# Forcing arguments and some degree-theoretic problems in higher type recursion theory

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Oxford - 1975

This note is an addition to Normann [4]. The starting point for the imbedding theory in [4] was the search for a notion of abstract k-section. This notion was not found in [4], but by Sacks [6]. The purpose with this note is to see how Sacks' result can be stated and proved inside the imbedding framework. We also adopt the method of Sacks to find a notion of abstract section for the super-jump. In the end we will try to throw some light on the extended plus-one hypothesis and on some degree-theoretic problems.

The use of forcing in characterisation-problems was introduced by Sacks [5], when he characterized the one-section of a normal type-2 functional. An alternative proof was given in Normann [3].

## 1. Reflecting properties.

A type-k-theory is said to be a Grilliot-theory if it satisfies the Grilliot selection principle, i.e. from every semirecursive set containing a type k element we are able to effectively select a nonempty recursive subset. MacQueen and Harrington [2] proved that recursion in a normal type k+2 functional gives a Grilliot-theory. Harrington [1] used this fact to prove the reflection principles listed below. These were verified in a more general setting by Kechris and Moldestad. We omit all proofs here, just formulate the various concepts in our terminology.

Lemma 1

Let $\Theta$ be a Grilliot-theory on type k (=I).

Let Spec $\Theta = \langle\langle M_a\rangle_{a\in I}, R\rangle$. Let $\mathcal{M}_a = \bigcup_{c\in tp(k-1)} M_{\langle a,c\rangle}$

Let $\varphi$ be a $\Delta_o$-formula in R with parameters from $M_a$.

Assume

$$\forall b \in I\ \exists x \in \mathcal{M}_{\langle a,b\rangle}\ \varphi(b,x)$$

Then

$$\exists f \in M_a \forall b \in I\ \exists x \in f(b)\ \varphi(b,x)$$

The conclusion in Lemma 1 may be regarded as a definition of Grilliot-selection on spectra. The only 'natural' proof of Grilliot selection is of recursion theoretic flavour. The consequences are, however, soft and can be proved in all reasonable frameworks.

Theorem 2 (Harrington [1]. Further reflection.)

Let $\Theta$ be a type-k-Grilliot-theory. Let $\langle\langle M_a\rangle_{a\in I}, R\rangle$ = Spec $\Theta$ .

Let C be a complete $\Sigma^*(R,a)$-subset of tp(k-1).

Let $\varphi$ be $\Delta_o(R)$-formula with parameters from $M_a$. Then

$$M_{a,C} \models \exists x\varphi \text{ if and only if } M_a \models \exists x\varphi .$$

## 2. Abstract k-sections.

Everything in this section is based on Sacks [6].

Definition 3

Let $A \subseteq V_I$. We say that $A$ is admissible with gaps if

i A is a rudimentary closed structure.

ii $A \models \Sigma_1$-collection.

In Sacks [6], i is given by: A is closed under pairing and union and satisfies $\Delta_o$-separation.

Remark. Given a nice family $\langle M_a \rangle_{a\in I}$, each individual $M_a$ will be admissible with gaps.

Definition 4

A is an abstract k+1-section if there is a B such that

i A and B are admissible with gaps.

ii $\forall x \in A(B)\ \exists \text{nice } \langle M_a \rangle_{a\in I}\ (x \in M_o \ \&\ M_o \subseteq A(B))$

In Sacks [6] this is called 'closed under recursion in ${}^{k+2}E$'

iii A and B are abstract structures. (See definition 2.11 of Normann [4].)

iv $A \in B, A <_1 B$ and A is countable in B.

Theorem 5

Let $F$ be a normal functional of type $\geq k+2$.

Let $R = \{\langle a,\alpha\rangle ; |a|_F = \alpha\}$.

Let $\langle M_a\rangle_{a\in I}$ be the least family nice relative to $R$.

Then $M_o$ is an abstract $k+1$-section.

Proof. Let $C$ be a complete $F$-r.e. subset of $\omega$. Then, by theorem 2, $M_o <_1 M_C$: Moreover there is an enumeration of $M_o$ in $M_C$.

Theorem 6 (Sacks [6])

Let $A$ be an abstract $k+1$-section. Then there exists an $R$ such that when $\langle M_a\rangle_{a\in I}$ is the least $R$-nice family, then $M_o = A$.

Proof: Let $B$ be as in the definition of abstract $k+1$-section. Define the set of conditions $\mathbb{P}$ by : $p \in \mathbb{P}_A(\mathbb{P}_B)$ if $p \subseteq I \times On$, $p \in A(B)$, rank(p) is 0,p-necessary (i.e. 0-necessary when $R$ is replaced by p.) $\mathbb{P}$ is ordered by $q < p$ if $q \cap I \times \text{rank}(p) = p$.

We say that for a $\Sigma_1$-formula $\exists x\varphi$ , $p \vdash \exists x\varphi \Leftrightarrow M_o^{\text{rank}(p)}(p) \models \exists x\varphi$ . The forcing relation is extended in the usual manner.

Claim 1 $p \Vdash$ 'n is an ordinal notation' is $\Delta_1$ over $A(B)$

Proof: $p \Vdash$ 'n is an ordinal notation' $\Leftrightarrow \exists\gamma \in M_o^{rn(p)}(p)$ (n is a notation for $\gamma$). The ordinal notations may for instance be as described in Normann [4] definition 5.5 with discussions.

Claim 2 Let $p \in \mathbb{P}_A$, $x \in A$. Then there is a $q \in \mathbb{P}_A$ such that $q \leq p$ and $x \in M_o^{rn(q)}(q)$.

Proof: Let $x$ and $p$ be given. Let $\langle N_a\rangle_{a\in I}$ be a nice family such that $N_o \subseteq A$ and $x,p \in N_o$. Let $Q$ be a code for $x$. Let

$q_o = \{<a, rn(p)>;\ a \in Q\}$. Then $q_o \in N_o$.

Let $q = q_o \cup p \cup \{<0, rn(p) + rn(x)>\}$. Then $q \in \mathbb{P}_A$ by the following argument : In the code $Q$, let for $a \in I$ $x_a$ be the set coded by $a$. By induction on rank $x_a$ and by $\Sigma^*$-collection one proves that $x_a \in N_a \cap M_a^{rn(p) + rn(x_a)}(q)$. The claim will follow.

Now, let $P$ be a $\mathbb{P}_A$-generic set. Since $A$ is countable in $B$ we may assume $P \in B$.

<u>Claim 3</u> If $P \in B$ is $\mathbb{P}_A$-generic and $<M_a>_{a \in I}$ is the least P-nice family, then $M_o \subseteq A$.

<u>Proof</u>: To obtain a contradiction, assume that this is not the case. Let $\alpha \in M_o \setminus A$ have a notation $m$, while for some $p \in P$, $p \Vdash$ 'm is not a notation' , i.e. $\forall q \leq p$ $\neg q \Vdash$ 'm is a notation'. Let $\beta$ be minimal such that $\alpha \in M_o^\beta(P)$. Let $P_o = P \cup \{<0, \beta>\}$. To prove that $\beta \in B$, we use that there is a nice family $<N_a>_{a \in I}$ such that $P \in N_o \subseteq B$. Then $M_o \subseteq N_o$. By definition of $\beta$, $\beta$ will be 0-necessary. Thus $P_o \in \mathbb{P}_B$ and

$$B \models \text{'m is a notation in } P_o\text{'}. \text{ Thus}$$

$$B \models \exists q \leq p \quad q \Vdash \text{'m is a notation'} .$$

Since $A <_1 B$, we have $A \models \exists q \leq p$ $q \Vdash$ 'm is a notation' .

