DIRICHLET FORMS AND DIFFUSION
PROCESSES ON RIGGED HILBERT SPACES [*]

by

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ABSTRACT

We study homogeneous symmetric Markov diffusion processes on
separable real rigged Hilbert spaces, with rigging provided by
locally convex complete vector spaces. The infinitesimal gene-
rators are given by Dirichlet forms associated with quasi in-
variant measures on the rigged Hilbert spaces. The processes
solve singular stochastic differential equations on these
spaces. We exhibit ergodic decompositions. We also prove
path continuity properties for the case of bounded measurable
drift and discuss briefly the relation with potential theory
on such spaces. The methods and results of the general theory
are then applied to models of local relativistic quantum fields
in two space-time dimensions, with polynomial or exponential
interactions. In particular we prove that the phycical vacuum,
restricted to the σ-algebra generated by the time zero fields,
is a quasi invariant analytic and strictly positive measure.

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1. Introduction

The main concern of this paper is the study of stochastic Markov processes and the corresponding stochastic differential equations on separable real rigged Hilbert spaces K, where the rigging $Q \subset K \subset Q'$ is given by a locally convex real complete vector space Q, densely contained in K, and its dual Q'. This study continues in other directions and extends our previous one [1]. [1])Some references concerning work from other points of view on stochastic differential equations, stochastic processes and their relations to differential operators in infinite dimensional spaces are e.g. [2] - [14] and references therein.

We shall now briefly discuss the content of the different sections of our present paper.

In section 2 we study Dirichlet forms on a rigged separable real Hilbert space K, the rigging $Q \subset K \subset Q'$ being as above. We recall that for any probability measure μ on Q' which is quasi invariant under translations by elements of Q two strongly continuous unitary representations $q \to U(q)$ and $q \to V(q)$ of Q in $L_2(d\mu)$ are defined, such that U and V satisfy the Weyl commutation relations. Such representations have been studied intensively before, see e.g. [15] - [23]. We have $(V(q)f)(\xi) = (\frac{d\mu(\xi+q)}{d\mu(\xi)})^{\frac{1}{2}}$ $f(\xi+q)$, for any $f \in L_2(d\mu)$, $q \in Q$, $\xi \in Q'$.

Let $i\pi(q)$ be the infinitesimal generator of the unitary group $V(tq)$, $t \in R$. Let $\mathscr{P}_1(Q')$ be the space of all quasi invariant probability measures on Q' with the property that the function 1 is in the domain of $\pi(q)$ for all $q \in Q$. $\mathscr{P}_1(Q')$ is the space of measures considered henceforth. The gradient $q \cdot \nabla$ in the direction q is defined in a natural way, hence also the closed map $f \to \nabla f$ from a

dense subset W^1 of $L_2(d\mu)$ into $K \otimes L_2(d\mu)$. The Dirichlet form we consider is then defined as the closed positive form $\int \bar{\nabla f} \cdot \nabla f \, d\mu$ in $L_2(d\mu)$. [2]

We study the correspondent self-adjoint operator H. In particular we exhibit its Q-ergodic decomposition induced by the ergodic decomposition of μ with respect to translations by elements in Q. We also give a definition of the Laplacian on $L_2(d\mu)$, for μ in a subspace of $\mathcal{P}_1(Q')$, as a self-adjoint positive operator. This is an alternative definition to ones given before ([27],[28],[6],[7]).

In Section 3 we start by proving that the semigroup e^{-tH}, $t \geq 0$ generated by H in $L_2(d\mu)$ is positivity preserving, i.e. it is a Markov semigroup. The proof is done by reduction to finitely many dimensions. In this case (as well as in the case of locally compact separable Hausdorff spaces) a general theory of symmetric processes generated by a class of symmetric bilinear forms has been given in a series of papers by Fukushima, see [29] and references therein. The class of forms considered by Fukushima consists of Dirichlet forms in a general sense related to potential theory and contain in particular, in the finite dimensional case, the closed Dirichlet forms considered by us in this paper and in [1]. Since the proof that our operator H generates a Markov semigroup can be reduced to the proof that an operator given by a Dirichlet form in finitely many dimensions generate a Markov semigroup, we can apply Fukushima's results to prove that our operator e^{-tH} is positivity preserving. Then we use the Markov semigroup to construct, by an adaptation of the standard Kolmogorov Theorem, the quasi invariant measure μ being regular, an homogeneous Markov process $\xi(t)$ with state space Q' and invariant measure μ. We then show that $\xi(t)$ solves, in

the sense of weak processes on Q' ([30]-[32]) the stochastic differential equation of a diffusion process

$$d\xi(t) = \beta(\xi(t))dt + dw(t) ,$$

where $w(t)$ is the standard Wiener process on K and the osmotic velocity $\beta(.)$, in the sense of [1], is such that $q \cdot \beta = 2\pi(q)1$.

This result covers in particular the /correspondent one mentioned in [1]. In the proof a suitable characterization of the standard Wiener process on R is used. Note that β is, in general, neither Lipschitz nor bounded.[3] We continue Section 3 by giving the time ergodic decomposition of the process ξ and of its generator H. We also compare the time-ergodic and Q-ergodic decompositions and show that the former is in general strictly finer than the latter. We also give a sufficient condition for the measure μ in order for the two ergodic decompositions to be equivalent. The condition, called strict positivity, is that the conditional measures obtained from μ by conditioning with respect to closed subspaces of codimension one be bounded away from zero on compacts of the corresponding one-dimensional subspaces. Two simple criteria for strict positivity of μ are then given. The first requires 1 to be an analytic vector for $\pi(q)$ and that $\pi(q)^n \cdot 1 \in D(q \cdot \nabla)$ for all $q \in Q'$. The second requires a gap at the bottom of the spectrum of H and a simple estimate involving the multiple commutators of $\pi(q)$ with H. These criteria find applications in Section 4.

We end Section 3 by proving continuity properties of the paths $\xi(t)$ in natural Banach norms, for a class of measures μ in $\mathscr{S}_1(Q')$. We use here results from Gross theory of abstract Wiener spaces (see e.g. [6],[7]). Our results on continuity properties give an extension of Stroock-Varadhan ones [35] to processes with infinite dimensional state space.

In Section 4 we apply the results of the previous sections to the case of two space-time dimensional quantum field theoretical models, continuing the discussion of Section 4 of Ref. [1]. For these applications the rigging is given by the real spaces $Q = \mathscr{S}(R)$, $K = L_2(d\mu)$, $Q' = \mathscr{S}'(R)$. We show in particular for the weakly coupled $P(\varphi)_2$ models ([38]),[39]), the $P(\varphi)_2$ model with Dirichlet boundary conditions and isolated vacuum (vacua) ([40] - [43]) and the exponential interaction models ([44],[45]), that the physical vacuum measure restricted to the σ-algebra generated by the time zero fields is an analytic, strictly positive quasi invariant measure μ on $\mathscr{S}'(R)$. Thus 1 is an analytic vector in $L_2(d\mu)$ for the canonical momentum $\pi(\varphi)$, $\varphi \in \mathscr{S}(R)$ and, with natural identifications for finitely based functions on $\mathscr{S}'(R)$,

$$\int f d\mu = \int_{R^n} f(x_1,\ldots,x_n)\rho(x_1,\ldots,x_n)dx_1 \ldots dx_n \quad \text{for any bounded con-}$$

tinuous f on R^n, with ρ strictly positive on any compact in R^n.

The restriction to time zero fields of the physical Hamiltonian of above Wightman field models coincides on the dense domain FC^2 of finitely based twice continuously differentiable functions with the Dirichlet operator H given by the Dirichlet form $\int \nabla \bar{f} \cdot \nabla f d\mu$, as well as with the diffusion operator given by μ in the terminology of Theorem 2.7 in Ref. [1]. The results of Sections 2, 3 apply and give in particular ergodic decompositions as well as the above mentioned stochastic differential equation for the Markov process $\xi(t,\cdot)$ with state space $\mathscr{S}'(R)$, infinitesimal generator H and invariant measure μ :

$$d\xi(x,t) = \beta(\xi(t))(x)dt + dw(x,t) ,$$

$w(\cdot,t)$ being the standard Wiener process on $\mathscr{S}'(R)$ and $\beta(\cdot) = 2i\,\pi(\cdot) \cdot 1$ the osmotic velocity corresponding to the measure μ.

2. The Dirichlet form and the Dirichlet operator

Let $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator as a self-adjoint operator in $L_2(R^n)$. There are well known conditions on a real-valued measurable function $V(x)$ such that $H = -\Delta + V(x)$ is essentially self-adjoint on $C_0^\infty(R^n)$. E.g. it suffices that $V \in L_p(R^n) + L_\infty(R^n)$, $p = 2$ for $n = 1,2,3$, and $p > n/2$ for $n \geq 4$ [46]. [4)] Let us now assume V is such a function and H has an eigenvalue E such that $H \geq E$. Again general conditions are known which are sufficient to ensure that E is a simple eigenvalue of H. E.g. it suffices that V has the form $V = V_1 + V_2$, with $0 \leq V_1 \in L_1^{loc}$ and $V_2 \in L_{n/2}(R^n) + L_\infty(R^n)$ for $n \geq 3$, and $L_{n/2}$ replaced by L_p, $p > 1$ for $n = 2$ and by L_1 for $n = 1$ [48] [5)]. In such a case, where E is a simple lowest eigenvalue of H, the corresponding normalized eigenfunction $\varphi(x)$ may be taken to be strictly positive almost everywhere.

In the case where V is smooth one has $\varphi(x) > 0$ for all $x \in R^n$ ([50]), so that $\rho(x)dx$ with $\rho(x) = \varphi(x)^2$ is a normalized probability measure equivalent to the Lebesgue measure. A simple calculation shows then that, for any f and g in $C^2(R^n)$, we have

$$(f\varphi, (H-E)g\varphi) = \sum_{i=1}^{n} \int \frac{\partial \bar{f}}{\partial x_i} \frac{\partial g}{\partial x_i} \rho \, dx \qquad (2.1)$$

with

$$(f\varphi, g\varphi) = \int \bar{f} \, g \, \rho \, dx \, . \qquad (2.2)$$

Moreover for any $f \in C^2(R^n)$ we have

$$(H-E)f\varphi = (-\Delta f - \beta \cdot \nabla f)\varphi \qquad (2.3)$$

where $\beta = \nabla \ln \rho$ and $\nabla f = \{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\}$. See also [1].

Hence we see that, in $L_2(\rho dx)$, $H - E$ is represented by

$$(H-E)f = -\Delta f - \beta \cdot \nabla f \tag{2.4}$$

for $f \in C^2(R^n)$, and the form given by $H - E$ is the Dirichlet form in $L_2(\rho dx)$ i.e.

$$(f, (H-E)g)_\rho = \sum_{i=1}^{n} \int \frac{\partial \bar{f}}{\partial x_i} \frac{\partial g}{\partial x_i} \rho dx \tag{2.5}$$

where $(,)_\rho$ is the inner product in $L_2(\rho dx)$. So that the Dirichlet form in $L_2(\rho dx)$ given by (2.5) defines an operator that is equivalent with the operator $H - E = -\Delta + V - E$ in $L_2(R^n)$ having $\varphi(x) = \rho(x)^{\frac{1}{2}}$ as an eigenfunction with lowest eigenvalue zero. The relation between ρ and V is of course given by

$$V - E = \frac{\Delta \varphi}{\varphi} \tag{2.6}$$

or

$$V - E = \frac{1}{4}\beta \cdot \beta + \frac{1}{2}\nabla \cdot \beta \tag{2.7}$$

where $\beta \cdot \beta = \sum_{i=1}^{n} \beta_i^2$ and $\nabla \cdot \beta = \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial x_i}$.

