A Gelfand - Neumark theorem for Jordan algebras

by

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Abstract

Let A be a Jordan algebra over the reals which is a Banach space with respect to a norm satisfying the requirements: (i) $\|a \circ b\| \leq \|a\| \, \|b\|$, (ii) $\|a^2\| = \|a\|^2$, (iii) $\|a^2\| \leq \|a^2 + b^2\|$ for $a, b \in A$.
It is shown that A possesses a unique norm closed Jordan ideal J such that A/J has a faithful representation as a Jordan algebra of self-adjoint operators on a complex Hilbert space, while every "irreducible" representation of A not annihilating J is onto the exceptional Jordan algebra M_3^8.

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To the memory of David Topping

§ 1. Introduction

One of the main results in Banach algebra theory is the Gelfand-Neumark theorem which asserts that an abstractly defined B*-algebra has a faithful isometric representation as a concrete C*- algebra. The proof, which is obtained by taking the direct sum of the GNS - representations due to all states of the algebra, fails for Jordan algebras because multiplication is non-associative. Indeed, the analogous result must be false for Jordan algebras, because it appears to be impossible in any reasonalbe way to exclude the exceptional Jordan algebra M_3^8 - the hermitian 3×3 matrices over the Cayley numbers, cf. Lemma 9.4 below.

The classical representation theorem, which takes care of the exceptional case M_3^8, was proved by Jordan, von Neumann, and Wigner in 1934 [15]. They classified the finite dimensional simple Jordan algebras over the reals, which were formally real, i.e. $a^2 + b^2 + \ldots + c^2 = 0$ implies $a = b = \ldots = c = 0$. Except for M_3^8 these algebras were all represented as Jordan algebras of self-adjoint operators acting on a complex Hilbert space.

The purpose of the present paper is to prove a Jordan Banach algebra version of the theorem of Jordan, von Neumann, and Wigner.

Our assumptions will be quite close to those of Segal [25], see also [5]. We shall assume the Jordan algebra A has identity, always denoted by 1, and is a Banach space with respect to a norm $\| \ \|$ having the following three algebraic properties: if $a, b \in A$ then

i) $\|a \circ b\| \leq \|a\| \, \|b\|$,

ii) $\|a^2\| = \|a\|^2$,

iii) $\|a^2\| \leq \|a^2+b^2\|$,

where $\circ$ denotes the Jordan product. An equivalent definition is order-theoretic and states that $(A,1)$ is a complete order-unit space such that $a^2 \geq 0$ for all $a \in A$, and $-1 \leq a \leq 1 \Rightarrow a^2 \leq 1$. In analogy with the name B*-algebra we shall call a Jordan algebra as above a JB-algebra. The analogues of concrete C*-algebras have been called JC-algebras by Topping [30], and are by definition norm closed Jordan algebras of self-adjoint operators on a complex Hilbert space. The structure of JC-algebras is quite well understood, and is close to that of C*-algebras, see [11, 27, 28, 29, 30, 31].

Our main result, Theorem 9.5, asserts that the study of JB-algebras can be reduced to that of JC-algebras and the exceptional one M_3^8. More formally it states that there is a Jordan ideal J in a JB-algebra A such that A/J has a faithful isometric representation as a JC-algebra, and every "irreducible" Jordan representation of A not annihilating J is onto M_3^8.

Our proof of this result follows well known paths, but is somewhat lengthy because we have to develop the necessary techniques on the way. The proof, and thus the paper, is divided into eight parts as follows.

In § 2 we give the formal definition of a JB-algebra A and prove the basic results. In § 3 we construct the enveloping JB-algebra $\tilde{A}$ of A, which is the analogue of the second dual so successfully used in C*-algebra theory. $\tilde{A}$ turns out to be a monotone complete JB-algebra with "sufficiently many" normal states. In the following sections we let M be a JB-algebra with the same properties as $\tilde{A}$. In § 4 we study commutativity in M and the projection lattice, and in § 5 the center of M. Of main interest is the case of JB-factors, i.e. the case when the center is the scalars. If ρ is a state of A, i.e. a positive linear functional such that $\rho(1) = 1$, then its central support $c(\rho)$ can be defined in $\tilde{A}$. If we cut down $\tilde{A}$ by $c(\rho)$ we obtain a map $\varphi_\rho : A \to \tilde{A}c(\rho)$, which is a Jordan homomorphism of A onto a dense JB-algebra. φ_ρ plays part of the role of the GNS-representation in C*-algebras. If ρ is pure, then the strong closure of $\varphi_\rho(A)$ is a JB-factor.

In § 6 we develop the necessary comparison theory for idempotents in a JB-factor with the aim of proving the important halving lemma, which states that except in the simplest cases the identity can be split into two equivalent idempotents.

From the theory of JC-algebras we know that the so-called spin factors, which are the JW-factors of type I_2, see [28] or [31], have to be treated separately. This we do in § 7. Then in § 8 the other possible kinds of JB-factors are studied, and we use the halving lemma to conclude they are all Jordan matrix algebras. Thus, except for the exceptional algebra M_3^8 and the spin factors, we can construct an "honest" GNS-representation for each pure state. As a consequence we show that if ρ is a pure state, then the strong closure of $\varphi_\rho(A)$ is isomorphic to a JC-algebra, unless it is the exceptional algebra.

In order to complete the proof of the main theorem, we begin § 9 by showing that the quotient of a JB-algebra by a norm closed Jordan ideal is itself a JB-algebra. Then the desired ideal is found by letting it essentially consist of those elements in the algebra which do not satisfy the so-called s-identities of Glennie [12]. In particular, it follows that A itself is (isometrically isomorphic to) a JC-algebra if and only if all elements of A satisfy the s-identities.

The authors are indebted to professor Richard Shafer for invaluable help with the proof of the halving lemma.

§ 2. Definition and basic properties of JB-algebras.

Definition. A JB-algebra is a Jordan algebra A over the reals with identity element 1 equipped with a complete norm satisfying the following requirements for $a,b \in A$:

(2.1) $$\|a \circ b\| \leq \|a\| \|b\|$$

(2.2) $$\|a^2\| = \|a\|^2$$

(2.3) $$\|a^2\| \leq \|a^2+b^2\|$$

We recall (cf.e.g.[3]) that an order-unit space is a partially ordered normed vector space with a distinguished order unit 1 which is Archimedean in that $na \leq 1$ for $n = 1,2,\ldots$ implies $a \leq 0$, and with norm given by

(2.4) $$\|a\| = \inf\{\lambda > 0 \mid -\lambda 1 \leq a \leq \lambda 1\} .$$

Theorem 2.1. If A is a JB-algebra, then the set A^2 of all squares in A is a proper convex cone organizing A to a (norm) complete order-unit space whose distinguished order unit is the multiplicative identity element and whose norm is the given one, and such that for $a \in A$:

(2.5) $$-1 \leq a \leq 1 \text{ implies } 0 \leq a^2 \leq 1 .$$

Conversely, if A is a complete order-unit space equipped with a Jordan product for which the distinguished order-unit acts as identity element and such that the requirement (2.5) is satisfied, then A is a JB-algebra in the order-unit norm (2.4).

Proof. 1. Suppose first that A is a JB-algebra.

For given $a \in A$ the polynomials in a will form an associative subalgebra (see e.g. [13;p.36]), and by (2.1) the closure of this algebra is a commutative Banach algebra: the Banach algebra $C(a)$ generated by a and 1.

By elementary theory of Banach algebras (binomial series for square roots), the following implication is valid for $b \in A$:

(2.6) $\|b\| \leq 1 \Rightarrow 1 + b = d^2$ for some $d \in C(b)$.

We claim that for $a \in A$ the following four statements are equivalent:

(2.7) $\|\alpha 1 - a\| \leq \alpha$ for all $\alpha \geq \|a\|$

(2.8) $\|\alpha 1 - a\| \leq \alpha$ for some $\alpha \geq \|a\|$

(2.9) $a = c^2$ for some $c \in C(a)$

(2.10) $a \in A^2$.

The implication (2.7) => (2.8) is trivial.

To prove (2.8) => (2.9) we suppose that $\|\alpha 1 - a\| \leq \alpha$ for given $\alpha \geq \|a\|$. Writing $b = \alpha^{-1}a - 1$ and applying (2.6), we get $1 + b = d^2$ with $d \in C(b) = C(a)$. Defining $c = \alpha^{\frac{1}{2}} d \in C(a)$, we obtain $a = \alpha(1+b) = \alpha d^2 = c^2$.

The implication (2.9) => (2.10) is again trivial.

To prove (2.10 => (2.7) we suppose $a = c^2$. Let $\alpha \geq \|a\|$ and define now $b = -\alpha^{-1}a$. By (2.6) $1 + b = d^2$ with $d \in C(a)$. Defining $f = \alpha^{\frac{1}{2}} d$, we obtain

$$\alpha 1 - a = \alpha(1+b) = \alpha d^2 = f^2 .$$

Hence $\alpha 1 = c^2 + f^2$. Using the equation above together with (2.3) and (2.2), we now find

$$\|\alpha 1 - a\| = \|f^2\| \leq \|c^2 + f^2\| = \alpha\|1\| = \alpha .$$

To prove that A^2 is a convex cone, we only have to verify that $A^2 + A^2 \subset A^2$. To this end we consider two elements a, b in A^2 , and we write $\alpha = \|a\|$, $\beta = \|b\|$. By (2.7)

$$\|(\alpha+\beta)1 - (a+b)\| \leq \|\alpha 1-a\| + \|\beta 1-b\| \leq \alpha+\beta ,$$

and since $\alpha+\beta \geq \|a+b\|$ we can apply (2.8) to conclude that $a+b \in A^2$.

It is easy to see that the cone A^2 is proper, i.e. $A^2 \cap (-A^2) = \{0\}$. In fact, if $a^2 = -b^2$ then $a^2 = 0$ by virtue of (2.3), and then $a = 0$ by (2.2).

A partial ordering is now defined on A by writing $a \leq b$ when $b-a \in A^2$. By (2.6) the inequalities $1+a \geq 0$ and $1-a \geq 0$ are valid when $\|a\| \leq 1$. Hence we have the implication

$$(2.11) \quad \|a\| \leq 1 \Rightarrow -1 \leq a \leq 1 ,$$

from which it follows that 1 is an order unit.

To prove Archimedicity we first note that A^2 is closed, since by the mutual equivalence of (2.7) - (2.10) it can be expressed as follows:

$$A^2 = \{a \in A \mid \|a\| \geq \|\|a\|1-a\|\} .$$

Now, if $na \leq 1$ for $n = 1,2,\ldots$, then $d_n = n^{-1}1-a \in A^2$, and so

$$\mathrm{dist}(A^2,-a) \leq \|d_n+a\| = n^{-1}\|1\| = n^{-1}$$

for $n = 1,2,\ldots$, Hence $-a \in (A^2)^- = A^2$, and so $a \leq 0$.

To verify (2.5) we assume $-1 \leq a \leq 1$. By definition of the ordering $a^2 \geq 0$; so we only have to prove $a^2 \leq 1$. Now $1-a^2 = (1-a)\circ(1+a)$ with the factors at the right side both positive. By the equivalence of (2.9) and (2.10) there exist elements c, d in $C(a)$ such that $1-a = c^2$ and $1+a = d^2$. By the associativity of $C(a)$, $1-a^2 = c^2 \circ d^2 = (c\circ d)^2$. Hence

$$(2.12) \qquad a^2 + (c \circ d)^2 = 1 .$$

Thus $a^2 \leq 1$, and (2.5) is proven.

Continuing from (2.12) and making use of (2.2) and (2.3), we also find

$$\|a\|^2 = \|a^2\| \leq \|a^2 + (c \circ d)^2\| = \|1\| = 1 .$$

Hence we have proved the implication

$$(2.13) \quad -1 \leq a \leq 1 \Rightarrow \|a\| \leq 1 .$$

By (2.11) and (2.13) the order-unit norm of A coincides with the given norm, and the first part of the proof is complete.

2. Suppose next that A is a complete order-unit space and a Jordan algebra for which the distinguished order-unit is identity element, and suppose also that (2.5) is satisfied.

Consider two elements a, b in A such that $\|a\| \leq 1$ and $\|b\| \leq 1$. Now $\|\frac{1}{2}(a+b)\| \leq 1$ and $\|\frac{1}{2}(a-b)\| \leq 1$. Hence $-1 \leq \frac{1}{2}(a+b) \leq 1$ and $-1 \leq \frac{1}{2}(a-b) \leq 1$. By (2.5) $0 \leq [\frac{1}{2}(a+b)]^2 \leq 1$ and $0 \leq [\frac{1}{2}(a-b)]^2 \leq 1$. Hence

$$-1 \leq (\tfrac{a+b}{2})^2 - (\tfrac{a-b}{2})^2 \leq 1 ,$$

and so

$$\|a \circ b\| = \|(\tfrac{a+b}{2})^2 - (\tfrac{a-b}{2})^2\| \leq 1 .$$

Now we have proved that $\|a\| \leq 1$ and $\|b\| \leq 1$ imply $\|a \circ b\| \leq 1$. From this (2.1) follows.

Assume next $\|a^2\| \leq 1$. Now $a^2 \leq 1$, and since all squares are positive by (2.5), we obtain

$$a = \tfrac{1}{2}[a^2 + 1 - (a-1)^2] \leq \tfrac{1}{2}[a^2+1] \leq 1$$

and

$$a = \tfrac{1}{2}[(a+1)^2 - a^2 - 1] \geq \tfrac{1}{2}[-a^2-1] \geq -1 ,$$

which gives $-1 \leq a \leq 1$, i.e. $\|a\| \leq 1$.

Now we have proved that $\|a^2\| \leq 1$ implies $\|a\|^2 \leq 1$. From this (2.2) follows.

Finally it follows from (2.4) and (2.5) that for arbitrary $a \in A$

$$\|a^2\| = \inf\{\lambda > 0 | 0 \leq a^2 \leq \lambda 1\}$$
$$\leq \inf\{\lambda > 0 | 0 \leq a^2 + b^2 \leq \lambda 1\} = \|a^2+b^2\| .$$

This establishes the inequality (2.3), and the proof is complete. □

<u>Corollary 2.2. If A is a JB-algebra, then A is formally real, i.e. $\sum_{i=1}^{n} a_i^2 = 0$ implies $a_i = 0$ for $i = 1,\ldots,n$</u> .

<u>Proof</u>. Suppose that $\sum_{i=1}^{n} a_i^2 = 0$ and let $1 \leq k \leq n$. Since A^2 is a convex cone, there exists $b \in A$ such that $\sum_{i \neq k} a_i^2 = b^2$. By (2.2) and (2.3)

$$\|a_k\|^2 = \|a_k^2\| \leq \|a_k^2 + b^2\| = 0 ,$$

and the corollary is proved. □

Note that our axiom (2.2) is analogous to the "B^*-condition" in the theory of involutive Banach algebras, and that the above verification that A^2 is a convex cone, is similar to the original proof by Kelley and Vaught for the corresponding statement for abstract B^*-algebras [18].

Note also that our axiom (2.3) has been used before, e.g. by Arens [5] and by Segal [25] (in a slightly different version involving sums with more than two terms). By this axiom one can never decrease the norm of a square by adding another square;

a fact which was used in an essential way in the above proof that every JB-algebra is formally real. However, there exist normed Jordan algebras which are formally real and satisfy all requirements for a JB-algebra except (2.3). One such example is the real subalgebra of the disk algebra consisting of functions with real values on the real axis.

It is possible to replace our axioms (2.1)-(2.3) by equivalent systems of axioms in various ways. One possibility is to keep (2.2) and to replace (2.1) and (2.3) by the following axiom (also used by Segal in [25]):

$$\|a^2 - b^2\| \leq \max(\|a^2\|, \|b^2\|) \quad (2.14)$$

Another possibility is to keep (2.1) and (2.2) and to replace (2.3) by the requirement that $1 + a^2$ be an invertible element in the Banach algebra $C(a)$ for all $a \in A$. The equivalence of the various approaches is proved by arguments similar to those in the proof of Theorem 2.1, and we omit the details.

Examples of JB-algebras are the so-called JC-algebras, i.e. the norm closed Jordan algebras of self-adjoint operators on a complex Hilbert space, and the exceptional algebra M_3^8 consisting of all hermitian 3×3-matirces over the Cayley numbers, see [26].

We will now establish some of the basic properties of JB-algebras.

Proposition 2.3. If A is a JB-algebra and M is a closed associative subalgebra containing 1, in particular if $M = C(a)$ for $a \in A$, then M is isometrically (order- and algebra-) isomorphic to $C(X)$ for some compact Hausdorff space X.

Proof. Note first that if a, b are positive elements of M, then it follows from the equivalence of (2.9) and (2.10) that there exist $c \in C(a) \subset M$ and $d \in C(b) \subset M$ such that $a = c^2$ and $b = d^2$. By the associativity of M $a \circ b = (c \circ d)^2$. Hence we have the implication:

(2.15) $a \geq 0,\ b \geq 0$ and $a, b \in M \Rightarrow a \circ b \geq 0$.