This is a contradiction and the claim is proved.

Theorem 6 now follows trivially from claims 1 to 3.

## 3. On abstract ${}^{k+3}S$-sections.

In this section we will use methods from section 2 to see that we may obtain similar results for the sections of some type k+2-functional F and the superjump. This result is based on § 7 of Normann [4], which again is based on Harrington [1]. In § 7 of Normann [4] we gave a definition of strongly impenetrable family which worked well for the spectra of theories. Here we will have to use a seemingly stronger definition. However, the results and proofs in § 7 work also for this stronger definition. The only place where we proved something to be strongly impenetrably was in lemma 7.5, and that argument works for this new concept too. Thus we do no harm if we use the following definition.

Definition 7.

A family $\langle M_b\rangle_{b\in I}$ is strongly impenetrable if for all $a \in I$ and all $\Delta^*_a$-functions f, if f is closed in $\langle M_{\langle a,b\rangle}\rangle_{b\in I}$, then there is a family $\langle N_b\rangle_{b\in I}$, nice relative to a, such that f is closed in $\langle N_b\rangle_{b\in I}$ and $\langle N_b\rangle_{b\in I} \in M_a$.

Definition 8 of abstract ${}^{k+3}S$-section.

a k = 0: A is an abstract ${}^{3}S$-section if

i A is an abstract 1-section (See Sacks [5] or Normann [3]).

ii If φ is a $\Delta_0$-formula and $\vec{y} \in A^n$ and $A \models \forall x \exists y \varphi(x,y,\vec{y})$ then there is an abstract 1-section N such that $\vec{y} \in N \in M$ and $N \models \forall x \exists y \varphi(x,y,\vec{y})$.

b k > 0: A is an abstract ${}^{k+3}S$-section if there exists a B such that $A \in B$, A is countable in B, $A <_1 B$ and both A and B have the following properties:

i They are admissible with gaps and satisfy the properties of abstract k+1-sections in definition 4.

ii $\forall x \in A(B) \exists \langle N_a \rangle_{a \in I} (x \in N_o \subseteq A(B)$ & $\langle N_a \rangle_{a \in I}$ is strongly impenetrable.).

Remark i and ii play the same role in both definitions, ii gives the appropriate variant of the Mahlo-property.

Theorem 9

Let F be of type k+2. Then $Str(k+1\text{-}sc(F,{}^{k+3}S))$ is an abstract ${}^{k+3}S$-section.

Proof: Let $\Theta = Th(F,{}^{k+3}S)$ be the Harrington-theory of F and ${}^{k+3}S$. Let $R_\Theta = \{\langle \sigma,\alpha \rangle ; |\sigma|_\Theta = \alpha\}$.

Case 1 k = 0 : We have that $A = L_\alpha^{R_\Theta}$ where $\alpha$ is the first $R_\Theta$-recursively Mahlo ordinal. If $A \models \forall x \exists y \varphi(x,y,\vec{y})$, define

$$g(\gamma) = \mu\beta : \forall x \in L_\gamma^{R_\Theta} \exists y \in L_\beta^{R_\Theta} \varphi(x,y,\vec{y}).$$

g is closed in an admissible ordinal $\alpha_o$ and $L_{\alpha_o}^{R_\Theta}$ will be an abstract 1-section.

Case 2 By a lemma to theorem 3.7 of Harrington [1], $\Theta$ will be a Grilliot-theory, and thus theorem 2 applies. Let C be a complete $\Theta$-r.e. subset of $\omega$, and let $B = k+1\text{-}sc(F,{}^{k+3}S,C)$. i is clearly satisfied. To see ii, we can let $\langle N_a \rangle_{a \in I}$ be $Spec(\Theta)$ or $Spec(\Theta[C])$.

Theorem 10

If A is an abstract ${}^{k+3}S$-section, then there is some normal type k+2-functional F such that A is the k+1-section of $F,{}^{k+3}E$.

Proof :

Case 1   $k = 0$ : By Normann [4] it is sufficient to find a $P$ such that $A$ is the least $P$-recursively Mahlo structure. Thus, let $p \subseteq On$ be a condition if $p \in A$ and no ordinal $\leq rn(p)$ is $p$-recursively Mahlo. $p \leq q$ if $q = p \cap rn(q)$. As in Normann [4] we may prove that if $P$ is generic, then

i   $L^{P}_{rn(p)} = A$

ii   $A$ is $P$-admissible.

By the way we defined the conditions, we see that no ordinal $< rn(p)$ will be $P$-recursively Mahlo. We will prove that $rn(p)$ is $P$-recursively Mahlo.

Assume $\langle A,P\rangle \models \forall x \exists y \varphi(x,y,\vec{y})$. This fact will be forced by some $p \subseteq P$, and thus

$$* \qquad \forall q \leq p \forall x \exists r \leq q \exists y\, r \Vdash \varphi(x,y,\vec{y})$$

Let $p$ be any condition forcing $\forall x \exists y \varphi(x,y,\vec{y})$. Let $N \in A$ be admissible such that $N$ is an abstract 1-section and

i   $p \in N$

ii   $N \models *$

Let $p'$ be an extention of $p$ generic over $N$. Then $\langle N,p'\rangle$ is admissible and $\langle N,p'\rangle \models \forall x \exists y \varphi(x,y,\vec{y})$.

Let $\alpha$ be the least ordinal such that $\alpha$ is $p'$-admissible and

$$\langle L^{p'}_{\alpha}, p' \cap \alpha\rangle \models \forall x \exists y \varphi(x,y,\vec{y}).$$

If $\alpha = rn(p)$ we have $\langle L^{P}_{\alpha}, P \cap \alpha\rangle \models \forall x \exists y \varphi(x,y,\vec{y})$, which is what we want to prove. If $\alpha > rn(p)$, we have $\alpha \leq rn(p')$. Let $p_1 = p' \cap \alpha$. By definition of $\alpha$, $p_1$ is a condition and $p_1 \leq p$. Thus $\langle L^{p_1}_{\alpha}, p_1\rangle \models \forall x \exists y \varphi(x,y,\vec{y})$. Since $P$ is generic, Mahloness is proved.

Case 2    k > 0: We are going to use the same proof as in the ordinary abstract k+1-section result, except that we want A to be the recursive part of the least strongly P-impenetrable family instead of the least P-nice family.

To definition 5.5 in Norman [4] we add

iii  $\alpha$ is a-necessary if $<M^{\alpha}_{<a,b>}(P)>_{b \in I}$ is penetrated, i.e. $<M^{\alpha}_{<a,b>}(P)>_{b \in I}$ is nice, and there is a $\Delta^*_a$-function g: On → On that is not closed in any family being an element of $M^{\alpha}_a(P)$.

We then use the same definition of the conditions as in the same part of section 2. In addition we will also have to prove that when $P \in B$ is generic over A, and when $<N_a>_{a \in I}$ is the least family strongly impenetrable in P, then $N_o = A$.

$A \subseteq N_o$ will follow by the definition of $\mathbb{P}$. Assume $N_o \not\subseteq A$. Then there has to be some ordinal, necessary by clause ii or iii, not in A. Anyhow, since $P \in B$, the ordinal will be in B. Assume $\alpha$ is necessary by clause iii, i.e. there is a $\Delta^*$-function f that is not closed in any nice family $<M^{\beta}_a(P)>_{a \in I}$ for any $\beta < \alpha$, while $<M^{\alpha}_a(P)>_{a \in I}$ is nice (due to the fact that we use clause iii.).

