The Geometric Weight System for Transformation Groups on Cohomology Product of Spheres

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Introduction.

In this paper we describe the cohomological "orbit structure" of the action of a torus G on a space X whose rational cohomology ring is isomorphic to $H^*(S^m \times S^n, Q)$, (with m and n even integers) from the equivariant cohomology of X. The basic approach follows ideas of Wu-Yi Hsiang, in particular we interpret his notion of "geometric weight system" as a set of invariants from the equivariant cohomology simple enough to be effictively computable, on the other hand strong enough to determine the cohomological orbit structure of X. This means the following: The connected orbit types G/G_x^o of X are determined by the identity components G_x^o of the isotropy subgroups. If $x \in X$ the F^o-variety of x, $F^o(x)$, is the connected component of x in the fixed point set of G_x^o. The structure of this network of F^o-varieties determines the orbit structure of X. Thus, in particular, the geometric weight system should determine all connected orbit types, the cohomological structure of the corresponding F^o-varieties and their "relative positions".

For some cases when $H^*(X,Q)$ has one generator, this program has been quite successful; see Hsiang (9) and Hsiang and Su(11).

The case with two generators is already considerably more complicated and shows interesting new features. As is demonstrated by many examples, the general case is no longer modelled on "linear actions". However, we obtain a complete description in terms of suitably defined geometric weight systems, and there is good correspondence between the theory and the examples which can be constructed explicitly.

The basic tool for setting up the geometric weight system is a linearity theorem for certain ideals associated to the equivariant cohomology algebra. This idea goes back to the "topological Schur lemma" of Wu-Yi Hsiang. (Hsiang (8)). In an early version of this work, (Tomter (15)), special cases of annihilator ideals of submodules of $H_G^*(X,X^G;\mathbb{Q})$ were studied. (Here H_G^* is the equivariant cohomology functor and X^G is the fixed point set of X .) A general structure theory for annihilator ideals of such submodules has been developed by T. Chang and T. Skjelbred (see Chang and Skjelbred (7)) and has found interesting applications. In our situation, however, it is necessary to consider the more general case of the primary decomposition of a quotient of two submodules of $H_G^*(X,X^G;\mathbb{Q})$.

In section one, after a few remarks on the basic notions and theorems of equivariant cohomology, we prove the relevant theorem for such ideals. This is applied to set up geometric weight systems in the second part. A number of examples show that practically all the phenomena predicted by the theory can occur. Under additional assumptions, however, many of the more complicated cases may be ruled out, for example fixed point sets of the type $P^2(h) + \{pt.\}$. (See section 2.4). On the other hand, consider the case $H^*(X^G;\mathbb{Q}) = H^*(S^p+S^q;\mathbb{Q})$ of section 2.3. It was shown in Tomter (16) that if $G = S^1$ it is possible that $p \neq q$. (Examples of this were known for $\mathbb{Z}_p$-transformation groups.) This is improved here to show that there exist tori of arbitrarily large rank acting on spaces with integral cohomology isomorphic to some $H^*(S^m \times S^n; \mathbb{Z})$ with $H^*(X^G; \mathbb{Z}) \simeq H^*(S^p+S^q; \mathbb{Z})$, $p \neq q$.

After the basic theory of the action of a torus is understood, it is possible to carry through systematic studies and computations for actions of simple, compact Lie groups by restricting to the maximal torus and using the Weyl group. Here we only include a simple example of such results, and leave a classification of principal isotropy subgroups, orbit types and dimension estimates for a later paper.

In this paper cohomology is taken with rational coefficients and is denoted by $H^*(X)$; hence we only get information on the connected orbit types. Cohomology with $\mathbb{Z}_p$-coefficients gives further information.

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§ 1. Structure Theorems in Equivariant Cohomology

Let G be a compact Lie group. All G-spaces X are assumed to be paracompact, of finite cohomological dimension and with a finite number of orbit types. $X \sim Y$ means that $H^*(X)$ is isomorphic to $H^*(Y)$ as a $\mathbb{Q}$-algebra. For standard constructions we refer to Bredon (4) or Hsiang (9). Thus X_G is the total space of the fibre bundle associated to the universal G-bundle : $E_G \to B_G$ by the given G-action on X . The equivariant cohomology of X is defined by $H_G^*(X) = H^*(X_G)$. If Y is an H-space; $\rho\colon G \to H$ is a homomorphism of compact Lie groups, and $f\colon X \to Y$ is ρ-equivariant, there is an induced homomorphism from $H_H^*(Y)$ to $H_G^*(X)$. We need more information on this homomorphism if $Y = X$ and f is the identity. G acts freely on $E_G \times E_H$ by $(e_1,e_2)\cdot g = (e_1\cdot g, e_2\cdot\rho(g))$; hence we may take $E_G \times E_H$ as the total space in a universal bundle for G . There is a well-defined map: $X_G = (E_G \times E_H)\times_G X \to E_H \times_{\rho(G)} X \to E_H \times_H X = X_H$ given by $(e_1,e_2,x) \to (e_2,x)$. The fibre of this map from X_G to X_H is easily seen to be H_G . When G is connected, the classifying space B_G is simply connected. The Eilenberg-Moore spectral sequence is a 2. quadrant spectral sequence (E_r,d_r) where $E_r \Rightarrow E_\infty = H_G^*(X)$ and $E_2 = \mathrm{Tor}_{RH}(RG, H_H^*(X))$. Here we denote $H^*(B_G)$ by RG ; RG and $H_H^*(X)$ are RH-modules through cup-product and the homomorphisms induced in cohomology from the commutative diagram of fibrations:

$$\begin{array}{ccccc} & & X & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ H_G & \longrightarrow & X_G & \longrightarrow & X_H \\ \downarrow & & \downarrow & & \downarrow \\ H_G & \longrightarrow & B_G & \longrightarrow & B_H \end{array}$$

If RG or $H_H^*(X)$ is a flat RH-module, it is well known that $\mathrm{Tor}^n{}_{RH}(RG, H_H^*(X)) = 0$ for $n \neq 0$ and $E_2 = \mathrm{Tor}^0{}_{RH}(RG, H_H^*(X)) = RG \otimes_{RH} H_H^*(X)$. Hence we have the following result:

Theorem 1.

If RG or $H_H^*(X)$ is a flat RH-module, the above Eilenberg-Moore spectral sequence collapses and $H_G^*(X) = H_H^*(X) \otimes_{RH} RG$; i.e. $H_G^*(X)$ is obtained from $H_H^*(X)$ by an extension of scalars corresponding to the canonical homomorphism $\rho^* : RH \to RG$.

The assumptions of the Theorem are satisfied in the following special cases:

a) G is a subgroup of H and X is totally non-homologous to zero in the fibration $X \to X_H \to B_H$. Then $H_G^*(X) = H_H^*(X) \otimes_{RH} RG$. If $G = (e)$ is the trivial subgroup, we get $H^*(X) = H_H^*(X) \otimes_{RH} \mathbb{Q}$.

b) G is a torus, K is a subtorus, and ρ is the epimorphism $G \to H = G/K$. Then $H_G^*(X) = H_{G/K}^*(X) \otimes_{R(G/K)} RG$.

c) G is a maximal torus in the compact, connected Lie group H. Then $H_G^*(X) = H_H^*(X) \otimes_{RH} RG$, and $H_H^*(X) = H_G^*(X)^W$ where W is the Weyl group.

Proof. In case a) it is obvious from the Serre spectral sequence of $X \to X_H \to B_H$ that $H_H^*(X)$ is a free RH-module; hence it is flat. In case b) it is easy to see that the fibre $H_G = E_G \times_G (G/K) \simeq B_K$. We may identify RG with the polynomial algebra $\mathbb{Q}[t_1, \dots, t_r]$ where the t_i's are identified with linear functionals on G ; i.e. elements of $H^1(G)$, via transgression in the universal bundle $G \to E_G \to B_G$. It is then obvious that RG is a free $R(G/K)$-module. For c) we notice that in general, if G is a subgroup of H, then E_H is also an E_G and there is a fibration from $H_G = E_H \times_G H$ to H_H with fibre H/G ; since $H_H \simeq B_{(e)}$ is acyclic, it follows from the Serre spectral sequence that $H^*_H(H_G) \simeq H^*(H/G)$. Let G be a maximal torus in H, let $N(G)$ be the normalizer of G in H and $W = N(G)/G$ the Weyl group. Then $H/N(G)$ is $\mathbb{Q}$-acyclic and the Serre spectral sequence of the fibration $H/N(G) \to X_{N(G)} \to X_H$ shows that $H^*_H(X) = H^*_G(X)^W$, $RH = RG^W$. Clearly $RG = RH \otimes_{\mathbb{Q}} H^*(H/G)$ is a free RH-module, hence it is flat, and the proof of Theorem 1 is complete.

Now if $x \in X$, let r_x be the canonical projection from RG to RG_x induced by inclusion of G_x in G . If S is a multiplicative subset of RG, let $X^S = \{x \in X; S \cap \ker(r_x) = \emptyset\}$. The basic localization theorem for equivariant cohomology is now well known.

Theorem 2.
The localized restriction homomorphism $S^{-1}H^*_G(X) \to S^{-1}H^*_G(X^S)$ is an isomorphism.

If S is the complement of a prime ideal P, we denote $S^{-1}H^*_G(X)$ by $H^*_G(X)_P$ and X^S by X^P . If $P = (0)$, $X^P = X^G = F$ is the fixed point set, and $H^*_G(X)_{(0)} = H^*_G(X) \otimes_{RG} R'G = H^*_G(F) \otimes_{RG} RG'$ $= (H^*(F) \otimes_{\mathbb{Q}} RG) \otimes_{RG} R'G = H^*(F) \otimes_{\mathbb{Q}} R'G$, where $R'G$ is the quotient field of RG .

From now on we assume that $G = T$ is a torus. There are examples of Hsiang which show that only in this case is there a strong relationship between the algebraic structure of the equivariant cohomology and the orbit structure of X . Let $\{x_i\}$ and $\{v_j\}$ be a set of even - and odd-dimensional generators of $H^*_T(X)_{(0)}$, respectively. Then there is a presentation of $H^*_T(X)_{(0)}$ given by an epimorphism p from the free, anti-commutative $R'T$-algebra $A_T = R'T[x_1,\dots,x_k] \otimes_{R'T} \wedge_{R'T}[v_1,\dots,v_l]$ to $H_T^*(X)_{(0)}$. Let p_j: $H_T^*(F)_{(0)} \to H_T^*(F^j)_{(0)}$ be induced from the inclusion of the j-th component F^j into F, let $I = \ker p$ and $I_j = \ker(p_j \circ p)$.

Theorem 3. (Hsiang (10)).

1. The radical of I is the intersection of s maximal ideals M_j whose varieties are rational points $a_j = (a_{j1},\dots,a_{jk}) \in (R'T)^k$; $i = 1,\dots,s$.
2. There is a natural bijection between the connected components F^j of F and the above points $\{a_j\}$, such that the restriction homomorphism of an arbitrary point $q_j \in F^j \subseteq X$ maps the even generator $x_i \in H_T^*(X)_{(0)}$ to $a_{ji} \in H_T^*(\{q_j\})_{(0)} \simeq R'T$.
3. $H^*(F^j) \otimes_{\mathbb{Q}} R'T \simeq A_T/I_j$, where $I_j = I_{M_j} \cap A_T$. Moreover $I = I_1 \cap \dots \cap I_s = I_1 \dots I_s$.

Let X be a cohomology manifold over $\mathbb{Q}$; then any component F^j of F is also a cohomology manifold over $\mathbb{Q}$. Let $w_i \in H^2(B_T)$ and let $H_i = (w_i^{\perp})$ be the corank one subtorus whose Lie algebra is the kernel of w_i interpreted as a linear functional. Let $X^{H_i} = G_i^1 + \dots + G_i^l$ with the G_i^k's connected; then each F^j is included in a unique $G_i^{i(j)}$. w_i is a local geometric weight at F^j if $\dim G_i^{i(j)} - \dim F^j > 0$, and the multiplicity is defined to be $\frac{1}{2}(\dim G_i^{i(j)} - \dim F^j)$. The local Borel formula asserts that the G_i^k's are transversal in the sense that $\dim X - \dim F^j = \sum_i (\dim G_i^{i(j)} - \dim F^j)$. Let $x \in X$ and $F^j \subseteq F^0(x)$; let $\{w_k, m_k\}$ be the local geometric weight system at F^j. Then $G_x^0 = (\cap H_k;\ H_k = (w_k^{\perp}) \supseteq G_x^0)^0$, and $\dim F^0(x) - \dim F^j = 2\Sigma m_k$ (sum over the k's such that $H_k \supseteq G_x^0$). This reveals the signigicance of the local geometric weight system.