Now the proposition will follow by application of Stone's Theorem on functional representation of partially ordered algebras (see [16; §3]) . □

Recall that an element a of a Jordan algebra A with identity is called invertible with b as an inverse if $a \circ b = 1$ and $a^2 \circ b = a$ (cf. [13;p.51]). This notion reduces to the customary one for special algebras, i.e. for Jordan algebras which can be embedded in an associative algebra with product ab in such a way that $a \circ b = \frac{1}{2}(ab+ba)$, by virtue of the following equivalence (proved in [13;p.51]):

(2.16) $a \circ b = 1,\ a^2 \circ b = a \iff ab = ba = 1$.

Proposition 2.4. Let a, b be elements of a JB-algebra A. Then the following are equivalent:

(2.17) a is invertible with inverse b in the Jordan algebra A,

(2.18) a is invertible with inverse b in the Banach algebra $C(a)$.

Proof. 1. Assume first (2.17). By the Shirshov-Cohn Theorem [13;p.48] the Jordan subalgebra M_0 generated by a, b

and 1 is special. By (2.16) $ab = ba = 1$; in particular a, b are commuting elements with respect to the associative product of the special algebra M_0 . By definition any two elements c, d of M_0 are ("Jordan-" and hence also "associative-") polynomials in a, b and 1 ; since a and b commute, the two polynomials c and d will also commute, i.e. $cd = dc$, and therefore $c \circ d = cd$. Hence the two products defined in M_0 will coincide, and M_0 must be an associative subalgebra of the given Jordan algebra A .

By continuity of the Jordan product (axiom (2.1)) the closure M of M_0 is also an associative subalgebra of A , and by Proposition 2.3 $M \cong C(X)$ for some compact Hausdorff space X . Now b is the inverse of a in the Banach algebra $M \cong C(X)$, and it follows by elementary theory of commutative Banach algebras that b is a norm limit of polynomials in a and 1 , i.e. $b \in C(a)$.

2. Assume next (2.18). Then $a \circ b = 1$, and by associativity of $C(a)$ also $a^2 \circ b = a \circ (a \circ b) = a$. This completes the proof. □

For a given element a of a JB-algebra A we define the spectrum of a to be the set $\sigma(a)$ of all $\lambda \in \mathbb{R}$ such that $a - \lambda 1$ is not invertible. By Proposition 2.4 $\sigma(a)$ is the same as the spectrum of a with respect to the Banach algebra $C(a)$. Hence the spectrum of an element a of a JB-algebra A will enjoy all properties of spectra in real Banach algebras isomorphic to $C(X)$. In particular $\sigma(a)$ is a non-empty compact subset of $\mathbb{R}$ such that:

(2.19) $$\|a\| = \sup_{\lambda \in \sigma(a)} |\lambda| ,$$

(2.20) $a \geq 0$ iff $\sigma(a) \subset \mathbb{R}^+$,

(2.21) for $a \geq 0$, a is invertible iff there exists $\lambda > 0$ such that $a \geq \lambda 1$.

Moreover, we can identify the compact set X in the isomorphism $C(X) \cong C(a)$ with the spectrum of the generator a. Then the isomorphic image of the identity function $(\xi \mapsto \xi)$ on $\sigma(a)$ will be a itself, and more generally the image of any polynomial π will be $\pi(a)$. For an arbitrary $\varphi \in C(\sigma(a))$ the isomorphic image of φ is denoted by $\varphi(a)$. Thus, we have a well behaved (continuous) functional calculus in A.

An important composition in a Jordan algebra is given by the <u>Jordan triple product</u> [13;p.36]:

$$(2.22) \quad \{abc\} = (a\circ b)\circ c - (c\circ a)\circ b + (b\circ c)\circ a ,$$

which reduces to the following if the algebra is special with $a\circ b = \frac{1}{2}(ab + ba)$:

$$(2.23) \quad \{abc\} = \tfrac{1}{2}(abc + cba) .$$

In particular, $\{aba\} = aba$ in a special Jordan algebra. In any Jordan algebra we shall denote the linear mapping $x \mapsto \{axa\}$ by U_a. Thus

$$(2.24) \quad U_a x = 2a\circ(a\circ x) - a^2\circ x .$$

The following two identities are valid in any Jordan algebra:

$$(2.25) \quad \{\{aba\}x\{aba\}\} = \{a\{b\{axa\}b\}a\} ,$$

$$(2.26) \quad \{bab\}^2 = \{b\{ab^2a\}b\} .$$

We shall indicate the proofs, since they provide an opporunity to present a general method which will be used repeatedly in the sequel. First one applies (2.13) to verify that the identities hold in any special algebra. Then one makes use of Macdonald's Theorem [13;p.41] by which every polynomial Jordan identity in three variables and 1 which is of degree at most one in one of these and which holds for all special Jordan algebras, is valid for all Jordan algebras.

For later references we state the following result whose proof involves (2.25) in an essential way (see [13;p.52] for details):

Proposition 2.5. An element a of a Jordan algebra is invertible iff the inverse operator U_a^{-1} exists; in this case a has the unique inverse $b = U_a^{-1}a$ and $U_b = U_a^{-1}$.

By (2.25) the operator identity $U_{\{aba\}} = U_a U_b U_a$ holds for every pair a, b of elements in a Jordan algebra, and therefore $U_{\{aba\}}^{-1}$ exists iff U_a^{-1} and U_b^{-1} exist. Hence we have the following:

Corollary 2.6. Let a, b be elements of a Jordan algebra. Then $\{aba\}$ is invertible iff a and b are both invertible.

Our next result will be an important tool in the sequel. But first some notation: The set of invertible elements of a JB-algebra A will be denoted by A_o, the set of positive elements of A by A^+ (in fact $A^+ = A^2$), and the set of positive elements of A_o by A_o^+. Note that A_o^+ is a convex subset of A by (2.21).

Proposition 2.7. For every element a of a JB-algebra A the operator U_a is positive, i.e. $U_a(A^+) \subset A^+$.

Proof. 1. We shall first prove that if $a \in A_o$, then $U_a(A_o^+) \subset A^+$.

Suppose not, then for some $a \in A_o$ there exists $b \in U_a(A_o^+)$ such that $b \notin A^+$. By (2.20) there exists $\lambda_o \in \sigma(b)$ such that

$\lambda_o < 0$. Now we can write 0 as a proper convex combination of λ_o and 1 , say $0 = t\lambda_o + (1-t)$ where $0 < t < 1$. Applying the linear function $\varphi(\lambda) = t\lambda + (1-t)$ to the scalar λ_o at the left side of the inclusion $\lambda_o \in \sigma(b)$ and also to the element $b \in A$ at the right side of the same inclusion, we find

$$0 = \varphi(\lambda_o) \in \sigma(\varphi(b)) = \sigma(tb + (1-t)1) .$$

Hence $tb + (1-t)1$ is not invertible.

At this point we note that $1 \in U_a(A_o^+)$. In fact, if c is the inverse of a , then $c \in C(a)$ by Proposition 2.4, and since $C(a)$ is associative we have

$$1 = (a \circ c) \circ (c \circ a) = \{ac^2a\} \in U_a(A_o^+) .$$

Since A_o^+ is convex and U_a is a linear map, the set $U_a(A_o^+)$ is also convex. Hence

$$tb + (1-t)1 \in U_a(A_o^+) \subset U_a(A_o) .$$

But it follows from Corollary 2.6 that for invertible a , $U_a(A_o) \subset A_o$. By the relation above $tb + (1-t)1$ must be invertible, a contradiction.

2. We shall next prove that if $a \in A_o$, then $U_a(A^+) \subset A^+$.

By the definition (2.24) and axiom (2.1), U_a is a continuous operator on A . By (2.21) A_o^+ is dense in A^+ . From this and from the first part of the proof the conclusion follows.

3. Now we consider an arbitrary $a \in A$, and again we shall first prove $U_a(A_o^+) \subset A^+$.

We consider an arbitrary element c of A_o^+ . Since c is positive and invertible, it has an invertible square root $b \in C(c) \cong C(X)$ (Propositions 2.3 and 2.4). Thus $c = b^2$, with $b \in A_o$.

By (2.26):

(2.27) $U_b U_a(b^2) = \{bab\}^2 \geq 0$.

Let $d \in A_o$ be inverse of b , By Proposition 2.5, U_b^{-1} exists and is equal to U_d . By the preceding part of the proof, U_d is a positive operator. Applying $U_b^{-1} = U_d$ to the inequality (2.27), we find $U_a(b^2) \geq 0$. Hence we have proved $U_a(c) \geq 0$, as desired.

4. Finally the general inclusion $U_a(A^+) \subset A^+$ for arbitrary $a \in A$, follows by continuity of U_a and density of A_o^+ in A^+ as in the second part of the proof. □

For the proof of our next proposition we shall need a general inequality which will also be useful later. Observe that from the relation $a^2 \leq \|a^2\|1$ and from the positivity of U_a we obtain the following relation valid for an arbitrary element a of a JB-algebra A :

$$a^4 = \{aa^2a\} \leq \|a^2\|\{a1a\} = \|a^2\|a^2 .$$

Hence for every positive element b of A we have the inequality

(2.28) $b^2 \leq \|b\|b$.

<u>Proposition 2.8.</u> <u>If a, b are positive elements of a JB-algebra A , then (2.29) and (2.30) are equivalent and imply (2.31)</u>:

(2.29) $\{aba\} = 0$,

(2.30) $\{bab\} = 0$,

(2.31) $a \circ b = 0$.

Proof. 1. Assume first (2.29). By positivity of U_a and by (2.28), we find

$$0 \leq \{ab^2a\} \leq \|b\|\{aba\} = 0 \ . \tag{2.32}$$

By the general identity (2.26), this implies $\{bab\}^2 = 0$, which gives (2.30).

By symmetry, (2.30) also implies (2.29).

2. In any associative algebra one has the identity

$$[\tfrac{1}{2}(ab+ba)]^2 = \tfrac{1}{4}[a(bab) + (bab)a + ab^2a + ba^2b] \ .$$

By Macdonald's Theorem the corresponding identity

$$(a \circ b)^2 = \tfrac{1}{4}[2a\circ\{bab\} + \{ab^2a\} + \{ba^2b\}] \tag{2.33}$$

will hold in any Jordan algebra.

Assume now that $\{aba\} = \{bab\} = 0$. Then also $\{ab^2a\} = \{ba^2b\} = 0$ by virtue of (2.32). Now it follows from (2.33) that $(a\circ b)^2 = 0$, which gives (2.31). □

If p is an idempotent element of a JB-algebra A , i.e. if $p^2 = p$, then

$$\{pap\} = 2p\circ(p\circ a) - p\circ a \tag{2.34}$$

for all $a \in A$. Hence we have the following:

Corollary 2.9. Let a be a positive element and p an idempotent in a JB-algebra A . Then $\{pap\} = 0$ iff $p\circ a = 0$.

For a given idempotent p of a Jordan algebra we denote the complementary idempotent by the symbol p' ; thus $p' = 1-p$. Now the following relations are easily proved by Macdonald's Theorem:

(2.35) $U_pU_p = U_p$, $U_{p'}U_p = 0$.

Corollary 2.10. Let a be a positive element and p an idempotent in a JB-algebra A . Then $U_pa = a$ iff $U_{p'}a = 0$.

Proof. If $U_pa = a$, then $U_{p'}a = U_{p'}U_pa = 0$. Conversely if $U_{p'}a = 0$, then $p'\circ a = 0$ by Corollary 2.9. Now $a = (p+p')\circ a = p\circ a$, so by (2.34)

$$U_pa = 2p\circ(p\circ a) - p\circ a = p\circ a = a\ . \qquad \square$$

Note that the equivalence stated in Corollary 2.10 will not subsist if the hypothesis $a \geq 0$ is omitted. One can give easy counterexamples where A is the self-adjoint part of a C*-algebra $\mathfrak{A}$. (It suffices to consider the case where $\mathfrak{A}$ is the 2×2 - matrix algebra).

From the definition of the Jordan triple product one can obtain the following identity valid for an arbitrary element a and an idempotent p in a Jordan algebra:

$$p\circ a = \tfrac{1}{2}(a + \{pap\} - \{p'ap'\})\ .$$

Denoting the multiplication operator determined by p by the symbol L_p , we can rewrite this as an operator identity:

(2.36) $L_p = \frac{1}{2}(I + U_p - U_{p'})$.

We recall that two elements a, b of a Jordan algebra are said to operator commute if L_a and L_b commute as operators, i.e. if $[L_a, L_b] = 0$ [13;p.320].

The following lemma gives useful criteria for operator commutativity. (Note that this lemma is valid for general Jordan algebras, and it is of course not new. In fact, it can be extracted from the proof of Lemma 1 in [13;p.320], but it is just as easy to give a direct proof).

Lemma 2.11. Let a be an arbitrary element and p an idempotent in a Jordan algebra. Then the following are equivalent:

(i) a and p operator commute,

(ii) $L_p a = U_p a$,

(iii) $a = (U_p + U_{p'})a$.

Proof. (i) => (ii). Assuming (i) we have $(L_p L_a - L_a L_p)p = 0$, from which we get $L_p(a \circ p) - a \circ p = 0$. Hence $L_p a = L_p L_p a$. Using (2.34) we find

$$U_p a = 2L_p L_p a - L_p a = L_p a \ .$$

(ii) => (iii) Substituting the expression (2.36) for L_p into (ii), we get (iii)

(iii) => (i) In the general Jordan identity

$$[L_{b\circ d}, L_c] + [L_{b\circ c}, L_d] + [L_{c\circ d}, L_b] = 0$$

(see (0_1) in [13;p.34]) we write $b = a$ and $c = d = p$, obtaining

$$(2.37) \quad 2[L_{p\circ a}, L_p] + [L_p, L_a] = 0 \ .$$

Assuming (iii) and writing $r = U_p a$, $s = U_{p'} a$, we have by (2.35) $U_p r = r$, $U_{p'} r = 0$, $U_p s = 0$, $U_{p'} s = s$. Hence by (2.36) $p \circ r = r$, so (2.37) gives $[L_r, L_p] = 0$. By (2.36) also $p \circ s = 0$, so (2.37) gives $[L_p, L_s] = 0$. Hence $[L_a, L_p] = [L_r + L_s, L_p] = 0$, and (i) is established. $\square$

Note ghat if A is the self-adjoint part of a C*-algebra, then an element a of A will "operator commute" with an idempotent p in A (a projection) exactly when $ap = pa$, i.e. when a and p commute in the customary sense.

The following result will be useful later.

Proposition 2.12. Let a be an arbitrary element and p an idempotent in a JB-algebra A . If p operator commutes with a , then p will operator commute with all elements of C(a).

Proof. By (ii) of Lemma 2.11 we can assume $\{pap\} = p\circ a$. Let M_0 be the Jordan subalgebra of A generated by a, p and 1. By the Shirshov-Cohn Theorem, M_0 is a special Jordan algebra, say that $c\circ d = \frac{1}{2}(cd+dc)$ for $c,d \in M_0$.

The hypothesis $\{pap\} = p\circ a$ can now be written

$$(2.38) \quad pap = \tfrac{1}{2}(pa+ap) .$$

Multiplying (2.38) from the left and from the right by p , we obtain in turn $pap = pa$ and $pap = ap$. Hence the two generators a, p of M_0 will commute. As in the proof of Proposition 2.4, we conclude from this that every pair c, d of elements of M_0 will commute, i.e. $cd = dc$, and hence $c\circ d = cd$. Thus M_0 is an associative subalgebra of A , and by the continuity of the Jordan product the closure M of M_0 will also be an associative subalgebra of A .

Now if $b \in C(a) \subset M$, then by (2.34)

$$\{pbp\} = 2p\circ(p\circ b) - p\circ b = p\circ b .$$

By Lemma 2.11, p operator commutes with b . □

§ 3. The enveloping algebra of a JB-algebra

Throughout this section we suppose that A is a fixed JB-algebra and we denote the state space of A by K; thus $\rho \in A^*$ belongs to K iff $\|\rho\| = \langle 1,\rho\rangle = 1$. Also we equip A^{**} with the ordering determined by K, i.e. $a \in (A^{**})^+$ iff $\langle a,\rho\rangle \geq 0$ for all $\rho \in K$. Then one can identify A^{**} with the ordered Banach space $A^b(K)$ of all bounded affine functions on K, and A with the space $A(K)$ of all w^*-continuous affine functions on K (cf. Theorems II.1.8, II.1.15 of [3]).

Recall that the Arens product on A^{**} is the unique bilinear extension of the (Jordan) product from A to A^{**} satisfying

(3.1) $\|a\circ b\| \leq \|a\|\circ\|b\|$ for $(a,b) \in A^{**}\times A^{**}$,

(3.2) $a \to a\circ b$ is w^*-continuous for $(a,b) \in A^{**}\times A^{**}$,

(3.3) $b \to a\circ b$ is w^*-continuous for $(a,b) \in A \times A^{**}$.

(Note that the construction of the Arens product is not symmetric in the two variables [6,Thm.3.2].)

It is not clear a priori that A^{**} with the Arens product becomes a JB-algebra. In particular, we do not a priori know that the (Arens) squares are positive, nor even that the product on $A^{**}\times A^{**}$ is commutative.

Lemma 3.1. Let M be a linear subspace of A^{**} such that a^2 is a positive element of A^{**} for all $a \in M$. Then for every $\rho \in K$ the function $a \to \langle a^2,\rho\rangle^{\frac{1}{2}}$ is a seminorm on M.