Since $p \in B$ we have $B \models \exists <N_a>_{a \in I} \cdot (<N_a>_{a \in I}$ is nice and f is closed in $<N_a>_{a \in I})$. Let $P' = P \cup \{<0,\alpha>\}$. Then $P' \in \mathbb{P}_B$, and $P' \Vdash \exists <N_a>_{a \in I} (<N_a>_{a \in I}$ is nice and f is closed in $<N_a>_{a \in I})$. Let $p \in P \cap \mathbb{P}_A$. Then $P' \leq p$. By reflection we find a $p' \leq p$ in A forcing the statement above. But this is a contradiction by the choice of f.

4. On the extended plus - one-hypothesis.

Sacks [6] formulated the extended plus - one-hypothesis as follows:

Definition 11

Let H be a normal object of type $\geq$ k+2. By the extended k+1-section of H we mean

$$\bigcup_{a \in tp(k)} k{+}1\text{-}sc(H,a)$$

The extended plus - one hypothesis is :

There exists a normal type k+2-functional F such that

extended k+1-section H = extended k+1-section F.

Sacks [6] states that the extended plus-one-hypothesis is correct when GCH holds. We will indicate the proof here. All ingenious parts are based on private information from Sacks.

Let $I = tp(k)$.

Definition 12 (GCH)

Let $M \subseteq V_I$. We say that M is an abstract extended k+1-section if

1. M is an abstract structure ($x \in M \Longleftrightarrow x$ has a code in M)
2. M is closed under subsets of cardinality $< \aleph_k$.
3. M is closed under full recursion in ${}^{k+2}E$. i.e. If $x \in M$ and $\langle N_a \rangle_{a \in I}$ is the least nice family such that $x \in N_o$, then $\bigcup_{a \in I} N_a \subseteq M$.

4. $\overline{\overline{M}} = \aleph_k$ .

<u>Theorem 13</u> (GCH) Let $M \subseteq V_I$ .

The following is equivalent :

<u>i</u> M is an abstract extended k+1-section.

<u>ii</u> For some normal F of type k+2, $M \cap \mathcal{P}(I) = \text{Ext.k+1-sc}(F)$.

<u>Proof</u>: <u>ii</u> ⇒ <u>i</u>. That the extended section of F has properties 1 to 4 is clear. For details see e.g. Normann [4]
To prove <u>i</u> ⇒ <u>ii</u> we need the following theorem of Moschovakis [8]:
Let F be a normal functional of type k+2. Let $A \subseteq I$ be co-semirecursive in F with index e. Then for some recursive function f independent of e there is a s.r.set B with index f(e) such that $a \in A \iff \exists b \langle b,a\rangle \in B$.

An alternative proof of this came out of the model theoretic considerations in Moldestad-Normann [9].
In our setting we will obtain:

<u>Proposition</u> (Moschovakis) : There is a $\Sigma^*(R)$-set B, uniformly definable in R, such that a is not a set-notation $\iff \exists b \langle b,a\rangle \in B$.

The construction of the functional will be by forcing. Let $p \in \mathbb{P}$ if $p \subseteq I \times On$ and if for some $a \in I$ rank p is a,p-necessary. $p \leq q \iff q \cap I \times \text{rank } p = p$.

<u>Claim 1</u>

If $\langle q_\beta\rangle_{\beta<\alpha}$ is an increasing sequence from $\mathbb{P}$ and $\bar{\bar{\alpha}} \leq \aleph_{k+1}$, then $p = \bigcup_{\beta<\alpha} q_\beta \in \mathbb{P}$.

<u>Proof</u> $p \in M$ since M is closed under subsets of cardinality $< \aleph_k$ and recursion in ${}^{k+2}E$. Let rank $q_\beta$ be $a_\beta$-necessary via formula no. $e_\beta$. $\langle a_p, e_\beta\rangle_{\beta<\alpha}$ may be coded as one element $a \in I$, and rank(p) will be a-necessary.

Claim 2

If $x \in M$ and $p \in \mathbb{P}$, there is a $q \leq p$ such that $x \in L^q_{rank(q)}$.

Proof This is proved exactly in the same way as in the abstract k+1-section case.

Claim 3

If $a \in I$ and $p \in \mathbb{P}$, there is a $q \leq p$ such that $q \Vdash a$ is a set notation, or $q \Vdash a$ is not a set notation in a way so that the truth of 'a is a set notation' will be settled there and then.

Proof Let $\langle N_b \rangle_{b \in I}$ be the least p-nice family, by the normal construction. If a is a set notation, let $\alpha$ be the first level $\geq$ rank (p) such that a is a set notation at level $\alpha$. Let $q = p \cup \langle 0, \alpha \rangle$

If a is not a set notation, let $\alpha$ be the first level $\geq$ rank (p) such that $\langle M^\alpha_b(p) \rangle_{b \in I} \models \exists b \langle b, a \rangle \in B(p)$.
Again let $q = p \cup \langle 0, \alpha \rangle$.
In both cases q is sufficiently like the trivial extention of p to guarantee that for any extension q' of q, a is a set-notation relative to q' if and only if it is so relative to the trivial extension of p.

So, let P be generic. Then a set is in the least family nice relative to P if and only if it has a set notation from a part of P that is in M if and only if it is in M. This proves the theorem.

## 5. Degrees of functionals, imbedding of order types.

Some notions of forcing have the advantage that they may be regarded as product forcing, with associate product lemmas. So is the case with the constructions of functionals we have made in this note. As a consequence we are able to split our generic functionals up in several recursively incomparable functionals, which again enables us to construct imbeddings of partial orderings into various orderings of degrees of functionals. The method is applicable whenever we have a generic functional of the kind described above, and thus the imbedding results will be finer the more spectra we are able to construct generic functionals over.

Theorem 14

Let $a \in tp(k)$, $f$ and $G \in tp(k+2)$. We say that $F \prec_a G$ if $F$ is Kleene-recursive in $a$, $G$ and ${}^{k+2}E$.

Let $\prec$ be a partial ordering with countable domain. Then $\prec$ may be imbedded in $\prec_a \restriction \langle {}^{k+2}E, {}^{k+2}E, c\rangle$ where $c$ is a complete ${}^{k+2}E$, a-r.e. subset of $\omega$.

Proof: To save notation, let $a$ be recursive in ${}^{k+2}E$. Identify $tp(k)$ and $\omega \times tp(k)$ in a recursive way, and let $\mathbb{P}$ be as in the proof of theorem 6. Let $P$ be generic over $k+1\text{-sc}^{k+2}E$ and assume $P$ is 'recursive' in ${}^{k+2}E,c$. Let $P_i = \{\langle a,\alpha\rangle : \langle\langle i,a\rangle,\alpha\rangle \in P\}$, and let $F_i$ be the associated functional. If $A \subseteq \omega$ is recursive in ${}^{k+2}E$, we have $F_i$ recursive in $\langle F_j\rangle_{j \in A}$ if and only if $i \in A$.

There exists a recursive partial ordering $\prec$ such that each other countable partial ordering may be imbedded in $\prec$. So, assume $\prec$ to be recursive. For $n \in \text{dom}\prec$, let $H_n = \langle F_j\rangle_{j \prec n}$. Then

$H_n$ is recursive in $H_m$ if and only if $m \prec n$.

To justify the sharpest formulations of our next result, we need the following :

Lemma 15 (GCH)

There exists a partial ordering $\prec$ of type $k$ such that every other partial ordering of type $k$ may be imbedded in $\prec$. Moreover, if $<_{tp(k)}$ is a minimal well-ordering of $tp(k)$, we may find $\prec$ recursive in $<_{tp(k)}$ and ${}^{k+2}E$.