After the proof of the Su conjecture this can be generalized to Poincaré duality spaces over $\mathbb{Q}$ (see Chang and Skjelbred (6)). A torus $L \subseteq T$ is said to be cohomology ineffective on X if $H^*(X,X^L) = 0$. T acts cohomology effectively if the only cohomology ineffective subtorus is the trivial subgroup. An F^o-variety in X with generic isotropy subgroup $K = K^o$ is then a component V of X^K such that the action of T/K on V is cohomology effective. Then the above statements hold in the more general setting of Poincaré duality spaces over $\mathbb{Q}$ when dimension is now interpreted as formal dimension. If X is a compact, orientable cohomology manifold, the two notions of local geometric weights coincide.

We will use the following observation: Let K be a subtorus of T and let P_K be the kernel of the homomorphism $r_K\colon RT \to RK$. The variety of the ideal P_K is the Lie algebra of K; this determines a bijective correspondence between subtori of T and linear subspaces of the Lie algebra of T which are rational with respect to the $\mathbb{Q}$-structure determined by the defining lattice of the torus T. It follows that to a given prime ideal P in RT there exists a unique minimal subtorus K in T such that $P_K \subseteq P$, hence $X^P = X^{P_K} = X^K$.

Let X be a T-space with $F = X^T \neq \emptyset$ and K a subtorus of T. Let M be a submodule of $H^*(F)$ and define $M_{T,K} = \partial(M \otimes RT) \subseteq H_T^*(X^K,F)$, where ∂ is the boundary operator in the exact sequence in the equivariant cohomology for the pair (X^K,F). If K is the trivial subgroup (e), we denote $M_{T,K}$ by M_T simply. Let ρ be the projection from T to $K' = T/K$. It follows from Theorem 1 that $H_T^*(X) \simeq H_{K'}^*(X) \otimes_{RK'} RT$, similarly for

F , so $H^*_T(X;F) \simeq H^*_{K'}(X,F) \otimes_{RK'} RT$.

Theorem 4.

Let X be a T-space with $F = X^T \neq \emptyset$. Let M and N be submodules of $H^*(F)$ with $N \subset M$. Then the prime ideals corresponding to a reduced primary decomposition of $\mathrm{Ann}(M_T/N_T)$ $= \{a \in RT;\ a\cdot M_T \subseteq N_T\}$ are linear ideals. The isolated primes $P_1,\dots,P_l$ are characterized as follows: A prime ideal P of RT is equal to one of the P_i, $i = 1,\dots,l$ if and only if $P = P_K$, where K is a maximal subtorus of T with respect to the property $M_{T,K} \neq N_{T,K}$.

We need a lemma for the proof.

Lemma 1.

Let K be a subtorus of T . Then all primary ideals associated with a reduced primary decomposition of $\mathrm{Ann}(M_{T,K}/N_{T,K})$ are contained in P_K .

Proof. RT is a flat RK'-module. ($K' = T/K$); hence it is easily seen that $M_{T,K} = M_{K',K} \otimes_{RK'} RT$ and $M_{T,K}/N_{T,K} \simeq (M_{K',K}/N_{K',K}) \otimes_{RK'} RT$. It is well known that in the flat case we must then have $\mathrm{Ann}(M_{T,K}/N_{T,K}) = \mathrm{Ann}(M_{K',K}/N_{K',K}) \otimes_{RK'} RT$. The generators in $H^2(B_{K'})$ represent linear functionals on T which vanish on K; hence $\rho^*(RK') \subseteq P_K$ and $\mathrm{Ann}(M_{T,K}/N_{T,K}) \subseteq P_K$. Let $\mathrm{Ann}(M_{K',K}/N_{K',K}) = \cap\, q_i$ be a reduced primary decomposition in RK' with associated prime ideals P_i . Again, since RT is flat as an RK'-module, it follows from Proposition 11 in Ch.IV, §2.6 in Bourbaki (2) that in order to prove that $\cap\, q_i \otimes_{RK'} RT$ is a reduced primary decomposition of $\mathrm{Ann}(M_{T,K}/N_{T,K})$, it is sufficient to show that all the ideals $P_i \otimes_{RK'} RT$ are prime. Let

$RK = \mathbb{Q}[t_1,\ldots,t_l]$, then it is clear that $RT \simeq RK'[t_1,\ldots,t_l]$. Here $RK'[t_1]/P_i[t_1] \simeq (RK'/P_i)[t_1]$; RK'/P_i and hence $(RK'/P_i)[t_1]$ is an integral domain, so $P_i[t_1]$ must be a prime ideal. By repetition we see that $P_i \otimes_{RK'} RT$ is prime in RT . Hence $P_i \otimes_{RK'} RT$ are the primes associated to a reduced primary decomposition of $\mathrm{Ann}(M_{T,K}/N_{T,K})$; since $P_i \subseteq RK'$ it follows that these are in P_K . q.e.d.

Proof of Theorem 4: Let $\mathrm{Ann}(M_T/N_T) = \cap\, q_i$ be a reduced primary decomposition and let $P_i = \sqrt{q_i}$. If P is a prime ideal in RT, $\mathrm{Ann}(M_T/N_T)_P \cap RT = \cap\, \{q_i;\ P_i \subseteq P\}$. Hence $P = P_i$ for one of the i's if and only if $\mathrm{Ann}(M_T/N_T)_P \cap RT \subsetneq \cap\, \mathrm{Ann}(M_T/N_T)_{P'} \cap RT$, the last intersection taken over those prime ideals P' with $P' \subsetneq P$. (Observed in Chang and Skjelbred (7)). Choose one of the P_i's and let K be the minimal subtorus with $P_K \subseteq P_i$. Let Q be any prime ideal such that the minimal subtorus L with $P_L \subseteq Q$ is equal to K . We have: $\mathrm{Ann}(M_T/N_T)_Q \cap RT = \mathrm{Ann}((M_T/N_T)_Q \cap RT$ $= \mathrm{Ann}[(M_T)_Q/(N_T)_Q] \cap RT = \mathrm{Ann}[(M_{T,K})_Q/N_{T,K})_Q] \cap RT$ $= \mathrm{Ann}[(M_{T,K}/N_{T,K})_Q] \cap RT = \mathrm{Ann}[M_{T,K}/N_{T,K}]_Q \cap RT$.
The first and the last equalities follow since we are dealing with finitely generated RT-modules. By the localization theorem $H_T^*(X,F)_Q \simeq H_T^*(X^K,F)_Q$; hence $(M_T)_Q \simeq (M_{T,K})_Q$ and the third equality follows. For the main step in the proof we apply Lemma 1. Since $P_K \subseteq Q$ it follows from Lemma 1 that $\mathrm{Ann}(M_{T,K}/N_{T,K})_Q \cap RT$ $= \mathrm{Ann}(M_{T,K}/N_{T,K})$. But if $P_K \neq P_i$, this contradicts the fact that $(\mathrm{Ann}\ M_{T,K}/N_{T,K})_{P_i} \cap RT \subsetneq \cap(\mathrm{Ann}\ M_T/N_T)_{P'} \cap RT$; $P' \subsetneq P_i$. Hence $P_i = P_K$; i.e. all the associated primes are linear. The isolated primes P_i are the minimal prime ideals P containing $\mathrm{Ann}(M_T/N_T)$, i.e. they are minimal with respect to the condition

that $\mathrm{Ann}(M_T/N_T)_P \cap RT \neq RT$. Again, letting K be the subtorus determined by P , we have $\mathrm{Ann}(M_T/N_T)_P \cap RT = \mathrm{Ann}(M_{T,K}/N_{T,K})$. Hence K is a maximal subtorus with respect to the condition that $M_{T,K} \neq N_{T,K}$; and this concludes the proof of Theorem 4.

Remark.

If $N = (0)$, we get the result of Chang and Skjelbred (7) for the submodule M_T of $H_T^*(X,F)$. In this case it follows directly that $(\mathrm{Ann}\, M_{T,K})_Q \cap RT = \mathrm{Ann}\, M_{T,K}$, since it is easily shown (Theorem 1) that the map $H_T^*(X^K,F) \to H_T^*(X^K,F)_Q$ is injective. This is not sufficient to conclude that $M_{T,K}/N_{T,K} \to (M_{T,K}/N_{T,K})_Q$ is injective, and we need Lemma 1 to see that $\mathrm{Ann}(M_{T,K}/N_{T,K})_Q \cap RT = \mathrm{Ann}\, M_{T,K}/N_{T,K}$.

If X is totally non-homologous to zero in X_T, we have $H_T^*(X) \otimes_{RT} R'T \simeq H^*(F) \otimes R'T$. If M is a submodule of $H^*(F)$, $\mathrm{Ann}\, M_T$ is always a principal ideal (generated by the least common multiple of the denominators when a set of generators of M are expressed as reduced RT-rational linear combinations of elements of $H_T^*(X)$). If $(0) \neq N \subseteq M$, however, there are several examples in section 2 showing that $\mathrm{Ann}(M_T/N_T)$ is not in general a principal ideal, and the general primary decomposition is needed.

The following corollary is known (Allday and Skjelbred (1)).

Proposition 1.

Let X be a Poincaré duality space over $\mathbb{Q}$ and let T act on X . Let $F_1,\dots,F_s$ be the connected components of $F = X^T$, let f_j be the fundamental cohomology class of F_j and 1_j the generator of $H^0(F)$. Let $M_1 = (f_j)$, $M_2 = (1_j)$. Then $\mathrm{Ann}\, M_1$ is a principal ideal whose generator is the product of the local geometric weights at F_j with multiplicities, and the isolated prime ideals of $\mathrm{Ann}\ M_2$ correspond to the generic isotropy subgroups of the minimal F^o-varieties connecting F_1 with other components of F.

§ 2. Geometric Weight Systems for Cohomology Product of Spheres

We use the theory developed in the last section to study the orbit structure of a cohomology effective action of a torus T on a space $X \sim S^m \times S^n$, where m and n are positive, even integers. From the Serre spectral sequence it is clear that all differentials are zero in this case; hence X is totally non-homologous to zero in the fibre bundle $X_T \to B_T$. We use j^* for the homomorphism in equivariant cohomology: $H_T^*(X) \to H_T^*(F)$ induced from the inclusion of $F = X^T$ in X, and i^* for the homomorphism: $H_T^*(X) \to H^*(X)$ induced from the inclusion of the fibre X in X_T; some times we use this notation also for the corresponding maps for invariant subspaces of X. Let x and y be generators in $H^m(X)$ and $H^n(X)$ respectively; it is easy to find $\hat{x} \in H_T^m(X)$ and $\hat{y} \in H_T^n(X)$ such that $i^*(\hat{x}) = x$, $i^*(\hat{y}) = y$ and $H_T^*(X) \simeq RT[\hat{x},\hat{y}]/I$, where I is the ideal generated by $\hat{x}^2 - c_1\hat{y} - d_1$ and $\hat{y}^2 - c_2\hat{x} - d_2$; $c_j, d_j \in RT$, $j = 1,2$. The variety of I consists of the intersection points of the parabolas $\hat{x}^2 = c_1\hat{y} + d_1$ and $\hat{y}^2 = c_2\hat{x} + d_2$, each intersection point corresponding to a component of the fixed point set with the intersection number of a point equal to the Euler characteristic of the corresponding component. (Tomter (16)).

Theorem 5.

Let $X \sim S^m \times S^n$ with m and n even, positive integers, and let T act on X. There are the following possibilities:

1. Both parabolas degenerate to double lines which intersect at the origin. $F \sim S^p \times S^q$ with p and q even, positive integers.
2. One parabola degenerates to a double line, the other is tangent to this at the origin. $F \sim P^3(h)$ with h an even, positive integer.
3. One parabola degenerates to a double line, the other intersects this in two distinct points. $F \sim S^p + S^q$ with p and q even, positive integers.
4. The parabolas have one transversal intersection point and a point of tangency with intersection number three. $F \sim P^2(h) + \{pt\}$ with h an even, positive integer.
5. The parabolas intersect at two simple points and are tangent at a third point. Then $F \sim S^p + \{pt\} + \{pt\}$ with p an even, positive integer.
6. The parabolas intersect transversally at four distinct points and F has four acyclic components.