Proof. By the assumption on M we can apply the standard proof of the Schwarz inequality to obtain

$$(3.4) \quad \langle \tfrac{1}{2}(a\circ b + b\circ a), \rho\rangle^2 \le \langle a^2,\rho\rangle\langle b^2,\rho\rangle$$

for $a,b \in M$ and $\rho \in K$. From this we get the triangle inequality. The other properties of a seminorm are trivial. □

For brevity we shall say that $a_\alpha \to a$ <u>weakly</u> in A^{**} when $\{a_\alpha\}$ converges to a in the w^*-topology (i.e. $\sigma(A^{**},A^*)$), and we shall refer to (3.2) and (3.3) as respectively <u>weak left continuity</u>, and <u>weak right continuity on $A \times A^{**}$</u>. If $M \subset A^{**}$ satisfies the hypothesis of Lemma 3.1, then the seminorms $a \to \langle a^2,\rho\rangle^{\frac{1}{2}}$ (with $\rho \in K$) define a locally convex Hausdorff topology, which we call <u>the strong topology on M</u>. Note that by the inequality $\langle a^2,\rho\rangle^{\frac{1}{2}} \le \|a\|$ (for $\rho \in K$), norm convergence will imply strong convergence. Note also that by (3.4) $\langle a,\rho\rangle^2 \le \langle a^2,\rho\rangle$ (for $\rho \in K$), and hence strong convergence will imply weak convergence. Note in particular that A itself satisfies the requirement on M in Lemma 3.1, so the notion of strong topology is defined on A.

<u>Definition</u>. $\tilde{A}$ is the set of all weak limits in A^{**} of norm bounded strong Cauchy nets in A.

<u>Proposition 3.2</u>. If $\{a_\alpha\}$ is a norm bounded strong Cauchy net in A which converges weakly to $a \in \tilde{A}$, then $\{a_\alpha^2\}$ converges weakly to a^2; in particular $a^2 \ge 0$ for all $a \in \tilde{A}$.

<u>Proof</u>. For arbitrary $\rho \in K$ we decompose

$$|\langle a^2 - a_\alpha^2, \rho\rangle| \le |\langle (a-a_\alpha)\circ a, \rho\rangle| + |\langle a_\alpha \circ (a-a_\alpha), \rho\rangle| .$$

By weak left continuity of the Arens product $\langle (a-a_\alpha)\circ a, \rho\rangle \to 0$. It remains to prove $\langle a_\alpha \circ (a-a_\alpha), \rho\rangle \to 0$. Let $N = \sup_\alpha \|a_\alpha\|$

and let $\epsilon > 0$ be arbitrary. We choose α_0 such that $\langle (a_\alpha - a_\beta)^2, \rho\rangle < \epsilon N^{-2}$ for $\alpha, \beta \geq \alpha_0$. Then by weak right continuity on $A \times A^{**}$ and by Schwarz inequality (for states on A), we have for $\alpha \geq \alpha_0$:

$$|\langle a_\alpha \circ (a - a_\alpha), \rho\rangle|^2 = \lim_{\beta \geq \alpha_0} |\langle a_\alpha \circ (a_\beta - a_\alpha), \rho\rangle|^2$$

$$\leq \limsup_{\beta \geq \alpha_0} \langle a_\alpha^2, \rho\rangle \langle (a_\beta - a_\alpha)^2, \rho\rangle \leq \epsilon,$$

which completes the proof. □

By Proposition 3.2 the notion of strong topology can be defined on $\tilde{A}$, so that it now makes sense to state:

<u>Corollary 3.3</u>. If $\{a_\alpha\}$ is a norm bounded strong Cauchy net in A such that $a_\alpha \to a \in \tilde{A}$ weakly, then $a_\alpha \to a$ strongly; in particular every $a \in \tilde{A}$ is strong limit of a norm bounded net from A.

<u>Proof</u>. Observe first that

(3.5) $\qquad a \circ b = b \circ a \qquad$ when $a \in A$, $b \in A^{**}$.

In fact, if $\{b_\alpha\}$ is a net in A and $b_\alpha \to b$ weakly, then by (3.3) $a \circ b_\alpha \to a \circ b$ weakly, and by (3.2) also $a \circ b_\alpha = b_\alpha \circ a \to b \circ a$ weakly.

Now by left continuity and by Proposition 3.2

$$\langle (a - a_\alpha)^2, \rho\rangle = \langle a^2, \rho\rangle - 2\langle a_\alpha \circ a, \rho\rangle + \langle a_\alpha^2, \rho\rangle \to 0. \quad \square$$

<u>Corollary 3.4</u>. The Arens product on $\tilde{A} \times \tilde{A}$ with values in A^{**} is commutative and weakly continuous in each variable separately.

<u>Proof</u>. By (3.2) it suffices to prove commutativity. Let $a, b \in \tilde{A}$ and choose $\{a_\alpha\}$ in A such that $a_\alpha \to a$ strongly. By

(3.2), $a_\alpha \circ b \to a \circ b$ weakly . By using (3.4) with $a - a_\alpha$ in place of a , we conclude that $a_\alpha \circ b + b \circ a_\alpha \to a \circ b + b \circ a$ weakly. Combining this with the preceeding statement, we see that $b \circ a_\alpha \to b \circ a$ weakly. By virtue of (3.5) $a_\alpha \circ b = b \circ a_\alpha$ for all α . Hence $a \circ b = b \circ a$. □

Note that by the positivity of a^2 for all $a \in \tilde{A}$ (Proposition 3.2) and by the commutativity of the Arens product on $\tilde{A} \times \tilde{A}$ (Corollary 3.4), the Schwarz inequality

$$(3.6) \qquad \langle a \circ b, \rho \rangle^2 \leq \langle a^2, \rho \rangle \langle b^2, \rho \rangle$$

holds for all $a, b \in \tilde{A}$ and $\rho \in K$.

We now state two auxiliary results valid for norm bounded nets $\{a_\alpha\}$, $\{b_\alpha\}$ in $\tilde{A}$. The first of these follows directly from (3.6), the second follows by applying the first and separate continuity to the terms at the right side of the equation $a_\alpha \circ b_\alpha = a_\alpha \circ (b_\alpha - b) + a_\alpha \circ b$:

(3.7) $a_\alpha \to 0$ strongly implies $a_\alpha \circ b_\alpha \to 0$ weakly.

(3.8) $a_\alpha \to 0$ weakly and $b_\alpha \to b \in \tilde{A}$ strongly implies $a_\alpha \circ b_\alpha \to 0$ weakly.

The next lemma is crucial.

<u>Lemma 3.5</u>. If $\{a_\alpha\}$ is a bounded net in A and $a_\alpha \to a \in \tilde{A}$ strongly, then $a^2 \in \tilde{A}$ and $a_\alpha^2 \to a^2$ strongly.

<u>Proof</u>. The proof proceeds in four steps.

1. First we assume that the net $\{a_\alpha\}$ is norm bounded, say with $\sup_\alpha \|a_\alpha\| = N$, and that it converges to zero strongly. We claim that in this case also $a_\alpha^2 \to 0$ strongly.

In fact, for every $\rho \in K$ the inequality (2.28) gives

$$\langle (a_\alpha^2)^2, \rho \rangle \leq N^2 \langle a_\alpha^2, \rho \rangle \to 0 .$$

2. We keep the assumptions imposed on $\{a_\alpha\}$ in part 1 of the proof, and we claim that if $\{b_\alpha\}$ is any norm bounded net in A such that $b_\alpha \to b \in \tilde{A}$ strongly, then $\{b_\alpha a_\alpha b_\alpha\} \to 0$ strongly.

To prove this, we write $M = \sup_\alpha \|b_\alpha\|$ and use the identity (2.26) together with positivity of the maps U_{a_α}, U_{b_α} to obtain

$$(3.9) \quad \{b_\alpha a_\alpha b_\alpha\}^2 = \{b_\alpha \{a_\alpha b_\alpha^2 a_\alpha\} b_\alpha\}$$

$$\leq M^2 \{b_\alpha \{a_\alpha 1 a_\alpha\} b_\alpha\} = M^2 (2 b_\alpha \circ (a_\alpha^2 \circ b_\alpha) - b_\alpha^2 \circ a_\alpha^2) .$$

By part 1 of the proof $a_\alpha^2 \to 0$ strongly; then by (3.7) $b_\alpha \circ a_\alpha^2 \to 0$ weakly, and then by (3.8) $b_\alpha \circ (b_\alpha \circ a_\alpha^2) \to 0$ weakly. Since $a_\alpha^2 \to 0$ strongly and $\|b_\alpha^2\| \leq M$ for all α, then by (3.7) $a_\alpha^2 \circ b_\alpha^2 \to 0$ weakly. Thus, the right side of (3.9) tends to zero weakly, and it follows that $\{b_\alpha a_\alpha b_\alpha\} \to 0$ strongly.

3. We keep the assumptions imposed on $\{a_\alpha\}$ and $\{b_\alpha\}$ in part 2 of the proof, but we now claim that $a_\alpha \circ b_\alpha \to 0$ strongly.

In fact, this follows from part 2 of the proof by means of the following general identity:

$$(3.10) \quad a \circ b = \tfrac{1}{2} [\{(1+b) a (1+b)\} - \{bab\} - a] .$$

4. We now assume that $\{a_\alpha\}$ is a norm bounded net in A and that $a_\alpha \to a \in \tilde{A}$ strongly, and we will show that $\{a_\alpha^2\}$ is strongly Cauchy. This will complete the proof by Proposition 3.2 and Corollary 3.3.

For given α, β we write $c_{\alpha,\beta} = a_\alpha - a_\beta$ and $d_{\alpha,\beta} = a_\alpha + a_\beta$. Then $\{c_{\alpha,\beta}\}$ and $\{d_{\alpha,\beta}\}$ are nets with the product ordering on the indices. Note that $c_{\alpha,\beta} \to 0$ strongly and $d_{\alpha,\beta} \to 2a$ strongly. By part 3 of the proof $a_\alpha^2 - a_\beta^2 = c_{\alpha,\beta} \circ d_{\alpha,\beta} \to 0$ strongly; hence

$\{a_\alpha^2\}$ is strongly Cauchy. □

Our next lemma, concerning the norm closure $(\tilde{A})^-$ of $\tilde{A}$ in A^{**}, is of a provisional nature; we shall eventually prove that $\tilde{A}$ itself is norm closed.

Lemma 3.6. $(\tilde{A})^-$ is a JB-algebra.

Proof. We will first show that $\tilde{A}$ enjoys all properties of a JB-algebra stated in Theorem 2.1, except possibly norm completeness. By Corollary 3.4 the product on $\tilde{A}\times\tilde{A}$ with values in A^{**} is commutative. By Lemma 3.5 $\tilde{A}$ is closed under squaring, and by the identity

$$(3.11) \qquad a\circ b = \tfrac{1}{2}[(a+b)^2 - a^2 - b^2]$$

$\tilde{A}$ is closed under products. Furthermore, if $\{a_\alpha\}$ and $\{b_\alpha\}$ are bounded nets in A such that $a_\alpha \to a \in \tilde{A}$ strongly and $b_\alpha \to b \in \tilde{A}$ strongly, then by (3.11) and Lemma 3.5 $a_\alpha\circ b_\alpha \to a\circ b$ strongly. By Corollary 3.3 every element in $\tilde{A}$ is strong limit of a bounded net from A; hence the defining Jordan identity

$$(a^2\circ b)\circ a = a^2\circ(b\circ a)$$

will hold in $\tilde{A}$.

Now observe that since $A^{**} \cong A^b(K)$ is an order-unit space, then $\tilde{A}$ is also.

We next verify the implication (2.5). By Proposition 3.2 $a^2 \geq 0$ for all $a \in \tilde{A}$. If $-1 \leq a \leq 1$, then $\|a\| \leq 1$, so by (3.1) $\|a^2\| \leq \|a\|^2 \leq 1$; thus $0 \leq a^2 \leq 1$.

Having proved that $\tilde{A}$ possesses all attributes of a JB-algebra except possibly norm completeness, we now turn to $(\tilde{A})^-$. By (3.1) the Arens product on $A^{**} \times A^{**}$ is jointly norm continuous. From this it follows easily that $(\tilde{A})^-$ is a JB-algebra. □

We shall need a result on joint strong continuity of multiplication on bounded sets. This could be proved by minor modifications of the proofs of Proposition 3.2, Corollary 3.3 and Lemma 3.5, but we prefer to give a direct proof.

Proposition 3.7. Let $A \subset M \subset A^{**}$ with M a JB-algebra for the norm and product inherited from A^{**}. Then multiplication is jointly strongly continuous on bounded sutsets of M.

Proof. Below $\{a_\alpha\}$ and $\{b_\alpha\}$ are norm bounded nets in M, and arrows indicate strong convergence. We will successively prove:

(i) $a_\alpha \to 0$ implies $a_\alpha^2 \to 0$,

(ii) $a_\alpha \to 0$ and $b_\alpha \to 0$ imply $a_\alpha \circ b_\alpha \to 0$,

(iii) $a_\alpha \to 0$ and $b \in M$ imply $a_\alpha \circ b \to 0$,

(iv) $a_\alpha \to a \in M$ and $b_\alpha \to b \in M$ imply $a_\alpha \circ b \to a \circ b$.

By (2.28) $0 \leq (a_\alpha^2)^2 \leq \|a_\alpha\| a_\alpha^2$, from which (i) follows. Then (ii) follows from (i) and the identity (3.11). To prove (iii) we assume $a_\alpha \to 0$ and $b \in M$. For any $c \in M$ the identity (2.26) gives

$$\{ca_\alpha c\}^2 = \{c\{a_\alpha c^2 a_\alpha\}c\} \leq \|c^2\| \{ca_\alpha^2 c\} = \|c^2\| U_c a_\alpha^2 .$$

By weak left continuity of the Arens product on A^{**}, the map $U_c : M \to M$ is weakly continuous (cf. the definition (2.24)). Hence $\{ca_\alpha c\}^2$ tends to zero weakly, and then $\{ca_\alpha c\}$ tends to zero strongly. By the identity (3.10) $a_\alpha \circ b \to 0$. Finally, (iv) follows from (ii) and (iii) and the identity

$$a\circ b - a_\alpha\circ b_\alpha = (a-a_\alpha)\circ b + (a_\alpha - a)\circ(b-b_\alpha) + a\circ(b-b_\alpha)\,. \quad \square$$

Corollary 3.8. Let M be as in Proposition 3.7 and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous, then the mapping $a \to \varphi(a)$ is strongly continuous on bounded subsets of M.

Proof. The function φ can be uniformly approximated by polynomials on compact subsets of $\mathbb{R}$. By Proposition 3.7 $a \to \pi(a)$ is strongly continuous on bounded subsets of M for every polynomial π, and from this the corollary follows. $\square$

Proposition 3.9. The unit ball A_1 of A is strongly dense in the unit ball $\tilde{A}_1$ of $\tilde{A}$.

Proof. Let $a \in \tilde{A}_1$ and choose a bounded net $\{a_\alpha\}$ in A converging strongly to a. Let $\varphi : \mathbb{R} \to [-1,1]$ be a continuous function such that $\varphi(\lambda) = \lambda$ for $|\lambda| \leq 1$. Then $\{\varphi(a_\alpha)\}$ is a net in A_1, and by Corollary 3.8 $\varphi(a_\alpha) \to \varphi(a) = a$ strongly. $\square$

Note that the proof above is similar to part of the original proof of Kaplansky's density theorem.

We recall that a state ρ on an order-unit space A is called normal if $\langle a_\alpha, \rho\rangle \searrow 0$ whenever $a_\alpha \searrow 0$, i.e. whenever $\{a_\alpha\}$ is a descending net in A with zero as g.l.b. in A. A set S of states on A is said to be full (cf. [17;p.180]) if it is convex and

(3.12) $a \geq 0$ iff $\langle a,\rho\rangle \geq 0$ all $\rho \in S$.

By a standard argument (see e.g. part 2 of the proof of Prop.II.1.7

of [3]) one can prove that if S is a full set of states on A, then

$$(3.13) \qquad \|a\| = \sup_{\rho \in S} |\langle a, \rho\rangle| .$$

In particular, every full set of states is point-separating.

If ρ is a state on a JB-algebra A, then for every $b \in A$ the functional $\rho_b : a \to \langle U_b a, \rho\rangle$ is positive. We say that a set S of states on A is invariant if $\rho \to \rho_b$ maps S into $\text{cone}\, S = \bigcup_{\lambda \geq 0} \lambda S$ for all $b \in A$.

Theorem 3.10. If A is any JB-algebra, then $\tilde{A}$ is a monotone complete JB-algebra. Furthermore, the notions of "order convergence", "weak convergence", and "strong convergence" will coalesce for monotone nets in $\tilde{A}$, and the states on A act as normal states on $\tilde{A} \subset A^{**}$; in particular they form an invariant full set of normal states on $\tilde{A}$.

Proof. By Proposition 3.9, A_1 is strongly dense in $\tilde{A}_1$. On the other hand every strong Cauchy net in A_1 converges strongly to an element in $\tilde{A}_1$. It follows that $\tilde{A}_1$ is strongly complete [7;Ch II, §3,Prop.9]).