We will not prove this in detail.

Step 1. By a construction of length $\aleph_k$ we may extend any partial ordering of cardinality $\aleph_k$ to one with the property *

* Let $A,B,C$ be disjoint subsets of $dom(\prec)$, all of cardinality $\aleph_{k-1}$. If it is consistent to assume the existence of an $a$ such that $A < a < B$ and $a$ and $C$ are incomparable, then there exists such $a$.

Step 2. Any two partial orderings of cardinality $\aleph_k$ satisfying * are isomorphic. This is a special case of a theorem in Sacks [7]. It is much the same as proving that dense, countable linear orderings are isomorphic.

Step 3. Starting with the empty ordering, see that the construction in step 1 can be done effectively in $<_{tp(k)}$ and ${}^{k+2}E$.

We will also use the lemma in section 8.

Recall $<_a$ from theorem 14. Let $F \leq G$ if $\exists a \in tp(k)$ $F \leq_a G$. We call the degrees derived from this inequality strong degrees.

Theorem 16 (GCH)

a Any partial ordering of cardinality $\aleph_k$ may be imbedded in the strong degrees.

b (V = L). Any partial ordering of cardinality $\aleph_k$ may be imbedded in the strong degrees of functionals F such that

$$\text{Full-}k+1\text{-section } F = \text{Full-}k+1\text{-section } {}^{k+3}E.$$

Remark: In section 8 we will strengthen b to functionals r.e. in ${}^{k+2}E$ and an individual.

Proof: Let $M$ = Full-section ${}^{k+3}E$, $<_{tp(k)}$, where $<_{tp(k)}$ is some minimal well-ordering of tp(k).

Let P be generic over M such that Full-sc P = Full-sc ${}^{k+3}E$, $<_{tp(k)}$ (See theorem 13). As in theorem 13 we may split P up, this time in $\{P_a : a \in tp(k)\}$. Let $\prec$ be a universal ordering from lemma 15. Define $H_a$ in analogy with the proof of theorem 14. Then $H_a < H_b \Leftrightarrow a \prec b$. Moreover, by genericity, for each a, Full-sc $P_a$ = Full-sc ${}^{k+3}E$, $<_{tp(k)}$, and if V = L, $<_{tp(k)}$ may be assumed to be recursive in ${}^{k+2}E$.

Corollary 17 (GCH)

Let the 'degrees' mean Kleene-degrees modulo ${}^{k+2}E$. Then a partial ordering $\prec$ of cardinality $\aleph_k$ is subordering of the degrees of type k+2 functionals if and only if each initial segment is countable.

Proof: Assume $\prec$ is on tp(k), and imbed $\prec$ as in theorem 16. Given $a \in dom(\prec)$, $\{b : b \preceq a\}$ is countable and recursive in $\prec$ and ${}^{k+2}E$. Then this set has an enumeration, and using

$<_k$ and ${}^{k+2}E$, we may pick one. But then $b \preceq a \Rightarrow b$ is recursive in $<_{tp(k)}, a, \prec$. Since $b \preceq a \Rightarrow H_a$ is recursive in $H_b$, a and $\prec$, we have $\langle H_a, a, \preceq, <_k\rangle$ is recursive in $\langle H_b, b, \preceq, <_k\rangle$ if and only if $a \preceq b$.

6. Degree on the individuals.

This section is devoted to the setting of terminology for sections 7 and 8. We assume V = L and let $<$ be the canonical wellordering of tp(k). $<$ is recursive in ${}^{k+2}E$ and of length $\aleph_k$. Whatever we are going to do in the next two sections, it will be modulo the subindividuals, so let $\langle M_a\rangle_{a\in I(=tp(k))}=\mathrm{Spec}({}^{k+2}E)$.

For $a \in I$, let $\mathcal{M}_a = \bigcup_{i\in tp(k-1)} M_{\langle a,i\rangle}$.

Let $a \in I$. $\{b : b \le a\}$ is recursive in a and ${}^{k+2}E$ and has cardinality $\aleph_{k-1}$. Then $\{c \in I : c$ 'enumerates' $\{b: b \le a\}\}$ is recursive in ${}^{k+2}E$, and the c of this form least in $\le$ will be recursive in a. But then each $b \le a$ is recursive in c and a subindividual (the one 'enumerating' b in c) and b is recursive in a, ${}^{k+2}E$ and some subindividual. This leads to

$$a < b \Rightarrow \mathcal{M}_a \subseteq \mathcal{M}_b.$$

We call a minimal if a is not recursive in ${}^{k+2}E$, a subindividual and any b such that $b < a$.

Let $K^a_{k-1} = \mathrm{Sup}(\mathrm{On} \cap \mathcal{M}_a)$. $\lambda^a_{k-1}$ = least ordinal not in $\mathcal{M}_a$ (which coincides with both the order type of the ordinals subconstructive in a and the supremum of a-recursive prewellorderings on tp(k-1).

Let a' = the first minimal point after a. Read a' as a-jump. a' will be element no. $\lambda^a_{k-1}$ in $\leq$ and this leads to the following:

$$\forall a \in I \; K^a_{k-1} \in \mathcal{M}_{a'} \; .$$

<u>Problem:</u> Let a be minimal. Will then

$$K^a_{k-1} > \sup\{K^b_{k-1} : b < a\} \; ?$$

Now, this is true for all jumps and for most limits of jumps. We have, however, verified neither the existence nor the non-existence of a counterexample.

We call a <u>bad</u> if a gives a negative answer to the problem.

By the recursive well-ordering we may from each recursive set pick a recursive element. Using simple and further reflection as well we see that

$$\forall a \in I \;\; M_a <_1 \mathcal{M}_{a'} \; .$$

We also have this <u>grand $\Delta_o$-Dependent Choice</u> :

Assume $c \in I$

$$\forall a \forall x \in \mathcal{M}_{a,c} \exists y \in \mathcal{M}_{a,c} \; \varphi(x,y,c) \quad \text{where } \varphi \text{ is } \Delta_o .$$

Then $\quad \exists \langle X_a \rangle_{a \in I} \in M_c \; \forall_a \in I \; \varphi(\langle X_b \rangle_{b<a}, X_a, c).$

<u>Proof</u>: Using our single-valued selection operator, we obtain

$$\forall a \forall x \in M_{a,c} \; \exists y \in M_{a,c} \; \varphi(x,y,c)$$

In this situation we may use Gandy's selection operator and $\Sigma^*$-collection to find the wanted sequence (as in the proof of ordinary $\Delta_o$-DC).

## 7. Recursion theories on spectra.

Companion theory on Spector-theories of $\omega$ gives rise to an infinite recursion theory, which is the natural theory on the admissible companion. We will here define a 'natural' recursion theory on the full section of a type-k-theory. By the lack of a recursive selection operator and admissibility, semirecursion in the thoery will not be $\Sigma_1$ over the underlying structure, but $\underset{\sim}{\Sigma}^*$ over the underlying spectrum.

### Definition 18

Let $\Theta$ be a type-k-theory, Spec $\Theta = \langle\langle M_a \langle_{a \in I}, R_\Theta\rangle = \mathcal{M}$.