Here $X \sim P^r(h)$ means that $H^*(X)$ has one generator u of dimension h which satisfies the relation $u^{r+1} = 0$.

Case 1.

Here $c_i = d_i = 0$, $i = 1,2$. Let u and v be generators of $H^p(F)$ and $H^q(F)$ respectively. Let U, V, and W be the submodules of $H^*(F)$ generated by $\{u,uv\}$, $\{v,uv\}$ and $\{uv\}$ respectively.

Theorem 6.

The ideals $\mathrm{Ann}(U_T/W_T)$ and $\mathrm{Ann}(V_T/W_T)$ are principal ideals.

The geometric weight system is defined by two generators $a = w_1^{k_1} \dots w_s^{k_s}$ and $b = w_1^{l_1} \dots w_s^{l_s}$ for these respective ideals, $w_i \in H^2(B_T)$. The connected components of the corank one isotropy subgroups are given by $H_i = w_i^{\perp}$; the structure of the corresponding corank one F^o-varieties are given by $X^{H_i} = F(H_i) \sim S^{p+2k_i} \times S^{q+2l_i}$, $i = 1,\dots,s$.

Proof: Let $j^*(\hat{x}) = a_1+u\otimes a_2+v\otimes a_3+uv\otimes a_4$, $a_i \in RT$. From $\hat{x}^2 = 0$ we have $a_1 = a_2a_3 = 0$, by renaming we may assume that $a_3 = 0$. Hence $j^*(\hat{x}) = u\otimes a_2+uv\otimes a_4$, and it follows easily that $j^*(\hat{y}) = v\otimes b_3+uv\otimes b_4$. Hence $\mathrm{Ann}(U_T/W_T) = (a_2)$ and $\mathrm{Ann}(V_T/W_T) = (b_3)$ are principal ideals. In this simple case, Theorem 4 implies that the factors of a_2 and b_3 are linear; i.e. we have generators a and b ehich are rationals multiples of a_2 and b_3 and which split as above. From the proof of theorem 4 it follows that the localization $\mathrm{Ann}(U_T/W_T)_{(w_i)} \cap RT = \mathrm{Ann}(U_{T,H_i}/W_{T,H_i}) = (w_i^{k_i})$, similarly $\mathrm{Ann}(V_{T,H_i}/W_{T,H_i}) = (w_i^{l_i})$. Obviously this implies that $X^{H_i} \sim S^{p+2k_i} \times S^{q+2l_i}$. q.e.d.

Remark 1. Ann W = $(a\cdot b)$; by Proposition 1 ab determines the local geometric weight system, i.e. it determines the local geometric weights w_i and the total dimension of $X^{H_i} \sim S^{m_i} \times S^{n_i}$, but to determine the individual sphere dimensions m_i and n_i we need the above refinement.

Remark 2. To compute $H^*(F((H_i \cap H_j)^o))$ one simply determines the weights which are in the two-dimensional subspace spanned by w_i and w_j , say $w_1,\dots,w_r$. Then

$F(H_i \cap H_j) \sim S^{p+2k_1+\dots+2k_r} \times S^{q+2l_1+\dots+2l_r}$. Similarly one can compute the cohomology of all the higher corank F^o-varieties $F(H_{i_1} \cap \dots \cap H_{i_k})$. The result shows that in case 1 a general torus action on X has the same cohomological orbit structure as the diagonal of two linear actions on S^m and S^n with weight systems a and b respectively.

We digress briefly in this case to consider a typical application to actions of classical groups.

Theorem 7.
Let $G = SU(l)$, $l \geq 4$ act on $S^m \times S^n$, let T be a maximal torus with $F(T) \sim S^p \times S^q$ and assume that $l(l-1) > m-2, n-2$. Then all orbits are finitely covered by complex Stiefel manifolds $SU(l)/SU(l-k)$.

Proof: Let $W_G = N(T)/T$, then $F(T)$ is easily seen to be W_G-invariant; hence there is a linear representation of W_G on each $H^k(F)$. Let $\theta_1, \dots, \theta_l$ with $\theta_1 + \dots + \theta_l = 0$ be the usual coordinates on T, then W_G is the symmetry group on $\{\theta_1, \dots, \theta_l\}$. Any representation of W_G of degree less that $l-1$ is trivial on the subgroup A_l of even permutations, so in our case A_l acts trivially on each $H^k(F)$. Since $j^*: H_T(X) \to H_T(F)$ is a W_G-morphism, it is clear that $\mathrm{Ann}(U_T/W_T)$ and $\mathrm{Ann}(V_T/W_T)$ are A_l-invariant; i.e. the weight systems $\{(\pm w_1; k_1), \dots, (\pm w_s; k_s)\}$ and $\{(\pm w_1; l_1), \dots, (\pm w_s; l_s)\}$ are invariant under even permutations of $\{\theta_1, \dots, \theta_l\}$. Let $w = n_1\theta_1 + \dots + n_l\theta_l$, $n_i \in \mathbb{Z}$. It is easily seen that the shortest A_l-orbit occurs if $w = \theta_i$ for some i, and the second shortest occurs if $w = \theta_i + \theta_j$, the latter has length $\frac{1}{2}l(l-1)$. From the dimension estimates $l(l-1) > m-2$ and $n-2$, it follows that only the shortest orbit can occur;

$a = \theta_1^{k_1} \ldots \theta_l^{k_l}$ and $b = \theta_1^{l_1} \ldots \theta_l^{l_l}$, i.e. the weight systems of $k_1(l_1)$ copies of the standard representation of $SU(l)$ on C^l . Now we can use the technique in Hsiang ( ) to reach the conclusion. By choosing a suitable point x on an arbitrary G-orbit of X , one may assume that the maximal torus T_1 of G_x^o is contained in T , i.e. there exist weights $w_{i_1}, \ldots, w_{i_k}$ such that $T_1 = T_x^o = w_{i_1}^{\perp} \cap \ldots \cap w_{i_k}^{\perp}$; one may as well assume $T_x^o = w_1^{\perp} \cap \ldots \cap w_k^{\perp}$. Let $\Delta(G)$ be the weight system of the adjoint representation, and $\Delta(G)| T_1$ the restriction of this to T_1 . The action of G along the orbit G/G_x has weight system $\Delta(G)|T_1 - \Delta(G_x^o)$, hence, if $\Omega = \{(\theta_1;k_1+l_1), \ldots, (\theta_l;k_1+l_1)\}$, then $\Delta(G_x^o) \supseteq \Delta(G)|T_1 - \Omega|T_1$. From this equation it is a Lie algebra computation to show that $G_x^o = SU(l-k)$. q.e.d.

Remark. By considering cohomology with $\mathbb{Z}_2$-coefficients and 2-weights, one can show that the orbits must actually be complex Stiefel manifolds. Obviously there are similar theorems for $SO(n)$ and $Sp(n)$.

Case 2.

Here $c_2 = d_2 = d_1 = 0$. Let u be a generator in $H^h(F)$. From the relations $\hat{x}^2 = c_1\hat{y}$, $\hat{y}^2 = 0$ we get: $j^*(\hat{x}) = u\otimes a_2+u^2\otimes a_3+u^3\otimes a_4$, $j^*(\hat{y}) = u^2\otimes b_3+u^3\otimes b_4$ with $a_i, b_i \in RT$ satisfying the relations: $a_2^2 = c_1b_3$, $2a_2a_3 = c_1b_4$, $j^*(\hat{x}\hat{y}) = u^3\otimes a_2b_3$. Let U, V and W be the submodules of $H^*(F)$ generated by $\{u,u^2,u^3\}$, $\{u^2,u^3\}$ and $\{u^3\}$ respectively. Then it follows from the above that $Ann(W_T) = (a_2b_3)$, $Ann(U_T/V_T) = (a_2)$.

Theorem 8.

The geometric weight system is defined by two splitting elements $a = w_1^{k_1} \dots w_s^{k_s}$ and $b = w_1^{l_1} \dots w_s^{l_s}$, with $w_i \in H^2(B_T)$, $0 < 2k_i \geq l_i$ for $i = 1,\dots,s$. The connected components of the corank one isotropy subgroups are given by $H_i = w_i^{\perp}$, $i = 1,\dots,s$. The structure of the corank one F^o-varieties are given by:

a) $l_i < 2k_i$: $F(H_i) \sim S^{h+2k_i} \times S^{2h+2l_i}$

b) $l_i = 2k_i$: $F(H_i) \sim P^3(h+2k_i)$.

Proof: Since $\mathrm{Ann}(W_T)$ is a principal ideal, it follows from Theorem 4 that the generator $a_2 b_3$ must split as the product of weights in $H^2(B_T)$, hence $a_2 = q_2 w_1^{k_1} \dots w_s^{k_s}$ and $b_3 = q_3 w_1^{l_1} \dots w_s^{l_s}$ with $q_2, q_3 \in Q$, $w_i \in H^2(B_T)$. Since u^3 is the fundamental cohomology class of F, it is actually clear from Proposition 1 that $w_1,\dots,w_s$ are the geometric weights. From $a_2^2 = c_1 b_3$ it follows that $2k_i \geq l_i$. Also $\dim F(H_i) = 3h+2k_i+2l_i$. We have the exact sequence $0 \to I \to R_T[\hat{x},\hat{y}] \to H_T^*(X) \to 0$; since $H_T^*(X)$ is a flat R_T-module it follows from Theorem 1 a) that $0 \to I \otimes_{RT} RH_i \to RH_i[\hat{x},\hat{y}] \to H_{H_i}^*(X) \to 0$ is exact. Here $\rho_i^*(c_1) \neq 0$ in RH_i iff $l_i = 2k_i$; i.e. in this case the H_i-action on X belongs to case 2, else it belongs to case 1. Thus, if $l_i < 2k_i$, $F(H_i) \sim S^{p_i} \times S^{q_i}$ with $p_i + q_i = 3h+2l_i+2k_i$. From Theorem 4 it follows that the localization $(a_2)_{(w_i)} \cap R_G = (w_i^{k_i}) = \mathrm{Ann}(U_{T,H_i}/V_{T,H_i})$. Applying the above discussion to the T-action on $F(H_i)$, it is then clear that one of the sphere dimensions p_i, q_i must equal $h+2k_i$. Hence $F(H_i) \sim S^{h+2k_i} \times S^{2h+2l_i}$. If $l_i = 2k_i$ the multiplicity of w_i is $3k_i$, from the above remarks it follows that $F(H_i) \sim P^3(h+2k_i)$.

q.e.d.

We can construct examples of Case 2 with the torus T of arbitrarily high rank. Let Q be the quaternions and S^7 the unit sphere in $Q^2 \simeq C^4$. We have the Hopf-bundle $S^3 \to S^7 \to QP(1) = S^4$, by taking the quotient by S^1 we get the bundle $S^2 \to \mathbb{C}P(3) \xrightarrow{\pi} S^4$ and the corresponding $\mathbb{R}^3$-bundle ξ over S^4. Let η be a C^d-bundle such that $\xi \oplus \eta$ is trivial, let S^1 act on η by complex multiplication and trivially on ξ. (This is the "Su" construction, see Bredon (4), p. 420) Let R be a representation of the r-dimensional torus T^r on $\mathbb{C}^e$ with weight system $\{(w_1;r_1),\ldots,(w_s;r_s)\}$, such that $\{w_1,\ldots,w_s\}$ are pairwise linearly independent. We may choose a weight w which is linearly independent of each w_i, $i = 1,\ldots,s$; let p be the corresponding homomorphism from T^r to S^1. Then T^r acts on the trivial bundle $\epsilon(e)$ over S^4 by R, and on $\xi \oplus \eta$ by p; hence $\xi \oplus \eta \oplus \epsilon(e)$ is a T^r-bundle over S^4 with unit sphere bundle $X = S^4 \times S^{2d+2e+2}$ and fixed point set $F = \mathbb{C}P(3)$. From the Serre spectral sequence it follows that the corank one F^o-varieties are given by $F(w_i^{\perp}) \sim S^4 \times S^{2r_i+2}$ for $i = 1,\ldots,s$ and $F(w^{\perp}) = S^4 \times S^{2d+2}$.

Case 3.