We now consider a net $\{a_\alpha\}$ in $\tilde{A}_1$ which converges in norm to an element a of A^{**}. By the inequality $\langle c^2, \rho\rangle \leq \|c\|^2$ valid for all $c \in \tilde{A}$ and $\rho \in K$, the net $\{a_\alpha\}$ is strongly Cauchy; hence it has a strong (and weak) limit $b \in \tilde{A}_1$. Since norm convergence in A^{**} implies weak convergence, $a = b \in \tilde{A}_1$. Hence $\tilde{A}_1$, and therefore $\tilde{A}$, is norm closed in A^{**}. Now it follows from Lemma 3.6 that $\tilde{A} = (\tilde{A})^-$ is a JB-algebra.

Next let $\{b_\alpha\}$ be an increasing net in $\tilde{A}$ bounded above by an element of $\tilde{A}$. Without loss of generality we assume $b_\alpha \geq 0$

for all α. Then there exists $b \in A^{**}$ given by $\langle b,\rho\rangle = \sup_\alpha \langle b_\alpha,\rho\rangle = \lim_\alpha \langle b_\alpha,\rho\rangle$ for all $\rho \in K$. We will prove that $b \in \tilde{A}$ and that $b_\alpha \to b$ strongly, which will show that $\tilde{A}$ is monotone complete and that "order", "weak", and "strong" convergence are equivalent for monotone nets in $\tilde{A}$.

Let α_0 be arbitrary and $\alpha_0 \leq \alpha \leq \beta$. Then by (2.28) and by the inequality $\|b_\gamma\| \leq \|b\|$ valid for all γ, the following relation holds for every $\rho \in K$:

$$\langle (b_\beta - b_\alpha)^2,\rho\rangle \leq \|b_\beta - b_\alpha\| \langle b_\beta - b_\alpha,\rho\rangle \leq 2\|b\| \langle b_\beta - b_\alpha,\rho\rangle$$

Hence $\{b_\alpha\}$ is strongly Cauchy. Thus $\{\|b\|^{-1} b_\alpha\}$ is a strong Cauchy net in $\tilde{A}_1$, and so it has a strong limit in $\tilde{A}_1$. Then $\{b_\alpha\}$ must converge strongly to an element of $\tilde{A}$, and this strong limit must coincide with the weak limit b. Hence $b \in \tilde{A}$ and $b_\alpha \to b$ strongly.

By the above argument, the supremum in $\tilde{A}$ of an increasing net bounded above in $\tilde{A}$, is the pointwise supremum (as functions on K). Hence all $\rho \in K$ act as normal states on $\tilde{A}$.

By definition, positivity of an element a of $\tilde{A}$ means exactly that $\langle a,\rho\rangle \geq 0$ for all $\rho \in K$; hence K is a full set of states on $\tilde{A}$.

It remains only to prove that K is an invariant set of states on $\tilde{A}$. To this end we consider an arbitrary $\rho \in K$ and $b \in \tilde{A}$, and we shall prove that there is an $\omega \in$ cone $K = (A^*)^+$ such that the linear functional $\rho_b : a \to \langle U_b a,\rho\rangle$ on $\tilde{A}$ is of the form $\rho_b(a) = \langle a,\omega\rangle$. Clearly $\rho_b|_A$ is a positive element of A^*. Hence there is an $\omega \in (A^*)^+$ such that

$$(3.14) \qquad \langle U_b a,\rho\rangle = \langle a,\omega\rangle \qquad \text{all } a \in A.$$

By left continuity of the Arens product in A^{**}, the map $U_b : \tilde{A} \to \tilde{A}$ is weakly continuous (cf. the definition (2.24)). By weak density of A in $\tilde{A}$, the equality (3.14) will subsist for all $a \in \tilde{A}$. Hence $\rho_b(a) = \langle a,\omega \rangle$. □

For a given JB-algebra A, the JB-algebra $\tilde{A}$ will be called the <u>enveloping monotone complete JB-algebra of A</u>, or briefly the <u>enveloping algebra of A</u>.

Finally it should be noted that there are two natural questions we have left open:

1.) Will $\tilde{A}$ be all of A^{**}?

2.) Will K contain all normal states on $\tilde{A}$?

By a modification of the arguments of Pedersen in [22] one can prove that $\tilde{A}$ is the smallest monotone closed subspace of A^{**} containing A, and from this it follows that the second question has an affirmative answer. The first question can probably also be solved to the affirmative by use of Theorem 9.5 below and results in [11]. However, this will not be needed in the sequel, and we will not pursue the questions above any further.

§ 4. Spectral theory

Throughout this section M will denote a monotone complete JB-algebra with an invariant full set of normal states K . Also we shall denote the linear span of K in M^* by V . Thus V consists of all $\omega = \lambda_1\rho_1 - \lambda_2\rho_2$ where $\lambda_i \in \mathbb{R}^+$ and $\rho_i \in K$ for i = 1,2 . The term "weak topology on M" refers to the weak topology defined by the natural duality of M and V (i.e. $\sigma(M,V)$); it will be the topology of pointwise convergence on K when the elements of M are interpreted as (affine) functions on K . Note that the invariance of K guarantees that each map $U_a : M \to M$ is weakly continuous. The functions $a \to \langle a^2, \rho\rangle^{\frac{1}{2}}$ where $\rho \in K$, are seen to be semi-norms on M (cf. the proof of Lemma 3.1), and we shall use the term "strong topology on M" with reference to the locally convex Hausdorff topology defined by these semi-norms. Clearly, norm convergence implies strong convergence, which in turn implies weak convergence. By Theorem 3.10 one may take M to be the enveloping algebra of any given JB-algebra A , and K to be the set of all states on A . Then the "weak" and "strong" convergence on M will have the same meaning as in § 3.

We will show that M has "many" idempotents and that they behave like the projections in a von Neumann algebra. In principle, this can be done by modifying existing results proved by various authors under slightly different hypotheses (see [21],[25],[30],[4]). However, we find it equally short and more informative to give direct proofs.

First we observe that the results on weak and strong convergence from § 3 will subsist in the present setting.

Lemma 4.1. For monotone nets in M the notions of "order", "weak", and "strong" convergence coincide. Multiplication in M is separately weakly continuous in each variable, and it is jointly strongly continuous on bounded subsets.

<u>Proof</u>. Let $\{a_\alpha\}$ be an increasing net in M, and assume without loss of generality that $a_\alpha \geq 0$ for all α. Since the ordering in M is pointwise on K (cf. the definition (3.12)) and K consists of normal states, an element a of M will be order limit of $\{a_\alpha\}$ iff it is pointwise, i.e. weak, limit. This in turn implies strong convergence to a, since by (2.28) for every $\rho \in K$:

$$\langle (a-a_\alpha)^2, \rho\rangle \leq \|a-a_\alpha\| \langle a-a_\alpha, \rho\rangle \leq 2\|a\| \langle a-a_\alpha, \rho\rangle \to 0 .$$

Observe next that separate weak continuity of multiplication follows from the weak continuity of the maps U_a by the general identity (3.10). Finally, joint strong continuity on bounded subsets follows as in the proof of Proposition 3.7, which depends on nothing more than weak continuity of the maps U_a. $\square$

For convenience we shall use the notations $a_\alpha \nearrow a$ and $a_\alpha \searrow a$ to express order (-weak and strong-) convergence of monotone nets in M. Also we shall say that a linear subspace N of M is <u>monotone closed</u> if $a_\alpha \in N$ for all α and $a_\alpha \nearrow a \in M$ implies $a \in N$. Recall that $C(a)$ denotes the norm closed subalgebra of M generated by a and 1. The weak closure of $C(a)$ in M will be denoted by $W(a)$. From Lemma 4.1 and Proposition 2.3 we immediately obtain the following:

<u>Lemma 4.2</u>. For each $a \in M$, $W(a)$ is a monotone closed associative subalgebra of M, isometrically isomorphic (as an ordered algebra) to a monotone complete $C(X)$.

From this lemma we obtain:

Proposition 4.3. For each $a \in M$ there exists a unique indexed set $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of idempotents in M such that

(4.1) $e_\lambda \leq e_\mu$ when $\lambda < \mu$,

(4.2) $e_\lambda \searrow e_\mu$ when $\lambda > \mu$ and $\lambda \to \mu$,

(4.3) $e_\lambda = 0$ for $\lambda < -\|a\|$ and $e_\lambda = 1$ for $\lambda > \|a\|$,

and such that for each $\omega \in K$ and $n = 1,2,\ldots$:

(4.4) $\langle a^n, \omega\rangle = \int \lambda^n \, d\langle e_\lambda, \omega\rangle$.

Moreover, $e_\lambda \in W(a)$ for all $\lambda \in \mathbb{R}$, and the Stieltjes sums $\sum_{i=1}^{n} \lambda_{i-1}(e_{\lambda_i} - e_{\lambda_{i-1}})$ converge in norm to a as the mesh of the partition $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ of $[-\|a\|, \|a\|]$ tends to zero.

Proof. The existence of an indexed family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ with the stated properties follows by calculation in $C(X)$. (For detailed proofs see [20;Thms 40.2,43.2]) . In particular we note that by Lemma 4.2, e_μ is greatest lower bound of $\{e_\lambda\}_{\lambda > \mu}$ in M and not only in $W(a)$.

To prove uniqueness, we suppose that $\{f_\lambda\}_{\lambda \in \mathbb{R}}$ is another indexed set of idempotents in M such that (4.1)-(4.4) hold. For given $\omega \in K$ the Borel measures on $\mathbb{R}$ with distribution functions $\lambda \mapsto \langle e_\lambda, \omega\rangle$ and $\lambda \mapsto \langle f_\lambda, \omega\rangle$ must coincide on all continuous functions by (4.4). Hence the two measures are equal, and so $\langle e_\lambda, \omega\rangle = \langle f_\lambda, \omega\rangle$ for all $\lambda \in \mathbb{R}$. □

For given $a \in M$ the indexed set of idempotents $\{e_\lambda\}$ described in Proposition 4.3, will be called the spectral family of a .

For given $a \in M$ the set of all real valued functions φ on $[-\|a\|,\|a\|]$ for which there exists $b \in W(a)$ such that

$$(4.5) \qquad \langle b,\omega\rangle = \int \varphi(\lambda)d\langle e_\lambda,\omega\rangle \quad \text{all} \quad \omega \in K ,$$

contains all continuous functions (by (4.4)), and it is pointwise monotone σ-complete (by the monotone convergence theorem and the monotone completeness of $W(a)$). Hence it contains all bounded Borel functions. For each bounded Borel function φ on $[-\|a\|,\|a\|]$ we now denote by $\varphi(a)$ the (unique) element b in M such that (4.5) holds. Thus $\varphi(a) \in W(a)$, and by definition

$$(4.6) \qquad \langle \varphi(a),\omega\rangle = \int \varphi(\lambda)d\langle e_\lambda;\omega\rangle \quad \text{all} \quad \omega \in K .$$

In this way we obtain a well behaved functional calculus in M for bounded Borel functions. In particular we note that for every $\lambda \in \mathbb{R}$:

$$(4.7) \qquad e_\lambda = \chi_{(-\infty,\lambda]}(a) .$$

If e_λ and e_μ are two members of the spectral family of $a \in M$, then by (2.24) and the associativity of $W(a)$:

$$\{e_\lambda e_\mu e_\lambda\} = 2e_\lambda \circ (e_\lambda \circ e_\mu) - e_\lambda \circ e_\mu = e_\lambda \circ e_\mu .$$

Hence it follows by Lemma 2.11 that every pair of members from the spectral family of a will operator commute.

<u>Lemma 4.4</u>. Let $\{e_\lambda\}$ be the spectral family of $a \in M$ and let $p \in M$ be an idempotent. Then p operator commutes with a if and only if p operator communtes with all e_λ .

<u>Proof</u>. Assume first that p operator commutes with a . Let $\{\varphi_n\}$ be a sequence of continuous functions on $\mathbb{R}$ with

values in [0,1] such that $\varphi_n \searrow \chi_{(-\infty,\lambda]}$. By Proposition 2.12, p operator commutes with all $\varphi_n(a)$. Hence by Lemma 2.11

$$\varphi_n(a) = (U_p + U_{p'})\varphi_n(a) \qquad n = 1,2,\ldots .$$

By weak continuity and by (4.7), this gives $e_\lambda = (U_p + U_{p'})e_\lambda$. Hence p operator commutes with e_λ .

Assume next that p operator commutes with all e_λ . Then p will operator commute with the Stieltjes sums of Proposition 4.3. Passing to the limit as above, we conclude that p operator commutes with a . □

Now let $a,b \in M$ and let $\{e_\lambda\}$ and $\{f_\mu\}$ be the spectral families of a and b , respectively. Then the following are equivalent by virtue of Lemma 4.4:

(4.8) a operator commutes with all f_μ ,

(4.9) b operator commutes with all e_λ ,

(4.10) all pairs e_λ, f_μ operator commute .

If these statements are valid, then we say that a and b are <u>compatible</u>. If a is compatible with all $c \in M$ compatible with b , then we say that a and b are <u>bicompatible</u>.

Clearly, every member of the spectral family of an element a of M will be bicompatible with a .

Now consider an idempotent p and an arbitrary element a in M . By Lemma 4.4, p is compatible with a iff p and a operator commute. Note also that p is bicompatible with a iff p operator commutes with all idempotents which operator commute with a .

By the above result, since $a \mapsto L_a$ is linear and isometric, two compatible dements of M will always operator commute.

For positive elements we have the following compatibility-criterion:

Lemma 4.5. If $a \in M^+$ and $p \in M$ is an idempotent, then a and p are compatible if and only if $U_p a \leq a$.

Proof. Assume first that a and p are compatible, or what is equivalent, that a and p operator commute. By Lemma 2.11, $a = (U_p + U_{p'})a \geq U_p a$.

Assume next $U_p a \leq a$. Now $a - U_p a \geq 0$, and since $U_p(a - U_p a) = 0$, we can apply Corollary 2.10 to get $U_{p'}(a - U_p a) = a - U_p a$. By (2.35) this gives $a = (U_p + U_{p'})a$, and now compatibility follows from Lemma 2.11. □

We recall that for given $a \in M^+$ the face of M^+ generated by a, is the set

$$\text{(4.11)}\quad \text{face}(a) = \{b \in M^+ \mid b \leq \lambda a \text{ some } \lambda \in \mathbb{R}^+\}.$$

Lemma 4.6. If $p \in M$ is an idempotent and $a \in \text{face}(p)$, then $a \leq \|a\| p$.

Proof. Applying $U_{p'}$ to all terms of the inequality $0 \leq a \leq \lambda p$, we obtain $0 \leq U_{p'} a \leq 0$. By Corollary 2.10, $U_p a = a$. Applying U_p to all terms of the inequality $0 \leq a \leq \|a\| 1$, we now obtain $0 \leq a \leq \|a\| p$. □

Consider an element a of M^+ with spectral family $\{e_\lambda\}$. From the isomorphism of $W(a)$ and $C(X)$ we conclude that $e_\lambda = 0$ for $\lambda < 0$. For $\lambda > 0$ and every $\omega \in K$

$$0 \leq \lambda \int_{(\lambda,\infty)} d\langle e_\mu, \omega\rangle \leq \int_{(\lambda,\infty)} \mu d\langle e_\mu, \omega\rangle \leq \int_{(0,\infty)} \mu d\langle e_\mu, \omega\rangle = \langle a, \omega\rangle ,$$

so $0 \leq \lambda(1-e_\lambda) \leq a$. Hence we have the general implication:

$$(4.12) \quad a \in M^+ , \lambda > 0 \Rightarrow 1-e_\lambda \in \text{face}(a) .$$

<u>Proposition 4.7</u>. If $a \in M^+$ and $\{e_\lambda\}$ is the spectral family of a , then $1-e_0$ is the smallest idempotent p in M such that $a \in \text{face}(p)$.

<u>Proof</u>. For every $\omega \in K$

$$\langle a,\omega\rangle = \int_{(0,\|a\|]} \lambda\, d\langle e_\lambda,\omega\rangle \leq \|a\| \int_{(0,\|a\|]} d\langle e_\lambda,\omega\rangle = \|a\|\langle(1-e_0),\omega\rangle$$

and so $a \in \text{face}(1-e_0)$.

Suppose now that $a \in \text{face}(q)$ for some idempotent q . Then $\text{face}(a) \subset \text{face}(q)$, so by (4.12) $1-e_\lambda \in \text{face}(q)$ for all $\lambda > 0$. By Lemma 4.6 $1-e_\lambda \leq \|1-e_\lambda\| q \leq q$ for all $\lambda > 0$, and by (4.2) $1-e_\lambda \nearrow 1-e_0$ when $\lambda > 0$ and $\lambda \to 0$. Hence $1-e_0 \leq q$, and the minimality is proved. □

<u>Definitions</u>. We denote the set of all idempotents in M by $\mathcal{P}$, and we use the symbols $\vee$ and $\wedge$ to denote the least upper bound and the greatest lower bound in $\mathcal{P}$ (whenever they exist). For given $a \in M^+$ we write

$$r(a) = 1-e_0 = \bigwedge\{p \in \mathcal{P} \mid a \in \text{face}(p)\} .$$

We will now show that $\mathcal{P}$ is a lattice. In fact, we will show that it is an <u>orthomodular lattice</u> under the map $p \mapsto p' = 1-p$, and we recall that this means that the following requirements are satisfied for $p,q \in \mathcal{P}$:

(4.13) $p'' = p$,

(4.14) $p \leq q \Rightarrow q' \leq p'$,

(4.15) $p \wedge p' = 0$ and $p \vee p' = 1$,

(4.16) $p \leq q \Rightarrow q = p \vee (q \wedge p')$ (orthomodular identity).