We will define a theory of partial functions $f : |\mathcal{M}|^n \to |\mathcal{M}|$ with indices in $|\mathcal{M}|$. The recursion is defined by 15 schemata, with indices :

i $\quad f(x_1,\cdots,x_n) = x_i$ $\qquad \langle 1,n,i\rangle$

ii $\quad f(x_1,\cdots,x_n) = x_i \setminus x_j$ $\qquad \langle 2,n,i,j\rangle$

iii $\quad f(x_1,\cdots,x_n) = \{x_i,x_j\}$ $\qquad \langle 3,n,i,j\rangle$

iv $\quad f(x_1,\cdots,x_n) \simeq \bigcup_{y\in x} h(y,x_2,\cdots,x_n)$ $\qquad \langle 4,n,e'\rangle$ where $e'$ is an index for $h$.

v $\quad f(x_1,\cdots,x_n) \simeq$ $\qquad \langle 5,n,m,e',e_1,\cdots;e_m\rangle$

$\quad h(g_1(x_1,\cdots,x_n),\cdots,g_m(x_1,\cdots,x_n)x_1,\cdots,x_n)$

vi $\quad f(x,i) = (x)_i$ $\qquad \langle 6\rangle$ (x varies over $I = tp(k)$ i over $\omega$)

vii $\quad f(x,y) = \langle x,y\rangle$ $\qquad \langle 7\rangle$ x,y vary over I

viii $\quad f(x) = C_i(x)$ $\qquad \langle 8,i\rangle$ x varies over I

<u>ix</u> $f(x,y) = (ev)^i(x,y)$ <9,i> x,y vary over I

<u>x</u> $f(x_1,\cdots,x_n) = x$ <10,n,x>

<u>xi</u> Induction scheme

$h(a,\vec{x}) \simeq f(a,\vec{x}) \quad a \in I \vee a = \emptyset$

$h(y,\vec{x}) \simeq g(\langle h(z,\vec{x})\rangle_{z\in y},\vec{x})$ $\langle 11,e_1,e_2\rangle$

<u>xii</u> Diagonalization

$f(e,\vec{x}) \simeq \{e\}(\vec{x})$ <12,n>

<u>xiii</u>

$$f(x) = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$$ <13>

<u>xiv</u>

$$f(x) = \begin{cases} 0 & \text{if } x \in R_\Theta \\ 1 & \text{if } x \notin R_\Theta \end{cases}$$ <14>

<u>xv</u> Permutations of $x_1,\cdots,x_n$ $\langle 15,n,\sigma\rangle$ $\sigma$ a permutation of n

This definition is probably not the most economic. All rudimentary functions will be recursive by schemate <u>i</u> - <u>xi</u>. Our program will be to prove that $A \subseteq I$ is semi-recursive in our theory with index in $M_a$ if and only if $A$ is $\Sigma^*_a$-definable over Spec($\Theta$) (if and only if $A$ is $\Theta$-s.r. in a).

<u>Lemma 19</u>

Let $a \in I$. If $e,x_1,\cdots,x_n \in M_a$ and $\{e\}(x_1,\cdots,x_n) \simeq x$, then $x \in M_a$. Moreover, the relation $\{e\}(x_1,\cdots,x_n) \simeq x$ is $\Sigma^*$.

Proof: The first claim is proved by induction on the length of the computations. To verify the latter, note that 'T is a computation tree for $<e,x_1,\cdots,x_n,x>$' is $\Delta_o$. We prove that if $e,x_1,\cdots,x_n$ are in $M_a$, then the computation tree is in $M_a$. This is also proved by induction on the length of the computation. Both inductions are fairly trivial; in cases iv and xi we use $\Sigma^*$-collection. The rest is straight forward.

Lemma 20

$S^n_m$-theorem.

Proof: Given e' and $x_1,\cdots,x_n$ we want to find an e, uniform in e' and x such that

$$\{e\}(y_1,\cdots,y_m) = \{e'\}(x_1,\cdots,x_n,y_1,\cdots,y_m)$$

First let $e_1 = <10,m,x_1>$, $e_{n+1} = <10,m,e'>$, i.e. the constants. We want

$$\{e\}(y_1,\cdots,y_m) = \{<12,n>\}(\{e_{n+1}\}(y_1,\cdots,y_m),\{e_1\}(y_1,\cdots,y_m),\cdots$$
$$\cdots\{e_n\}(y_1,\cdots,y_m),y_1,\cdots,y_m)$$

So let $e = <5,m,n+1,<12,n>,e_{n+1},e_1,\cdots,e_n>$:

As usual the recursion theorem follows from the $S^n_m$-theorem. Recall from Normann[4] that we have a canonical well-ordering $<_a$ on each $M_a$ induced by the partial constructibility.

Lemma 21

'$x<_a y$' is a recursive relation in x,y and a.

Proof: First we see that the function $f(\alpha) = L_\alpha^{R_\theta}$

is recursive. We use the induction scheme, and it suffices to prove that DEF is a recursive function.

We give an informal description of how to compute DEF(X) given X. First, given $\varphi$, there is a canonical index $e \in \omega$ such that $\{y \in X : \varphi_X(y,\vec{x})\} = \{e\}(X,\vec{x})$. e will be an index for a rudimentary function. The set of such indices is recursive. Call it A. Then

$$DEF(X) = X \cup \Big( \bigcup_{\substack{e \in A \\ x \in X^n}} \{\{e\}(X,\vec{x})\}\Big).$$

$S^a_\alpha(R_\Theta)$, the part of $L^{R_\Theta}_\alpha(I)$ proved to be in $M_a$, will be recursive as a function of a and $\alpha$, since it is $\Delta_0$-definable from $L^{R_\Theta}_\alpha(I)$ and a.

Note that the $\mu$-operator on the ordinals will be recursive. So given $x,y \in M_a$, let $\alpha = \mu\alpha(x,y \in S^a_\alpha(R_\Theta)$. In $\alpha$ we may effectively decide whether $x <_a y$, $x = y$ or $y <_a x$.

<u>Lemma 22</u>

Our theory admits a selection operator in the following sence: There is a recursive function $\varphi$ such that when $e \in M_a$ and $\exists x \in M_a \{e\}(x){\downarrow}$, then $\{e\}(\varphi(e,a)){\downarrow}$.

<u>Proof</u>: Let $\varphi'(e,a) = \mu\alpha(\exists T \in S^\alpha_a(R_\Theta))$ (T is a computation tree for $\{e\}(x) \simeq y$.) In $<_a$, pick the least such T, and let $\varphi(e,a)$ be the argument in the actual computation.

Theorem 23

a  A set $X \subseteq M$ is semicomputable with an index in $M_a$

$\iff X$ is $\Sigma^*_a$-definable

b  A set $X \subseteq M$ is computable with an index in $M_a$

$\iff X$ is $\Delta^*_a$-definable.

Proof: $\Rightarrow$ in a and b follow from lemma 21.
$\Leftarrow$ in a. Let $y \in X \iff \exists z \in M_{y,a}\psi(z,y,a)$, where $\psi$ is $\Delta_o$. There is an index $e$ such that $\{e\}(z,y,a) \simeq 0 \iff \psi(z,y,a)$. Then $y \in X \iff \varphi(e,\langle y,a\rangle) \simeq 0$.

Remark: In lemma 22 we defined $\varphi$ on arguments in $I$ only. There is no harm in doing constructibility relative to $y$ and thereby extending it to all kinds of arguments.

$\Leftarrow$ in b. By the selection operator used in $\Leftarrow$ a it is not hard to see that when both $X$ and $M \setminus X$ are $\Sigma^*_a$, then $X$ is computable.