Here $F = F_1+F_2 \sim S^p+S^q$. Let u and v be generators in $H^p(F)$ and $H^q(F)$ respectively, and let 1_i be the generator of $H^o(F_i)$, $i = 1,2$. From the relations $\hat{x}^2 = c_1\hat{y}+d_1$ and $\hat{y}^2 = 0$ it follows easily they $j^*(\hat{y}) = u\otimes a + v\otimes b$, $j^*(\hat{x}) = u\otimes c + 1_1\otimes d + v\otimes e - 1_2\otimes d$, where we have the relations (i) $d_1 = d^2 \neq 0$. (ii) $ac_1 = 2cd$. (iii) $bc_1 = -2ed$. (iv) $bc + ae = 0$.

Let U, V and W be the submodules of $H^*(F)$ generated by $\{u\}$, $\{v\}$ and $\{u,v\}$ respectively. Then $\mathrm{Ann}(W_T/V_T) = (a)$,

$\mathrm{Ann}(W_T/U_T) = (b)$. Let $M = H^*(F)$, then $\mathrm{Ann}(M_T/W_T) = (d)$. Since $j^*(\hat{x}\hat{y} + d\hat{y}) = u \otimes 2ad$ and $j^*(\hat{x}\hat{y}+d\hat{y}) = -v \otimes 2bd$, we have $\mathrm{Ann}(U_T) = (ad)$, $\mathrm{Ann}(V_T) = (bd)$, hence ad and bd determine the local geometric weight systems around F_1 and F_2 respectively.

The complexity of the orbit structure depends on whether the first parabola $\hat{x}^2 = c_1\hat{y} + d_1$ degenerates to two parallell lines or not. We treat the simpler case first.

a) $c_1 = 0$.

Theorem 9.

Let the equivariant cohomology of X be given by the ideal I of relations generated by $\hat{x}^2 = d_1 \neq 0$ and $\hat{y}^2 = 0$. The geometric weight system is then given by the generators of the above three annihilator ideals: $a = q_1 w_1^{k_1} \ldots w_s^{k_s}$, $b = q_2 w_1^{l_1} \ldots w_s^{l_s}$, and $d = q_3 w_1^{p_1} \ldots w_s^{p_s}$, where $q_i \in \mathbb{Q}$ and $w_i \in H^2(B_T)$. The structure of the corank one F^0-varieties are given as follows:

Let $H_i = w_i^{\perp}$. Then $F(H_i) \sim S^{p+2k_i} + S^{q+2l_i}$ if $p_i = 0$.

and $F(H_i) \sim S^{2p_i} \times S^{p+2k_i}$ if $p_i > 0$.

Proof: By Theorem 4 the generators of annihilator ideals which are principal ideals must split into linear factors as above. If $p_i = 0$, we know from Theorem 4 that the localization $(d)_{(w_i)} \cap RT = RT = \mathrm{Ann}(M_{T,H_i}/W_{T,H_i})$, i.e. $1_1 - 1_2$ is in the image of $j^*\colon H_T^*(F(H_i)) \to H_T^*(F)$; hence $F(H_i)$ has two components. From the multiplicities of w_i in the local geometric weight systems ad and bd it then follows that $F(H_i) \sim S^{p+2k_i} + S^{q+2l_i}$. On the other hand, if $p_i > 0$, $I \otimes_{RT} RH_i$ is generated by $\hat{x}^2$ and $\hat{y}^2$; from the exact sequence $0 \to I \otimes_{RT} RH_i \to RH_i[\hat{x},\hat{y}] \to H^*_{H_i}(X) \to 0$ it follows that $F(H_i) \sim S^{m_i} \times S^{n_i}$. Here $(d)_{(w_i)} \cap RT = (w_i^{p_i})$;

it is then an easy corollary of Theorem 4 that one of the individual sphere dimensions must be $2p_i$. By counting dimensions in the local geometric weight systems, it follows that the other is $p+2k_i = q+2l_i$. q.e.d.

b) $c_1 \neq 0$.

Theorem 10.

Let the equivariant cohomology of X be given by the ideal I of relations defined by $\hat{x}^2 = c_1\hat{y}+d_1$, $\hat{y}^2 = 0$, where $c_1, d_1 \neq 0$. Let N be the submodule of $H^*(F)$ generated by $1_1 - 1_2$. Then the geometric weight system is given by the above three annihilator ideals defined by $a = q_1 w_1^{k_1} \ldots w_s^{k_s}$, $b = q_2 w_1^{l_1} \ldots w_s^{l_s}$, $d = q_3 w_1^{p_1} \ldots w_s^{p_s}$ together with $\mathrm{Ann}(N_T) = (w_1^{r_1} \ldots w_s^{r_s})$. The structure of the corank one F^o-varieties are given as follows: Let $H_i = (w_i^{\perp})$. Then:

$F(H_i) \sim S^{p+2k_i} + S^{q+2l_i}$ if $p_i = 0$.

$F(H_i) \sim P^3(2p_i)$ if $p_i > 0$, $r_i = 3p_i$.

$F(H_i) \sim S^{2p_i} \times S^{p+2k_i}$ if $p_i > 0$, $r_i < 3p_i$.

There is at least one corank one F^o-variety of type $S^{2p_i} \times S^{p+2k_i}$.

Proof: The same proof as in a) gives the splitting of a, b and d and the structure of $F(H_i)$ when $p_i = 0$. We compute $\mathrm{Ann}(N_T)$. We may consider $\partial(1_1 - 1_2)$ as the generator of $H^1(X;F)$ and x, y, xy as elements of $H^*(X;F)$. From Theorem 2 we know that $H_T^*(X;F)$ is a torsion RT-module; hence, in the Serre spectral sequence for the pair of fibrations $X_T \to B_T$, $F_T \to B_T$; one of the E_2-elements x, y, xy must transgress to $\partial(1_1 - 1_2) \otimes g$. Since we are in the lowest filtration degree, this expression becomes zero in $H^*(X_T;F_T)$, and it follows that $\mathrm{Ann}(N_T) = (g)$. From the cohomology exact sequence of the pair (X_T, F_T) this means that

$(1_1 - 1_2 \otimes g$ is in the image of $j^*: H_T^*(X) \to H_T^*(F)$. We claim that it is xy which transgresses to $\partial(1_1-1_2) \otimes g$. For otherwise $j^*(a_1\hat{x}+a_2\hat{y}) = (1_1-1_2) \otimes g$, with a_1 or a_2 equal to 1. This gives $a_1c+a_2a = a_1e+a_2b = 0$. Substituting the relation (iv), we get $2a_2b = 0$, i.e. $a_2 = 0$, $a_1 = 0$ which is a contradiction. We have $j^*(\hat{x}\hat{y} + bde^{-1}\hat{x}) = (1_1-1_2) \otimes bd^2e^{-1}$; it follows that e must divide bd and g is a rational multiple of bd^2e^{-1}. Now if $p_i > 0$, it follows as before that $F(H_i)$ is connected. From relation (iii) we see that the multiplicity of w_i in c_1 is $l_i + 2p_i - r_i + p_i - l_i = 3p_i - r_i$. Hence $\rho_i^*(c_1) = 0$ if and only if $r_i < 3p_i$, where ρ_i^* is the canonical homomorphism $RT \to RH_i$. If $r_i = 3p_i$, we have $I \otimes_{RT} RH_i = (\hat{y}^2, \hat{x}^2 - \rho_i^*(c_1)\hat{y})$; if $r_i < 3p_i$, we have $I \otimes_{RT} RH_i = (\hat{y}^2, \hat{x}^2)$. Hence from the exact sequence $0 \to I \otimes_{RT} RH_i \to RH_i[\hat{x},\hat{y}] \to H_{H_i}^*(X) \to 0$ it again follows that these correspond to Case 2 and Case 1 for the H_i-action on X, respectively. The dimensions in these cases are computed by the multiplicities of w_i in the local geometric weight systems and by localizing the ideal (d) as in Theorem 9. If all connected corank one F_o-varieties were of type $P^3(2p_i)$, we would have $m+n = \dim g = 3 \dim d = 3m$, i.e. $\hat{x}^2 = c_1\hat{y}+d_1$, $c_1 \in \mathbb{Q}$ hence $x^2 = c_1 y$, which would mean that X was a cohomology projective space. q.e.d.

If T acts linearly on S^m and S^n with fixed point sets S^p $(p>0)$ and S^o respectively, then the diagonal action of T on $S^m \times S^n$ gives an example of a) with $p = q$, ("linear examples"). We now construct an example with equivariant cohomology as in b) (non-degenerate parabola); it is sufficient to show that corank one F^o-varieties can occur as cohomology projective spaces. Let $S^7 \subseteq Q^2 = \mathbb{C}^4$, let $S^2 \to \mathbb{C}P(3) \xrightarrow{\pi} QP(1) = S^4$ be the "Su bundle" considered in Case 2, and let ξ be the associated $\mathbb{R}^3$-bundle. Consider a linear action of a torus T on Q^2 with weight system $\{(w;2)\}$, i.e. $g\cdot(x_1,x_2) = (\exp(2\pi i\langle w,g\rangle x_1, \exp(2\pi i\langle w,g\rangle)x_2)$, this projects to a linear action on $QP(1) = S^4$ with $F = S^2$ and the local representation of T around F given by the weight system $\{(0;1),(-2w;1)\}$. On the other hand, viewing Q^2 as $\mathbb{C}^4$, this also induces a linear action on $\mathbb{C}P(3)$ with complex weitht system $(\pm w;2)$ and fixed point set $F_1 + F_2 = S^2 + S^2$. The local representations around fixed points now have weight systems $\{(0;1),(2w;2)\}$ (see Tomter (16)). Let R be a representation of T on $\mathbb{C}^n$ with weight system $\{(w_1;r_1),\dots,w_s;r_s)\}$ and let ε be the corresponding trivial T-bundle on S^4 (for the given T-action on the base space). This defines a T-structure on the unit sphere bundle X of the Whitney sum of ξ and ε. From the Serre spectral sequence of this bundle it is clear that $X \sim S^4 \times S^{2n+2}$. We may assume that the weight vectors $\{w,w_1,\dots,w_s\}$ are pairwise linearly independent. Then $F = X^T = S^2 + S^2$, and the corank one F^0-varieties are $F(w^\perp) = \mathbb{C}P(3)$, $F(w_i^\perp) \sim S^2 \times S^{2r_i}$, $i = 1,\dots,s$.

From this it is clear that Case 3 b) can occur with tori T of arbitrarily large rank. By a variation of this construction we can obtain the following improvement of Theorem 2 in Tomter (16) for circle actions, also see Chang and Comenetz (5), Theorem 3.

Theorem 11.

For any torus T it is possible to find a space X whose integral cohomology is isomorphic to $H^*(S^4 \times S^n ; \mathbb{Z})$ for some even integer n and an effective action of T on X such that the fixed point set $F = S^p + S^q$ with $p \neq q$.

Proof: Consider the Su bundle $\mathbb{C}P(3) \overset{\pi}{\to} QP(1) = S^4$ and ξ as in the last example. Now take a linear T-action on Q^2 with weight system $\{(w;1),(0;1)\}$, this defines a linear action on $QP(1)$ with fixed points P_1 and P_2 and local weight systems $\{(\pm w;1)\}$ and $\{(-w;2)\}$ respectively. On $\mathbb{C}P(3)$ the induced action has complex weight system $\{(\pm w;1),(0;2)\}$ and fixed point set $F_1 + F_2 + F_3$, where F_1 and F_2 are points and $F_3 = S^2$. The corresponding local representations have weight systems $\{(2w;1),(w;2)\}$, $\{(-2w;1),(-w;2)\}$ and $\{(0;1),(\pm w;1)\}$ respectively. Here the local representation around F_1 has weight $2w$ along the fibre of the Su bundle and $\{(w;2)\}$ transversally to the fibre, similarly for F_2 and F_3 . We have $\pi(F_1) = \pi(F_2) = P_1$, $\pi(F_3) = P_2$. Let R be a faithful representation of T on $\mathbb{C}^n$ with weight system $\{(0;r_o),(w_1;r_1),\ldots,(w_s;r_s)\}$ such that the weight vectors $w,w_1,\ldots,w_s$ are peirwise linearly independent. Let ϵ be the corresponding trivial T-bundle on S^4, and proceed to construct X as in the previous example. Then the Serre spectral sequence actually shows that $X \sim S^4 \times S^{2n+2}$ with $\mathbb{Z}$-coefficients. Furthermore $F = X^T = S^{2r_o} + S^{2r_o+2}$, and the corank one F^o-varieties are given by $F(w^\perp) \sim S^4 \times S^{2r_o+2}$ and $F(w_i^\perp) \sim S^{2r_o+2r_i} + S^{2r_o+2r_i+2}$ for $i = 1,\ldots,s$. q.e.d.