If we now let ρdx be an arbitrary probability measure which is absolutely continuous with respect to the Lebesgue measure, then first of all, by a well known theorem, ρdx is an arbitrary probability measure which is quasi invariant under translations in R^n and moreover, by what is said above, the Dirichlet form (2.5) is a natural generalization of the forms given by operators of the type $-\Delta + V - E$.

We shall say that a real separable Hilbert space K is rigged if there is a real locally convex complete vector space Q such that

$$Q \subset K \subset Q' , \qquad (2.8)$$

where Q' is the dual space of Q and such that Q is densely contained in K and Q' respectively and the inner product $(,)$ in K coincides on $Q \times K$ with the dualization between Q and Q'.

In this case the inner product $(,)$ on $Q \times K$ extends by continuity in the last variable to $Q \times Q'$ and this extension coincides with the dualization between Q and Q'. Hence we shall denote the dualization between Q and Q' by (q,ξ), $q \in Q$, $\xi \in Q'$.

Let now $\mathscr{M}(Q')$ be the space of bounded measures defined on the σ-algebra generated by the weak *-topology, and let $\mathscr{P}(Q')$ be the subset of probability measures. We shall say that $\mu \in \mathscr{P}(Q')$ is quasiinvariant if, for any $q \in Q$, $d\mu(\xi)$ and $d\mu(\xi+q)$ are equivalent as measures, and we shall let $\mathscr{P}_o(Q')$ denote the subset of quasi invariant probability measures.

Let now $\mu \in \mathscr{P}(Q')$, then on $L_2(d\mu)$ we have a representation $U(q)$ of Q by unitary operators with the cyclic vector $\Omega(\xi) \equiv 1$, given by

$$(U(q)f)(\xi) = e^{i(q,\xi)}f(\xi) . \qquad (2.9)$$

We have easily that $q \to U(q)$ is a strongly continuous representation of Q, because for $f \in L_\infty(d\mu)$ we have

$$\|(U(q)-1)f\|_2^2 \leq 2\|f\|_\infty^2 (1-\mathrm{Re} \int_{Q'} e^{i(q,\xi)}d\mu(\xi)) , \qquad (2.10)$$

which shows that $U(q)$ is strongly continuous since $L_\infty(d\mu)$ is dense in $L_2(d\mu)$.

If moreover $\mu \in \mathscr{P}_o(Q')$, then we also have another representation of Q. Since $d\mu(\xi+q)$ and $d\mu(\xi)$ are equivalent we know that

$$\alpha(\xi,q) = \frac{d\mu(\xi+q)}{d\mu(\xi)} \tag{2.11}$$

is a non negative L_1-function, and if we define

$$(V(q)f)(\xi) = \alpha^{\frac{1}{2}}(\xi,q)f(\xi+q) \tag{2.12}$$

then $q \to V(q)$ is again a unitary representation of Q on $L_2(d\mu)$, which is not necessarily continuous. However, it is always ray continuous and moreover if Q is a Fréchet space or a strict inductive limit of Fréchet spaces then $q \to V(q)$ is also strongly continuous i.e. for any $f \in L_2(d\mu)$ the mapping $q \to V(q)f$ is strongly continuous. (For this last result see [20].)

It follows now easily that U and V satisfy the Weyl-commutation relation

$$V(p)U(q) = e^{i(p,q)}U(q)V(p) . \tag{2.13}$$

We have obviously that (q,ξ) is the infinitesimal generator for $U(tq)$, and we shall use the convention $(q,\xi) = \xi(q)$. Let $\pi(q)$ be the infinitesimal generator of the unitary group $V(tq)$, and let $\Omega \in L_2(d\mu)$ be the function $\Omega(\xi) = 1$.

We shall say that $\mu \in \mathscr{P}_0(Q')$ is n-times differentiable if Ω is in the domain of $\pi(q_1),\ldots,\pi(q_n)$ for all n-tuples $q_1,\ldots,q_n$ in Q, and the subset of n-times differentiable probability measures will be denoted by $\mathscr{P}_n(Q')$. [6] We shall also say that $\mu \in \mathscr{P}_0(Q')$ is analytic if Ω is an analytic vector for $\pi(q)$, for all $q \in Q$. Let now $\mu \in \mathscr{P}_1(Q')$ then

$$\beta \cdot q = 2i\pi(q)\Omega \tag{2.14}$$

is a linear mapping from Q into $L_2(d\mu)$, and we denote by $\beta(\xi) \cdot q$ the value of the image function at the point $\xi \in Q'$.

Remark: The mapping $q \to \beta \cdot q$ is not necessarily continuous. We have though by Prop. 2.3 and Prop. 2.5 of ref. [1] that if Q is a countably normed space then $q \to \beta \cdot q$ is continuous, and if Q is a nuclear space then β is actually given by a measurable mapping $\beta(\xi)$ from Q' to Q' so that $(q, \beta(\xi))$ is the value of $\beta \cdot q$ at the point ξ.

Let now R be a finite dimensional subspace of Q. Then the orthogonal projection P_R in K with range R extends by continuity to a continuous projection from Q' into Q with range R. This because if $r_1, \ldots, r_n$ is an orthonormal base in R then for any $k \in K$ we have that

$$P_R k = \sum_{i=1}^{n} (r_i, k) r_i ,$$

which obviously extends by continuity.

We shall say that a measurable function f on Q' is finitely based if there is a finite dimensional subspace R of Q such that $f(\xi) = f(P_R \xi)$. Moreover we shall say that a finitely based function f is in $FC^n(Q')$ if its restriction to its base R is in $C^n(R)$ i.e. n-times continuously differentiable. This definition is obviously independent of the choice of R.

We shall say that a function $f \in C(Q')$ is in $C^n(Q')$ if, for any $\xi \in Q'$ and any $q \in Q$, $f(\xi+tq)$ is n-times continuously differentiable functions of t and at $t = 0$ all the derivatives are in $C(Q')$. If $f \in C^1$ we define

$$(q \cdot \nabla f)(\xi) = \frac{d}{dt} f(\xi+tq)\big|_{t=0} . \qquad (2.15)$$

We see that if $\mu \in \mathscr{P}_1(Q')$ then $C^1(Q')$ is contained in the domain of $\pi(q)$ for all $q \in Q$ and for $f \in C^1$ we have

$$\pi(q)f = \frac{1}{i}(q\cdot\nabla f + \frac{1}{2}\beta\cdot qf) . \tag{2.16}$$

Now the operator $q\cdot\nabla$ is defined on C^1 and it has a densely defined adjoint, namely $-q\cdot\nabla - \beta\cdot q$, whose domain contains C^1 which is obviously dense in $L_2(d\mu)$. Hence $q\cdot\nabla$ is closable and we shall denote its closure also by $q\cdot\nabla$.

Let now $f \in D(q\cdot\nabla)$ for all $q \in Q$. For any finite dimensional subspace $R \subset Q$ we define

$$(f,f)_1^R = \sum_{i=1}^{n} \|e_i\cdot\nabla f\|_2^2 \tag{2.17}$$

where $e_1,\ldots,e_n$ is an orthonormal basis for R. It is evident that (2.17) is independent of the particular basis $e_1,\ldots,e_n$.

We have obviously that if $R_1 \subset R_2$ then $(f,f)_1^{R_1} \leq (f,f)_1^{R_2}$. Hence the limit of $(f,f)_1^R$ over all finite dimensional subspaces exists and we denote this by $(f,f)_1$. It follows immediately by taking the limit over the subspaces spanned by $\{e_1,\ldots,e_n\}$, where $\{e_i\}^\infty$ is an orthonormal base in K of elements in Q, that

$$(f,f)_1 = \sum_{i=1}^{\infty} \|e_i\cdot\nabla f\|_2^2 , \tag{2.18}$$

so by construction (2.18) is basis independent. We shall also use the notation

$$(f,f)_1 = \int \nabla f\cdot\nabla f \, d\mu , \tag{2.19}$$

and we call $(f,f)_1$ the Dirichlet form given by μ.[7]

Let R be a finite dimensional subspace of Q and let E_R be the conditional expectation with respect to the σ-subalgebra generated by the functions (q,ξ) with $q \in R$. We then have the following formula for any $f \in D(q\cdot\nabla) \cap L_\infty$ and any $q \in Q$

$$q \cdot \nabla E_R f = E_R q \cdot \nabla f + E_R ((\beta \cdot q - E_R \beta \cdot q) f) . \qquad (2.20)$$

In particular we have that E_R maps $D(q \cdot \nabla) \cap L_\infty$ into $D(q \cdot \nabla)$. This follows from a simple computation. From (2.20) we find that $E_R f$ converges to f strongly in the graph norm of $q \cdot \nabla$. It is also easy to see that $D(q \cdot \nabla) \cap L_\infty$ is dense in $D(q \cdot \nabla)$ in the graph norm. Hence if we let FL_∞ denote the finitely based functions in L_∞ we have the following lemma

Lemma 2.1

If $\mu \in \mathscr{P}_1(Q')$ then $FL_\infty \cap D(q \cdot \nabla)$ is a core for $q \cdot \nabla$, i.e. it is dense in $D(q \cdot \nabla)$ in the graph norm. ☐

If f is in the domain of $q \cdot \nabla$ for all $q \in Q$ and the Dirichlet form $(f,f)_1$ is finite we define ∇f as an element in $K \otimes L_2(d\mu)$ and obviously we have then

$$\|\nabla f\|^2 = \int \nabla \bar{f} \cdot \nabla f \, d\mu = (f,f)_1 . \qquad (2.21)$$

Lemma 2.2

Let $\mu \in \mathscr{P}_1(Q')$, then the mapping ∇ from $L_2(d\mu)$ into $K \otimes L_2(d\mu)$ is closable.

Proof: Let R be a finite dimensional subspace of Q, and let $e_1, \ldots, e_n$ be an orthonormal basis in R. Let $h \in R \otimes L_2(d\mu)$ have the decomposition $h = \{h_1, \ldots, h_n\}$, $h_i \in L_2(d\mu)$, with respect to the basis $e_1, \ldots, e_n$. Assume that the h_i are in $D(q \cdot \nabla)$ for all q in Q. Then such h are dense in $K \otimes L_2(d\mu)$ and the adjoint of ∇ is applicable to such h and we have

$$\nabla^* h = \sum_{i=1}^{n} e_i \cdot \nabla h_i + \beta \cdot e_i . \qquad (2.22)$$

Therefore ∇ has a densely defined adjoint, hence it is closable. This proves the lemma. $\square$

From now on we shall denote the closure also by ∇, so that in what comes ∇ is a closed map from $L_2(d\mu)$ into $K \otimes L_2(d\mu)$ with domain equal to $W^1(d\mu)$ consisting of those $f \in L_2(d\mu)$ which have finite Dirichlet norm, and in fact (2.21) holds for all $f \in W^1$. The adjoint ∇^* of ∇ is also densely defined and closed. We have thus proved the following theorem.

Theorem 2.1

Let $\mu \in \mathscr{P}_1(Q')$, then the Dirichlet form

$$(f,f)_1 = \int \nabla \bar{f} \cdot \nabla f \, d\mu$$

is a closed form in $L_2(d\mu)$ and its associated operator is given by the selfadjoint operator

$$H = \nabla^* \nabla .$$

For $f \in FC^2(Q')$ we have that

$$Hf = -\Delta f - \beta \cdot \nabla f ,$$

where $\Delta = \sum_{i=1}^{n} (e_i \nabla)^2$ and $\beta \cdot \nabla = \sum_{i=1}^{n} (\beta \cdot e_i)(e_i \cdot \nabla)$, where $e_1, \ldots, e_n$ is an orthonormal base in R, f being finitely based in R. $\square$

In the same way as for the Dirichlet form we get that (2.17) defines a closed form $(f, f_1)^R$ for any finite dimensional subspace $R \subset Q$, and the corresponding selfadjoint operator H^R is given by

$$H^R = (\nabla_R)^* \nabla_R \tag{2.23}$$

where ∇_R is the corresponding map from $L_2(d\mu)$ into $R \otimes L_2(d\mu)$.