Lemma 4.8. If $p, q \in \mathcal{P}$, then the following are equivalent:

(i) $p \circ q = 0$,

(ii) $p + q \in \mathcal{P}$,

(iii) $p + q \leq 1$,

(iv) $U_p U_q = 0$.

Proof (i) => (ii). If $p \circ q = 0$, then $(p+q)^2 = p^2 + q^2 = p + q$, so $p + q$ is an idempotent.

(ii) => (iii) This implication is trivial since every idempotent r satisfies $\|r\| \leq 1$, and then also $r \leq 1$.

(iii) => (iv) If $p + q \leq 1$, then $U_p p + U_p q \leq p$. Hence $p + U_p q \leq p$, and therefore $U_p q = 0$. For arbitrary $a \in M^+$, we have $0 \leq a \leq \|a\| 1$. Therefore $0 \leq U_q a \leq \|a\| q$, and in turn $0 \leq U_p U_q a \leq \|a\| U_p q = 0$. Hence we have shown $U_p U_q = 0$.

(iv) => (i) If $U_p U_q = 0$, then $U_p q = U_p U_q 1 = 0$, and by Corollary 2.9, $p \circ q = 0$. □

Remark. Clearly one can replace $U_p U_q = 0$ by the symmetric statement $U_q U_p = 0$ in (iv).

Definition. We say that two idempotents p, q are orthogonal, and we write $p \perp q$, if the equivalent statements (i)-(iv) above are valid.

<u>Proposition 4.9</u>. The set $\mathcal{P}$ of idempotents in M is a complete orthomodular lattice where $\bigvee_i p_i = r(\sum_i p_i)$ for every finite set $\{p_1,\dots,p_n\} \in \mathcal{P}$; in particular

$$\bigvee_i p_i = \sum_i p_i \quad \text{if} \quad p_i \perp p_j \quad \text{for} \quad i \neq j \ .$$

<u>Proof</u>. Let $p_1,\dots,p_n \in \mathcal{P}$. Clearly, $p_j = r(p_j) \leq r(\sum_i p_i)$ for $j = 1,\dots,n$. Now suppose $q \in \mathcal{P}$ and $p_j \leq q$ for $j = 1,\dots,n$. Then $\sum_i p_i \in \text{face}(q)$, so $r(\sum_i p_i) \leq q$. This proves $\bigvee_i p_i = r(\sum_i p_i) \in \mathcal{P}$.

If $p_1,\dots,p_n$ are mutually orthogonal, then it follows from Lemma 4.8 that $\sum_i p_i \in \mathcal{P}$, and so $r(\sum_i p_i) = \sum_i p_i$. Hence $\bigvee_i p_i = \sum_i p_i$ in this case.

If $\{p_\alpha\}$ is an increasing net from $\mathcal{P}$, then there exists $p \in M$ such that $p_\alpha \nearrow p$. By Lemma 4.1, p is an idempotent. Hence $\bigvee_\alpha p_\alpha = p \in \mathcal{P}$.

Since we have an order reversing 1-1 map $p \mapsto p'$ of $\mathcal{P}$ onto itself, we conclude that $\mathcal{P}$ is a complete lattice.

The requirements (4.13)-(4.15) are trivially satisfied. To prove (4.16), we suppose that $p \leq q$. Then $p + q' = p + 1 - q \leq 1$, so $p \perp q'$ (by Lemma 4.8). Thus, by the above results, $q' \vee p = q' + p$. Since $p \mapsto p'$ is order-reversing, we now find

$$q \wedge p' = (q' \vee p)' = 1 - (q' + p) = q - p \ .$$

In particular, $(q \wedge p') + p = q \leq 1$, so $(q \wedge p') \perp p$. Hence

$$p \vee (q \wedge p') = p + (q - p) = q \ . \qquad \square$$

By weak continuity of U_p , there exists for every idempotent $p \in M$ a map $U_p^* : V \to V$ defined by $\langle a, U_p^*\omega\rangle = \langle U_p a, \omega\rangle$ for all $a \in M$. Clearly, U_p^* will map V^+ into itself, but U_p^* will not map K into itself in general. We shall now prove that U_p^* maps an element of K into K only if it is invariant under U_p^*. This is an important property of the maps U_p^*. (E.g. it is used in one of the proofs of the existence of polar decompositions for normal states of a von Neumann algebra, see [10; Thm.12.2.4]; and it characterizes the "neutral projections" studied in [4]).

Lemma 4.10. Let $p \in M$ be idempotent and let $\rho \in K$. Then $\|U_p^*\rho\| = 1$ if and only if $U_p^*\rho = \rho$.

Proof. To prove the non trivial part of the equivalence, we suppose $\|U_p^*\rho\| = 1$, or what is equivalent (since M^* is a base-norm space, cf. e.g. [3;Ch II.§1]), that $\langle 1, U_p^*\rho\rangle = 1$. We now apply (2.36) with p' in place of p; then we get for arbitrary $a \in M$:

$$(4.17)\quad \langle a,\rho\rangle = \langle U_p a,\rho\rangle - \langle U_{p'} a,\rho\rangle + 2\langle p'\circ a,\rho\rangle .$$

We will show that the last two terms of this equation vanish. Without loss of generality we assume $a \geq 0$.

By the assumption on ρ:

$$\langle p',\rho\rangle = 1 - \langle p,\rho\rangle = 1 - \langle U_p 1,\rho\rangle = 1 - \langle 1, U_p^*\rho\rangle .$$

The desired conclusion now follows from the implication

$$\langle p',\rho\rangle = 0 \Rightarrow \langle p'\circ a,\rho\rangle = \langle U_{p'} a,\rho\rangle = 0 ,$$

which in turn follows by Schwarz's inequality and the relation

$$0 \leq \langle U_{p'} a,\rho\rangle \leq \|a\|\langle U_{p'} 1,\rho\rangle = \|a\|\langle p',\rho\rangle = 0 . \quad \square$$

For given idempotent $p \in M$, we denote by M_p the image of M under U_p, i.e. $M_p = U_p(M)$. Since U_p is an idempotent map (cf. (2.35)), an element a of M belongs to M_p iff $U_p a = a$. Also we denote by K_p the set of all those $\rho \in K$ whose restriction to M_p is a (positive and) normalized linear functional on M_p, i.e.

$$\sup\{|\langle a,\rho\rangle| \mid a \in M_p, \|a\| \leq 1\} = 1 .$$

This implies that $\|U_p^*\rho\| = 1$. Hence it follows from Lemma 4.10 that K_p consists of exactly those $\rho \in K$ for which $U_p^*\rho = \rho$.

<u>Proposition 4.11</u>. If $p \in M$ is an idempotent, then M_p is weakly closed in M. Moreover, M_p is a monotone complete JB-algebra and the (restrictions of) elements of K_p form an invariant full set of normal states on M_p.

<u>Proof</u>. It follows by weak continuity of U_p that M_p is weakly closed. By monotone completeness of M (and by Lemma 4.1), M_p is also monotone complete. Clearly also M_p is a norm closed linear subspace of M.

For every $a \in M$ we have by (2.26)

$$(U_p a)^2 = U_p\{ap^2a\} \in U_p(M) .$$

Hence M_p is closed under squares, and by (3.11) also under Jordan products. Clearly the norm conditions (2.1)-(2.3) will prevail in M_p : Hence M_p is a JB-algebra.

By definition, $\rho | M_p$ is a positive linear functional of norm one, hence a state on M_p for every $\rho \in K_p$. Clearly, $\rho | M_p$ is a normal state on M_p since ρ is a normal state on M.

Since K is a full set of states on M, we have the following series of equivalences for $a \in M_p$

$$a \geq 0 \iff \langle a,\rho\rangle \geq 0 \quad \text{all} \quad \rho \in K$$

$$\iff \langle U_p a,\rho\rangle \geq 0 \quad \text{all} \quad \rho \in K$$

$$\iff \langle a,\omega\rangle \geq 0 \quad \text{all} \quad \omega \in K_p$$

Hence, the set of all ρ/M_p where $\rho \in K_p$ is a full set of states on M_p . It is also easily seen to be invariant, and the proof is complete. □

§ 5. The center

We use the notation of the previous section and let M denote a monotone complete JB-algebra with an invariant full set K of normal states. We shall study the center of M and then construct representations of a JB-algebra A into a subalgebra of its enveloping algebra $\tilde{A}$ for each state of A.

If $X \subset M$ we denote by $Z(X)$ the set of elements in M which are compatible with all elements in X. If $a \in M$ we write $Z(a)$ for $Z(\{a\})$

Lemma 5.1. For each $b \in M$ $Z(b)$ is a weakly closed subalgebra of M containing b.

Proof. If $\{e_\lambda\}$ is the spectral family of b then $Z(b) = \bigcap_{\lambda \in \mathbb{R}} Z(e_\lambda)$. By Lemma 4.4 $Z(e_\lambda)$ is the set of elements in M which operator commute with e_λ, so by Proposition 2.12 $Z(e_\lambda)$ is a weakly closed subalgebra of M. □

Lemma 5.2. A subset X of M consists of mutually compatible elements if and only if X is contained in a weakly closed associative subalgebra containing the identity.

Proof. Suppose all elements in X are compatible. Then $X \subset Z(X)$, so that $Z(X) \supset Z(Z(X)) \supset X$. Thus $Z(Z(X))$ is by Lemma 5.1 a weakly closed subalgebra of M consisting of mutually compatible elements. As remarked after Lemma 4.4 mutually compatible elements operator commute. Thus so do all elements in $Z(Z(X))$, which implies that if $a, b, c \in Z(Z(X))$ then $a \circ (b \circ c) = (a \circ b) \circ c$. Thus $Z(Z(X))$ is the desired associative subalgebra.

The converse is an immediate consequence of the spectral theorem. □

We define the center of M to be the set $Z(M)$. Since $Z(M) = Z(Z(1))$ it follows as in the proof of Lemma 5.2 that $Z(M)$ is a weakly closed associative subalgebra of M containing 1. Also it is immediate from the preceding that $a \in Z(M)$ if and only if a operator commutes with each idempotent in M. Recall that an operator $s \in M$ is a symmetry if $s^2 = 1$. Then we have the following characterization of the center.

Lemma 5.3. $a \in Z(M)$ if and only if $U_s a = a$ for all symmetries $s \in M$.

Proof. There is a one-one correspondence between the set of idempotents in M and the set of symmetries in M, given by $p \to s = 2p-1$. Furthermore it is easily verified that $U_s = U_{2p-1} = 2U_p + 2U_{1-p} - I$. Thus $a \in Z(M)$ if and only if $(U_p + U_{1-p})(a) = a$ for each idempotent $p \in M$ (see Lemma 2.11) if and only if $U_s = a$ for each symmetry $s \in M$. □

From Proposition 4.11 we know that if p is an idempotent in M then $M_p = U_p(M)$ is a monotone complete JB-algebra. If p is central we compute the center of M_p.

Lemma 5.4. If p is a central idempotent in M then $Z(M_p) = Z(M)_p$.

Proof. Clearly $Z(M)_p \subset Z(M_p)$. In order to prove the converse inclusion it suffices to show that each idempotent $e \in Z(M_p)$ belongs to $Z(M)_p$. Let $a \in M^+$. Then by Lemma 2.11 $a = (U_p + U_{1-p})a$, so $U_e a = U_e(U_p + U_{1-p})a = U_e U_p a \leq U_p a \leq a$ using Lemma 4.5 twice.

Again by Lemma 4.5 e is compatible with a, and $e \in Z(M)$. In particular $e = U_p e \in Z(M)_p$. $\square$

Let ρ be a state in the set K. Since the projections in M form a complete lattice there is a smallest projection $\mathrm{supp}(\rho)$ in M with the property $\rho(\mathrm{supp}(\rho)) = 1$. Supp ρ is called the <u>support</u> of ρ. If we apply this to the restriction of ρ to $Z(M)$ we obtain the support $c(\rho)$ of $\rho|Z(M)$, called the <u>central support</u> of ρ.

We say M is a JB-<u>factor</u> if $Z(M) = \mathbb{R}1$.

<u>Lemma 5.5</u>. If ρ is an extreme point of K then $M_{c(\rho)}$ is a JB-factor.

<u>Proof</u>. Suppose e is an idempotent in the center of $M_{c(\rho)}$ such that $0 \neq e \neq c(\rho)$. By Lemma 5.4 $e \in Z(M)$, so $(U_e + U_{1-e})a = a$ for all $a \in M$. Let $\rho_1 = \langle e,\rho\rangle^{-1} U_e^* \rho$ and $\rho_2 = \langle 1-e,\rho\rangle^{-1} U_{1-e}^* \rho$. Then $\rho_1, \rho_2 \in K$ and $\rho = \langle e,\rho\rangle \rho_1 + \langle 1-e,\rho\rangle \rho_2$ is a convex combination of ρ_1 and ρ_2. Thus $\rho_1 = \rho_2 = \rho$, which is impossible by choice of e. $\square$

<u>Proposition 5.6</u>. Let A be a JB-algebra, ρ a state of A and $c(\rho)$ its central support in $\tilde{A}$. Let φ_ρ denote the map $\varphi_\rho : A \to \tilde{A}_{c(\rho)}$ defined by $\varphi_\rho(a) = U_{c(\rho)}(\tilde{a})$, where $\tilde{a}$ is the image of a in $\tilde{A}$. Then φ_ρ is a Jordan homomorphism such that $\varphi_\rho(A)$ is strongly dense in $\tilde{A}_{c(\rho)}$. Furthermore, if ρ is a pure state then the strong closure of $\varphi_\rho(A)$ in $\tilde{A}$ is a JB-factor.

<u>Proof</u>. Let $M = \tilde{A}$ and K be the state space of A considered as a full set of invariant state of M. Since U_b is

strongly continuous for $b \in M$ it is clear that $\varphi_\rho(A)$ is strongly dense in $M_{c(\rho)}$, and by Lemma 5.5 that the strong closure of $\varphi_\rho(A)$ in $\tilde{A}$ is a JB-factor whenever ρ is a pure state. It remains to show that φ_ρ is a Jordan homomorphism, or what amounts to the same, to show that the map U_e on M is a Jordan homomorphism for each central idempotent e in M. Let e be one. Then $I = U_e + U_{1-e}$, so $L_e = \frac{1}{2}(I + U_e - U_{1-e}) = U_e$. In particular $L_e^2 = U_e^2 = U_e = L_e$; thus if $a, b \in M$ we have $L_e(a \circ b) = L_e L_a(b) = L_a L_e(b) = a \circ L_e(b)$. Applying L_e again and using that L_e is an idempotent we have

$$\begin{aligned} L_e(a \circ b) &= L_e(L_e(a \circ b)) = L_e(a \circ L_e(b)) \\ &= L_e(L_e(b) \circ a) = L_e(b) \circ L_e(a) \\ &= L_e(a) \circ L_e(b) , \end{aligned}$$

completing the proof. □

If A is a JB-algebra and φ a Jordan homomorphism of A onto a strongly dense Jordan subalgebra of a JB-factor, we say φ is a <u>factor representation</u> of A.

<u>Corollary 5.7</u>. A JB-algebra has a faithful family of factor representations.

<u>Proof</u>. A faithful family is given by the set of φ_ρ with ρ a pure state. □

§ 6. Comparison theory

Throughout this section M denotes a JB-factor, and $\mathcal{P}$ its lattice of idempotents. Our main purpose is to show that if $\mathcal{P}$ has no minimal elements then there is $e \in \mathcal{P}$ and a symmetry $s \in M$ such that $U_s e = 1-e$. Note that for a symmetry s the map U_s is a Jordan automorphism of M, and U_s restricts to a lattice automorphism of $\mathcal{P}$. We say two idempotents p and q in $\mathcal{P}$ are <u>equivalent</u> and write $p \sim q$ if there exists a finite family $s_1,\dots,s_n$ of symmetries in M such that

$$\{s_n\{s_{n-1}\{\dots\{s_1 p s_1\}\dots\}s_{n-1}\}s_n\} = q ,$$

i.e. $U_{s_n}U_{s_{n-1}}\dots U_{s_1}(p) = q$. We say $p \sim q$ <u>via s</u> if $U_s p = q$. We write $p \precsim q$ if $p \sim r \le q$ for some $r \in \mathcal{P}$. We say p and q in $\mathcal{P}$ are <u>related</u> if there exist nonzero $p_1, q_1 \in \mathcal{P}$ with $p_1 \le p$, $q_1 \le q$, and $p_1 \sim q_1$.

<u>Lemma 6.1</u> If $0 \ne q \in \mathcal{P}$ then $\bigvee\{p \in \mathcal{P} : p \precsim q\} = 1$.

<u>Proof</u>. Let $e = \bigvee\{p \in \mathcal{P} : p \precsim q\}$. Since M is a JB-factor it suffices to show e is central. Let s be a symmetry in M. Then $U_s e = \bigvee\{U_s p : p \in \mathcal{P}, p \precsim q\}$. Now $p \precsim q$ implies $U_s p \precsim q$, so we have $U_s e \le e$. But then $e = U_s^2 e \le U_s e$, so $U_s e = e$. By Lemma 5.3 e is central. □

<u>Lemma 6.2</u> Let $p,q \in \mathcal{P}$. Then there exists a symmetry s in M such that $U_s\{pqp\} = \{qpq\}$.

