardinals $\aleph$, $<^o_\aleph = \bigcup_{\aleph' < \aleph} <^o_{\aleph'}$. $\aleph_0$ is treated like 0^+.

If $\aleph^+ < 2^{\aleph_0}$ and $<^o_\aleph$ is constructed let

$A = \overline{\{c; c < a_{\aleph^+}\}}$ field $(<^o_\aleph)$

$\bar{\bar{A}} = \aleph^+$ and there is a t recursive in $a_{\aleph^+}$ such that

$A = \{c; c < a_{\aleph^+}\} * t$.

We order A by $c_1 <_A d_1$ if $\exists c < a_{\aleph^+} \forall d < a_{\aleph^+}$

$$\{c_1\} = \{c\}*t\ \&\ (\{d_1\} = \{d\}*t \Rightarrow c < d)$$

We extend $<^{0}_{\aleph}$ to $<^{0}_{\aleph^+}$ by adding A wiłh this ordering at the end.

If $\aleph^{+} = 2^{\aleph_0}$, we let $A = I \setminus \mathrm{field}(<^{0}_{\aleph})$ and $<_A = < \restriction A$. We then proceed as above.

The construction is effective and the result as required. $\square$

From now on assume that $<$ has the properties $<^{0}$ has by construction.

Lemma 5.13 (MA, $<$)

For each $a \in I$ $\mathcal{M}_a = \bigcup_{i<\mathrm{card}(a)} M_{a,i}$

Proof: Clearly $\bigcup_{i<\mathrm{card}(a)} M_{a,i} \subseteq \mathcal{M}_a$.

To prove the converse, pick the least t such that $\{c; c<a\} = \{i; i < \mathrm{card}(a)\} * t$. Then each $c < a$ is recursive in t, a and some $i < \mathrm{card}(a)$. $\square$

We will define $<^1$, $<^2$ and $<_\sigma$ as in section 2. Lemma 2.4 then reads:

Lemma 5.14 (MA, $<$)

a For any $x \in M$, $\{<_\sigma | x; \sigma \in On\}$ has cardinality $< 2^{\aleph_0}$.

b If $x \in \mathcal{M}_a$ then $\forall \sigma \geq K^{a}_{<a}$ $(<_\sigma \restriction x = <_{K^{a}_{<a}} \restriction x)$

c For any x, $\{\|x\|_\sigma; \sigma \in On\}$ is finite. $\square$

We may now state and prove the main result of this section:

Theorem 5.15 (MA, $<$)

Let $\langle M_a \rangle_{a \in I} = \mathrm{Spec}(F)$ and use the terminology from this section. Replace $V = L$ in assumptions by (MA, $<$). Then

a Theorem 3.1 relativized to F will hold.

b Theorem 4.1 relativized to F will hold.

We may obtain the same corollaries as in sections 3.1 and 4.1.

Proof: With few modifications we may use the proofs given in the $V = L$-case:

1. Coding of the a-conditions: We let i,j vary over $\{c; c<\mathrm{card}(a)\}$ instead of over the subindividuals, and then define a-conditions as before. We use the ordering on $\{c; c < \mathrm{card}(a)\}$ to order the a-conditions. (Here we use lemma 5.12)

2. Changes in notation: At all places in the proof, replace $\aleph_k$ by $2^{\aleph_0}$, 'At most $\aleph_{k-1}$' with $< 2^{\aleph_0}$, K^a_{k-1} with $K^a_{<a}$.

3. New proofs: At some points in the proof we used that $\aleph_{k-1}$ is regular and that the cofinality of $K^a_{k-1} = \aleph_{k-1}$. At these points we must give a new proof. A typical example is claim 2 of theorem 3.1 of which we give a new proof.

New proof of Claim 2, Th. 3.1.

After $\mathrm{Sup}(K^b_{<b}; b<a)$ we will only realize candidates for c-conditions where $c \geq a$. There are $< \|\mathrm{card}(a)\|$ such conditions of higher priority than ν. So, let $\nu = \langle i,a\rangle$, $i < \mathrm{card}(a)$. Then $X = \{\xi;$ we make a change on a condition $\leq \nu$ at position ξ, and

$$\mathrm{Sup}\{K^b_{<b}; b<a\} \leq \xi < K^a_{<a}\}$$

has cardinality $< \|\mathrm{card}(a)\|$, and for some $j < a$, $\|j\|$ is the ordertype of X, $j \in \mathcal{M}_a$, and X will be definable from i, j, a by Σ^*-collection. Then $X \in \mathcal{M}_a$ and cannot be cofinal in $\mathcal{M}_a$. The claim is proved.

This method can be used whenever we in the original proofs used that $\aleph_{k-1}$ was regular.

With the modifications given above, the proofs of Theorems 3.1 and 4.1 are proofs of Theorem 5.15 a and b. $\square$

Remark

Also higher order versions of Martin's axiom has been studied (Baumgartner, Laver) and the following will be consistent with ZFC.

Assumption

Let $k > 1$. For $i < k-1$, $2^{\aleph_i} = {}_{i+1}$. Let $I = tp(k)$. There is a formula φ 1. order over I such that when A, B are subsets of I and $\bar{\bar{A}} \leq \bar{\bar{B}} < 2^{\aleph_{k-1}}$, then there is a $t \in I$ such that

$$\forall a \in I \; (a \in \bar{\bar{A}} \Longleftrightarrow \varphi(a,t,B))$$

From this assumption we may give the same proofs as we did in this section for $k = 1$.

This consistency result was told me by Keith Devlin.

Remark

From a model for ZF + DC + AD we may construct a model for ZFC + CH +

If $A \subseteq tp\,(2)$ is r.e. in ${}^3E$ and some individual, and $B \subseteq I$ is complete r.e. $({}^3E)$, then either is A weakly recursive in ${}^3E$ and some individual or B is recursive in A, ${}^3E$ and some individual.

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