Theorem 2.2

$R \to H^R$ is a monotone map from the ordered set of finite dimensional subsets of Q into the ordered set of positive selfadjoint operators. Moreover e^{-tH^R} converges strongly to e^{-tH} uniformly on finite t-intervals as $R \to Q$ through the net of finite subsets.

Proof: In the paragraph following (2.17) we already observed that $R \to H^R$ is monotone. The monotone form convergence follows from (2.18) and this implies the strong semigroup convergence by the theorem on convergence from below of symmetric semibounded forms. (Theorem 3.13), Ch. VIII, Ref. [51].) □

We shall call the operator H of theorem 2.1 the Dirichlet operator.[*]

We shall say that a measure $\mu \in \mathcal{P}(Q')$ is Q-ergodic iff the only measurable subsets of Q' which are Q-invariant, i.e. invariant under translations by arbitrary $q \in Q$, have μ-measure zero or one.

There exists a compact Hausdorff space Z with a regular measure dz such that μ has a unique Q-ergodic decomposition

$$\mu = \int_Z \mu_z \, dz \, . \tag{2.24}$$

Z is simply the maximal ideal space for the subalgebra of $L_\infty(d\mu)$ consisting of Q-invariant functions. Hence a continuous function on Z is in natural correspondence with a translation invariant function in L_∞ and dz is just the restriction of $d\mu$ to the translation invariant functions.

So that in fact μ_z is the conditional probability measure, conditioned with respect to the subalgebra of Q-invariant subsets. (See also [20]).

Lemma 2.3

If $z_1 \neq z_2$ then $\mu_{z_1} \perp \mu_{z_2}$.

Proof: If $z_1 \neq z_2$, since Z is Hausdorff, there are two open non intersecting sets A_1 and A_2 such that $z_1 \in A_1$ and $z_2 \in A_2$. Now for any μ-measurable set $B \subset Q'$ and any A open in Z we have by definition that

$$\int_A \mu_z(B)dz = \mu(\tilde{A} \cap B) \tag{2.25}$$

where $\tilde{A}$ is a Q-invariant measurable set such that its characteristic function is represented on Z by the characteristic function of A. So that if $z \in A$ then, by (2.25), μ_z has support in $\tilde{A}$. Since $A_1 \cap A_2 = \emptyset$ we have that $\tilde{A}_1$ and $\tilde{A}_2$ may be chosen such that $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$. This proves the theorem. $\Box$

Theorem 2.4

Let $\mu \in \mathscr{P}_1(Q')$ then zero is a simple eigenvalue of H if and only if μ is Q-ergodic. Moreover the eigenspace of eigenvalue zero is exactly the subspace of $L_2(d\mu)$ consisting of Q-invariant functions.

In fact the decomposition (2.24) gives a direct decomposition of $L_2(d\mu)$ of the form

$$L_2(d\mu) = \int_Z L_2(d\mu_z)dz$$

and with respect to this decomposition H decomposes as

$$H = \int H_z \, dz$$

such that, for each z, H_z has zero as a simple eigenvalue.

Proof: Let $f \in D(H)$ such that $Hf = 0$. Then

$$(f,Hf) = (f,f)_1 = \int \nabla \bar{f} \cdot \nabla f \, d\mu = 0 .$$

By the definition of $(f,f)_1$ this implies that the derivative of f in any Q-direction is zero, hence f is Q-invariant. On the other hand if f is Q-invariant, then obviously $(f,f)_1 = 0$ so that $f \in W^1 = D(H^{\frac{1}{2}})$ and $H^{\frac{1}{2}}f = 0$, which implies that $f \in D(H)$ and $Hf = 0$.

The direct decomposition of $L_2(d\mu)$ follows from the fact that $\mu_{z_1} \perp \mu_{z_2}$ for $z_1 \neq z_2$. That H decomposes and that H_z has a simple eigenvalue follows from the corresponding decomposition of W^1. $\square$

We shall say that a $\mu \in \mathscr{P}_1(Q')$ is in $\mathscr{P}_D(Q')$ if $\beta \in K \otimes L_2(d\mu)$ i.e. if the Dirichlet norm of μ,

$$\|\mu\|_D^2 = \|\beta\|^2 = \sum_{i=1}^{\infty} \|\beta_i\|_2^2 \tag{2.26}$$

is finite, where $\beta_i = \beta \cdot e_i$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal base in K of elements in Q. Similarly as in lemma 2.1 we have

Lemma 2.4

If $\mu \in \mathscr{P}_D(Q')$ then $FL_\infty \cap W^1$ is dense in $W^1 = D(\nabla)$ in the graph norm of ∇, i.e. in the Dirichlet norm $(\|f\|^1)^2 = (f,f)_1 + (f,f)$.

Proof: Let $f \in W^1$ and set $f^k(\xi) = f(\xi)$ if $|f(\xi)| \leq k$ and equal to $\pm k$ if $f(\xi)$ is larger than k (smaller than $-k$). Then $f^k \to f$ in $L_2(d\mu)$ and

$$(f-f^k, f-f^k)_1 = \int_{|f(\xi)|>k} \nabla \bar{f} \cdot \nabla f \, d\mu$$

which goes to zero since $\nabla \bar{f} \cdot \nabla f \in L_1$. So that $W^1 \cap L_\infty$ is dense in W^1 . Let now $f \in W^1 \cap L_\infty$ and let R be a finite dimensional subspace of Q with its corresponding conditional expectation E_R . Then by (2.20)

$$\nabla E_R f = E_R \nabla f + E_R [(\beta - E_R \beta) f] . \qquad (2.27)$$

So from the triangle inequality in $K \otimes L_2(d\mu)$ we have

$$\|\nabla E_R f\| \leq \|E_R \nabla f\| + \|E_R [(\beta - E_R \beta) f]\|$$

and thus, since E_R is a projection in L_2 ,

$$\|\nabla E_R f\| \leq \|\nabla f\| + \|f\|_\infty \|\beta\|$$

so that $E_R f \in W^1$. Consider now $f - E_R f$, which obviously goes to zero in $L_2(d\mu)$ as $R \to Q$. On the other hand

$$\|\nabla (f - E_R f)\| \leq \|\nabla f - E_R \nabla f\| + \|E_R [(\beta - E_R \beta) f]\|$$

$$\leq \|\nabla f - E_R \nabla f\| + \|f\|_\infty \|\beta - E_R \beta\| .$$

Since $E_R \to 1$ in $L_2(d\mu)$ we have that $1 \otimes E_R \to 1$ in $K \otimes L_2(d\mu)$, hence the right hand side of the previous inequality goes to zero. This proves the lemma. $\square$

Let now $\mu \in \mathscr{P}_2(Q')$ and let us also assume that, for an orthonormal basis $\{e_n\}$ in K of elements in Q ,

$$V = -\sum_{n=1}^{\infty} \pi(e_n)^2 \Omega \qquad (2.28)$$

converges in $L_2(d\mu)$, where $\Omega(\xi) \equiv 1$. In that case the Laplacian

$$\pi^2 = \sum_{i=1}^{\infty} \pi(e_i)^2 \qquad (2.29)$$

is defined on FC^2 , it is obviously non negative and we shall denote by π^2 also its Friedrichs extension. Although (2.29) looks

basis dependent, we may see in the following way that it is not.

Let R be a finite dimensional subspace of Q and

$$\pi_R^2 = \sum_{i=1}^{n} \pi(r_i)^2 , \qquad (2.30)$$

where $r_1, \ldots, r_n$ is an orthonormal base in R . We denote also by π_R^2 its Friedrichs extension. It is easy to see that (2.30) is basis independent. Moreover $R \rightarrow \pi_R^2$ is monotone from the directed set of finite dimensional subspaces into the directed set of non negative operators. π^2 is then simply the limit, by theorem 3.13 of Ch. VIII of ref. [51] , of π_R^2 as $R \rightarrow Q$. This shows that the Friedrichs extension of (2.29) is basis independent.

We have obviously that on $FC^2(Q')$

$$H = \pi^2 + V , \qquad (2.31)$$

where H is the Dirichlet operator. We can also give the L_2-function V directly in terms of β if we assume $\|\mu\|_D < \infty$. Since

$$i\pi(q)\Omega = \tfrac{1}{2}\beta \cdot q$$

we see that

$$-\pi^2\Omega = \sum_{i=1}^{\infty} (\tfrac{1}{2}\nabla_i\beta_i + \tfrac{1}{4}\beta_i \cdot \beta_i) . \qquad (2.32)$$

Now, if $\|\mu\|_D < \infty$, $\sum_{i=1}^{\infty} \beta_i^2$ converges in L_2 so, by the assumption that (2.28) converges, we get that

$$V = \tfrac{1}{2}\mathrm{div}\beta + \tfrac{1}{4}\beta \cdot \beta , \qquad (2.33)$$

where

$$\beta \cdot \beta = \sum_{i=1}^{\infty} \beta_i^2 \quad \text{and} \quad \mathrm{div}\beta = \sum_{i=1}^{\infty} \nabla_i\beta_i \qquad (2.34)$$

and $\beta_i = \beta \cdot e_i$ and $\nabla_i = e_i \cdot \nabla$, $\{e_i\}$ being an orthonormal base in K of elements in Q .

Remark: The Laplacian $-\pi^2$ on $L_2(d\mu)$ as introduced here has no relation to the Laplacian studied by Gross in [6] or the Laplacian studied by Lévy in [27].

It is not immediately obvious that the class of quasi invariant measures so that (2.28) converges in $L_2(d\mu)$ is non empty. So we shall therefore give a simple example.

Example: Let A be a positive invertible trace class operator on a real separable Hilbert space K. Consider now the Gaussian measure $d\mu_A$ with covariance A^{-1}, it is

$$E_A[(x,\xi)(y,\xi)] = (x,A^{-1}y) .$$

Let Q be the Hilbert space $Q = D(A^{-1})$ with its natural norm. Then we have that Q' is the completion of K in the norm $\|Ax\|$. It is well known that, since A is of trace class, $d\mu_A$ is a measure on Q', which is quasi invariant under translations by all $q \in Q$, in fact by all $q \in Q'$ such that $(q,Aq) < \infty$. In this case

$$\beta q = -Aq \tag{2.35}$$

so that $\mu_A \in \mathscr{P}_D(Q')$ namely

$$\|\mu_A\|_D = \int \|A\xi\|^2 d\mu_A(\xi) = \operatorname{tr} A \tag{2.36}$$

and

$$(\pi^2\Omega)(\xi) = -\|A\xi\|^2 + \operatorname{tr} A . \tag{2.37}$$

So that with

$$V = \|A\xi\|^2 - \operatorname{tr} A$$

we have that

$$\|V\|_2^2 = E_A[(\xi A^2\xi)^2 - 2\operatorname{tr}A(\xi A^2\xi) + (\operatorname{tr}A)^2]$$

$$= \operatorname{tr} A^2 + (\operatorname{tr} A)^2 - 2(\operatorname{tr}A)^2 + (\operatorname{tr}A)^2 ,$$

i.e.

$$\|V\|_2^2 = \operatorname{tr} A^2 \tag{2.38}$$

which is finite since A is of trace class. We see in fact that
we may do with the weaker condition that A is a Hilbert-Smith
operator , because (2.38) still holds and also in this case $d\mu_A$
is a measure on Q' .