<u>Proof</u>. Let $a = p+q-1$, so $a^2 = 2p\circ q-p-q+1$. Since $L_p = \frac{1}{2}(I+U_p-U_{1-p})$ we have

$$a^2 = (U_p-U_{1-p})q-p-1 .$$

By Lemma 2.11 M_p and M_{1-p} are contained in $Z(p)$, so that a^2 is compatible with p. Similarly $a^2 \in Z(q)$. In particular $|a| = (a^2)^{\frac{1}{2}} \in Z(p) \cap Z(q)$. Therefore we have by Lemma 2.11

$$\{|a|\, p\, |a|\} = p \circ |a|^2 = p \circ a^2 = p \circ (2p \circ q - p - q + 1)$$

$$= 2p \circ (p \circ q) - p \circ q = \{pqp\} .$$

By spectral theory there is a symmetry s in the associative strongly closed JB-algebra $W(a)$ generated by a such that $s \circ a = |a|$. Since all elements in $W(a)$ operator commute it follows that U_s and $U_{|a|}$ commute, and $U_s U_{|a|} = U_{s \circ |a|}$, see [13,p.38,eq.(66)]. Thus

$$U_s\{pqp\} = U_s U_{|a|}(p) = U_{s \circ |a|}(p) = U_a(p) =$$

$$= 2(p+q-1) \circ [(p+q-1) \circ p] - (2p \circ q - p - q + 1) \circ p$$

$$= \{qpq\} . \quad \square$$

<u>Lemma 6.3</u> Every pair of non orthogonal idempotents p and q in M dominate nonzero idempotents e and f in M respectively such that $e \sim f$ via a symmetry.

<u>Proof</u>. Note $\{pqp\} \leq p$, so $r(\{pqp\}) \leq p$. Similarly $r(\{qpq\}) \leq q$. Furthermore $p \circ q \neq 0$, so by Lemma 4.8 $\{pqp\} \neq 0 \neq \{qpq\}$. By Lemma 6.2 there is a symmetry s in M such that $U_s\{pqp\} = \{qpq\}$. Since U_s is a lattice automorphism of $\mathcal{P}$ it follows that $U_s r(\{pqp\}) = r(\{qpq\})$. Thus $e = r(\{pqp\})$ and $f = r(\{qpq\})$ are the desired idempotents. $\square$

<u>Lemma 6.4</u> Every pair of nonzero idempotents in M are related.

Proof. Let p and q be nonzero elements in $\mathcal{P}$. If p and q are not related, then by Lemma 6.3 p is orthogonal to every idempotent $r \lesssim q$. Thus $r \leq 1-p$, whenever $r \lesssim q$. By Lemma 6.1 $p = 0$. □

The next result almost shows that whenever e and f are orthogonal and $e \sim f$ then $e \sim f$ via a symmetry. We are greatly indebted to Richard Schafer for showing us the proof.

Lemma 6.5 Let e and f be orthogonal idempotents in M . Suppose there exist symmetries s and t in M such that $U_t U_s e = f$. Then $e \sim f$ via a symmetry.

Proof. Let $a = 2\{est\}$ (recall that $\{bcd\} = (b \circ c) \circ d + (c \circ d) \circ b - (d \circ b) \circ c)$. We will show

(6.1) $a^2 = e+f$

(6.2) $a \circ e = \frac{1}{2}a$

(6.3) $a \circ f = \frac{1}{2}a$

(6.4) $e \sim f$ via the symmetry $1+a-a^2$.

We first establish

(6.5) $\{est\} = \{swt\} = \{stf\}$,

where $w = \{ses\} = \{tft\}$. By [13,p.57,eq.87] we have the identity

$$\{x\{bcb\}y\} = \{\{xbc\}by\} + \{\{cby\}bx\} - \{\{xby\}bc\} .$$

Thus we have, since $\{bbd\} = b^2 \circ d$ and $\{bcd\} = \{dcb\}$,

$$\begin{aligned}\{swt\} &= \{s\{ses\}t\} \\ &= \{\{sse\}st\} + \{\{est\}ss\} - \{\{sst\}se\} \\ &= \{est\} + \{est\} - \{tse\} \\ &= \{est\} .\end{aligned}$$

Symmetrically $\{tws\} = \{fts\}$, which proves (6.5).

Now let $e_1 = e$, $e_2 = f$, $e_3 = 1 - e - f$. Then e_1, e_2, e_3 are pairwise orthogonal idempotents with sum 1. Let $M = M_{11} \oplus M_{22} \oplus M_{33} \oplus M_{12} \oplus M_{13} \oplus M_{23}$ be the Pierce decomposition corresponding to them, i.e.

$$M_{ij} = \begin{cases} U_{e_i}(M) & \text{if } i = j \\ 2U_{e_i,e_j}(M) & \text{if } i < j \end{cases},$$

where $U_{e_i,e_j}(x) = \{e_i x e_j\}$, see [13,p.120].

By the multiplication rules for Pierce components [14,p. 2.5] $\{e_1 bc\} \in M_{11} + M_{12} + M_{13}$ for all $b,c \in M$. Thus $\frac{1}{2}a = \{est\} \in M_{11} + M_{12} + M_{13}$. But by (6.5) $\frac{1}{2}a = \{fts\}$, so $\frac{1}{2}a \in M_{22} + M_{12} + M_{23}$. Therefore

$$a \in (M_{11} + M_{12} + M_{13}) \cap (M_{22} + M_{12} + M_{23}) = M_{12}.$$

But $M_{ij} = \{x : e_i \circ x = e_j \circ x = \frac{1}{2}x\}$ by [13,p.120,eq.(13)]. Thus we have established (6.2) and (6.3).

From Glennie's result [12], [13,p.51] that there are no identities of degree ≤ 7 in three variables, which hold for all special Jordan algebras, but fail for some others, we have the identity

$$4\{xby\}^2 = 4\{x\{b(x \circ y)b\}y\} + \{x\{by^2b\}x\} + \{y\{bx^2b\}y\} - 2\{xbx\} \circ \{yby\}.$$

Thus we have by (6.5)

$$\begin{aligned} a^2 &= 4\{swt\}^2 = 4\{s\{w(s \circ t)w\}t\} + \{s\{wt^2w\}s\} \\ &\quad + \{t\{ws^2w\}t\} - 2\{sws\} \circ \{twt\} \\ &= 4\{s\{w(s \circ t)w\}t\} + e + f - 2e \circ f \\ &= 4\{s\{w(s \circ t)w\}t\} + e + f \end{aligned}$$

Therefore to show $a^2 = e + f$ we must show $\{s\{w(s \circ t)w\}t\} = 0$.

This will be accomplished as soon as we have shown

(6.6) $\{w(s\circ t)w\} = \{s\{e\{s(s\circ t)s\}e\}s\}$

(6.7) $\{s(s\circ t)s\} = s\circ t$

(6.8) $\{e(s\circ t)e\} = 0$.

The identity (6.6) follows from eq. 2.25. To establish (6.7) we use the identity

$$L_{c^3} = 3L_c L_{c^2} - 2L_c^3$$

[13,p.35,eq. 56], which implies $L_s^3 = L_s$. Thus

$$\{s(s\circ t)s\} = (2L_s^2 - L_{s^2})L_s t = (2L_s^3 - L_s)t = s\circ t .$$

To prove that (6.8) holds note that $s\circ t = \{1st\} = \{e_1 st\} + \{e_2 st\} + \{e_3 st\}$, so $\{e(s\circ t)e\} = \{e_1(s\circ t)e_1\} = \{e_1\{e_1 st\}e_1\} = \{e_1 \tfrac{1}{2}ae_1\} = 0$, where we have used that $a \in M_{12}$, $\{e_2 st\} \in M_{22} + M_{12} + M_{23}$, and $\{e_3 st\} \in M_{33} + M_{13} + M_{23}$. Thus we have shown (6.1).

To show (6.4) let $h = 1 + a - a^2 = a + (1-e-f)$. Then by (6.1) - (6.3)

$$\begin{aligned} h^2 &= a^2 + 2a\circ(1-e-f) + (1-e-f)^2 \\ &= e + f + 0 + 1 - e - f = 1 , \end{aligned}$$

so h is a symmetry. Finally

$$U_h e = 2h\circ(h\circ e) - e = 2h\circ(\tfrac{1}{2}a) - e = a^2 - e = f .$$

The proof is complete. □

Let e and f be orthogonal idempotents in M, and $M = \sum_{i<j} M_{ij}$ the corresponding Pierce decomposition for $e, f, 1-e-f$. Then e and f are said to be <u>strongly connected</u> if there is $a \in M_{12}$ such that $a^2 = e + f$. By virtue of (6.1) - (6.3) of the previous proof we have the following corollary.

Corollary 6.6 If e and f are orthogonal idempotents in M and $e \sim f$ via a symmetry, then e and f are strongly connected.

Lemma 6.7 If e and f are orthogonal idempotents and $e \sim f$ then e and f dominate non-zero idempotents p and q respectively such that $p \sim q$ via a symmetry.

Proof. By assumption there exist symmetries $s_1, \ldots, s_n$ in M and idempotents $e_1 = e, e_2, \ldots, e_{n+1} = f$ such that $e_i \sim e_{i+1}$ via s_i, $i = 1, \ldots, n$. We use induction on n. If $n = 1$ the lemma is trivial, and if $n = 2$ it follows from Lemma 6.5. Assume $n > 2$ and that the lemma holds for all smaller values of n.

If e_1 and e_n are orthogonal then by induction there exist nonzero idempotents $p \leq e_1$ and $r \leq e_n$ with $p \sim r$ via a symmetry. Let $q = \{s_n r s_n\}$. Then $q \leq e_{n+1} = f$, $r \sim q$ via s_n, and p and q are orthogonal. By Lemma 6.5 $p \sim q$ via a symmetry.

If e_1 and e_n are not orthogonal then by Lemma 6.3 there exist nonzero idempotents $p \leq e_1$, $r \leq e_n$ such that $p \sim r$ via a symmetry. Now proceed as in the preceeding paragraph. $\square$

Lemma 6.8 Let $\{e_\alpha\}$ and $\{f_\alpha\}$ be indexed sets of pairwise orthogonal idempotents. Let $e = \vee e_\alpha$, $f = \vee f_\alpha$ and assume e and f are orthogonal. If $e_\alpha \sim f_\alpha$ via a symmetry then $e \sim f$ via a symmetry.

Proof. Let p and q be orthogonal idempotents with $p \sim q$ via a symmetry. From the proof of Lemma 6.5 applied to the case $t = 1$ we have that $p \sim q$ via a symmetry $h = a + (1-p-q)$ where

$a^2 = p+q$. Thus $p \sim q$ via $-h = -a-(1-p-q)$. Since $-a \in U_{p+q}(M)$, $-a = 2r-(p+q)$ where r is an idempotent, $r \leq p+q$. Thus $p \sim q$ via $2r-1$.

We can thus for each pair e_α, f_α choose an idempotent $p_\alpha \leq e_\alpha + f_\alpha$ such that $e_\alpha \sim f_\alpha$ via $2p_\alpha - 1$. Let $p = \vee p_\alpha$. We show $e \sim f$ via $2p-1$. For this we establish

(6.9) $p - p_\alpha$ is orthogonal to e_α for all α

(6.10) $\{(2p-1)e_\alpha(2p-1)\} = f_\alpha$ for all α

By assumption $e_\alpha + f_\alpha$ is orthogonal to $e_\beta + f_\beta$ for all $\alpha \neq \beta$. Thus p_α is orthogonal to p_β for all $\alpha \neq \beta$, and $p = \bigvee_{\beta\neq\alpha} p_\beta + p_\alpha$. Therefore

$$p - p_\alpha = \bigvee_{\beta\neq\alpha} p_\beta \leq \bigvee_{\beta\neq\alpha}(e_\beta + f_\beta) \leq 1-(e_\alpha+f_\alpha) ,$$

which proves (6.9). In order to show (6.10) it suffices to show

(6.11) $\{(2p-1)e_\alpha(2p-1)\} = \{2p_\alpha-1)e_\alpha(2p_\alpha-1)\}$.

Now by (6.9)

$$L_{2p-1}e_\alpha = (L_{2p_\alpha-1} + L_{2(p-p_\alpha)})e_\alpha = L_{2p_\alpha-1}e_\alpha .$$

Since $p \geq p_\alpha$, p and p_α operator commute so

$$L^2_{2p-1}e_\alpha = L_{2p-1}L_{2p_\alpha-1}e_\alpha = L_{2p_\alpha-1}L_{2p-1}e_\alpha = L^2_{2p_\alpha-1}e_\alpha .$$

Thus

$$\begin{aligned}\{(2p-1)e_\alpha(2p-1)\} &= (2L^2_{2p-1} - I)e_\alpha \\ &= (2L^2_{2p_\alpha-1} - I)e_\alpha = \{(2p_\alpha-1)e_\alpha(2p_\alpha-1)\}\end{aligned}$$

as asserted. Thus (6.11) and therefore (6.10) follows.

By (6.10) we have that for finite subsets $\{e_{\alpha_1},\dots,e_{\alpha_n}\}$ of $\{e_\alpha\}$ we have

$$U_{2p-1}(\bigvee_1^n e_{\alpha_i}) = U_{2p-1}(\sum_1^n e_{\alpha_i}) = \sum_1^n f_{\alpha_i} = \bigvee_1^n f_{\alpha_i} .$$

By Lemma 4.1 we have $U_{2p-1}(e) = f$, as asserted. □

Theorem 6.9 (The halving lemma). If the JB-factor M has no minimal idempotents then every idempotent e in M can be halved, i.e. $e = p + q$ where p and q are idempotents in M such that $p \sim q$ via a symmetry.

Proof. We may assume $e \neq 0$. Let $\{e_\alpha\}$ and $\{f_\alpha\}$ be maximal collections of idempotents satisfying the hypotheses of Lemma 6.8 and $e_\alpha + f_\alpha \leq e$ for all α. Let $p = \vee e_\alpha$, $q = \vee f_\alpha$. By Lemma 6.8 $p \sim q$, $p + q \leq e$. We show $p + q = e$. If not then $0 \neq r = e - (p+q)$. By assumption r is not a minimal idempotent so there exist nonzero idempotents r' and r'' in M with sum r. By Lemma 6.4 r' and r'' are related, say $0 \neq r_1' \leq r'$ and $0 \neq r_1'' \leq r''$ are equivalent. Since r_1' and r_1'' are orthogonal they have by Lemma 6.7 non-zero subprojections r_2' and r_2'' with $r_2' \sim r_2''$ via a symmetry. But then $\{r_2'\} \cup \{e_\alpha\}$ and $\{r_2''\} \cup \{f_\alpha\}$ are families satisfying the conditions of Lemma 6.8, contradicting the maximality of $\{e_\alpha\}$, $\{f_\alpha\}$. Thus we conclude that $p + q = e$. □

We say a JB-factor is of type I if it contains a minimal idempotent. Notice that if p is a minimal idempotent in a JB-factor M then every idempotent q in M_p is an idempotent in M with $0 \leq q \leq p$. By minimality of p it follows that $M_p \cong \mathbb{R}$.

Theorem 6.10 Let M be a JB-factor of type I. Then all minimal idempotents p, q in M satisfy $p \sim q$ via a symmetry, and $1 = \vee p_\alpha$ for a suitable orthogonal family $\{p_\alpha\}$ of minimal idempotents.

Proof. The first statement follows from Lemma 6.3 if p and q are non orthogoal, and from Lemmas 6.4 and 6.7 if they are orthogonal.

For the second statement let $\{p_\alpha\}$ be a maximal family of orthogonal minimal idempotents. Suppose $p = \vee p_\alpha < 1$. Then for any β p_β and $1-p$ are related by Lemma 6.4, say $p_\beta \sim p_0 \leq 1-p$ via a symmetry s. Since U_s is a lattice automorphism of the lattice $\mathcal{P}$, p_0 is a minimal idempotent. This contradicts the maximality of the family $\{p_\alpha\}$. Thus $\vee p_\alpha = 1$. $\square$

We shall say that a JB-factor is of type I_n, $1 \leq n \leq \infty$, if n is the least upper bound of the number of pairwise orthogonal non-zero idempotents.

§ 7. Spin factors

We show that every JB-factor of type I_2 is an abstract spin factor as defined by Topping [30], and thus isometrically isomorphic to a JC-algebra [31].

Let H be a real Hilbert space of dimension at least 3 and e a distinguished unit vactor in H. Let $N = \{e\}^{\perp}$, so $H = \mathbb{R}e \oplus N$. Then H becomes an abstract spin factor when equipped with the Jordan product

$$(7.1)\quad (\alpha e+a)\circ(\beta e+b) = (\alpha\beta+(a,b))e + (\alpha b+\beta a), \qquad \alpha,\beta \in \mathbb{R},\ a,b \in N.$$

Proposition 7.1 Every JB-factor M of type I_2 admits an inner product making it an abstract spin factor. Thus every JB-factor of type I_2 is isometrically isomorphic to a JC-algebra.

Proof. Let N be the linear span of the symmetries in M different from ± 1. Then $M = \mathbb{R}1 + N$. Indeed, since M is of type I_2 if $a \in M$ then there are minimal orthogonal idempotents $p,q \in M$ with sum 1 and $\alpha,\beta \in \mathbb{R}$ such that $a = \alpha p + \beta q$, hence

$$(7.2)\quad a = \tfrac{1}{2}(\alpha+\beta)1 + \tfrac{1}{2}(\alpha-\beta)(p-q) \in \mathbb{R}1 + N.$$

Thus $M = \mathbb{R}1 + N$.