From $\alpha$-recursion theory we borrow the following concept : Let $X,Y \subseteq M$. We say that $X < Y$ is there exists an index $e$ such that for all $x,y$

$$x \subseteq X \;\&\; y \cap X = \emptyset \iff \exists u,v \in \mathcal{M}_{x,y,e}(\{e\}(x,y,u,v) \simeq 0 \;\&\; u \in Y \;\&\; v \cap Y = \emptyset)$$

The intuition behind the definition is this: To decide finite information about $X$ we only need equally finite information about $Y$. This definition can then only be justified when 'Y-finite' means the same as 'finite'. In $\alpha$-recursion theory this is the case for regular and hyper-regular sets. We have not found a striking formulation of a good substitute for regular and hyper-regular. However, our discussions below indicate what it should be like.

Let, for $X \subseteq M, [X] = \{A : A$ codes an element in $X\}$ If $X$ is $\Sigma^*_a$, $[X]$ will be $\Sigma^*_a$. $[X]$ and $X$ are $\Delta^*$ in each other over any nice family. Note that $\Sigma^*$-subsets of $\mathscr{P}(tp(k))$ are semi-recursive. Section 8 will be devoted to the proof of the following :

Theorem 24

Let $V = L$ and let $\langle M_a \rangle_{a \in tp(k)}$ $(=I)$ $= Spec({}^{k+2}E)$. Then there exists a $\Sigma^*$-subset $Q$ of $I \times M$ such that when $Spec(Q, {}^{k+2}E) = \langle N_a \rangle_{a \in I}$ we have for all minimal $a$ that are not bad that $\mathcal{M}_a = \mathcal{N}_a$. Moreover, let $Q_b = \{x : \langle b,x \rangle \in Q\}$. Let $Q_{-b} = \langle Q_c \rangle_{c \neq b}$. For any minimal, not bad $a$, if $b$ is recursive in $a$ and a subindividual, $Q_b \cap \mathcal{M}_a$ is not reducible to $Q_{-b} \cap \mathcal{M}_a$ via an index in $\mathcal{M}_a$.

We end this paragraph by proving two corollaries of this theorem.

Corollary 25 (Post's Problem) $(V = L)$

There exist two subsets $A$ and $B$ of $tp(k+1)$ such that both are Kleene-semirecursive in ${}^{k+2}E$ and neither is Kleene-recursive in the other modulo ${}^{k+2}E$ and any individual.

Proof: Let $a \neq b$, both recursive in ${}^{k+2}E$. Let $A = [Q_a]$ and $B = [Q_b]$. Since $Q_a$ is reducible to $Q_{-b}$ and vice versa, $Q_a$ and $Q_b$ will not be reducible in each other modulo any type-k-element. This must also hold for $A$ and $B$ then.

Assume $A$ is Kleene-recursive in $B$, ${}^{k+2}E$, $c$, $i$

$(i \in tp(k-1))$. We may assume $c$ to be minimal and not bad.

Let $x \subseteq [A]$, $y \cap [A] = \emptyset$ and assume $x,y \in \mathcal{M}_c$. Then there will be $B$, ${}^{k+2}E$, $c$, $i$-recursive sets $z,u$ such that $z$ is the part of $B$ used positive and $u$ is the part of $V \setminus B$ used negative to verify $x \subseteq A$ and $y \cap A = \emptyset$. Then, since $\mathcal{N}_c = \mathcal{M}_c$, $z$ and $u$ will be in $\mathcal{M}_c$. For disjoint $z,u$, let

$$\sigma(z,u)(v) = \begin{cases} 0 & \text{if } v \in z \\ 1 & \text{if } v \in u \end{cases}$$

Define

$$\langle X,Y,Z,u\rangle \in R \iff \forall v \in X\{i\}^{{}^{k+2}E,\alpha(z,u),c}(v)=0 \;\&\; \forall v \in y\{i\}^{{}^{k+2}E,\alpha(z,v),c}(v)=1$$

Clearly $R$ is $\Sigma^*_{c,i}$ and $A \cap \mathcal{M}_c$ will be reducible to $B \cap \mathcal{M}_c$ via $R$. But this was impossible.

<u>Corollary 26</u> $(V = L)$

Any partial ordering on $tp(k)$ can be imbedded in the strong r.e$({}^{k+2}E)$-degrees.

<u>Proof</u>: The strong degrees are defined after the proof of lemma 17. By lemma 17 it is sufficient to imbed a partial ordering recursive in ${}^{k+2}E$, so let $\prec$ be an ordering on $tp(k)$ recursive in ${}^{k+2}E$. Let $Q_{[a]} = \langle Q_b\rangle_{b \preceq a}$. Then $Q_{[a]}$ is reducible to $Q_{[b]}$ if and only if $a \preceq b$. As in corollary 25 we prove that $[Q_{[a]}]$ is Kleene-recursive in $[Q_{[b]}]$, ${}^{k+2}E$ and an individual if and only if $a \preceq b$. Since $[Q_{[a]}]$ is r.e in ${}^{k+2}E$ and $a$, the corollary is proved.

<u>Remark:</u> Harrington [1] proved that Post's problem always has a solution when 'recursive in an individual' is replaced by

'recursive in a subindividual'. Thus there will exist two type k+2-functionals semirecursive in $^{k+2}E$ such that none of them is Kleene-recursive in the other modulo $^{k+2}E$ and a subindividual. By the same method he will be able to prove theorem 24 for subindividuals.

8. Proof of theorem 24.

For reasons of convenience we enumerate all r.e.-sets by a pair of a type-k-element and a subindividual; if $x \in \mathcal{M}_a$, there is a subindividual i and an index n for a code of x as an i, a-recursive set. If n is an i,a-index for a code for x, we let

$$R_{<<n,i>,a>} = \{\vec{x}:\{x\}\ (\vec{x})\downarrow\}.$$ We let

$$R^{\sigma}_{<<n,i>,a>} = \{\vec{x}:\{x\}\ (\vec{x})\downarrow \text{ in less than } \sigma\text{-steps and } \mathcal{M}^{\sigma}_a \models n$$

is an i,a-index for x$\}$.

To obtain simplicity in formulas, we contract <n,i> to one a-index for x. The definition of $R^{\sigma}_{<j,a>}$ is meaningful for all $j \in tp(k-1)$, $a \in tp(k)$, so we let

$$R_{<j,a>} = \bigcup_{\sigma \in On} R^{\sigma}_{<j,a>}.$$

If $B \subseteq M$, let $x \subseteq [i,a]^B$ if $(\exists y,z \in \mathcal{M}_{x,a})(y \subseteq B \ \&\ z \cap B = \emptyset \ \&\ R_{<i,a>}(x,y,z))$. We will only regard the cases where we have 'if and only if' above. If b is recursive in a via subindividual i, denote b by $[i]^a$.

We now start on the details of the proof. We are led to the following conditions :

1.<i,j>,a $\quad \mathcal{M}_a \setminus Q_{[i]^a} \neq [j,a]^{Q_{-[i]^a} \cap \mathcal{M}_a}$

or, in English : if b is recursive in a, then $\mathcal{M}_a \setminus Q_b$ is not recursive in $Q_{-b} \cap \mathcal{M}_a$ via subindividual j.

2.<n,i>,a $\quad$ If $\exists x\ \varphi_n(x,i,a,Q)$ then $\exists x \in \mathcal{M}_a\ \varphi_n(x,i,a,Q)$.

Each condition may be viewed as a pair of a subindividual and an individual. Using the minimal recursive well-ordering, we wellorder the conditions in the antilexicographical ordering. Each condition then receives a position $\nu < \aleph_k$.

The construction is going by induction on the pair $<\sigma,\nu>$ in the lexicographic ordering, $\sigma < K_n^{k+2}E$, $\nu \in$ Positions. The pair $<\sigma,\nu>$ is called a stage in the construction. Stages will be denoted by $\xi,\xi'$ etc. The conditions are given priority from the ordering on them.