3. The diffusion process generated by the Dirichlet operator.

We have from the previous section that the Dirichlet operator $H = \nabla^*\nabla$ is a self adjoint operator in $L_2(d\mu)$ which is the limit in the strong resolvent sense of the operators

$$H_R = \nabla_R^*\nabla_R \tag{3.1}$$

where ∇_R is the gradient in the direction of the finite dimensional subspace R , i.e. $\nabla_R = (P_R \otimes I)\nabla$, where P_R is the orthogonal projection in K with range R . The limit is to be taken over the filter of all finite dimensional subspaces. From the strong resolvent convergence we then have that $e^{-tH_R} \to e^{-tH}$ strongly.

We say that a contraction semigroup T_t in $L_2(d\mu)$ is a Markov semigroup if for any $f \in L_2(d\mu)$ with $f \geq 0$ we have that $T_t f \geq 0$. From the strong convergence above we get that if e^{-tH_R} is Markov, then so is e^{-tH} . We shall now see that e^{-tH_R} is Markov if $\mu \in \mathscr{P}_1(Q')$.

We have seen in the previous section that since R is finite dimensional P_R extends by continuity to a continuous projection defined on all of Q' and with range R . We shall denote this extension still by P_R . The decomposition of the identity on Q' given by

$$I = P_R + (I-P_R) \tag{3.2}$$

gives a direct decomposition of Q' of the form

$$Q' = R \oplus R^\perp , \tag{3.3}$$

where $R^\perp$ is the annihilator of R in Q' . Since P_R is continuous on Q' , so is $I - P_R$, hence for $x \in R$ and $\eta \in R^\perp$ we have that $(x,y) \to x \oplus y$ is one to one and bicontinuous. Hence

Q' and $R \times R^{\perp}$ are equivalent as measure spaces. Therefore we may consider μ as a measure on the product space $R \times R^{\perp}$. Let now $\mu(x|\eta)$ be the conditional probability measure on R conditioned with respect to $R^{\perp}$. Thus for any measurable set $A \subset R$ we have that $\mu(A|\eta)$ is a positive measurable function on $R^{\perp}$ such that, for any measurable set B in $R^{\perp}$,

$$\int_B \mu(A|\eta)d\nu(\eta) = \mu(A \times B) , \qquad (3.4)$$

where ν is the measure induced on $R^{\perp}$ by μ. Let now $A \subset R$ and $B \subset R^{\perp}$. The quasi invariance of μ under translations by elements in Q gives us that $\mu(A \times B)$ as a function of A for fixed B is a quasi invariant measure on R, and therefore by (3.4) we get that $\mu(A|\eta)$ is a quasi invariant measure on R for ν-almost all $\eta \in R^{\perp}$. Thus we have

$$\mu(A|\eta) = \int_A \rho(x|\eta)dx = \int_A \varphi^2(x|\eta)dx \qquad (3.5)$$

with $\varphi(x|\eta)$ and $\rho(x|\eta)$ different from zero almost everywhere in the sense of Lebesgue. From (3.4) we now easely get

$$L_2(d\mu) = \int_{R^{\perp}} L_2(d\mu(\cdot|\eta)d\nu(\eta) \qquad (3.6)$$

where the integral is taken in the sense of a direct integral of Hilbert spaces. We see that the operator H_R of (3.1) is reducible with respect to the direct integral decomposition (3.6) and in fact with respect to that decomposition we have

$$H_R = \int_{R^{\perp}} H_{\eta} \, d\nu(\eta) , \qquad (3.7)$$

where H_{η} is the Dirichlet operator in $L_2(R; d\mu(\cdot|\eta)$.

Hence

$$e^{-tH_R} = \int_{R^\perp} e^{-tH_\eta} \, d\nu(\eta) \ . \tag{3.8}$$

Therefore if we can prove that e^{-tH_η} is a Markov semigroup, then e^{-tH_R} is a Markov semigroup. Hence we have reduced the problem of whether e^{-tH} is Markov or not to a corresponding finite dimensional problem.

Let now $\{X, dm\}$ be a σ-finite measure space. Let ε be a closed non negative symmetric form on the real L^2-space $L^2(X, dm)$ with domain of definition $D(\varepsilon)$ which is dense in $L^2(X, dm)$.

We shall say that every unit contraction operates on ε if for any $u \in D(\varepsilon)$ the function $v = (0 \vee u) \wedge 1$ is again in $D(\varepsilon)$ and

$$\varepsilon(v,v) \leq \varepsilon(u,u) \ . \tag{3.9}$$

The following theorem is proved in section 3 of ref. [29] .

Theorem 3.1 [Fukushima]

Let X be a locally compact separable Hausdorff space with a Radon measure dm . Let ε be a closed non negative symmetric form on real $L^2(X, dm)$ with a dense domain of definition $D(\varepsilon)$. If every unit contraction operates on ε , then the semigroup e^{-tH_ε} generated by the self adjoint operator H_ε associated with the closed form ε is a Markov semigroup. Moreover if e^{-tH_ε} is a Markov semigroup, then every unit contraction operates on ε .   ☐

Since H_η in (3.7) is the Dirichlet operator in R and R is finite dimensional and H_η is the operator associated with the Dirichlet form in $L_2(R, \rho(x|\eta)dx)$ we have only to check that every unit contraction operates on the corresponding Dirichlet

form. However with $v = (0 \vee u) \wedge 1$ we have that

$$(v,v)_1 = \int\limits_{0 \le v \le 1} |\nabla u|^2 \rho(x|\eta) dx \le \int |\nabla u|^2 \rho(x|\eta) dx = (u,u)_1 \, . \qquad (3.10)$$

Hence we see that the condition of theorem 3.1 is satisfied so that e^{-tH}_η is Markov. Thus we have proved the following theorem.[9]

Theorem 3.2

Let $\mu \in \mathscr{P}_1(Q')$, then the corresponding Dirichlet operator H generates a contraction semigroup e^{-tH} which is Markov. $\square$

Since μ is a regular measure on Q' and e^{-tH} is a Markov semigroup, it gives rise to a Markov process $\xi(t)$ on Q' which is homogeneous in t such that μ is an invariant measure for $\xi(t)$ and for any $f \in L_2(d\mu)$ we have that

$$e^{-tH} f = E_0 f(\xi(t)) \, , \qquad (3.11)$$

where E_0 is the conditional expectation with respect to the subalgebra generated by the linear functions $(q,\xi(0))$ for $q \in Q$.

Let now $(X,d\omega)$ be the underlying probability space for the process $\xi(t)$ induced by the Markov semigroup by the Kolmogorov construction. Then we have the natural inclusion $L_2(d\mu \subset L_2(X,d\omega)$ as the subspace of L_2-functions measurable with respect to the subalgebra generated by $q \cdot \xi(0)$. Moreover the time translation $\xi(s) \to \xi(s+t)$ induces in a natural way a strongly continuous unitary group T_t in $L_2(X,d\omega)$, and with this notation (3.11) takes the form

$$e^{-tH} = E_0 T_t E_0 \qquad (3.12)$$

where E_0 is the projection onto the $\xi(0)$ measurable functions, i.e. onto $L_2(d\mu)$. Let now $f \in L_2(d\mu)$, then of course

$f(\xi(t)) = T_t f(\xi(0)) T_{-t}$ so that $f(\xi(t)) \in L_2(X, d\omega)$ and depends strongly continuously on t .

Since $\mu \in \mathscr{P}_1(Q')$ we have that $q \cdot \beta \in L_2(d\mu)$ so that $q \cdot \beta(\xi(t)) \in L_2(X, d\omega)$, and this depends strongly continuously on t . Hence it is strongly integrable and $\int_0^t q \cdot \beta(\xi(\tau)) d\tau \in L_2(X, d\omega)$ is actually strongly differentiable with respect to t . Consider now the real valued process

$$q \cdot w(t) = q \cdot \xi(t) - \int_o^t q \cdot \beta(\xi(\tau)) d\tau . \qquad (3.13)$$

We have obviously that $q \cdot w(t)$ is well defined for all $q \in Q$ and as a function on the probability space $(X, d\omega)$ it is linear in q . In short $w(t)$ is a weak process on Q' [30]-[32].We shall se that it is actually the restriction to Q of the standard weak Wienerprocess on K . Consider for this

$$e^{i\alpha q \cdot w(t)} = e^{i\alpha q \cdot \xi(t)} \cdot e^{-i\alpha \int_o^t q \cdot \beta(\xi(\tau)) d\tau} . \qquad (3.14)$$

From (3.12) we get that if $f \in L_2(d\mu)$ is in the domain of definition of the Dirichlet operator H then $E_o f(\xi(t))$ is strongly differentiable in $L_2(X, d\omega)$ with respect to t and for all $t \geq 0$

$$\frac{d}{dt} E_o f(\xi(t)) = - E_o(Hf)(\xi(t)) .$$

By homogeneity we therefore get that

$$\frac{d}{dt} E_s f(\xi(t)) = - E_s(Hf)(\xi(t)) \qquad (3.15)$$

for all $s \geq 0$ and $t \geq s$. For $t = s$ the derivative above is the one sided derivative. Since $e^{i\alpha q \cdot \xi} \in D(H)$ we have by (3.14) that $E_s e^{i\alpha q \cdot w(t)}$ is strongly differentiable with respect

to t for $t \geq s$ and since $H e^{i\alpha q \cdot \xi} = (\alpha^2 q^2 - i\alpha q \cdot \beta(\xi))e^{i\alpha q \cdot \xi}$ we have for $t \geq s$

$$\frac{d}{dt} E_s e^{i\alpha q w(t)} = -\alpha^2 q^2 e^{i\alpha q \cdot w(t)} . \qquad (3.16)$$

Hence for any function $f \in S(R)$ we get that $E_s f(q \cdot w(t))$ is strongly $L_2(X, d\omega)$-differentiable and

$$\frac{d}{dt} E_s f(q \cdot w(t)) = q^2 E_s(\Delta f(q \cdot w(t))) \qquad (3.17)$$

where $q^2 = (q,q)$. By lemma 3.1 below we then have that $q \cdot w(t)$ is the Wiener process with diffusion $2q^2$ on R . Hence we have proven that $w(t)$ given by (3.13) is the standard weak Wiener process on K . We have thus proven the following theorem.[10]

Theorem 3.3

Let $\xi(t)$ be the Markov process given by the Markov semigroup of theorem 3.2. Then $\xi(t)$ satisfies the following stochastic differential equation in the sense of weak processes on Q'

$$d\xi(t) = \beta(\xi(t))dt + dw(t) ,$$

where $w(t)$ is the standard weak Wiener process on K . □ In the proof above we ma e use of the following lemma.

Lemma 3.1

Let $\eta(t)$, $t \geq 0$ be a real valued stochastic process, i.e. a real valued measurable function $\eta(t,\omega)$ from $([0,\infty) \times X, d\lambda \times d\omega)$ into R where $(X, d\omega)$ is a probability space and λ is the Lebesgue measure. For any measurable function f on R we define the forward derivative

$$(D_+ f)(\eta(t)) = \lim_{h \searrow 0} \frac{1}{h} E_t[f(\eta(t+h)) - f(\eta(t))]$$

where E_t is the conditional expectation with respect to the

subalgebra generated by $\eta(\tau)$ for $0 \leq \tau \leq t$, whenever this limit exists in the strong $L_2(X,d\omega)$ sense. $D_+f(\eta(t))$ is thus a function in $E_t L_2(X,d\omega)$ whenever it exists.