Case 4.

The equivariant cohomology is defined by the ideal $I = (\hat{x}^2 - c_1\hat{y} - d_1, \hat{y}^2 - c_2\hat{x} - d_2)$ with c_1, c_2, d_1 and d_2 non-zero elements of RT ; the variety of I consits of the two intersection points $(a, -2a^2c_1^{-1})$ and $(-3a, 6a^2c_1^{-1})$ corresponding to the fixed point components $F_1 \sim P^2(h)$ and $F_2 \sim \{pt\}$ respectively. An easy computation gives the relations (i) $8a^3 = c_1^2 c_2$, $d_1 = 3a^2$ and $d_2 c_1^2 = 12a^4$.

Let u, 1_1 and 1_2 be generators in $H^h(F_1)$, $H^o(F_1)$ and $H^o(F_2)$ respectively. Let $x' = \hat{x} - a$, $y' = \hat{y} + 2a^2 c_1^{-1}$. Then $j^*(x') = u^2 \otimes \alpha + u \otimes \beta - 1_2 \otimes 4a$, $j^*(y') = u^2 \otimes \gamma + u \otimes \delta + 1_2 \otimes 8a^2 c_1^{-1}$, with $\alpha, \beta, \gamma, \delta \in RT$. Straightforward computations give $\gamma = (2\alpha a + \beta^2) c_1^{-1}$, $\delta = 2\beta a c_1^{-1}$. Let M and N be the submodules of $H^*(F)$ generated by u^2 and 1_2 respectively. Then $\mathrm{Ann}(M_T) = (\beta^2 a c_1^{-1})$ and $\mathrm{Ann}(N_T) = (a^3 c_1^{-1}) = (c_1 c_2)$. By Proposition 1 these define the local geometric weight systems around F_1 and F_2 , and the elements a, c_1, c_2, β must all split as the product of weights in $H^2(B_T)$.

In this case we can describe the orbit structure from the equivariant cohomology as follows:

Theorem 12.

The geometric weight system is given by the splitting elements $a = q_1 w_1^{k_1} \cdots w_l^{k_l}$, $\beta = q_2 w_1^{h_1} \cdots w_l^{h_l} w_{l+1}^{h_{l+1}} \cdots w_{l+s}^{h_{l+s}}$ and $c_1 = q_3 w_1^{p_1} \cdots w_l^{p_l}$, where $k_i = h_i + \frac{1}{2}h$ for $i = 1, \ldots, l$ and $\sum_{j=l+1}^{l+s} 2h_j = (l-1)h$. Let $H_i = (w_i^{\perp})$, $i = 1, \ldots, s$. The corank one F^o-varieties are given as follows:

a) Let $1 \le i \le l$. Then $0 \le p_i \le \frac{3}{2}k_i$.

If $0 < p_i < \frac{3}{2}k_i$: $F(H_i) \sim S^{2k_i} \times S^{4k_i - p_i}$.

If $p_i = 0$: $F(H_i) \sim P^3(h+2k_i)$. If $p_i = \frac{3}{2}k_i$: $F(H_i) \sim P^3(h+k_i)$.

b) Let $i > l$. Then $F(H_i) \sim P^2(h+2k_i) + F_2$.

Proof: For $1 \leq i \leq l$ the two intersection points of the parabolas are joined to zero by restricting the action to H_i, hence $F(H_i)$ is connected. Comparing multiplicities of w_i in $Ann(M_T)$ and $Ann(N_T)$ we get $k_i = h_i + \frac{1}{2}h$. From (i) it follows that $2p_i \leq 3k_i$. If $0 < p_i < \frac{3}{2}k_i$ it follows that $\rho_i^*(c_2) = 0$ in RH_i, hence $F(H_i) \sim S^{m_i} \times S^{n_i}$. Let U and V be the submodules of $H^*(F)$ generated by $(u^2,u,1)$ and (u^2,u) respectively. Then $Ann(U_T/V_T) = (a, a^2c_1^{-1})$ and $Ann(U_T,V_T)_{(w_i)} \cap R_T = (w_i^{l_i})$ with $l_i = \min(k_i, 2k_i - p_i)$. From Theorem 4 it follows that $2l_i = \min(m_i,n_i)$, from the local geometric weitht systems $m_i+n_i = 6k_i-2p_i$; hence $F(H_i) \sim S^{2k_i} \times S^{4k_i-2p_i}$.

If $p_i = 0$, $\rho_i^*(c_1) \neq 0$ in RH_i, hence the H_i-action on X is Case 2 and $F(H_i) \sim P^3(h+2k_i)$. If $p_i = \frac{3}{2}k_i$, $\rho_i^*(c_2) \neq 0$ and $F(H_i) \sim P^3(h+k_i)$ as is easily seen by checking dimensions.

If $i > l$, the intersection points of the parabolas remain separated when restricting to H_i, by a dimension check $F(H_i) \sim P^2(h+2k_i) + \{pt\}$. q.e.d.

Case 4 can occur only under rather special circumstances. It is possible only in the dimension range $n < 2m < 4n$. In Tomter (16) an example was constructed for a circle action on $X \sim S^4 \times S^4$, the construction has been extended to circle actions with other dimensions in Chang and Comenetz (5). In a recent paper of Skjelbred (12), he applies a theorem by Sylvester and Grünwald on affine dependence relations of points in the plane to prove that if $F = F_1 + F_2$ with F_2 acyclic, then $rk\, T \leq 3$ (for an arbitrary Poincaré duality space X with $\dim H^*(X) = \dim H^*(F)$ and $\dim F_1 > \dim F_2$). Hence Case 4 cannot occur for tori of large rank; we do not know of examples with $rk\, T = 2$ or $rk\, T = 3$.

Case 5.

Let (a_i,b_i) , $i = 1,2,3$ be the intersection points of the parabolas $\hat{x}^2 = c_1\hat{y}+d_1$ and $\hat{y}^2 = c_2\hat{x}=d_2$ with (a_1,b_1) the point of tangency. We may assume that c_1,d_1,d_2 are non-zero. Let $F = F_1+F_2+F_3 \sim S^p+\{pt\}+\{pt\}$, let u and 1_i be generators of $H^p(F_1)$ and $H^o(F_i)$ respectively, $i = 1,2,3$. Then $j^*(\hat{x}) = u\otimes c + 1_1\otimes a_1 + 1_2\otimes a_2 + 1_3\otimes a_3$, $j^*(\hat{y}) = u\otimes d + 1_1\otimes b_1 + 1_2\otimes b_2 + 1_3\otimes b_3$. Straightforward computation gives the relations:

(i) $(a_i-a_j)(a_i+a_j) = c_1(b_i-b_j)$, $(b_i-b_j)(b_i+b_j) = c_2(a_i-a_j)$.

(ii) $2a_1c = c_1d$. $2b_1d = c_2c$. $4a_1b_1 = c_1c_2$.

(iii) $(a_i+a_j)(b_i+b_j) = c_1c_2$.

Again, the simplest orbit structure occurs when the second parabola degenerates to two parallell lines.

a) $c_2 = 0$.

This implies $a_1 = 0$, $a_2 = -a_3$, $b_1 = -b_2 = -b_3$. Let $a = a_2$ and $b = b_1$. Then $j^*(\hat{x}) = u\otimes c + 1_2\otimes a - 1_3\otimes a$. From $j^*(\hat{y}^2) = j^*(d_2) = d_2 \otimes (1_1+1_2+1_3)$ it then follows that $j^*(\hat{y}) = 1_1\otimes b - 1_2\otimes b - 1_3\otimes b$.

Proposition 3. Let M and M_i be the submodules of $H^*(F)$ generated by u and 1_i respectively. Then $\mathrm{Ann}(M_T) = (bc)$, $\mathrm{Ann}(M_{1T}) = (b)$ and $\mathrm{Ann}(M_{2T}) = \mathrm{Ann}(M_{3T}) = (ab)$.

Proof: $j^*(\hat{x}\hat{y}+b\hat{x}) = u\otimes 2bc$, $j^*(\hat{x}\hat{y}-b\hat{x}+a\hat{y}-ab) = 1_2\otimes(-4ab)$, $j^*(\hat{x}\hat{y}-b\hat{x}-a\hat{y}+ab) = 1_3\otimes 4ab$, $j^*(\hat{y}+b) = 1_1\otimes 2b$.

Theorem 13.

The geometric weight system in Case 5 a) is given by the three splitting elements $a = q_1w_1^{k_1}\dots w_s^{k_s}$, $b = q_2w_1^{l_1}\dots w_s^{l_s}$ and

$c = q_3 w_1^{p_1} \dots w_s^{p_s}$, $q_i \in \mathbb{Q}$. The structure of the corank one F^o-varieties are given as follows: Let $H_i = w_i^{\perp}$.

(a) $2k_i > l_i > 0$: $F(H_i) \sim S^{2k_i} \times S^{2l_i}$.

(b) $2k_i = l_i > 0$: $F(H_i) \sim P^3(2k_i)$.

(c) $2k_i > l_i = 0$: $F(H_i) \sim S^{p+2p_i} + S^{2k_i}$.

(d) $k_i = l_i = 0$: $F(H_i) \sim S^{p+2p_i} + \{pt\} + \{pt\}$.

Proof: From (i) we get $a^2 = -2c_1 b$, $d_1 = -c_1 b$.

(a) $\rho_i^*(c_1) = \rho_i^*(d_1) = \rho_i^*(d_2) = 0$ in RH_i , hence $I \otimes_{RT} RH_i = (\hat{x}^2, \hat{y}^2)$, from the exact sequence $0 \to I \otimes_{RT} RH_i \to RH_i[\hat{x}, \hat{y}] \to H^*_{H_i}(X) \to 0$ it follows that the restriction to the H_i-action on X is Case 1 and $F(H_i) \sim S^{m_i} \times S^{n_i}$. By Proposition 3 the local geometric weight systems around F_2 and F_3 are given by ab , hence $m_i + n_i = 2k_i + 2l_i$. By Theorem 4 $\mathrm{Ann}(M_{1T})_{(w_i)} \cap RT = (w_i^{l_i}) = \mathrm{Ann}(M_{1T,H_i})$, hence one of the individual sphere dimensions is $2l_i$ and $F(H_i) \sim S^{2k_i} \times S^{2l_i}$.

(b) $\rho_i^*(d_1) = \rho_i^*(d_2) = 0$, $\rho_i^*(c_1) \neq 0$; hence the H_i-action on X is Case 2, and from the local geometric weight system $F(H_i) \sim P^3(2k_i)$.

(c) $\rho_i^*(c_1) = \rho_i^*(d_1) = 0$, $\rho_i^*(d_2) \neq 0$ and the H_i-action on X is Case 3 a).

(d) $\rho_i^*(c_1)$, $\rho_i^*(d_1)$ and $\rho_i^*(d_2)$ are all non-zero, and the H_i-action on X is Case 5 a).

The dimension in (c) and (d) follow from the local geometric weight systems.

q.e.d.

b) $c_2 \neq 0$.

Proposition 4.

$I_1 = \mathrm{Ann}(M_T) = (c(a_1-a_2)(a_1-a_3)c_1^{-1}) = (d(b_1-b_2)(b_1-b_3)c_2^{-1})$.

$I_2 = \mathrm{Ann}(M_{2T}) = ((a_2-a_3)(a_2-a_1)^2c_1^{-1}) = ((b_2-b_3)(b_2-b_1)^2c_2^{-1})$.

$I_3 = \mathrm{Ann}(M_{3T}) = ((a_2-a_3)(a_3-a_1)^2c_1^{-1}) = ((b_2-b_3)(b_3-b_1)^2c_2^{-1})$.

These determine the local geometric weight systems around F_1, F_2 and F_3 respectively.