We define a function $f(\xi,\nu)$ indicating what we want kept out of Q to meet condition $\nu$ at stage $\xi$. Let $f(b,\xi,\nu) = (f(\xi,\nu))_b = \{x: <b,x> \in f(\xi,\nu)\}$.
We also define $Q^\xi = <Q_a^\xi>$ at each stage $\xi$.

When we believe to have met a condition $\nu$, we put up a marker at $\nu$. When we have no reason to believe it any more, we take the marker down. At limit stages $\xi$, define
$f(b,\xi,\nu) = \lim_{\xi' \to \xi} \bigcap_{\xi'<\xi''<\xi} f(b,\xi'',\nu)$ in the discrete topology except when $\xi = <\sigma,\nu>$ and $f(b,\xi,\nu)$ defined this way is empty.

Then let $f(b,\xi,\nu) \in M_{\xi,b}$ be something nonempty and disjoint from the other $f(b,\xi,\nu')$ and from $Q^{\xi}_{b}$ where $Q^{\xi} = \bigcup_{\xi'<\xi} Q^{\xi'}$. It will follow from the construction that the limit above always exists.

Remark. When we as above say: Let $f(b,\xi,\nu) \in M_{\xi,b}$ be ... we may always find such value effectively by a selection operator. Thus the instructions for the construction will give a single-valued construction.

The construction.

Step $\xi$. Let $\xi = <\sigma,\nu>$

Case 1 $\nu = 1,<i,j>,a$

Question 1 Is there a marker at $\nu$ ? If that is so, procede to 'no' under question 2. If not, ask :

Question 2 $\exists y \exists z(y,z \in \mathcal{M}^{\xi}_{a}$ & $\mathcal{M}^{\xi}_{a} \models [i]^{a}$ is total $(=b)$ & $f(b,\xi,\nu) \in \mathcal{M}^{\xi}_{a}$ & $R^{\xi}_{<j,a>}(f(b,\xi,\nu),y,z)$ & $y \subseteq Q^{\xi}_{-b}$ & $z \cap Q^{\xi}_{-b} = \emptyset)$ ?

If yes, put up a marker at $\nu$ and remove all markers at $\nu'$ for $\nu' > \nu$. These conditions are then injured. Select a pair $y,z$ in $\mathcal{M}^{\xi}_{a}$.

Let $Q^{\xi+1} = Q^{\xi} \cup \{b\} \times f(b,\xi,\nu)$.

For $\nu' < \nu$, let $f(\xi+1,\nu') = f(\xi,\nu')$.

Let $f(\xi+1,\nu)$ be the part of $z$ not in any $f(\xi+1,\nu')$ for $\nu' < \nu$ or in $\{b\} \times M$. Let for $\nu' > \nu + 1$ $f(\xi+1,\nu') = f(\xi,\nu') \setminus (Q^{\xi+1} \cup f(\xi+1,\nu))$. If for $c \in I$ $f(c,\xi+1,\nu+1)$ is nonempty when defined in this way, it is OK. Else find something nonempty in

$\mathcal{M}_{\xi,c}$ disjoint from $Q_c^{\xi+1}$ and the other $f(c,\xi+1,\nu')$ for $\nu' \neq \nu+1$ .

If no, let $Q^{\xi+1} = Q^{\xi}$. For $\nu' \neq \nu+1$, let $f(\xi+1,\nu') = f(\xi,\nu')$. If for $c \in I$, $f(c,\xi,\nu+1) \neq \emptyset$, let $f(c,\xi+1,\nu+1) = f(c,\xi,\nu+1)$. If not, let $f(c,\xi+1,\nu+1) \in M_{\xi,c}$ be as above.

<u>Case 2</u> $\nu = 2, \langle n,i\rangle, a$

<u>Question 1</u> Is there a marker at $\nu$ ? If yes, set $y = \emptyset$ and procede to *.

If no, ask :

<u>Question 2</u> $\exists x \in \mathcal{M}_a^{\xi}(Q^{\xi})\ [\varphi_n(x,i,a,Q^{\xi})]$. If no, set $y = \emptyset$ and procede to *. If yes, ask :

<u>Question 3</u> Is this verifiable using negative information about $Q^{\xi}$ collected in $\cup\{f(\xi,\nu');\ \nu' < \nu\}$ ? If yes, set $y = \emptyset$ and procede to *. If no, let $\delta$ be the least ordinal such that $\exists x \in \mathcal{M}_a^{\delta}(Q^{\xi})[\varphi_n(x,i,a,Q^{\xi})]$ and let $y = L_{\delta}^{Q^{\xi}}[I] \setminus Q^{\xi}$.

* :

Let $Q^{\xi+1} = Q^{\xi}$. Let $f(\xi+1,\nu) = f(\xi,\nu) \cup (y \setminus \bigcup_{\nu'<\nu} f(\xi,\nu')$.
Let for $\nu' < \nu$ $f(\xi+1,\nu') = f(\xi,\nu')$.
Let for $\nu' > \nu+1$ $f(\xi+1,\nu') = f(\xi,\nu') \setminus y$.
If for $b \in I$, $f(b,\xi,\nu+1) \setminus y_b \neq \emptyset$, let $f(b,\xi+1,\nu+1) = f(b,\xi,\nu+1) \setminus y_b$.
Else let $f(b,\xi+1,\nu+1) \in M_{\xi,b}$ be nonempty and disjoint from all other $f(b,\xi+1,\nu')$ for $\nu' \neq \nu+1$ and from $Q_b^{\xi+1}$ .

This ends the construction. Now it just remains to prove that it works.

A condition is said to be met at stage $\xi$ if it is either marked for ever at stage $\xi$, or if it after $\xi$ will never be marked.

Let $f_\nu(\xi) = f(\xi,\nu)$.

We say that we do a change on condition $\nu$ at stage $\xi$, if we put on or remove a marker or $f(\xi+1,\nu) \neq f(\xi,\nu)$.

<u>Claim 1</u> The number of changes on a condition $\nu$ has at most cardinality $\aleph_{k-1}$.

<u>Proof:</u> By induction on $\nu$. When all changes on all conditions $\nu' < \nu$ are done, there is at most one change to do, i.e. if we want to put a marker on. Since $\nu < \aleph_k$, there will at most be $\aleph_{k-1} \times \aleph_{k-1} + 1 = \aleph_{k-1}$ changes on $\nu$.

<u>Corollary</u>.

All conditions will be met.

<u>Proof</u>: Since the cofinality of our construction is $\aleph_k$, this is immediate from the claim.

If a condition $\nu$ is of the form $\langle i,a \rangle$ where $i \in tp(k-1)$, $a \in I$, $\nu$ is said to be an <u>a-condition</u>. We also divide the conditions in type-1-conditions and type-2-conditions (as in case 1 and 2 above).

<u>Claim 2</u> If $a < b$ and $b \in \mathcal{M}_a$, then all b-conditions are met when all a-conditions are met. (We may, by formulation, assume a to be minimal.)

<u>Proof:</u> We will first see that when all a-conditions are met, then for each $y \in TC(\mathcal{M}_a)$ we have decided whether $y \in Q$ or not.

Let $x \in \mathcal{M}_a$ be such that $y \in x$. We regard the sentence $\forall y \in x(y \in A \vee y \notin A)$. Since $x \in \mathcal{M}_a$ will have a code, this is formally a $\Sigma_1^{a,i}$-sentence for some $i \in tp(k-1)$, leading to an a-condition of type 2. This condition will be met with a marker put on it at stage $\xi$. Then $x \setminus Q^\xi$ is kept out of $Q$ for ever.