If for any $f \in S(R)$, the Schwartz test function space, we have that $f(\eta(t))$ is strongly $L_2(X,d\omega)$ differentiable and

$$(D_+f)(\eta(t)) = \frac{\sigma}{2}(\Delta f)(\eta(t)) ,$$

then $\eta(t)$ is a Wiener process on R with diffusion σ, i.e. $\eta(t)$ is a Markov process and if ν is the distribution of $\eta(0)$, then the distribution of $\eta(t)$ is

$$(2\pi\sigma t)^{-\frac{1}{2}} \int e^{-\frac{1}{2\sigma t}(x-y)^2} d\nu(y) .$$

Proof. Since obviously $E_s \cdot E_{s+t} = E_s$ for s and t positive, we have by the assumptions of the lemma that $E_s f(\eta(t+s))$ is strongly $L_2(X,d\omega)$ differentiable in t for $t \geq 0$, since E_s is a strongly $L_2(X,d\omega)$ continuous projection, and

$$\frac{d}{dt} E_s f(\eta(t+s)) = E_s(\frac{\sigma}{2}\Delta f(\eta(t+s))). \qquad (3.18)$$

Therefore since $\Delta f, \Delta^2 f, \ldots$ are again in $S(R)$ we get from (3.18) that, for all $t \geq 0$,

$$\frac{d^n}{dt^n} E_s f(\eta(t+s)) = E_s((\tfrac{1}{2}\sigma)^n \Delta^n f(\eta(t+s))) \qquad (3.19)$$

where we must remember that for $t = 0$ the derivatives are the one sided derivatives. Hence for $f \in S(R)$ with $\hat{f}$ of bounded support we easily get by Sobolev inequalities that there is a constant c such that

$$\|\Delta^n f\|_\infty \leq c^n . \qquad (3.20)$$

But then $\|\Delta^n f(\eta(t+s)))\|_\infty \leq c^n$ so that

$$\left\|\frac{d^n}{dt^n} E_s f(\eta(t+s))\right\|_\infty \le (\tfrac{1}{2}\sigma c)^n . \tag{3.21}$$

From this it follows that $E_s f(\eta(t+s))$ is strongly $L_\infty(X, d\omega)$ analytic in t so that for all $t \ge 0$ we have

$$E_s f(\eta(t+s)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\tfrac{1}{2}\sigma)^n E_s \Delta^n f(\eta(s)) . \tag{3.22}$$

Since $\tfrac{1}{2}\Delta$ is the infinitesimal generator of the semigroup $e^{\frac{1}{2}t\Delta}$ with kernel

$$e^{\frac{1}{2}t\Delta}(x,y) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{1}{2t}(x-y)^2} \tag{3.23}$$

we have that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}(\tfrac{1}{2}\sigma)^n \Delta^n f(x) = (2\pi\sigma t)^{-\frac{1}{2}} \int e^{-\frac{1}{2t\sigma}(x-y)^2} f(y)\,dy, \tag{3.24}$$

where the sum is strongly L_∞ convergent. From (3.22) and the strong L_∞ convergence of (3.24) we get

$$E_s f(\eta(t+s)) = (T_t f)(\eta(s)) , \tag{3.25}$$

where

$$(T_t f)(x) = (2\pi\sigma t)^{-\frac{1}{2}} \int e^{-\frac{1}{2t\sigma}(x-y)^2} f(y)\,dy . \tag{3.26}$$

In particular

$$E_0 f(\eta(t)) = (T_t f)(\eta(0)) . \tag{3.27}$$

Since T_t is a semigroup (3.27) proves that $\eta(t)$ is a Markov process and from (3.26) we get that the conditional distribution of $\eta(t)$ given the condition $\eta(0) = 0$ is

$$(2\pi\sigma t)^{-\frac{1}{2}} e^{-\frac{1}{2t\sigma}(x-y)^2}\,dx . \tag{3.28}$$

This then proves the lemma. $\square$

In what follows we shall also need the following lemma of Frobenius type. [11])

Lemma 3.2

Let A be a bounded operator on an L_2-space, such that $\|A\| \leq 1$ and A is positivity preserving, i.e. $f \geq 0 \Rightarrow Af \geq 0$. If 1 is an eigenvalue for A, then 1 is a simple eigenvector if the only multiplication operators that commute with A are the constants. Moreover if 1 is a simple eigenvalue, then the corresponding eigenfunction may be taken non negative, and if the only multiplication operators that commutes with A are the constants, then the corresponding eigenfunction is positive almost everywhere.

Proof: Let us assume that 1 is a eigenvalue of A with a corresponding eigenfunction φ. Since A is positivity preserving, we have, if $\|\varphi\| = 1$, that

$$1 = (\varphi, A\varphi) \leq (|\varphi|, A|\varphi|)$$ so that $|\varphi|$ is an eigenfunction to the eigenvalue 1, since $\|A\| \leq 1$. Hence if 1 is simple, we may take $|\varphi|$ as the corresponding eigenfunction. On the other hand if 1 is not simple, we have at least another one, ψ, which is orthogonal to $|\varphi|$. Since A is positivity preserving, the real and imaginary parts of ψ will also be eigenfunctions and both will be orthogonal to $|\varphi|$, so we may for this reason take ψ to be real. If $\psi = \pm|\psi|$, then $|\varphi|$ and $|\psi|$ are orthogonal, and if ψ and $|\psi|$ are not proportional, then $|\psi| \pm \psi$ are two positive orthogonal eigenfunctions. Hence if 1 is not simple, we can always find a non negative eigenfunction v corresponding to the eigenvalue 1 such that the characteristic function χ of its support is not a constant.

As a multiplication operator χ is a projection of $L_2(X, d\omega)$

onto $L_2(\chi X, d\omega)$. Obviously the functions $f \in L_2(X, d\omega)$ such that $|f| \leq c \cdot v$ for some constant c are dense in the range of χ. Since $Av = v$ and A is positivity preserving we have, for any $-cv \leq f \leq cv$, that $-cv \leq Af \leq cv$, so that A takes a dense subspace of the range of χ into itself. By continuity A then takes the range of χ into it self, i.e. A commutes with χ. Suppose now that the only multiplication operators that commute with A are the constants. Then 1 is a simple eigenvalue and it follows from above that the characteristic function to its support commutes with A. If this characteristic function is to be constant, then the eigenfunction must be positive almost everywhere. □

Let now $u \in \mathscr{P}_1(Q')$ and let H be the corresponding Dirichlet operator in $L_2(Q', d\mu)$. By $L_\infty(V)$ we shall understand the subalgebra of $L_\infty(Q', d\mu)$ of multiplication operators which commute with e^{-tH} for all $t > 0$. Since $L_\infty(V)$ is a commutative C^*-algebra, we have that it is equal to all the continuous functions on some compact space which we shall denote V. Let dv be the measure induced on V by the integral induced on $L_\infty(V)$ by $d\mu$. It is then easy to see that $L_\infty(V)$ is also isomorphic with $L_\infty(V, dv)$. The spectral decomposition of $L_2(d\mu)$ with respect to the commutative algebra of operators $L_\infty(V)$ is then given by

$$L_2(Q', d\mu) = \int_V L_2(d\mu(\cdot|v))dv , \qquad (3.29)$$

where $d\mu(\cdot|v)$ is the conditional probability measures conditioned with respect to the σ-subalgebra generated by the functions in $L_\infty(V)$. Since all the elements in $L_\infty(V)$ commute with e^{-tH}

we have that H is reduced by the direct decomposition (3.29) and

$$H = \int_V H_v \, dv \, . \tag{3.30}$$

Thus H_v is a self adjoint operator for almost all v . By the
corresponding reduction of the Dirichlet form we get that

$$(f, H_v f)_v = \int_{Q'} \nabla f \cdot \nabla f \, d\mu(\xi | v) \, . \tag{3.31}$$

Hence we get that the Dirichlet form in $L_2(d\mu(\cdot | v)$ is closed,
and the corresponding Dirichlet operator is H_v . We should here
bear in mind that $d\mu(\xi | v)$ is not necessarily quasi invariant
under translations by elements in q , but nevertheless the corre-
sponding Dirichlet form (3.31) is closed.

By the decomposition (3.29) we have that the only multipli-
cation operators which commute with all e^{-tH_v} in $L_2(d\mu(\cdot | v))$
are the constants. Hence, by lemma 3.2, 0 is a simple eigenvalue
of H_v . We have thus proved the following theorem.

Theorem 3.4

Let $u \in \mathcal{P}_1(Q')$ and let

$$L_2(d\mu) = \int_V L_2(d\mu(\cdot | v)) dv$$

be the spectral decomposition with respect to the subalgebra
$L_\infty(V)$ of multiplication operators which commutes with e^{-tH}
for all $t > 0$, then $u(\cdot | v)$ is the conditional probability
measure conditioned with respect to the σ-subalgebra generated
by $L_\infty(V)$ and the Dirichlet forms in $L_2(d\mu(\cdot | v))$ are closed
for almost all v . If

$$H = \int_V H_v \, dv$$

is the corresponding direct decomposition of H , then H_v are the self adjoint operators in $L_2(d\mu(\cdot\,|v))$ given by the Dirichlet forms in $L_2(\cdot\,|v))$. Zero is a simple eigenvalue for H_v and the corresponding eigenfunction is positive almost everywhere, for almost all v . Moreover the zero eigenspace for H is the closure of $L_\infty(V)$ in $L_2(d\mu)$.

__Proof__: That the zero eigenvalue for H_v is positive almost everywhere follows from the fact that the only multiplication operators that commute with e^{-tH_v} for all $t > 0$ are the constants, in a similar way as in lemma 3.2. Now obviously $L_\infty(V)$ is in the zero eigenspace for H since it is invariant under e^{-tH} . Suppose now $e^{-tH}f = f$ for all t , and let us assume f real. Then of course we have also that $e^{-tH}(f-\lambda) = f - \lambda$ and by the proof of lemma 3.2 $|f-\lambda|\pm(f-\lambda)$ is also invariant. In the same way as in lemma 3.2 we then also get that the support of $|f-\lambda|\pm(f-\lambda)$ has a characteristic function which is invariant. Hence the characteristic function of any set of form $\lambda_1 \le f < \lambda_2$ is invariant under e^{-tH} . But then f is obviously in the L_2-closure of $L_\infty(V)$. This proves the theorem. $\square$

The Markov semigroup e^{-tH} is said to be ergodic if the only multiplication operators that commute with e^{-tH} are the constant. We see from above that this is equivalent with 1 being a simple eigenvalue which again is equivalent with the condition that if $f \ge 0$ and $g \ge 0$, then $(f, e^{-tH}g) = 0$ for all t implies that $f = 0$ or $g = 0$. Take f and g to be characteristic functions for measurable sets A and B . Then for $s \le t$

$$\Pr\{\xi(s) \in A \ \& \ \xi(t) \in B\} = (\chi_A, e^{-(t-s)H}\chi_B) . \qquad (3.32)$$

Now we have that if (3.32) is zero for all t, then either A or B has measure zero which is to say that the stochastic process $\xi(t)$ is ergodic. We also get that if $\xi(t)$ is ergodic, then e^{-tH} is ergodic.

Since in the decomposition

$$e^{-tH} = \int_V e^{-tH_v} \, dv \tag{3.33}$$

the semigroup e^{-tH_v} is ergodic, (3.33) gives the ergodic decomposition of the Markov semigroup e^{-tH}. But by what is said above we then have that

$$\mu(A) = \int_V \mu(A|v) dv \tag{3.34}$$

is the ergodic decomposition of the measure μ with respect to the action of the Markov process $\xi(t)$.