Let $s,t \in N$ be symmetries different from ± 1. Then $s\circ t \in \mathbb{R}1$, Indeed, let p,q be minimal orthogonal idempotents such that $s = p-q$. Let $M = M_{11} \oplus M_{22} \oplus M_{12}$ be the Pierce decomposition of M for p,q, and let $t = \alpha p + \beta q + r$ be the corresponding decomposition of t. Then $p\circ r = q\circ r = \tfrac{1}{2}r$, so that

$$1 = t^2 = (\alpha^2 p+\beta^2 q+r^2) + \tfrac{1}{2}(\alpha+\beta)r.$$

By properties of the Pierce decomposition $r^2 \in M_{11} \oplus M_{22}$ [13,p.119, Lemma 1], and $(\alpha+\beta)r = 0$. If $r = 0$ then $\alpha^2 = \beta^2 = 1$ so $\alpha = -\beta$ since $t \neq \pm 1$. If $r \neq 0$ then $\alpha + \beta = 0$. Thus in either case $t = \alpha(p-q) + r$, so that

$$s \circ t = (p-q) \circ (\alpha(p-q)+r) = \alpha 1 \in \mathbb{R} 1 ,$$

and the assertion follows.

We now show $\mathbb{R} 1 \cap N = \{0\}$, thus showing that $M = \mathbb{R} 1 \oplus N$. Suppose $1 = \sum_1^n \lambda_i s_i$, with $s_1, \ldots, s_n$ symmetries $\neq \pm 1$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then for $j \in \{1, \ldots, n\}$

$$s_j = s_j \circ 1 = \Sigma \lambda_i s_j \circ s_i \in \mathbb{R} 1 ,$$

contrary to the assumption that $s_j \neq \pm 1$, so $1 \notin N$, and the sum is direct.

To construct the real Hilbert space let ρ be the linear functional on M which is 1 on 1 and 0 on N. Define a bilinear form on M by

$$(a,b) = \rho(a \circ b) .$$

By (7.2) $a \in M$ can be written as $a = \lambda 1 + \mu s$ with s a symmetry in N. Then if $a \neq 0$

$$(a,a) = \rho(a^2) = \lambda^2 + \mu^2 \neq 0 ,$$

so that $(,)$ is an inner product on M. Note $\{1\}^\perp = \{a : \rho(a \circ 1) = 0\} = N$.

By (7.2) and $M = \mathbb{R} 1 \oplus N$ every element of N is a multiple of a symmetry; thus if $a,b \in N$ then $a \circ b \in \mathbb{R} 1$, and therefore $(a,b)1 = \rho(a \circ b)1 = a \circ b$. Thus we have

$$(\alpha 1+a) \circ (\beta 1+b) = (\alpha\beta+(a,b))1 + (\alpha b+\beta a) ,$$

which shows that (7.1) holds. It therefore remains to show that M

is complete with respect to the norm $|||a||| = \rho(a^2)^{\frac{1}{2}}$. But if $a = \alpha p + \beta q \in M$ with p and q orthogonal minimal idempotents then with $\|a\| = \max(|\alpha|, |\beta|)$ the JB-norm, we have from (7.2)

$$2^{-\frac{1}{2}}\|a\| \leq 2^{-\frac{1}{2}}(\alpha^2+\beta^2)^{\frac{1}{2}} = |||a||| \leq \|a\| ,$$

so the two norms are equivalent. The proof is complete.

§ 8. Jordan matrix algebras

Let $\mathfrak{A}$ be any algebra over $\mathbb{R}$ with identity 1 and involution $*$. Let $\mathfrak{A}_n$ denote the $n \times n$ matrices over $\mathfrak{A}$ with $A \to A^*$ the usual involution (apply $*$ to each entry and then transpose). Let $H(\mathfrak{A}_n)$ denote the hermitian matrices in $\mathfrak{A}_n$ $(A = A^*)$ with the product $A \circ B = \frac{1}{2}(AB+BA)$. If $H(\mathfrak{A}_n)$ is a Jordan algebra then we say $H(\mathfrak{A}_n)$ is a Jordan matrix algebra.

Theorem 8.1 Every JB-factor M (except those of type I_2) is isomorphic to a Jordan matrix algebra $H(\mathfrak{A}_n)$. If in addition M is not of type I_3 then $\mathfrak{A}$ is associative.

Proof. It is known that a Jordan algebra with identity is isomorphic to a Jordan matrix algebra $H(\mathfrak{A}_n)$ with $n \geq 3$ if and only if the identity 1 is the sum of n strongly connected idempotents [13, Theorem 5, p.133]. Furthermore $\mathfrak{A}$ will be associative if $n \geq 4$ [13, Theorem 1, p.127]. We apply this result to the different types of JB-factors.

1. If M is of type I_n, $3 \leq n < \infty$, then by Corollary 6.6 and Theorem 6.10 the identity is the sum of n strongly connected idempotents.

2. Suppose M is of type I_∞. If the identity is the supremum of an infinite set of orthogonal minimal idempotents, divide these idempotents into four sets of equal cardinality $\{p^i_\alpha\}$, $i = 1,2,3,4$, and let $p^i = \vee_\alpha p^i_\alpha$. Then $\sum_1^4 p^i = 1$ and by Corollary 6.6, Lemma 6.8 and Theorem 6.10 the p^i are all strongly connected. If it should happen that no such infinite set of idempotents exist, then 1 is the sum of arbitrarily large finite subsets of orthogonal minimal idempotents, so 1. applies

3. Suppose M is not of type I. By Theorem 6.9 $1 = p + q$ with $p \sim q$ via a symmetry s. Applying Theorem 6.9 again to p we have $p = p_1 + p_2$ with $p_1 \sim p_2$ via a symmetry t. Define $q_i = U_s p_i$. Then $q_1 + q_2 = q$, and so $1 = p_1 + p_2 + q_1 + q_2$ with $q_1 \sim p_1$ via s, $p_1 \sim p_2$ via t, $p_2 \sim q_2$ via s. By Lemma 6.5 and Corollary 6.6 each pair among $\{p_1, p_2, q_1, q_2\}$ are strongly connected, concluding the proof. ∎

Lemma 8.2 Let $H(\mathcal{a}_n)$, $n \geq 3$, be a Jordan matrix algebra which is also a JB-algebra. Then for each $a \in \mathcal{a}$, $a^*a = aa^* = 0$ implies $a = 0$.

Proof. Let $\{E_{ij}\}$ be the matrix units for $\mathcal{a}_n$. If $a \in \mathcal{a}$ let $A = a^*E_{12} + aE_{21}$. Then $A \in H(\mathcal{a}_n)$ and $A^2 = a^*aE_{11} + aa^*E_{22}$. Thus if $a^*a = aa^* = 0$ then $A^2 = 0$, hence $A = 0$, since $H(\mathcal{a}_n)$ is a JB-algebra. Thus $a = 0$. ∎

Proposition 8.3 JB-factor M of type I_n $(3 \leq n < \infty)$ is finite dimensional, and thus the JB-factors of these types are precisely the $n \times n$ Jordan matrix algebras over the reals, complexes, or the quaternions, or the exceptional algebra M_3^8 - the 3×3 Jordan matrix algebra over the Cayley numbers.

Proof. By Theorem 8.1 we can identify M with $H(\mathcal{a}_n)$. We will show that if a Jordan matrix algebra $H(\mathcal{a}_n)$, $n \geq 3$, is also a JB-algebra then $\mathcal{a}$, and hence $H(\mathcal{a}_n)$, is finite dimensional.

We will use a result of Albert [1], which says that an alternative quadratic algebra over $\mathbb{R}$ is finite dimensional. (A quadratic algebra is an algebra with identity in which every element satisfies a quadratic over $\mathbb{R}$ and every element generates a subalgebra which is also a field. An algebra is alternative if the identities $(a^2)b = a(ab)$ and $b(a^2) = (ba)a$ hold).

We first show that each $a \in \mathcal{A}$ satisfies a quadratic. Note that the set $H(\mathcal{A})$ of hermitian elements in $\mathcal{A}$ equals $\mathbb{R}1$. Indeed, if $\{E_{ij}\}_{1\leq i,j\leq n}$ are the matrix units in $\mathcal{A}_n$ then since $\Sigma E_{ii} = 1$ and $H(\mathcal{A}_n)$ is a JB-factor of type I_n, each E_{ii} is a minimal idempotent. Thus

$$\{aE_{ii} : a \in H(\mathcal{A})\} = \{E_{ii} M E_{ii}\} = \mathbb{R} E_{ii} ,$$

and $H(\mathcal{A}) = \mathbb{R}1$, as asserted. Therefore, if $a \in \mathcal{A}$ there is $\lambda \in \mathbb{R}$ such that $a - \lambda 1$ is skew adjoint, hence $(a-\lambda 1)^2$ is hermitian so in $\mathbb{R}1$. Say $(a-\lambda 1)^2 = \mu 1$. Then a satisfies the quadratic $a^2 - 2\lambda a + (\lambda^2 - \mu) = 0$.

It is known [13, Theorem 1, p.127] that if $H(\mathcal{A}_n)$ is Jordan, $n \geq 3$, then $\mathcal{A}$ is alternative. There remains to show that for each $a \in \mathcal{A}$, $\mathbb{R}1 + \mathbb{R}a$ is a field. Suppose a satisfies the quadratic $a^2 - \alpha a - \beta 1 = 0$. If $\beta \neq 0$ then clearly a is invertible in $\mathbb{R}1 + \mathbb{R}a$. If $\beta = 0$ then $a^2 = \alpha a$. But then either $a = 0$ or $a = \alpha 1$. Indeed, since $\mathcal{A}$ is alternative

(8.1) $\quad (a^*a)a = a^*(a^2) = a^*(\alpha a) = \alpha(a^*a)$

(8.2) $\quad a(aa^*) = (a^2)a^* = (\alpha a)a^* = \alpha(aa^*)$

Both aa^* and a^*a are hermitian, hence in $\mathbb{R}1$. If both are zero then by Lemma 8.2 $a = 0$, and if one of them is non-zero then by (8.1) and (8.2) $a = \alpha 1$. This completes the proof that M is finite dimensional.

Finally, the last statement of the proposition follows from the Jordan-von Neumann-Wigner classification of finite dimensional formally real Jordan algebras [15]. An alternative proof is provided by the fact that finite dimensional alternative division algebras over $\mathbb{R}$ must be either the reals, complexes, quaternions, or the Cayley numbers, see e.g. [19, p.234]. $\square$

Lemma 8.4 Let $H(\mathcal{A}_n)$, $n \geq 3$, be a Jordan matrix algebra which is also a JB-algebra. Assume $\mathcal{A}$ is associative. Then $A^*A = 0$ if and only if $A = 0$, and $A^*A \geq 0$ for all $A \in \mathcal{A}_n$.

Proof. We assert that if $A \in \mathcal{A}_n$ then $\sigma(A^*A) \cup \{0\} = \sigma(AA^*) \cup \{0\}$. Let $\mathcal{U}$ be any associative algebra over $\mathbb{R}$ with identity, and let $\mathcal{U}^+$ be $\mathcal{U}$ with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Then for elements of $\mathcal{U}^+$, Jordan inverses in $\mathcal{U}^+$ coincide with associative inverses [13, p.51]. Thus if $B \in H(\mathcal{A}_n) \subset \mathcal{A}_n^+$ then B is Jordan invertible in $\mathcal{A}_n^+$ if and only if B is invertible in $\mathcal{A}_n$. Thus $\sigma(B) = \{\lambda \in \mathbb{R} : \lambda I - B$ is not invertible in $\mathcal{A}_n\}$. Now a standard argument shows that in any associative algebra over $\mathbb{R}$ with identity 1, if $0 \neq \lambda \in \mathbb{R}$ then $ab - \lambda 1$ is invertible if and only if $ba - \lambda 1$ is invertible. The assertion follows.

As a consequence of the preceeding it follows that if $a \in \mathcal{A}$ and $a^*a = 0$ then $a = 0$. Indeed, if $\{E_{ij}\}$ are the matrix units in $\mathcal{A}_n$ and $A = aE_{11}$, then $A^*A = 0$. By the above $0 = AA^* = aa^*E_{11}$, so Lemma 8.2 implies $a = 0$.

We assert that if $A^*A \leq 0$ then $A = 0$. Indeed, if $a \in \mathcal{A}$ then

$$0 \leq E_{ii}(a^*E_{ij} + aE_{ji})^2 E_{ii} = a^*aE_{ii} ,$$

so if $A = \Sigma a_{ij} E_{ij}$ then $A^* = \Sigma a_{ji}^* E_{ij}$, and

$$0 \geq E_{ii}(A^*A)E_{ii} = \sum_k (a_{ki}^* a_{ki})E_{ii} \geq 0 .$$

Thus $a_{ki}^* a_{ki} E_{ii} = 0$, and therefore by the preceeding paragraph

$a_{ki} = 0$ for all k,i. Thus $A = 0$, as asserted.

Note that in particular $A^*A = 0$ implies $A = 0$.

Let $A \in \mathfrak{A}_n$. By spectral theory there exist $B_1, B_2 \geq 0$ in $H(\mathfrak{A}_n)$ such that $A^*A = B_1 - B_2$ and $B_1B_2 = 0$. Now $(AB_2)^*(AB_2) = B_2A^*AB_2 = -B_2^3 \leq 0$, so by the preceeding paragraph $AB_2 = 0$, hence $B_2 = 0$, and $A^*A \geq 0$. □

We are now in the position where we can construct the GNS-representation due to a state on a JB-algebra which is a Jordan matrix algebra over an associative algebra.

__Lemma 8.5__ Let $H(\mathfrak{A}_n)$, $n \geq 3$, be a Jordan matrix algebra which is also a JB-algebra. Assume $\mathfrak{A}$ is associative. Let ρ be a state on $H(\mathfrak{A}_n)$. Then there exist a complex Hilbert space H_ρ, a Jordan homomorphism π_ρ of $H(\mathfrak{A}_n)$ into the self-adjoint operators on H_ρ, and a unit vector ξ_ρ in H_ρ, such that for $A \in H(\mathfrak{A}_n)$, $\langle A, \rho \rangle = (\pi_\rho(A)\xi_\rho, \xi_\rho)$.

__Proof.__ Extend ρ to all of $\mathfrak{A}_n$ by defining it to be zero on skew-adjoint elements of $\mathfrak{A}_n$. By Lemma 8.4 ρ is a linear functional on $\mathfrak{A}_n$ such that $\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}_n$. If $\mathfrak{A}_n^+$ is the set of operators A^*A, $A \in \mathfrak{A}_n$ then $\mathfrak{A}_n^+$ is a cone and the map $B \to A^*BA$ maps $\mathfrak{A}_n^+$ into itself. Thus if we define $\|A\| = \|A^*A\|^{\frac{1}{2}}$ then $\|AB\|^2 = \|B^*A^*AB\| \leq \|B^*(\|A^*A\|)B\| \leq \|A\|^2\|B\|^2$, and

$$\|A+B\|^2 = \|(A^*+B^*)(A+B)\| = \sup_{\omega \in K}\langle A^*A+B^*B+B^*A+A^*B, \omega\rangle$$

$$\leq \sup_{\omega \in K}[\langle A^*A,\omega\rangle + \langle B^*B,\omega\rangle + 2\langle A^*A,\omega\rangle^{\frac{1}{2}}\langle B^*B,\omega\rangle^{\frac{1}{2}}]$$

$$\leq \|A\|^2 + \|B\|^2 + 2\|A\|\|B\| = (\|A\|+\|B\|)^2,$$

where K is the state space of $H(\mathfrak{A}_n)$.

Since the norm completeness of $H(\mathcal{A}_n)$ implies $\mathcal{A}_n$ is norm complete, $\mathcal{A}_n$ with the norm $\| \ \|$ is a real Banach algebra. Thus the usual GNS-construction is applicable to $\mathcal{A}_n$, see [10, Proposition 2.4.4] which is stated for algebras over the complexes, but whose proof is valid for real Banach algebras. We can therefore find a real Hilbert space H, a $*$-homomorphism π of $\mathcal{A}_n$ into the bounded operators on H, and a unit vector ξ in H such that $\langle A, \rho \rangle = (\pi(A)\xi, \xi)$ for all $A \in \mathcal{A}_n$.

Finally, let H_ρ be the complexification of H. Then the injection of H into H_ρ induces an isometric imbedding of the bounded operators on H into those on H_ρ. The injection is also a $*$-isomorphism, so the image of $\pi(H(\mathcal{A}_n))$ consists of self-adjoint operators. (It is a JC-algebra by Lemma 9.3 below.) Let π_ρ be the composition of π and the injection of H into H_ρ, and ξ_ρ the image of ξ. $\square$

<u>Theorem 8.6</u> Every JB-factor M except M_3^8 is isomorphic to a JC-algebra.

<u>Proof</u>. We have already shown that I_2-factors are isomorphic to JC-algebras, Proposition 7.1. It is a classical result of Albert-Paige [2] that M_3^8 is not special, and so cannot be isomorphic to a JC-algebra. All other JB-factors are by Theorem 8.1 and Proposition 8.3 isomorphic to a Jordan matrix algebra $H(\mathcal{A}_n)$ with $\mathcal{A}$ associative. We will show that such JB-factors are isomorphic to JC-algebras.