Proof: We compute I_1. Let $x' = \hat{x} - a_3$, $y' = \hat{y} - b_3$. Since $j^*(x'y'+Ax'+By') = u \otimes D$ for $A,B,C,D \in RT$ we get:

(1) $A(a_i-a_3) + B(b_i-b_3) + (a_i-a_3)(b_i-b_3) = 0$, $i = 1.2$. From (i) we get $A(a_i-a_3) + Bc_1^{-1}(a_i^2-a_3^2) + c_1^{-1}(a_i-a_3)^2(a_i+a_3) = 0$. For the intersection points $a_i \neq a_j$ and $b_i \neq b_j$ in case 5 b) if $i \neq j$. Hence $c_1A = -(a_i+a_3)B - a_i^2 + a_3^2$ for $i = 1,2$; by subtraction we get $B = -(a_1+a_2)$. Similarly $A = -(b_1+b_2)$. Substitution of A in (1) gives $-(b_2+b_3)(a_1-a_3) + B(b_1-b_3) = 0$, using (i) we obtain $B = c_2^{-1}(b_1+b_3)(b_2+b_3) = -(a_1+a_2) = -c_1c_2(b_1+b_2)^{-1}$, hence $(b_1+b_2)(b_1+b_3)(b_2+b_3) = -c_1c_2^2$, similarly $(a_1+a_2)(a_1+a_3)(a_2+a_3) = -c_1^2c_2$. Now $D = c(b_1-b_2) + d(a_1-a_3) - c(b_1+b_2) - d(a_1+a_2)$
$= -c(b_2+b_3) - d(a_2+a_3) = -cc_1c_2(a_2+a_3)^{-1} - 2a_1cc_1^{-1}(a_2+a_3)$
$= cc_1^{-1}(a_1+a_2)(a_1+a_3) - 2a_1cc_1^{-1}(a_2+a_3) = cc_1^{-1}(a_1-a_2)(a_1-a_3)$, and this gives the formula for I_1. By similar computations we obtain the others. q.e.d.

The description of the orbit structure is more complicated in this case. However since there are examples of such torus actions, we state the result with a short proof.

Theorem 14.

The geometric weight system in Case 5 b is determined by the following annihilator ideals: $I_1 = \mathrm{Ann}\, M_T = (w_1^{g_1} \ldots w_s^{g_s})$, $I_2 = \mathrm{Ann}\, M_{2T} = (w_1^{h_1} \ldots w_s^{h_s})$, $I_3 = \mathrm{Ann}\, M_{3T} = (w_1^{j_1} \ldots w_s^{j_s})$, $I_4 = \mathrm{Ann}((M+M_2+M_3)_T/(M_2+M_3)_T) = (c,d) = (w_1^{k_1} \ldots w_s^{k_s})$. There are the following possibilities for corank one F^o-varieties $(H_i = w_i^{\perp})$:

(a) $g_i > 0,\ h_i = j_i = 0$. $\quad F(H_i) \sim S^{p+2g_i} + \{pt\} + \{pt\}$.

(b) $h_i > 0,\ j_i = 0,\ g_i = h_i - \frac{p}{2}$. $\quad F(H_i) = F' + F_3 \sim P^2(h_i) + \{pt\}$.

$\quad j_i > 0,\ h_i = 0,\ g_i = j_i - \frac{p}{2}$. $\quad F(H_i) = F' + F_2 \sim P^2(j_i) + \{pt\}$.

(c) $h_i = j_i > 0,\ k_i = g_i$. $\quad F(H_i) \sim S^{p+2g_i} + S^{2h_i}$.

(d) $h_i = j_i = g_i + \frac{p}{2} > 0$. $k_i < g_i$. $F(H_i) \sim S^{p+2k_i} \times S^{2(h_i-k_i)-p}$ for $3p + 6k_i \neq 2h_i$ and $F(H_i) \sim P^3(p+2k_i)$ for $3p+6k_i = 2h_i$.

Proof. (a) and (b) follow by inspecting the local geometric weight systems I_1, I_2, I_3. Now $(I_4)_{(w_i)} \cap RT = (w_i^{k_i})$, $(I_1)_{(w_i)} \cap RT = (w_i^{g_i})$. Hence one of the generators of $H^*(F(H_i))$ has dimension $p + 2k_i$ and the dimension of $F(H_i)$ around F_1 is $p + 2g_i$. If these are equal, $F(H_i)$ is disconnected, and in (c) we get $F(H_i) \sim S^{p+2g_i} + S^{2h_i}$ from I_1, I_2, I_3. Conversely, in (d) $F(H_i)$ must be connected. If $F(H_i) \sim P^3(p+2p_i)$, we would have $\mathrm{Ann}((M+M_2+M_3)_{T,H_i}/(M_2+M_3)_{T,H_i}) = (w_i^{p_i}) = (w_i^{k_i})$, hence $p_i = k_i$ and $3p + 6k_i = 2h_i$. Thus, if $3p + 6k_i \neq 2h_i$, $F(H_i) \sim S^{m_i} \times S^{n_i}$, by the localization of I_4 again one of the sphere dimensions must be $p + 2k_i$; i.e. $F(H_i) \sim S^{p+2k_i} \times S^{2(h_i-k_i)-p}$. It remains only to prove that if $3p + 6k_i = 2h_i$, $F(H_i) \sim S^{p+2k_i} \times S^{2p+4k_i}$ is impossible. From the dimensions it is clear that in this case the T-action on $F(H_i)$ must be Case 5 a). Let $x_i \in H_T^{p+2k_i}(F(H_i))$ and $y_i \in H_T^{2p+4k_i}(F(H_i))$ be generators, then $x_i^2 \in RT$ and from

5 a) we would have $\mathrm{Ann}((M\oplus M_2\oplus M_3)_{T,H_i}/(M_2\oplus M_3)_{T,H_i}) = (w_i^{2k_i+\frac{p}{2}})$,
which is a contradiction.

q.e.d.

Again, 5 b) can occur only inrather special cases. It is possible only in the dimension range $n < 2m < 4n$. If $\mathrm{rk}\ T > 4$, then $\mathrm{rk}\ H_i > 3$ and (b) in Theorem 14 cannot occur. The local geometric weight systems around F_2 and F_3 must then be the same, hence $a_2 - a_1 = q_1(a_3-a_1)$, $b_2 - b_1 = q_2(b_3-b_1)$ with $q_1,q_2 \in \mathbb{Q}$, from (i) we get $a_3 = q_3a_1$, similarly for a_2,b_2, etc.
$I_1 = (ca_1^2c_1^{-1}) = (cb_1)$, $I_2 = I_3 = (a_1^3c_1^{-1}) = (a_1b_1)$, just as in 5 a).
Hence it is only for $\mathrm{rk}\ T \leq 4$ and $n < 2m < 4n$ that the more complicated description of Theorem 14 is necessary. We give examples which shows that it can occur for tori of rank two.

First we obtain an example of 5 a). Let ξ be the Su bundle over S^4 and let T act on $Q^2 = \mathbb{C}^4$ with weight system $\{(0;1),(w;1)\}$. Let ϵ be the trivial $\mathbb{C}^n$ bundle over the T-space $S^4 = QP(1)$ corresponding to a representation of T with weight system $\{(w_1;r_1),\dots,(w_s,r_s)\}$ with $\{w,w_1,\dots,w_s\}$ pairwise linearly independent. (See Case 3) Let X be the unit sphere bundle of $\xi \oplus \epsilon$, then $X \sim S^4 \times S^{2+2n}$ and $F = X^T = S^2+\{pt\}+\{pt\}$. Instead of computing the equivariant cohomology, we just observe that if $n > 8$, we must be in Case 5 a . The corank one F^0-varieties: $F(w^{\perp}) = \mathbb{C}P(3)$, $F(w_i^{\perp}) \sim S^{2+2r_i} + S^{2r_i}$. This shows that case (b) of Theorem 13 can occur for tori of large rank.
Next we give exapmples which shows that both parabolas may be nondegenerate. Let T have rank two, and let a_1 and a_2 be linearly independent weights on T . Let T act on $X_1 = QP(2)$ and $X_2 = QP(2)$ with weight systems $\{(0;1),(a_2-a_1;1),(-a_2-a_1;1)\}$ and

$\{(a_2;2),(a_1;1)\}$ respectively. Then $X_1^T = F^1 + F^2 + F^3 = \{pt\}+\{pt\} + \{pt\}$ and $X_2^T = F^4 + F^5 = S^2 + \{pt\}$. The local representations of T around F^1 and F^5 have complex weight systems $\{\pm(a_2-a_1),\pm(a_2+a_1)\}$ and $\{(\pm a_2-a_1;2)\}$ respectively; hence there are disc neighbourhoods around F^1 and F^5 which are equivariantly diffeomorphic under an orientation-preserving diffeomorphism. Let X be the equivariant connected sum $X_1 \# X_2$ as in Tomter (16), then $X \sim S^4 \times S^4$ and $X^T = X_1^T \# X_2^T = F^4 + F^2 + F^3 = S^2 + \{pt\}+\{pt\}$. The local representations around fixed points have weight systems: F^4: $\{0,-2a_2,\pm a_1-a_2\}$. F^2: $\{(a_1-a_2;2),2a_1,-2a_2\}$. F^3: $\{(a_1+a_2;2),2a_1,2a_2\}$. Then the corank one F^o-varieties of X are the following: $F((a_1-a_2)^\perp) = (S^4+F^3) \# \mathbb{C}P(2) = \mathbb{C}P(2) + F^3$. $F((a_1+a_2)^\perp) = (S^4+F^2) \# \mathbb{C}P(2) = \mathbb{C}P(2) + F^2$. $F(a_1^\perp) = (F^1+S^2) \# (S^2+F^5) = S^2 + S^2$. $F(a_2^\perp) = (F^1+S^2) \# (S^4+F^5) = S^4 + S^2$. The equivariant cohomology can be computed explicitly [illegible]. Since there are corank one F^o-varieties as in Theorem 14 (b), it is clear that this is Case 5 b). The weithts a_1 and a_2 correspond to (c) of Theorem 14. By other choices of weight vectors other cases of the theorem can be illustrated. For example, consider the quaternionic weight system $\{(2a_1;2),(2a_2;1)\}$ for X_1 and $\{(a_1-a_2;2),(3a_1+a_2;1)\}$ for X_2, with $X_1^T = F^1 + F^2 = S^2 + \{pt\}$ and $X_2^T = F^3 + F^4 = S^2 + \{pt\}$. The local representation around F^1 has weight system $\{0,-4a,2(a_3-a_1),-2(a_3+a_1)\}$ and around F^3 it is $\{0,2a_1-2a_3,2a_1+a_3,-4a_1\}$. Hence we can take the equivariant connected sum; this time with respect to points on the two spheres F^1 and F^3 such that the torus T acts on $X \sim S^4 \times S^4$ with $X^T = S^2 + \{pt\}+\{pt\}$. The corank one F^o-varieties are now $F((a_1-a_2)^\perp) = \mathbb{C}P(2) + \{pt\}$, $F((a_1+a_2)^\perp) = \mathbb{C}P(2) \# \mathbb{C}P(2) \sim S^2 \times S^2$, and $F(a_1^\perp) = \mathbb{C}P(2) + \{pt\}$, corresponding to b) and d) of Theorem 14.

Case 6.

The two defining parabolas have four simple intersection points (a_i,b_i), $i = 1,\dots,4$ corresponding to four acyclic compnents of $F = F_1+F_2+F_3+F_4$. Let 1_i be the generator of $H^o(F_i)$, then $j^*(\hat{x}) = 1_1\otimes a_1 + 1_2\otimes a_2 + 1_3\otimes a_3 + 1_4\otimes a_4$ and $j^*(\hat{y}) = 1_1\otimes b_1 + 1_2\otimes b_2 + 1_3\otimes b_3 + 1_4\otimes b_4$, and substitution of the defining equations of the parabolas gives the relations (i) $(a_i-a_j)(a_i+a_j) = c_1(b_i-b_j)$, $(b_i-b_j)(b_i+b_j) = c_2(a_i-a_j)$, $i,j = 1,2,3,4$. Again the complexity of the orbit structure depends on the shape of the parabolas, and there are three possibilities.

a) both parabolas degenerate, $c_1 = c_2 = 0$. The intersection points are of the form $(\pm a,\pm b)$, and, after renumbering, we may write $j^*(\hat{x}) = 1_1\otimes a + 1_2\otimes a - 1_3\otimes a - 1_4\otimes a$. $j^*(\hat{y}) = 1_1\otimes b - 1_2\otimes b + 1_3\otimes b - 1_4\otimes b$.

Theorem 15.