Example 3.1

Let K be one dimensional, i.e. $Q = K = Q' = R$ (the real line) and let $d\mu = (\pi)^{-\frac{1}{2}} P_2(x)^2 e^{-x^2} dx$ where $P_2(x)$ is the properly normalized second Hermite polynomial, i.e.

$$d\mu = \tfrac{1}{2} \pi^{-\frac{1}{2}} (2x^2-1)^2 e^{-x^2} dx$$

and $\mu \in \mathscr{P}_1$. We then have that $d\mu = \varphi^2 \, dx$, where φ is the third lowest eigenfunction of the operator $-\Delta + x^2$. In fact $(-\Delta + x^2)\varphi = 5\varphi$ so that by (2.6)

$$H = -\Delta + x^2 - 5$$

when applied to functions of the form $f \cdot \varphi$ with smooth f. Since φ has simple zeros at $x = \pm \tfrac{1}{2}\sqrt{2}$, we actually find also that $H = -\Delta_0 + x^2 - 5$ in $L_2(dx)$, where Δ_0 is the Laplacian with Dirichlet boundary conditions on $x = \pm \tfrac{1}{2}\sqrt{2}$. Hence

$$d\mu = \theta(-\tfrac{1}{2}\sqrt{2}-x)\varphi^2 dx + \theta(x+\tfrac{1}{2}\sqrt{2})\theta(\tfrac{1}{2}\sqrt{2}-x)\varphi^2 dx + \theta(\tfrac{1}{2}\sqrt{2}-x)\varphi^2 dx \quad (3.35)$$

is the ergodic decomposition of $d\mu$ given by (3.33) in this case. The corresponding decomposition of H and e^{-tH} is given by

$$L_2(dx) = L_2(-\infty,-\tfrac{1}{2}\sqrt{2}) \oplus L_2(-\tfrac{1}{2}\sqrt{2},\tfrac{1}{2}\sqrt{2}) \oplus L_2(\tfrac{1}{2}\sqrt{2},\infty), \quad (3.36)$$

where in each component $H = -\Delta_0 + x^2 - 5$, Δ_0 being the Laplacian with Dirichlet boundary conditions for each component.

We shall call the ergodic decomposition (3.34) of μ with respect to the action of the Markov process $\xi(t)$ "the T-ergodic decomposition". Thus we have that the T-ergodic decomposition of μ is just the decomposition of μ into its conditional probability measures $\mu(\cdot/v)$ conditioned with respect to the σ-algegra generated by the functions which are eigenfunctions with eigenvalue zero for H.

Since we know already that the Q-invariant functions are eigenfunctions with eigenvalue zero for H , we see that the T-ergodic decomposition is a finer decomposition than the Q-ergodic decomposition given in (2.24), and the example 3.1 indicates that normally the T-ergodic decomposition is strictly finer than the Q-ergodic decomposition.

Let now μ be a Q-quasi invariant probability measure on Q'. Let P_q be the orthogonal projection onto q in K , and let $\rho(x|\eta)$ for $x \in P_q K$ and $\eta \in (1-P_q)Q'$ be the conditional probability density in (3.5). We may identify $P_q K$ with the real line R . So that for $A \subset P_q K$ and $B \subset (1-P_q)Q'$ we have

$$\mu(A \times B) = \int_B (\int_A \rho(x|\eta)dx)d\nu(\eta) . \quad (3.37)$$

We shall say that μ is <u>strictly positive</u> if $\rho(x|\eta)$ are bounded

away from zero on compacts in R for ν-almost all η .

Theorem 3.5

If u is strictly positive and $u \in \mathscr{P}_1(Q')$ then the T-ergodic decomposition and the Q-ergodic decompositions are identical.

Proof

Let $A \subset Q'$ be a subset that is measurable with respect to the σ-subalgebra generated by the eigenfunctions corresponding to the eigenvalue zero of H . Then as we have seen the characteristic function χ_A is an eigenfunction of eigenvalue zero of H. Since $H = \nabla^*\nabla$ we therefore have that $\chi_A \in D(\nabla)$ and $\nabla\chi_A = 0$. In particular $q \cdot \nabla\chi_A = 0$, so that

$$\int |q \cdot \nabla\chi_A|^2 d\mu = 0 \ . \tag{3.38}$$

Let now $q^2 = 1$ and $\chi_A = \chi_A(xq \oplus \eta)$ with $(q,\eta) = 0$. Since

$$\int |q \cdot \nabla f|^2 \, d\mu = \int_{R^\perp} (\int_R | \tfrac{d}{dx} f \, (xq + \eta)|^2 \, \rho \, (x|\eta)dx)d\nu(\eta) , \tag{3.39}$$

we see by (3.38) and the fact that $\rho(x|\eta)$ is bounded away from zero on compacts that $\chi_A(xq + \eta)$ is independent of x for ν-almost all η . Since q was arbitrary we have that χ_A and therefore A is invariant under translations by elements in Q . Hence we have proved that the σ-algebra generated by the zero eigenfunctions is contained in the σ-algebra of Q-invariant subsets. The other direction was proved in theorem 2.4. This proves the theorem. $\qquad\square$

We shall say that a quasi invariant probability measure μ is analytic if 1 is an analytic vector for $\pi(q)$ for any $q \in Q$, and with this notation we have the following criteria.[12)]

Theorem 3.6

Let μ be analytic, and such that $\pi(q)^n \cdot 1$ is in the domain o $q \cdot v$. Then we have that u is strictly positive.

Proof: That 1 is an analytic vector for $\pi(q)$ is by definition to say that there are some $r > 0$ depending on q, such that

$$\|\pi(q)^n \cdot 1\| \leq r^{-n} \, n! \, . \tag{3.40}$$

Let now $q \in Q$ with $q^2 = 1$ and $\rho(x|\eta)$ for $(\eta, q) = 0$ be given by (3.37). Then we have the direct decomposition

$$L_2(Q', d\mu) = \int_{R^\perp} L_2(\rho(x|\eta) dx) d\nu(\eta) \, , \tag{3.41}$$

where $R^\perp$ is the subspace of Q' orthogonal to q. This decomposition reduces $V(tq)$ and therefore also $\pi(q)$, so that

$$\pi(q) = \int_{R^\perp} \pi_\eta(q) d\nu(\eta) \, . \tag{3.42}$$

We shall see that 1 is an analytic function also for $\pi_\eta(q)$, inasmuch as

$$(f, V(tq)1) = \int_{R^\perp} [\int \overline{f}(xq+\eta)\varphi(x|\eta) \cdot \varphi(x+t|\eta) dx] d\nu(\eta) \tag{3.43}$$

where $\varphi(x|\eta) = \rho(x|\eta)^{\frac{1}{2}}$.

(3.42) is analytic in t for $|t| < r$, so let Γ be any smooth closed curve in the disk $|z| < r$. Then the integral of (3.43) with respect to t around Γ is zero. So by the Fubini theorem

$$\int_{R^\perp} [\int_R \overline{f}(xq + \eta)\varphi(x|\eta)(\int_\Gamma \varphi(x+t|\eta) dt) dx] d\nu(\eta) = 0 \, . \tag{3.44}$$

Since f is arbitrary in $L_2(d\mu)$ and $\varphi(x|\eta) > 0$ for almost

all x and ν - almost all η we have that

$$\int_\Gamma \varphi(x + z \,|\, \eta)\,dz = 0 \qquad\qquad (3.45)$$

for almost all x and ν-almost all η . Hence $\varphi(x + z \,|\, \eta)$ is analytic for $|z| < r$ for all x and ν-almost all η, where r is given in (3.40). So that $\varphi(z \,|\, \eta)$ is analytic in a strip of with $2r$ around the real z axis.

Further more we have that $\pi(q)^n \cdot 1$ is in the domain of $q \cdot \nabla$. Using now the direct decomposition (3.42) we have

$$\int \| q \cdot \nabla_\eta \pi_\eta(q)^n \cdot 1 \|^2 d\nu(\eta) = \| q \cdot \nabla \cdot \pi(q)^n 1 \|^2 , \qquad (3.46)$$

for
so that,$/\nu$-almost all η $\| q \cdot \nabla_\eta \pi_\eta(q)^n \cdot 1 \| < \infty$.

However

$$\| q \nabla_\eta \pi_\eta(q)^n \cdot 1 \|^2 = \int_R \left(\left(\frac{\varphi^{(n)}(x \,|\, \eta)}{\varphi(x \,|\, \eta)} \right)' \right)^2 \varphi^2(x \,|\, \eta)\,dx$$

$$\qquad\qquad (3.47)$$

$$= \int_R \left| \frac{\varphi^{(n+1)}(x \,|\, \eta)}{\varphi(x \,|\, \eta)} - \frac{\varphi'(x \,|\, \eta)}{\varphi(x \,|\, \eta)} \cdot \frac{\varphi^{(n)}(x \,|\, \eta)}{\varphi(x \,|\, \eta)} \right|^2 \varphi^2(x \,|\, \eta)\,dx .$$

Now we have that $\dfrac{\varphi^{(n+1)}}{\varphi} \in L_2(\varphi^2(\cdot \,|\, \eta))$ for ν-almost all η since

$$\int_{R^\perp} \int_R \left(\frac{\varphi^{(n+1)}(x \,|\, \eta)}{\varphi(x \,|\, \eta)} \right)^2 \varphi(x \,|\, \eta)^2 dx \; d\nu(\eta) = \| \pi(q)^{n+1} \cdot 1 \|^2 , \qquad (3.48)$$

which is finite by assumption. Since by (3.47) and (3.46) the difference is in $L_2(\varphi^2(x \,|\, \eta)\,dx)$ for ν-almost all η we have that

$$\frac{\varphi'(x \,|\, \eta)}{\varphi(x \,|\, \eta)} \cdot \frac{\varphi^{(n)}(x \,|\, \eta)}{\varphi(x \,|\, \eta)} \in L_2(\varphi^2(x \,|\, \eta)\,dx) \qquad\qquad (3.4\;)$$

for ν-almost all η . Now since $\varphi(x \,|\, \eta)$ is analytic in x we have that the zeros of φ are isolated and of finite order.

Let $\varphi(a|\eta) = 0$, then there is an n such that $\varphi^{(n)}(a|\eta) \neq 0$.

By (3.49) $\frac{\varphi'(x|\eta)}{\varphi(x|\eta)} \cdot \varphi^{(n)}(x|\eta) \in L_2(dx)$ and since $\varphi^{(n)}(x|\eta) \neq 0$

in a neighborhood of a , we get that $\frac{\varphi'(x|\eta)}{\varphi(x|\eta)}$ is in $L_2(dx)$

near a . Let n be the lowest value such that $\varphi^{(n)}(a|\eta) \neq 0$,

then $\varphi(x|\eta) \sim c(x-a)^n$ near a . From this we get that

$$\frac{\varphi'(x|\eta)}{\varphi(x|\eta)} \notin L_2(dx)$$

near $x = a$. Hence we have that $\varphi(x|\eta) > 0$ for all x , and

this proof goes for ν-almost all η . Since $\varphi(x|\eta)$ is ana-

lytic in x , it is therefore bounded away from zero on compacts.

This proves the theorem. $\square$

Theorem 3.7

Let $u \in \mathscr{P}_1(Q')$ and assume that zero is separated from the

rest of the spectrum of H by a positive distance $m > 0$, where

H is the corresponding Dirichlet operator.

Let $ad\,\pi(q)(H) = [\pi(q),H]$ and let us assume that for any

$q \in Q$ there is a constant $c_q > 0$ depending only on q such that

$$\| (H+1)^{-\frac{1}{2}}\, ad^n\, \pi(q)(H)(H+1)^{-\frac{1}{2}} \| \leq c_q^n$$

for all $n = 1,2,3,\ldots$. Then for any vector v such that

$Hv = 0$ we have that

$$\| (H+1)^{\frac{1}{2}}\, \pi(q)^n\, v \| \leq n! \, (\frac{m+1}{m}\, ec_q)^n \| v \| \ .$$

In particular we get that v is an analytic vector for $\pi(q)$,

and u is analytic and strictly positive.