Let K be the state space of M and H the Hilbert space direct sum $\sum_{\rho \in K} \oplus H_\rho$, where H_ρ is given by Lemma 8.5. Let $\pi = \sum_{\rho \in K} \pi_\rho$ be the direct sum of the representations π_ρ on H_ρ found in Lemma 8.5. Then for $A \in \mathcal{A}_n$

$$\|\pi(A)\| = \sup_{\rho \in K} \|\pi_\rho(A)\| \leq \|A\|$$

On the other hand $\|\pi_\rho(A)\xi_\rho\|^2 = \rho(A^*A)$, so

$$\|\pi_\rho(A)\|^2 \geq \rho(A^*A)$$

Thus

$$\|A\|^2 \geq \|\pi(A)\|^2 \geq \sup_{\rho \in K} \rho(A^*A) = \|A^*A\| = \|A\|^2 ,$$

so π is an isometry. Therefore the image of M in $B(H)$ - the bounded operators on H - is a JC-algebra. $\square$

For pure states we can sharpen the above result. The following proposition will not be needed in the sequel.

<u>Proposition 8.7</u> Let A be a JB-algebra and ρ a pure state on A. Let φ_ρ be the Jordan homomorphism of A into $\tilde{A}$ defined by ρ in Proposition 5.6. Then the strong closure of $\varphi_\rho(A)$ in $\tilde{A}$ is a JB-factor of type I.

<u>Proof.</u> Let M be the strong closure of $\varphi_\rho(A)$ in $\tilde{A}$. By Proposition 5.6 M is a JB-factor. If $M \cong M_3^8$ then M is of type I; otherwise M is isomorphic to a JC-algebra by Theorem 8.6. In particular, by [29, Theorem 7.1] a state ω on M is pure if and only if its kernel $I_\omega = \{a \in M : \omega(a^2) = 0\}$ is a maximal quadratic ideal (a quadratic ideal is a linear subspace I of M with $U_a b \in I$ whenever $a \in I$, $b \in M$). Since every state of M majorized by a multiple of ρ is itself strongly continuous, ρ is a pure state on M. Thus I_ρ is a strongly closed maximal quadratic ideal. Since I_ρ is a Jordan algebra, it has an identity e, which equals

$1 - \mathrm{supp}\,\rho$. Since M_p is a quadratic ideal for each idempotent p in M, the maximality of I_ρ implies that $\mathrm{supp}\,\rho$ is a minimal idempotent. Thus M is of type I. $\square$

§ 9. Ideals and representations

In this section we prove our main representation theorem for JB-algebras.

Let A be a JB-algebra. By a <u>Jordan ideal</u> in A we shall mean a norm closed linear subspace J of A such that $a \in J$, $b \in A$ implies $a \circ b \in J$. From the identity $U_b a = 2b \circ (b \circ a) - b^2 \circ a$ it follows that $U_b a \in J$ whenever a or b belongs to J. We say a family (u_α) in J is an <u>increasing approximate identity</u> for J if i) $0 \leq u_\alpha \leq 1$, ii) $\alpha \leq \beta$ implies $u_\alpha \leq u_\beta$, iii) $\lim_\alpha \|a - u_\alpha \circ a\| = 0$ for all $a \in J$. If A/J is given the quotient norm we let $a \to a + J$ be the canonical homomorphism of A onto A/J.

We shall first show that A/J is a JB-algebra. The proof is modelled on the analogous one for C*-algebras, as found in [10].

<u>Lemma 9.1</u> Let J be a Jordan ideal in the JB-algebra A. Then J has an increasing approximate identity (u_α) such that for all $a \in A$ we have

$$\|a + J\| = \lim_\alpha \|a - u_\alpha \circ a\| = \lim_\alpha \|U_{u_\alpha - 1} a\| .$$

<u>Proof</u>. Let Λ be the set of finite subsets of J ordered by inclusion. For $\alpha = \{a_1, \ldots, a_n\} \in \Lambda$ let $v_\alpha = \sum_{i=1}^n a_i^2$, $u_\alpha = v_\alpha \circ (\frac{1}{n} 1 + v_\alpha)^{-1}$. Then $u_\alpha \in J$, and by spectral theory $0 \leq u_\alpha \leq 1$. On the other hand

$$\sum_{i=1}^n U_{u_\alpha - 1}(a_i^2) = U_{u_\alpha - 1}(v_\alpha) = n^{-2} v_\alpha (\tfrac{1}{n} 1 + v_\alpha)^{-2} \leq 1/4n .$$

In particular

$$(9.1) \qquad U_{u_\alpha - 1}(a_i^2) \leq 1/4n , \qquad i = 1, \ldots, n .$$

Now if $a,b \in A$ then

$$\|U_a b^2\| = \|U_b a^2\| . \tag{9.2}$$

Indeed, by eq. 2.25

$$\begin{aligned}\|U_a b^2\|^2 &= \|\{a\{b\{ba^2b\}b\}a\}\| \\ &\leq \|\{ba^2b\}\| \; \|\{ab^2a\}\| ,\end{aligned}$$

so that $\|U_a b^2\| \leq \|U_b a^2\|$, and (9.2) follows by symmetry.

Let $a,b \in A$ with $b \geq 0$. Then we have

$$\|a \circ b\|^2 \leq \|U_a b\| \; \|b\| . \tag{9.3}$$

Indeed, $(a \circ b)^2 = \frac{1}{2}(U_a b) \circ b + \frac{1}{4} U_a b^2 + \frac{1}{4} U_b a^2$, so by (9.2) we have

$$\begin{aligned}\|a \circ b\|^2 &\leq \tfrac{1}{2}\|U_a b\| \; \|b\| + \tfrac{1}{2}\|U_a b^2\| \\ &\leq \tfrac{1}{2}\|U_a b\| \; \|b\| + \tfrac{1}{2}\|U_a b\| \; \|b\| ,\end{aligned}$$

and (9.3) follows. In particular by (9.1)

$$\|(u_\alpha - 1) \circ a_i^2\|^2 \leq \|U_{u_\alpha - 1}(a_i^2)\| \; \|a_i^2\| \leq \tfrac{1}{4}n \|a_i\|^2$$

Thus for all $a \in J^+$, and therefore by linearity for all $a \in J$, $\|u_\alpha - 1) \circ a\| \to 0$ with α. In particular (u_α) is an approximate unit. By spectral theory it follows, see [10,p.16], that (u_α) is an increasing approximate unit.

Let $b \in J$. Then $u_\alpha \circ b \to b$. Therefore if $a \in A$

$$\begin{aligned}\overline{\lim}\|a - u_\alpha \circ a\| &= \overline{\lim}\|a - u_\alpha \circ a + b - u_\alpha \circ b\| \\ &= \overline{\lim}\|(1 - u_\alpha) \circ (a+b)\| \\ &\leq \|a+b\|\end{aligned}$$

Thus

$$\begin{aligned}\|a+J\| &\geq \overline{\lim}\|a - u_\alpha \circ a\| \geq \underline{\lim}\|a - u_\alpha \circ a\| \\ &\geq \inf_{b \in J}\|a+b\| = \|a+J\| ,\end{aligned}$$

so the first equality in the lemma follows. To show the second let $a \in A$. Then

$$(9.4) \qquad U_{u_\alpha - 1}(a) = a + c$$

where $c = U_{u_\alpha}(a) - 2u_\alpha \circ a \in J$. If $b \in J$ then by (9.1) $U_{u_\alpha - 1}(b) \to 0$. Thus

$$(9.5) \qquad \overline{\lim}\|U_{u_\alpha - 1}(a)\| = \overline{\lim}\|U_{u_\alpha - 1}(a+b)\| \leq \|a+b\| ,$$

where the last inequality follows since $\|u_\alpha - 1\| \leq 1$, so the norm of $\|U_{u_\alpha - 1}\| \leq 1$. Thus by (9.4) and (9.5)

$$\|a+J\| \leq \underline{\lim}\|U_{u_\alpha - 1}(a)\| \leq \overline{\lim}\|U_{u_\alpha - 1}(a)\|$$
$$\leq \inf_{b \in J}\|a+b\| = \|a+J\| .$$

The proof is complete. $\square$

<u>Lemma 9.2</u> Let J be a Jordan ideal in a JB-algebra A. Then A/J with its natural Jordan product and quotient norm is a JB-algebra.

<u>Proof.</u> We have to show that if $a,b \in A$ then

i) $\|a \circ b + J\| \leq \|a+J\| \, \|b+J\|$,

ii) $\|a^2 + J\| = \|a+J\|^2$,

iii) $\|a^2 + J\| \leq \|a^2 + b^2 + J\|$.

Let $a,b \in A$. Then

$$\begin{aligned} \|a \circ b + J\| &= \inf_{c \in J}\|a \circ b + c\| \\ &\leq \inf_{c,d \in J}\|(a+c) \circ (b+d)\| \\ &\leq \inf_{c,d \in J}\|a+c\| \, \|b+d\| \\ &= \|a+J\| \, \|b+J\| . \end{aligned}$$

Thus i) follows, and in particular $\|a^2+J\| \leq \|a+J\|^2$.
To prove the converse inequality in ii) let $a \in A$. Note that if $b \in A$, $\|b\| \leq 1$, then $(U_b a)^2 = U_b\{ab^2a\} \leq U_b(a^2)$. Thus $\|U_b(a)\|^2 \leq \|U_b(a^2)\|$. In particular if (u_α) is the approximate identity found in Lemma 9.1

$$\|a+J\|^2 = \lim_\alpha \|U_{u_\alpha^{-1}}(a)\|^2 \leq \lim_\alpha \|U_{u_\alpha^{-1}}(a^2)\| = \|a^2+J\|,$$

and ii) is proved.

Finally we show iii). Since for $a \in A$, $U_{u_\alpha^{-1}}(a^2) \geq 0$, $U_{u_\alpha^{-1}}(a^2)$ is itself a square. Thus by Lemma 9.1, if $a,b \in A$

$$\begin{aligned}\|a^2+b^2+J\| &= \lim_\alpha \|U_{u_\alpha^{-1}}(a^2+b^2)\| \\ &= \lim_\alpha \|U_{u_\alpha^{-1}}(a^2) + U_{u_\alpha^{-1}}(b^2)\| \\ &\geq \lim_\alpha \|U_{u_\alpha^{-1}}(a^2)\| \\ &= \|a^2+J\| . \qquad \square\end{aligned}$$

<u>Lemma 9.3</u> Let A and B be JB-algebras and $\varphi : A \to B$ a Jordan homomorphism such that $\varphi(1) = 1$. Then $\varphi(A)$ is a JB-algebra, and if φ is injective then φ is isometric.

<u>Proof</u>. Let $a \in A$ and $C(a)$ be the JB-subalgebra of A generated by a and 1. By Proposition 2.3 $C(a)$ is identified with a real $C(X)$, so if φ is injective it follows from well known arguments that $\|a\| = \|\varphi(a)\|$, hence φ is isometric. In the general case let J be the kernel of φ. By Lemma 9.2 A/J is a JB-algebra and the induced homomorphism $\bar{\varphi} : A/J \to B$ is an isomorphism onto $\varphi(A)$. By the above $\bar{\varphi}$ is isometric, so its image is complete, hence is a JB-algebra. $\square$

It is known that no set of identities exist characterizing special Jordan algebras among all Jordan algebras [13,Thm.2,p.11]. However, there do exist identities satisfied by all special Jordan algebras but not by all Jordan algebras, "s-identities". In what follows $f(a,b,c) = 0$ will be any such s-identity in three variables not satisfied by M_3^8 (cf. [13,Thm.12,p.51] for an example).

Lemma 9.4 For a JB-algebra A the following are equivalent:

i) A is special,

ii) $f(a,b,c) = 0$ for all $a,b,c \in A$,

iii) A is isomorphic to a JC-algebra.

Proof. The implications iii) $\Rightarrow$ i) $\Rightarrow$ ii) are trivial. We show ii) $\Rightarrow$ iii). Let ρ be a pure state of A, $c(\rho)$ its central support in $\tilde{A}$, and φ_ρ the associated factor representation (cf. Proposition 5.6). Since $\varphi_\rho(A)$ is strongly dense in $\tilde{A}_{c(\rho)}$ the Kaplansky density theorem (Proposition 3.9) shows that the unit ball in $\varphi_\rho(A)$ is strongly dense in that of $\tilde{A}_{c(\rho)}$. Since $f(\varphi_\rho(a),\varphi_\rho(b),\varphi_\rho(c)) = \varphi_\rho(f(a,b,c))$ for all $a,b,c \in A$, it follows from the strong continuity of multiplication on bounded sets (Proposition 3.7) that the identity holds in $\tilde{A}_{c(\rho)}$. By Theorem 8.6 $\tilde{A}_{c(\rho)}$ is isomorphic to a JC-algebra, hence Lemma 9.3 shows $\varphi_\rho(A)$ is isomorphic to a JC-algebra.

Let $B = \Sigma\oplus\varphi_\rho(A)$ be the direct sum of the algebras $\varphi_\rho(A)$, ρ a pure state (i.e. pointwise operations with $\|\Sigma(a)_\rho\| = \sup_\rho\|(a)_\rho\|$). Clearly B is isomorphic to a JC-algebra. The map $a \to \Sigma\varphi_\rho(a)$ is an isomorphism since the pure states separate points, so by Lemma 9.3 A is isomorphic to the JB-algebra which is the image of A

in B . Thus A is isomorphic to a JC-algebra. □

Theorem 9.5 Let A be a JB-algebra. Then there is a unique Jordan ideal J in A such that A/J has a faithful isometric Jordan representation as a JC-algebra, and every factor representation of A not annihilating J is onto the exceptional algebra M_3^8.

Proof. Let J be the Jordan ideal generated by $\{f(a,b,c) : a,b,c \in A\}$. Note that if $\varphi : A \to B$ is a homomorphism then the identity $f(a,b,c) = 0$ holds in $\varphi(A)$ if and only if $J \subset \ker\varphi$. In particular this identity holds in A/J , so by Lemma 9.4 A/J is isomorphic to a JC-algebra.

If φ is any factor representation not annihilating J then the identity $f(a,b,c) = 0$ fails in $\varphi(A)$, so the strong closure of $\varphi(A)$ must equal M_3^8 by Theorem 8.6, hence $\varphi(A) = M_3^8$.

To prove uniqueness suppose J' is another Jordan ideal with the same properties. Since A/J' is special, J' must contain J by Lemma 9.4. Now each factor representation φ of A/J induces a factor representation $\bar{\varphi}$ of A. Since each such φ is not onto M_3^8, $\bar{\varphi}$ must annihilate J'. Since A/J admits a faithful family of factor representations, $J' \subset J$ follows. □

Remark 9.6 Instead of using s-identities as in Lemma 9.4 we could prove Theorem 9.5 by using structure space techniques. We then let the structure space Prim A consist of the kernels of all factor representations equipped with the hull-kernel topology. The crucial lemma is to show that the set $C = \{\ker\varphi \in \text{Prim}\,A : \varphi(A)$ is a JB-factor of type $I_n, n \leq 3\}$ is closed in Prim A .

The ideal J in Theorem 9.5 is defined as kernel T, where $T = \{\ker\varphi \in \text{Prim}\,A : \varphi(A)$ is isomorphic to a JC-algebra$\}$. If $S = \{\ker\varphi \in \text{Prim}\,A : \varphi(A) \cong M_3^8\}$ we have $J \cap \ker S = (0)$ by Theorem 8.6. Since C is closed the proof is easily completed.

Remark 9.7 In the case that A is a separable JB-algebra, then the proof of Theorem 9.5 can be greatly simplified. For every relatively exposed state ρ of a general JB-algebra A, the weak closure of the representation $\varphi_\rho(A)$ can be seen to be a JB-factor of type I. If A is separable, then by a theorem of Mazur and Milman [23,p.57] the exposed (and a fortiori the relatively exposed) states of A will be w*-dense among all pure states. Thus, in this case it will suffice to do all our general analysis of JB-factors for those of type I, and to restrict attention to minimal idempotents. Note that the Mazur-Milman Theorem can not be generalized to the non-separable case, not even if the term "exposed" is replaced by "relatively exposed". We are indebted to R. Phelps for this observation, which is based on Proposition 2.1 of [8]. (By the Hahn-Banach Theorem the algebraic exposed points of [8;Prop. 2.1] are the same as the relatively exposed points). Hence it is not possible to use a "relativized" Mazur-Milman Theorem to prove Theorem 9.5 in the general case. However, it will follow from Theorem 9.5 and [29;Thm.7.1] that all pure states of a JB-algebra are relatively exposed (cf. Proposition 8.7).

Remark 9.8 It might be expected that Theorem 9.5 could be improved in the sense that A is the direct sum of A/J and J. This is not true, as the following example shows.

For $n = 1,2,\ldots$, let $A_n = M_3^8$. Let $A \subset \Sigma \oplus A_n$ consist of all convergent sequences (a_n), where $a_n = (x_{ij}^n)_{i,j=1,2,3} \in M_3^8$, and $x_{ij}^n \to 0$ for $i \neq j$. Then it is easy to show that with pointwise operations A is a JB-algebra, $J = \{(a_n) : a_n \to 0\}$, and A/J is 3-dimensional and associative.

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