If the equivariant cohomology of X is defined by the ideal $I = (\hat{x}^2-d_1,\hat{y}^2-d_2)$ with d_1 and d_2 non-zero in RT, the geometric weight system is given by $a = q_1w_1^{k_1}\dots w_s^{k_s}$ and $b = q_2w_1^{l_1}\dots w_s^{l_s}$; $q_i \in \mathbb{Q}$, $w_i \in H^2(B_T)$, with $d_1 = a^2$ and $d_2 = b^2$. Let $H_i = w_i^{\perp}$; $i = 1,\dots,s$. The corank one F^o-varieties are given by:

(a) $k_i > 0$, $l_i > 0$: $F(H_i) \sim S^{2k_i} \times S^{2l_i}$.

(b) $k_i > 0$, $l_i = 0$: $F(H_i) = X_1+X_2 \sim S^{2k_i} + S^{2k_i}$ with $X_1 \supset F_1+F_3$, $X_2 \supset F_2+F_4$.

(c) $k_i = 0$, $l_i > 0$: $F(H_i) = X_1+X_2 \sim S^{2l_i} + S^{2l_i}$ with $X_1 \supset F_1+F_2$, $X_2 \supseteq F_3+F_4$.

Remark. The geometric weight system then determines the relative position of F in the F^o-varieties. Notice that the components

F_i do not enter symmetrically here, e.g. a geometric weight w_i can join F_1 to either F_2 or F_3 into a cohomology sphere (2 and 3), but not to F_4.

Proof: For (a) $\rho_i^*(d_1) = \rho_i^*(d_2) = 0$ in RH_i, $I \otimes_{RT} RH_i = (\hat{x}^2, \hat{y}^2)$, and the restriction to the H_i-action on X must be Case 1, with $F(H_i) \sim S^{m_i} \times S^{n_i}$. Let M be the submodule of $H^*(F)$ spanned by (1_1+1_2), then $\mathrm{Ann}(M_T) = (a)$, and $\mathrm{Ann}(M_{T,H_i}) = (a)_{(w_i)} \cap RT = (w_i^{k_i})$. It follows that one of the sphere dimensions must be $2k_i$. We have $j^*(\hat{x}\hat{y}+b\hat{x}+a\hat{y}+ab) = 1_1 \otimes 4ab$, and it is easily seen that the local geometric weight systems around all components F_i are given by (ab). Hence $m_i + n_i = 2k_i + 2l_i$, and the conclusion of (a) follows. In (b) $\rho_i^*(d_1) = 0$, $\rho_i^*(d_2) \neq 0$, $I \otimes_{RT} RH_i = (\hat{x}^2, \hat{y}^2 - \rho_i^*(d_2))$ and the H_i-action is Case 3. The dimensions follow from the local geometric weight system, the same argument applies to (c). q.e.d.

Let T act on S^m and S^n with weight systems a and b respectively and assume that there are no zero weights. Then the diagonal action of T on $S^m \times S^n$ belongs to 6 a), and Theorem 15 shows that the general case 6 a) is modelled after this "linear" example.

b) One parabola is degenerate: $c_1 \neq 0$, $c_2 = 0$. Now $a_1 = -a_2$, $a_3 = -a_4$, $b_1^2 = b_2^2 = b_3^2 = b_4^2 = d_2$, and we can write: $j^*(\hat{x}_1) = 1_1 \otimes a_1 - 1_2 \otimes a_1 + 1_3 \otimes a_3 - 1_4 \otimes a_3$, $j^*(\hat{y}) = 1_1 \otimes b + 1_2 \otimes b - 1_3 \otimes b - 1_4 \otimes b$.
The relations (i) reduce to (ii) $(a_1 - a_3)(a_1 + a_3) = 2c_1 b$.
Denote the submodule generated by elements $z_1, \ldots, z_r$ of $H^*(F)$ by $[z_1, \ldots, z_r]$. Define $M_i = [1_i]$, $M = [1_2, 1_3, 1_4]$, $U = [1_2, 1_3]$ and $V = [1_2, 1_4]$.

Proposition 5.

$I_1 = \mathrm{Ann}\, M_{1T} = \mathrm{Ann}\, M_{2T} = (a_1 b)$. $I_2 = \mathrm{Ann}\, M_{3T} = \mathrm{Ann}\, M_{4T} = (a_3 b)$.
$I_3 = \mathrm{Ann}(M_1+M_2)_T = (b)$. $I_4 = \mathrm{Ann}(M_T/U_T) = (a_3+a_1, b)$.
$I_5 = \mathrm{Ann}(M_T/V_T) = (a_3-a_1, b)$.

This is proved by straightforward computations.

I_1 and I_2 determine the local geometric weight systems. The next theorem shows that $I_1,\dots,I_5$ determine the whole cohomological orbit structure.

Theorem 16.

Let the equivariant cohomology of X be given by the ideal $I = (\hat{x}^2-c_1\hat{y}-d_1, \hat{y}^2-d_2)$ with c_1, d_1 and d_2 non-zero in RT . Then the cohomological orbit structure of X is determined by the above five ideals $I_1,\dots,I_5$. Let $w_i \in H^2(B_T)$ and $H_i = w_i^{\perp}$. Define the indices r_{ij} by the localization $(I_j)_{(w_i)} \cap RT = (w_i^{r_{ij}})$. Then the corank one F^o-varieties are given as follows:

(a) $r_{i3} = 0$. $F(H_i) = X_1+X_2 \sim S^{2r_{i1}} + S^{2r_{i2}}$ with $F_1+F_2 \subseteq X_1$, $F_3+F_4 \subseteq X_2$

(if $r_{i2} = 0$, $F(H_i) \sim S^{2r_{i1}} + \{pt\} + \{pt\}$, etc.)

(b) $0 < r_{i1} = r_{i2} = r_{i3}$. $F(H_i) = X_1+X_2 \sim S^{2r_{i1}} + S^{2r_{i1}}$.

Here either $r_{i5} = r_{i1}$, $r_{i4} = 0$ and $F_1+F_3 \subseteq X_1$, $F_2+F_4 \subseteq X_2$
or $r_{i4} = r_{i1}$, $r_{i5} = 0$ and $F_1+F_4 \subseteq X_1$, $F_2+F_3 \subseteq X_2$.

(c) $r_{i1} = r_{i2} > r_{i3} > 0$. Either $3r_{i3} \neq 2r_{i1}$ and $F(H_i) \sim S^{2r_{i3}} \times S^{2(r_{i1}-r_{i3})}$
or $3r_{i3} = 2r_{i1} = 6r_{i4} = 6r_{i5}$ and $F(H_i) \sim P^3(r_{i3})$
or $3r_{i3} = 2r_{i1} < 3r_{i4} + 3r_{i5}$ and $F(H_i) \sim S^{r_{i3}} \times S^{2r_{i3}}$.

Proof: (a) $\rho_i^*(d_2) = \rho_i^*(b^2) \neq 0$ in RH_i . Hence $F(H_i)$ is not connected; since $(I_3)_{(w_i)} \cap RT = RT = \mathrm{Ann}(M_1+M_2)_{T,H_i}$ by Theorem 4,

it is clear that F_1 and F_2 is in the same component of $F(H_i)$.
(b): $\rho_i^*(d_2) = 0$, from $a_1^2 = c_1b+d_1$ it follows that $\rho_i^*(d_1) \neq 0$, hence $F(H_i)$ has two components. From (ii) it follows that w_i divides exactly one of a_3-a_1, a_3+a_1 at least r_{i3} times. Now $I_{4(w_i)} \cap RT = (w_i^{m_i})$, where m_i is the minimum of the multiplicities of w_i in b and a_3+a_1 , hence $r_{i5} = r_{i1}$, $r_{i4} = 0$ or $r_{i4} = r_{i1}$, $r_{i5} = 0$. In the first case $\rho_i^*(a_1) = \rho_i^*(-a_3)$; hence the first and the fourth intersection points are joined in RH_i , also the second and the third; i.e. $F_1+F_4 \subseteq X_1$, $F_2+F_3 \subseteq X_2$.
(c) $\rho_i^*(d_1) = \rho_i^*(d_2) = 0$, hence $F(H_i)$ is connected. One of the generators of $H^*(F(H_i))$ has dimension $2r_{i3}$ (from I_3's localization). If $r_{i1} = 3r_{i3}$, it follows from (ii) that $\rho_i^*(c_1) = 0$, hence $F(H_i) \sim S^{2r_{i3}} \times S^{2(r_{i1}-r_{i3})}$. The same conclusion must then also hold if $3r_{i3} \neq 2r_{i1}$. If $3r_{i3} = 2r_{i1}$, we have $r_{i4} \geq \min(r_{i3}, r_{i1}-r_{i3}) = \frac{1}{2}r_{i3}$, similarly for r_{i5} . From (ii) it follows that $\rho_i^*(c_1) \neq 0$ precisely if $r_{i4} = r_{i5} = \frac{1}{2}r_{i3}$, i.e. in this case $F(H_i) \sim P^3(r_{i3})$. q.e.d.

Consider again the Su bundle ξ over S^4 , and let T act on $Q^2 = \mathbb{C}^4$ with quaternionic weight system $((w_1;1),(w_2,1))$, where w_1 and w_2 are linearly independent. This defines a T-action on $\mathbb{C}P(3)$ with four isolated fixed points F_1, F_2, F_3, F_4 and two fixed points. $F^1 = \pi(F_1) = \pi(F_2)$ and $F^2 = \pi(F_3) = \pi(F_4)$ in the base space S^4 . (See Case 3.) The complex weight systems for the local representations are: F_1: $2w_1, w_1-w_2, w_1+w_2$. F_2: $-2w_1, -w_1-w_2, -w_1+w_2$. F_3: w_2-w_1, w_2+w_1, $2w_2$. F_4: $-w_2-w_1$, $-w_2+w_1, -2w_2$, and around F^1: $\pm w_1-w_2$, around F^2: $\pm w_2-w_1$. Add a trivial T bundle ϵ over the T-space S^4 corresponding to a representation of T on $\mathbb{C}^n$ with weight system

$\{(w_3;k_3),\ldots,(w_s;k_s)\}$ where $\{w_1,w_2,w_1-w_2,w_1+w_2,w_3,\ldots,w_s\}$ are pairwise linearly independent. Then $X = S(\xi\oplus\epsilon) \sim S^4 \times S^{2n+2}$, $F = X^T = F_1 + F_2 + F_3 + F_4$, and the corank one F^o-varieties are: $F(H_1) = X_1 + F_3 + F_4 \sim S^2 + \{pt\}+\{pt\}$, $F(H_2) = F_1 + F_2 + X_1 \sim \{pt\}+\{pt\}+S^2$. $F((w_1-w_2)^\perp) = X_1 + X_2$
$\sim S^2 + S^2$ where $F_1 + F_3 \subseteq X_1$, $F_2 + F_4 \subseteq X_2$. $F((w_1+w_2)^\perp) = X_1+X_2$
$\sim S^2 + S^2$ where $F_1 + F_4 \subseteq X_1$, $F_2 + F_3 \subseteq X_2$. $F(H_i) = X_1 + X_2$
$\sim S^{2k_i} + S^{2k_i}$ for $i \geq 3$. We see that F_1 may be linked to each of F_2, F_3, F_4 in cohomology spheres in various corank one F^o-varieties. By theorem 15 this is not possible for Case 6 a). Choosing $n+2 > 8$ we are outside the dimension range where both parabolas can be non-degenerate; hence we must be in Case 6 b). The weights w_1 and w_2 correspond to Theorem 16 (a), $w_1 \pm w_2$ corresponds to (b). If $w_3 = w_1-w_2$, $F(H_3) \sim S^2 \times S^{2k_3}$ gives an example of (c).

c) Both parabolas are non-degenerate, $c_1 \neq 0$, $c_2 \neq 0$. In this case, which is possible only for $n < 2m < 4n$, the orbit structure can be more complicated. We give a description of the orbit structure from the equivariant cohomology and outline the computations. It is easily seen that $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Furthermore we have the relations (iii) $(a_i+a_j)(b_i+b_j) = c_1c_2$ for $i \neq j$,

$$(a_i+a_j)(a_i+a_k)(a_i+a_l) = c_1^2c_2 , \quad (b_i+b_j)(b_i+b_k)(b_i+b_l) = c_1c_2^2$$

for $\{i,j,k,l\} = \{1,2,3,4\}$, and $a_1+a_2=a_3+a_4 = b_1+b_2+b_3+b_4$. Using these the local geometric weight systems can be computed:
$I_j = \mathrm{Ann}([1_j]_T) = (c_1^{-1}(a_j-a_i)(a_j-a_k)(a_j-a_l)) =$
$= (c_2^{-1}(b_j-b_i)(b_j-b_k)(b_j-b_l))$ with $\{i,j,k,l\} = \{1,2,3,4\}$, (also see the computation in Proposition 4). Let $M_1 = [1_2,1_3,1_4]$, $M_2 = [1_3,1_4]$, $M_3 = [1_2,1_4]$ and $M_4 = [1_2,1_3]$ and let

$L_j = \mathrm{Ann}(M_{1T}/M_{jT}) = (a_j - a_1, b_j - b_1)$, $j = 2,3,4$. Define the indices r_{ij} and l_{ij} by the localizations $(I_j)_{(w_i)} \cap RT = (w_i^{r_{ij}})$ and $(L_j)_{(w_i)} \cap RT = (w_i^{l_{ij}})$, then l_{ij} is the minimum of the multiplicities of w_i in $a_j - a_1$ and $b_j - b_1$.