Proof: For any n we have the following algebraic relation

$$H\pi(q)^n = \pi(q)^n H - \sum_{j=1}^{n} \binom{n}{j} ad^j\, \pi(q)(H)\pi(q)^{n-j} \ . \qquad (3.50)$$

So if $Hv = 0$ we get by the assumptions of the theorem that

$$\| (H+1)^{-\frac{1}{2}} H\pi(q)^n v \| \leq \sum_{j=1}^{n} \binom{n}{j} c_q \| (H+1)^{\frac{1}{2}} \pi(q)^{n-j} v \| . \qquad (3.51)$$

Let $m > 0$ be the separation of zero from the rest of the spectrum of H, then, since $H\pi(q)^n v$ is in the subspace orthogonal to the zero eigenspace and $(H+1)H^{-1}$ is normbounded by $\frac{m+1}{m}$ on that subspace, we have

$$\| (H+1)^{\frac{1}{2}} \pi(q)^n v \| \leq \frac{m+1}{m} \sum_{j=1}^{n} c_q^j \binom{n}{j} \| (H+1)^{\frac{1}{2}} \pi(q)^{n-j} v \| . \qquad (3.52)$$

Let us now assume that

$$\| (H+1)^{\frac{1}{2}} \pi(q)^k v \| \leq k! \left(\frac{m+1}{m} e\, c_q \right)^k \| v \| , \qquad (3.53)$$

which is obviously true for $k = 0$. Then by (3.52) we have that

$$\| (H+1)^{\frac{1}{2}} \pi(q)^{k+1} v \| \leq \frac{m+1}{m} c_q^{k+1} \left(\frac{m+1}{m} \right)^k e^k (k+1)! \sum_{j=1}^{k+1} \frac{1}{j!} \| v \|$$

$$\| (H+1)^{\frac{1}{2}} \pi(q)^{k+1} v \| \leq \left(c_q \frac{m+1}{m} \cdot e \right)^{k+1} (k+1)! \| v \| . \qquad (3.54)$$

Hence the inequality of the theorem is proved by induction. Take $v = 1$, then the inequality gives us that u is analytic and $\pi(q)^n 1$ is in the domain of $H^{\frac{1}{2}}$ hence also in the domain of $q \cdot v$. This proves the theorem. $\qquad \square$

Let dn be the normal measure associated with the real separable Hilbert space K, i.e. the integral with respect to dn is defined for all functions on K which are continuous bounded and for which $f(x) = f(Px)$ for some finite dimensional projection P, and

$$\int e^{i(x,y)}\, dn(x) = e^{-\frac{1}{2}(y,y)} . \qquad (3.5\,)$$

It is well known that the integral above is not given by a

countable additive measure on K , but however there exist suitable compactifications of K such that the finitely based continuous functions can be continued onto the compactification and the integral (3.5) is given by a countably additive measure dn on this compactification. However there is no natural choice of such a compactification, and a class of compactifications were given by Leonard Gross in the following way. For reference see [6],[7],[54],[55].

A seminorm $p(x)$ on K is said to be measurable if for any $\epsilon > 0$ there is a finite dimensional orthogonal projection P_0 such that, for any finite dimensional projection P orthogonal to P_0, we have that

$$\int_{p(Px)>\epsilon} dn(x) < \epsilon .$$
(3.55)

It is a consequence of (3.55) that $p(x)$ is a continuous seminorm on K . Moreover Gross proves that if B' is the completition of K with respect to a measurable norm, then the integral (3.54) is given by a regular measure n on B' . In fact we have the following theorem due to Gross.

Theorem 3.8

Let B be the completition of K with respect to a measurable norm. Then the integral (3.54) is given by a regular measure on B' . Moreover if $|x|$ is any measurable norm on K, then $|x|$ is a continuous norm on K and if $w(t)$ is the standard weak Wiener process on K and if B' is the completition of K with respect to $|x|$, then $w(t)$ may be realized as a stochastic process on B' with continuous paths.

For the proof of this theorem and more details about the Wiener process associated with a real separable Hilbert space see

ref. [6],[7], [54],[55]. We also remark that in fact it follows from the proof of theorem 3.8 that the standard weak Wiener process on K is continuous with respect to any measurable seminorm.

Let us now consider a Banach rigging

$$B \subset K \subset B' \qquad (3.56)$$

of the real separable Hilbert space K where B is a real separable Banach space dense in its dual B' such that B' is the completition of K with respect to a measurable norm on K . We shall refer to the rigging (3.56) shortly as a measurable Banach rigging of K . Let now $u \in \mathscr{P}_1(Q')$ where $Q \subset K \subset Q'$ is the original rigging of K , and let $q \cdot \beta(\xi)$ be the corresponding osmotic velocity, i.e.

$$\pi(q) \cdot 1 = \tfrac{1}{2} q \cdot \beta(\xi) . \qquad (3.57)$$

Let $| \ |'$ be the norm in B' , then since B is separable

$$|\beta(\xi)|' = \sup_n \frac{|q_n \cdot \beta(\xi)|}{|q_n|} \qquad (3.58)$$

is a measurable function,where $\{q_n\}$ is a dense countable set in Q that is dense in B , and $| \ |$ is the norm in B .

We then have the following theorem:

Theorem 3.9

Let $B \subset K \subset B'$ be a measurable rigging of K such that $Q \subset B \subset K \subset B' \subset Q'$. Let $u \in \mathscr{P}_1(Q')$ and $\beta(\xi)$ and $\xi(t)$ the corresponding osmotic velocity and Markov process and let us assume that $|\beta(\xi)|'$ is bounded, where $| \ |'$ is the norm in B' . Then $\xi(t)$ is continuous in the B' norm, i.e.,for any t and s , $\xi(t) - \xi(s)$ is in B' and $|\xi(t) - \xi(s)|' \to 0$ as $t \to s$, for almost all paths.

Remark: We may conclude that $\xi(t)$ is a continuous process with values in B' only in the case where B' has u-measure 1 .

Proof: From theorem 3.3 we know that in a weak sense

$$\xi(t) - \xi(s) = \int_s^t \beta(\xi(\tau))d\tau + w(t) - w(s) . \qquad (3.59)$$

By theorem 3.8 we have that $|w(t) - w(s)|'$ goes to zero as $t \to s$, and the conclusion of the theorem then follows by the triangle inequality for the norm. This proves the theorem. $\square$

Remark: A corresponding theorem for the finite dimensional case was given by Stroock and Varadhan in [35].

4. Applications to two-dimensional quantum fields theoretical models

In Section 4 of Ref. [1] we considered the so called weak coupling $P(\varphi)_2$ models of quantum field theory. We shall here continue that discussion, using results and methods from the previous sections of this paper. General references for these $P(\varphi)_2$ models are e.g. [38],[39],[56]. The models are given by a measure μ^*, the so called physical vacuum, on the space $\mathscr{S}'(R^2)$ of tempered distributions over R^2. Let $\langle \xi^*, \delta_0 \vee \varphi \rangle$, where δ_0 is the δ-distribution concentrated at $t = 0$ and $\varphi \in \mathscr{S}(R)$, be the time zero field. In the usual way we identity it with the distribution $\langle \xi, \varphi \rangle$, $\varphi \in \mathscr{S}(R)$, $\xi \in \mathscr{S}'(R)$ and identify the restriction μ^* to the σ-algebra generated by the time zero fields $\langle \xi^*, \delta_0 \times \varphi \rangle$ with a measure μ on the Borel subsets of $\mathscr{S}'(R)$. In this way the closed subspace $E_0 L_2(d\mu^*)$ of $L_2(d\mu^*)$ spanned by the time zero fields is identified with $L_2(d\mu)$.

For the applications in this section the rigging $Q \subset K \subset Q'$ of the preceding sections is to be taken with Q, K, Q' equal respectively to the real subspaces of $\mathscr{S}(R)$, $L_2(R)$, $\mathscr{S}'(R)$, as in Section 4 of Ref. [1]. Thus in this case we have a nuclear rigging.

In Theorem 4.2 of Ref. [1] we proved that $u \in \mathscr{P}_1(\mathscr{S}'(R))$, i.e. the function 1 in $L_2(d\mu)$ is in the domain of the infinitesimal generator $\pi(\varphi)$ of the one parameter unitary group of translations by $t\varphi$, $t \in R$, $\varphi \in \mathscr{S}(R)$ in $L_2(d\mu)$. $\pi(\varphi)$ is the so called time zero canonical field momentum. Moreover it follows from Theorem 4.2 of Ref. [1] that the physical Hamiltonian, i.e. the infinitesimal generator of time translations for the Wightman models considered, restricted to $L_2(d\mu)$ coincides[13] on the dense subset

FC^2 with the Dirichlet operator $H = \nabla^*\nabla$ associated with the Dirichlet form $\int \nabla\bar{f} \cdot \nabla f d\mu$ given by μ, according to Theorem 2.1. This is so since H coincides on FC^2 with the diffusion operator given by μ, in the sense of Theorem 2.7 of Ref. [1].[14]

For the Dirichlet operator H the results of Sections 2, 3 hold, in particular Theorems 2.1, 2.2, 2.4, 3.2, 3.3, 3.4. We shall now see that the measure μ is strictly positive, so that in particular, by Theorem 3.5, the T-ergodic and the Q-ergodic decompositions for the Markov semigroup e^{-tH} and Markov diffusion process generated by H are equivalent. The proof of the strict positivity of μ is as in Theorem 3.7, so that we first derive the estimates used in that theorem.

Let H_ℓ be the Hamiltonian of the space cut-off weak coupling $P(\varphi)_2$ model i.e.

$$H_\ell = H_o + V_\ell \; .\qquad\qquad (4.1)$$

H_o is the free Hamiltonian, shown in Theorem 4.0 of Ref. [1] to coincide with the Dirichlet operator associated with the Gaussian measure μ_o on the real $\mathscr{S}'(R)$ space, with Fourier transform

$$\exp[-\frac{1}{4} \int_R (p^2+m^2)^{-\frac{1}{2}} |\hat{\varphi}(p)|^2 dp] \; ,$$

with $\qquad \hat{\varphi}(p) = (2\pi)^{-\frac{1}{2}} \int e^{-ipx} \varphi(x)dx \; .\qquad\qquad (4.2)$

V_ℓ is the space cut-off interaction i.e. the real function in $L_2(d\mu_o)$ given by

$$V_\ell = :p: (\chi_\ell) \; ,\qquad\qquad (4.3)$$

where χ_ℓ is the characteristic function of the interval $[-\ell,+\ell]$ and $p(x)$ is the real polynomial

$$p(x) = \sum_{k=o}^{2n} a_k x^k\qquad\qquad (4.4)$$

and

$$:p:(h) = \sum_{k=0}^{2n} a_k :\xi^k:(h) \tag{4.5}$$

for any $h \in L_2(R)$, $:\xi^k:$ being the k-th Wick power of ξ.

In the weak coupling $P(\varphi)_2$ models it is assumed $a_{2n} > 0$ and sufficiently small coefficients a_k. It is well known that $H_o + V_\ell$ is essentially self-adjoint, bounded from below, with an isolated simple eigenvalue E_ℓ such that $H_\ell \geq E_\ell$, see e.g. [56]. It was shown in Lemma 4.1 of Ref. [1], using that the operation of taking derivatives with respect to the fields commutes with the Wick ordering, that

$$i[\pi(\varphi), H_\ell] = :p': (\chi_\ell \varphi) + \langle \xi, (-\Delta+m^2)\varphi \rangle , \tag{4.6}$$

where p' is the derivative of the polynomial p. By the same methods we obtain further

$$i^n ad^n \pi(\varphi)(H_\ell) = :p^{(n)}: (\chi_\ell \varphi^n) + L_n(\varphi) , \tag{4.7}$$

with $p^{(n)}$ the n-th derivative of p and $L_n(\varphi) = 0$ for $n \geq 2$, $L_2(\varphi) = \langle \varphi, (-\Delta+m^2)\varphi \rangle$, $L_1(\varphi) = \langle \xi, (-\Delta+m^2)\varphi \rangle$.