Theorem 17.

In Case 6 c) the cohomological orbit structure of X is determined by the above ideals I_j and L_k of RT, $j = 1,2,3,4$; $k = 2,3,4$. The connected components of the corank one isotropy sucgroups are given by $H_i = w_i^{\perp}$ (where $r_{ij} > 0$ for some j) and the cohomological structure of the corank one F^o-varieties are given as follows: (let $\{j(1),j(2),j(3),j(4)$ be a permutation of $\{1,2,3,4\}$):

(a) $r_{ij(1)} = r_{ij(2)} > r_{ij(3)} = r_{ij(4)} \geq 0$. $F(H_i) = X_1 + X_2 \sim S^{2r_{ij(1)}} + S^{2r_{ij(3)}}$, where $F_{j(1)} + F_{j(2)} \subseteq X_1$, $F_{j(3)} + F_{j(4)} \subseteq X_2$.

(b) $r_{ij(1)} = r_{ij(2)} = r_{ij(3)} > r_{ij(4)} = 0$,

$F(H_i) = X_1 + F_{j(4)} \sim P^2(\frac{2}{3} r_{ij(1)}) + \{pt\}$.

(c) $r_{ij(1)} = r_{ij(2)} = r_{ij(3)} = r_{ij(4)} > 0$, $l_{ik(2)} > 0$,

$l_{ik(3)} = l_{ik(4)} = 0$. (Here $\{k(2),k(3),k(4)\}$ is a permutation of $\{2,3,4\}$.) Then $F(H_i) = X_1 + X_2 \sim S^{2r_{ij(1)}} + S^{2r_{ij(1)}}$, with $F_1 + F_{k(2)} \subseteq X_1$.

(d) $r_{ij(1)} = r_{ij(2)} = r_{ij(3)} = r_{ij(4)} > 0$. $l_{ik(2)}, l_{ik(3)}, l_{ik(4)} > 0$. If $l_{ik(2)} = l_{ik(3)} = l_{ik(4)} = \frac{1}{3} r_{ij(1)}$, $F(H_i) \sim P^3(l_{ik(2)})$. Otherwise $F(H_i) \sim S^{2l_{ik(2)}} \times S^{2(r_{ij(1)} - l_{ik(2)})}$.

Proof: Here cases (a) and (b) follow directly from the local geometric weight systems I_j; (c) follows once we observe that the intersection points (a_1,b_1) and $(a_{k(2)},b_{k(2)})$ of the parabolas are linked to the same intersection point under change of rings from RT to RH_i, but the other points are not linked to this. This is seen from the localizations $(L_{k(2)})_{(w_i)} \cap RT = (w_i^{l_{ik(2)}}) \neq RT$, and $(L_{k(3)})_{(w_i)} \cap RT = (L_{k(4)})_{(w_i)} \cap RT = RT$. In (d) $(L_k)_{(w_i)} \cap RT \neq RT$ for $k = 2,3,4$; hence all intersection points (a_j,b_j) are linked to the origin under change of ring from RT to RH_i, and $F(H_i)$ is connected. From Theorem 4 it follows that the dimension of one of the generators of $H^*(F(H_i))$ must be $2l_{ik(2)}$. If $F(H_i) \sim P^3(2k_i)$ and $\hat{x}_i$ is a suitable generator for $H^{2k_i}(F(H_i))$, then $j^*(\hat{x}_i) = 1_2 \otimes q_2 w_i^{k_i} + 1_3 \otimes q_3 w_i^{k_i} + 1_4 \otimes q_4 w_i^{k_i}$. The roots of the characteristic equation $\hat{x}_i(\hat{x}_i - q_2 w_i^{k_i})(\hat{x}_i - q_3 w_i^{k_i})(\hat{x}_i - q_3 w_i^{k_i}) = 0$ correspond to the components F_j of F by Theorem 3; hence each root has multiplicity one, and the q_j are different, non-zero rational numbers, $j = 2,3,4$. $(L_k)_{(w_i)} \cap RT = (w_i^{k_i})$, $k = 2,3,4$, and by dimension counting $6k_i = \dim F(H_i) = 2r_{ij(1)}$. On the other hand, suppose that the H_i-action is Case 1 and that $3l_{ik(2)} = 3l_{ik(3)} = 3l_{ik(4)} = r_{ij(1)}$; then $F(H_i) \sim S^{2l_{ik(2)}} \times S^{4l_{ik(2)}}$. Let $m(w_i, a_1 - a_2)$ denote the multiplicity of w_i as a factor of $a_1 - a_2$, etc.; then $l_{ik(2)} = \min(m(w_i, a_1 - a_{k(2)}), m(w_i, b_1 - b_{k(2)}))$. Since $m(w_i, c_1) > 0$ and $I_1 = (c_1^{-1}(a_1 - a_2)(a_1 - a_3)(a_1 - a_4))$, it is clear that $m(w_i, a_1 - a_j) > l_{ik(2)}$ for at least one $j = 2,3,4$; by a symmetric argument $m(w_i, b_1 - b_j) > l_{ik(2)}$ for at least one j. We may then assume that $m(w_i, a_1 - a_2) = m(w_i, a_1 - a_3) = l_{ik(2)}$, $m(w_i, b_1 - b_4) = l_{ik(2)}$, $m(w_i; a_1 - a_4) = l_{ik(2)} + m(w_i, c_1)$. From (i): $(a_1 - a_4)(a_1 + a_4) = c_1(b_1 - b_4)$, hence $m(w_i, a_1+a_4) = 0$, which contradicts the assumption that all the intersection points (a_j, b_j) are linked to the origin under the change of rings $\rho_i^* : RT \to RH_i$. q.e.d.

We finish this investigation by giving various examples of Case 6 c). We notice that this case allows much more freedom than cases 6 a) and 6 b). For example, it follows from Theorems 15 and 16 that if $m \geq 2n$, there is only one other component that can be linked with F_2 to a cohomology sphere in an F^o-variety of the type $S^p + \{pt\} + \{pt\}$, (this component is called F_1 in Theorem 16). We give an example of a torus of rank 3 acting such that F_2 can be linked with all the other components in corank one F^o-varieties of the type $S^p + \{pt\} + \{pt\}$.

Let T act linearly on $X_1 = QP(2)$ with quaternionic weight system $(2a_1, 2a_2, 2a_3)$ and on $X_2 = QP(2)$ with $(-a_2+a_3, 2a_1-a_2-a_3, 2a_1+a_2+a_3)$, where a_1, a_2, a_3 are linearly independent weight vectors on T. Then $X_1^T = F^1 + F^2 + F^3 = \{pt\}+\{pt\}+\{pt\}$ and $X_2^T = F^4 + F^5 + F^6 \sim \{pt\} + \{pt\} + \{pt\}$.

The local representations around F^1 has complex weight system $\{\pm 2a_2-2a_1, \pm 2a_3-2a_1\}$ and around F^4: $\{2a_1-2a_3, -2a_1+2a_2, 2a_1+2a_2, -2a_1-2a_3\}$. Taking equivariant connected sum around F^1 and F^4 we have: $X = X_1 \# X_2 \sim S^4 \times S^4$ and $X^T = X_1^T \# X_2^T = F^2 + F^3 + F^5 + F^6$.

By computing the weights of the local representations around F^2, F^3, F^5, F^6, we get the following corank one F^o-varieties:

$F((a_1-a_2)^\perp) = (S^2+F^3) \# (S^2+F^6) = S^2 + F^3 + F^6$.

$F((a_1+a_2)^\perp) = (S^2+F^3) \# (S^2+F^5) = S^2 + F^3 + F^5$.

$F((a_2-a_3)^\perp) = (S^2+F^1) \# (F^4+F^5+F^6) = S^2 + F^5 + F^6$.

$F((a_1+a_3)^\perp) = (S^2+F^2) \# (S^2+F^5) = S^2 + F^2 + F^5$.

$F((a_2+a_3)^\perp) = (S^2+F^1) \# (F^4+S^2) = S^2 + S^2$.

$F((a_1)^\perp) = (F^1+F^2+F^3) \# (F^4+S^2) \sim S^2 + F^5 + F^3$.

$F((a_1-a_3)^\perp) = (S^2+F^2) \# (S^2+F^6) \sim S^2 + F^2 + F^6$.

The equivariant cohomology can be computed explicitly (Tomter ( )). The three first F^0-varieties give the desired linkings of F_2 to other components, and show that this must be Case 6 c).

There exists a map $f: S^{2n-1} \to S^n$ of Hopf-invariant 2, such that the adjunction space $X = D^{2n} \cup_f S^n \sim P^2(n)$. (Steenrod-Epstein ( )). Let φ be an orthogonal representation of T on $\mathbb{R}^n$, let T act on D^{2n} by $\varphi \oplus \varphi$ and on $S^n \subseteq \mathbb{R}^{n+1}$ by $\varphi \oplus 1$. Then f is equivariant, and there is an induced T-action on X. Equivariant connected sums of such spaces gives only trivial exapmles of Case 6 a). There is a more interesting example if we let one of the spaces come from the usual Hopf fibiation. Thus let $\mathrm{rk}\, T = 2$ and let φ be a faithful representation of T on $\mathbb{C}^2 = \mathbb{R}^4$ with complex weights w_1, w_2. Consider the induced action on $X_1 = D^8 \cup_f S^4 \sim P^2(4)$, then $X_1^T = F_1 + F_2 + F_3$, where F_1 corresponds to the origin in D^8. The local weight system at F_1 is $\{(w_1;2),(w_2;2)\}$. Let $X_2 = QP(2)$ and let T act by a quaternionic linear action with weight system $\{(0;1),(w_1;1),(w_2;1)\}$ and $X_2^T = F_4 + F_5 + F_6$. The local weights around F_4 are $\{\pm w_1, \pm w_2\}$. Locally, X_1 is a manifold aroung F_1; taking equivariant connected sum around F_1 and F_4 we obtain $X = X_1 \# X_2 \sim S^4 \times S^4$. The corank one F^0-varieties are easily computed to be:

$F(w_1^\perp) = (D^4 \cup_f S^2) \# S^4 + F_6 \sim P^2(2) + \{pt\}$.

$F(w_2^\perp) = (D^4 \cup_f S^2) \# S^4 + F_5 \sim P^2(2) + \{pt\}$.

$F((w_1-w_2)^\perp) = (F_1+F_2+F_3) \# (F_4+S^2) \sim S^2 + \{pt\} + \{pt\}$.

$F((w_1+w_2)^\perp) = (F_1+F_2+F_3) \# (F_4+S^2) \sim S^2 + \{pt\} + \{pt\}$.

This gives an example of (b) in Theorem 17.

Remarks.

Throughout the discussion we have determined the structure of the corank one F^o-varieties. In Case 1 it was also shown how to determine the higher corank F^o-varieties and the whole cohomological orbit structure from the geometric weight system. It is clear that this can be done in the same way for the other cases.

The discussion given here indicates that torus actions on spaces with more complicated cohomology must be expected to have complicated orbit structures and geometric weight systems in general; in this case practically all possibilities allowed by general principles of cohomology theory for transformation groups can occur. However, by using the geometirc weight system, it is possible to exploit additional information (the dimensions of the cohomology groups, the dimension of the torus, Weyl group invariance, etc.) to rule out the more complicated cases.

Clearly Theorem and its counterparts for the other classical groups have analogues for all the other cases. Moreover there are applications to classification of principal orbit types, degree of symmetry, etc., for action of compact, simple Lie groups on cohomology products of spheres.

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