By the estimates of Ref. [57] and a resolution of the identity we then have the estimate

$$\pm i^n ad^n \pi(\varphi)(H_\ell) \leq C_1^n(H_\ell - E_\ell + 1) , \tag{4.8}$$

where C_1 is independent of ℓ. This estimate, for $n = 1$, was used in Theorem 4.1 of Ref. [1] to prove that the measure μ_ℓ is in $\mathcal{P}_1(\mathcal{S}'(R))$, where $d\mu_\ell = g_\ell^2 d\mu_o$, with g_ℓ such that $\Omega_\ell = g_\ell \Omega_o$, Ω_ℓ being the eigenvector to the eigenvalue E_ℓ of H_ℓ. We also recall, incidentally, that the corresponding osmotic velocity has uniformly bounded $L_2(d\mu_\ell)$ norms.

The left hand side of (4.8) being independent of ℓ for large ℓ

and φ of bounded support, by a standard procedure, see [58],
the estimate (4.8) extends to the infinite volume limit $\ell \to \infty$,
so that

$$\pm \; i^n \; ad^n \; \pi(\varphi)(H_{ph}) \leq C_1^n(H_{ph}+1) \; , \tag{4.9}$$

where H_{ph} is the physical Hamiltonian of the infinite volume
weak coupling $P(\varphi)_2$ model. Since moreover, by a well known re-
sult of Glimm, Jaffe and Spencer [38], H_{ph} has a gap $m^* > 0$
at the bottom of its spectrum , we see from the proof of
Theorem 3.7 that (4.9) implies

$$\|(H_{ph}+1)^{\frac{1}{2}} \; \pi(\varphi)^n\Omega\| \leq n! \, (\frac{m^*+1}{m^*} \, e \, C_1)^n \; , \tag{4.10}$$

where $\Omega(\xi) \equiv 1$ and $\| \; \|$ is the norm in $L_2(d\mu)$.
In particular we obtain that the physical vacuum is an analytic
vector for $\pi(\varphi)$ for all $\varphi \in \mathscr{S}(R)$. In addition we remark that
the estimates (4.9), (4.10) hold also with H_{ph} replaced by the
diffusion operator H_d associated to μ in the sense of Theorem
2.7 of Ref. [1].
(We recall that H_d is the Friedrichs extension of the Dirichlet
operator on FC^2.) This is so since $H_{ph} = H_d$ on FC^2, so that
(4.9) holds for H_d on FC^2, and thus by continuity to the whole
form domain of H_d. Then from (4.10) written for H_d instead of
H_{ph} we obtain that $\pi(\varphi)^n\Omega$ is in the domain of $\varphi \cdot \nabla$ for all
$\varphi \in \mathscr{S}(R)$. From Theorem 3.6 we then have that the measure μ
is strictly positive.
We now remark that all the arguments leading to this result can be
repeated in the case of the $P(\varphi)_2$ models with Dirichlet boundary
conditions ([40]-[43]) which are such that the physical Hamiltonian has zero as
an isolated but not necessarily simple eigenvalue. Examples of

such interactions are given by $p(x) = ax^4 + bx^2 - \mu x$ with $a > 0$, $\mu \neq 0$ (no restriction on the size of the coefficients) [42]. The basic reason why such models can be treated by the same method is that the estimate used for proving (4.8) is also available for such models [59].

We have thus proven the following

Theorem 4.1

For the weak coupling $P(\varphi)_2$ models and for the $P(\varphi)_2$ models with Dirichlet boundary conditions and such that zero is an isolated but not necessarily simple eigenvalue for the energy, the physical vacuum (as a measure on $\mathscr{S}'(\mathbb{R}^2)$) restricted to the σ-algebra generated by the time zero fields, is an analytic strictly positive $\mathscr{S}(\mathbb{R})$-quasi invariant measure in the sense of sections 2, 3, with respect to the nuclear rigging $\mathscr{S}(\mathbb{R}) \subset L_2(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$. The Dirichlet operator $\nabla^*\nabla$ in $L_2(d\mu)$ associated with the measure μ coincides on the dense domain FC^2 with the physical Hamiltonian. For this Dirichlet operator the results of Sections 2, 3 hold, in particular Theorems 2.2, 2.4, 3.2, 3.3, 3.4. Especially, μ is the invariant measure of a Markov diffusion process $\xi(t)$ solving on $\mathscr{S}'(\mathbb{R})$ the stochastic differential equation

$$d\xi(t) = \beta(\xi(t))dt + dw(t) ,$$

where β is the osmotic velocity to μ and w is the standard Wiener process on $L_2(\mathbb{R})$. The $\mathscr{S}(\mathbb{R})$-ergodic and time-ergodic decompositions for the measure μ, the Dirichlet form, the Dirichlet operator and the Markov process are equivalent. $\square$

We shall now derive the correspondent results for the case of the exponential interactions models of [44], [45]. In these models

the space cut-off interaction is given by

$$V_\ell = \, :v: \, (\chi_\ell) \tag{4.11}$$

with $$v(x) = \int e^{\alpha x} \, d\nu(\alpha) \tag{4.12}$$

and $:v:(h) = \int :e^{\alpha\xi}:(h) d\nu(\alpha)$ for any $h \in L_2(R) \cap L_1(R)$, where $d\nu$ is an arbitrary even finite measure with support in $[-\alpha_o, \alpha_o]$ with $\alpha_o < \sqrt{2\pi}$. It was shown by one of us in [44] that V_ℓ is positive, $H_\ell = H_o + V_\ell$ essentially self-adjoint on a common domain of H_o and V_ℓ and that H_ℓ has an isolated eigenvalue $E_\ell \geq 0$ with eigenvector Ω_ℓ at the bottom of the spectrum. Similarly, as in the proof of Lemma 4.1 of Ref. [1] and of the formula (4.7) above we have, using the method of [45] for the convergence of series involved

$$i^n \, \text{ad}^n \, \pi(\varphi)(H_\ell) = \, :v^{(n)}:(\chi_\ell \varphi^n) + L_n \, , \tag{4.13}$$

where $v^{(n)}$ is the n-th derivative of the function v , so that

$$:v^{(n)}:(\chi_\ell \varphi^n) = \int \alpha^n :e^{\alpha\xi}:(\chi_\ell \varphi^n) d\nu(\alpha) \tag{4.14}$$

We have now, using that $:e^{\alpha\xi}:(h) \geq 0$ for $h \geq 0$ in $L_2(R) \cap L_1(R)$ ([44],[45]):

$$\pm \, :v^{(n)}:(\chi_\ell \varphi^n) \leq C_1^n (H_\ell - E_\ell + 1) \, , \tag{4.15}$$

with $C_1^n = \alpha_o^n \, \|\varphi\|_\infty^n$.

We remark that, for $n = 1$, this estimate yields the correspondent of Theorem 4.1 of [1], i.e. that $\mu_\ell \in \mathcal{P}_1(\mathcal{S}'(R))$, where $d\mu_\ell$ is the measure corresponding to Ω , and that the components of the corresponding osmotic velocity β_ℓ have $L_2(d\mu_\ell)$ norms bounded uniformly in ℓ . By the same reasons as for (4.8), the bound (4.15) extends to infinite volume limit $\ell \to \infty$, proven to exist

in [45] for any $\alpha_o < \frac{4}{\sqrt{\pi}}$, so that:

$$\pm i^n \, ad^n \, \pi(\varphi)(H_{ph}) \leq C_1^n(H_{ph} + 1) \, , \qquad (4.16)$$

where H_{ph} is the physical Hamiltonian, i.e. the infinitesimal generator of time translations, of the Wightman theory constructed in [45].

We can now proceed in the same way as we did for the case of $P(\varphi)_2$ interactions, from (4.9) on. This is so since H_{ph} has a mass gap (actually larger than the free mass occurring in μ_0), as proven in [45], and Theorem 4.2 of Ref. [1] holds for the case of the exponential interactions also, since it follows from Theorem 4.1 of Ref. [1] and general theorems of [1], and we already verified that Theorem 4.1 of [1] carries over to our present case of exponential interactions. We have thus proven the following

Theorem 4.2

For the exponential interaction models of [44], [45], the physical vacuum μ restricted to the σ-algebra generated by the time zero fields is an analytic strictly positive $\mathscr{S}(R)$-quasi invariant measure, with respect to the nuclear rigging

$$\mathscr{S}(R) \subset L_2(R) \subset \mathscr{S}'(R) \, .$$

The restriction to FC^2 of the physical Hamiltonian of the Wightman theory given by these models coincides with the Dirichlet operator generated by μ , in the sense of Theorem 2.1 above, and with the diffusion operator of Theorem 2.7 of Ref. [1]. Moreover, all results of Theorem 4.1 above hold also for these interactions.

Remark 1: All results of Theorem 4.2 above and of Section 4 of Ref. [1] hold also for the case where μ is replaced by the corresponding

measure μ_ℓ of the space cut-off exponential interactions (4.11),
with α_0 only restricted by $\alpha_0 < \sqrt{2\pi}$. The corresponding osmotic
velocities β_ℓ have $L_2(d\mu_\ell)$ norms uniformly bounded in ℓ .

Remark 2: The results of Theorem 4.2 hold also for the exponential
interactions with Dirichlet boundary conditions, also considered
in Ref. [45], [41].

Remark 3: Further results about the circle of problems discussed
in this section are given in Ref. [60].

Footnotes

1) The present paper together with the paper under Ref. [60] below constitute the reference number 4 in Ref. [1].

2) Contexts where related Dirichlet forms appear are [1], [17], [22] - [26].

3) In fact the drift coefficient β is, even for K finite dimendional, more singular (just L_2) than the ones usually considered in the theory of stochastic differential equations. See e.g. [33] - [37].

4) See also e.g. [47].

5) See also e.g. [49].

6) The measures μ in $\mathscr{P}_1(Q')$ were called "measures with regular first order derivatives" in Ref. [1].

7) The restriction of this form to $FC^2(Q')$ is what was called Dirichlet form in [1]. Note that in [1] we used the notation F^n for FC^n and $(f,f)_1$ was denoted by $(f,f)_\mu$.

8) The Friedrichs extension of the restriction of the Dirichlet operator $\nabla^* \nabla$ to the dense domain $FC^2(Q')$ of $L_2(d\mu)$ is what was called "the diffusion operator given by μ" in Ref.[1] (Th. 2.7). It is still an open question whether $FC^2(Q')$ is a core for the Dirichlet operator, in which case Dirichlet operator and diffusion operator would coincide on their whole domain.

9) This is the correspondent for the Dirichlet operator of Th. 2.7 in [1].

10) Note that about the osmotic velocity β we only used what follows from the assumption $\mu \in \mathscr{P}_1(Q')$, namely $q \cdot \beta(\xi(t)) \in L_2$. Thus, the remark in footnote 4) applies. Cases where β is linear, Lipschitz continuous or smooth were considered in [3] - [13]. One reason for our interest in results of above generality, with singular β, is that in the applications to interacting quantum fields such cases actually arise, see section 4 below and in Ref. [1].

11) This is well known, but we give nevertheless a proof for introducing methods also used later on. For references to the theorem, see e.g. § 10 of Ref. [47].

12) These criteria find applications e.g. to quantum fields in the infinite volume limit, see Section 4. For smoother cases, applicable to space cut-off or polaron models, related results are in [52], resp. [53].

13) In the sense that $(f, E_o H_{ph} E_o g) = (f, H g)$ for all f, g in FC^2.

14) It is open whether FC^2 is a core for $E_o H_{ph} E_o$ and H. If it would be so, these operators would of course coincide with the diffusion operator given by μ.

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