This gives us that for b-conditions of type 1 our claim is formally clear, since each effort on meeting this conditions will be injured by some a-condition, and when all a-conditions are met, we will be unbable to put any marker on any b-condition of type 1 due to the demand that $f(c,\xi,\nu) \in \mathcal{M}_b^\xi$ (where here $c = [i]^b$).

What we really want to achieve is that $Q_c$ is not b-recursive in $Q_{-c}$. But if $b$ is recursive in $a$ and we obtain that $Q_c$ is not a-recursive in $Q_{-c}$, $Q_c$ cannot be b-recursive in $Q_{-c}$ either. Now assume we are in case 2 and

$$\exists x \in \mathcal{M}_b^\xi(Q^\xi)[\varphi_n(x,i,b,Q^\xi)].$$

Since $b$ is recursive in $a$ via a subindividual, this may be viewed as a $\Sigma_1^{a,j}$-formula no. m, adding a description of $b,i$ from $a,j$. Since all c-conditions for $c < b$ are assumed to be met, nothing would be added to $Q$ to interfere with the fact $\varphi_n(x,i,b,Q^\xi)$ until next time we come back to the b-conditions. Let $\nu'$ be the position of the condition 2, $\langle m,j\rangle,a$. Since $\varphi_n(x,i,b,Q^{\xi'})$ holds, where $\xi'$ is the associated stage to $\nu'$, $\Sigma_1^{a,j}$-formula no. m will also hold. Since this condition is already met, it would have received its final marker at a stage $\xi''$ before $\xi$. Then at stage $\xi''$, $\exists x \in \mathcal{M}_b^{\xi''})(Q^{\xi''})[\varphi_n(x,i,b,Q^{\xi''})]$ is true and will remain so. Then we will answer yes to question 3 and do nothing.

What claim 2 actually shows is that we positively try to meet the conditions for minimal a, and when this is done, starts directly on a'-conditions. Also, if an a-condition of type 1 is injured cofinally many times in $K^a_{n-1}$ there is no hope in meeting it in the way we want. In this case, which is actual when a is bad, we may try to meet a'-conditions of type 1, and these will never be injured by any a-conditions of type 2.

This leads to the following :

Claim 3 If a is minimal and not bad, all a-conditions of type 1 are met inside $\mathcal{M}_a$. i.e. at a stage in $\mathcal{M}_a$. If a is bad, they will be met at stage $K^a_{n-1}$. This also holds for the last injury of any a-condition.

That we have a similar pattern for the meeting of conditions of type 2 will follow from the next claim.

There is a notation system for the elements in the least Q-nice family, see for instance Normann [4]. Each element in $\mathcal{M}_a(Q)$ will have a notation [a,i], and the $\Sigma_1$-formula $\exists x$ (x has notation [a,i]) is complete $\Sigma^*$. Thus we restrict ourselves to this formula what concerns meeting of conditions of type 2.

Claim 4 Let $a,c \in I$, $\delta \in \mathcal{M}_{c,a}$ be an ordinal. Assume $x \in \mathcal{M}_a(Q)$ has a Q-notation [a,i]. Let $\nu$ be the position of the type-2 condition associated with the formula $\exists x$(x has notation [a,i].) Then there is a $\sigma > \delta$, $\sigma \in \mathcal{M}_{c,a}$ such that

$$\exists x \in \mathcal{M}_a^{<\sigma,\nu>} (Q^{<\sigma,\nu>}) \text{ [x has } Q^{<\sigma,\nu>}\text{-notation [a,i]]}.$$

Proof: We prove this on rank x. From the first part of the proof of claim 2 it is clear that each $\mathcal{M}_a$ is rudimentary closed in Q. Thus the crucial point is when x is constructed by $\Sigma^*$-collection. So assume

$$\forall b \in I\ \exists x_b \in M_{a,i,b}(Q)\ \varphi(b,x_b,a,i)$$

and that x is the collection of these $x_b$'s. The first such $x_b$ will have a notation [<e,i>,<b,a>] uniform in i,b with associated position $\nu_b$:

<u>Subclaim</u>

$$\forall c \forall \gamma \in M_{a,c} \forall b \in I \exists \sigma_b \in M_{<c,a,b,i>}(\sigma_b > \gamma\ \&\ \forall d \leq b \exists y \in \mathcal{M}_{d,a}^{<\sigma_b,\nu_d>}(Q^{<\sigma_b,\nu_d>})$$

(y has notation [<e,i>,<d,a>])).

<u>Proof of subclaim</u>: Note that the induction hypothesis means that the claim shall hold for all c and $\delta$. We will find $\sigma_b$ in $\mathcal{M}_{<c,a,b>'}$ and by reflection find it in $M_{<c,a,b,i>}$.

After $K_{n-1}^{<c,a,b>}$ none of the d-conditions of type 2 can be injured for $d \leq b$ by the proof of claim 2. Now, by the induction hypothesis

$$\forall \gamma > K_{n-1}^{<c,a,b>} \forall d \leq b(\gamma \in \mathcal{M}_{<c,a,b>'} \Rightarrow \exists \delta_o > \gamma(\delta_o \in \mathcal{M}_{<c,a,b>'}$$

$$\&\ \exists y_d \in \mathcal{M}_{d,a}^{<\delta_o,\nu_d>}(Q^{<\delta_o,\nu_d>})\ (y_d \text{ has notation } [<e,i>,<d,a>]))$$

By our extended $\Delta_o$-DC we find a sequence $<\delta_d>_{d \leq b} \in \mathcal{M}_{<c,a,b>'}$, and since no injuries can be done, at each stage $<\delta_d,\nu_d>$ we secure the fact that $y_d$ has notation [<e,i>,<d,a>] in Q.

Let $\delta = \mathrm{Sup}(\delta_d : d \leq b)$. Since nothing can be injured, $\delta$ must have the wanted property of $\sigma_b$ except being in $M_{<c,a,b,i>}$

But $\delta$ is in $\mathcal{M}_{<c,a,b>'}$, so by $\Sigma_1$-reflection we find an element in $M_{<c,a,b,i>}$ having the same properties.

(End of proof of subclaim).

Now let everything before position $\nu$ have calmed down at level $<\gamma',0>$. We may choose $\gamma' = K^{a,c}_{n-1} + 1$. Let $\gamma = \max(\gamma',\delta)$. Let $\sigma_b$ come from the subclaim. We may assume that $d < b \Rightarrow \sigma_d < \sigma_b$. Let $\lambda = \mathrm{Sup}(\sigma_b : b \in I)$, $\lambda \in \mathcal{M}_{<a,c>'}$ by the extended $\Delta_0$-DC. Let $\xi = \mathrm{Sup}\{<\sigma_b,\eta> : b \in I$ & $\eta \in$ Positions$\}$: $\xi = <\lambda,0>$. $\xi$ will have cofinality $\aleph_k$. We will now regard the construction up to stage $\xi$. We say that a condition is 'met under $\xi$ ' at stage $\xi'$ if it undergoes no changes between $\xi'$ and $\xi$. By the proof of claim 1 and its corollary every condition is 'met under $\xi$'.

By our subclaim each condition $\nu_b$ will be marked after any injury at any stage under $\xi$. Thus $\nu_b$ will be 'met under $\xi$ ' by a marker.
Thus $\forall b\ \exists y_b \in \mathcal{M}_{<a,b,c>}(Q^{\xi})(y_b$ has $Q^{\xi}$-notation $[<e,i>,<b,a>]$ ). By the choice of $\gamma'$ nothing will happen between $<\lambda,0>$ and $<\lambda,\nu>$. Thus

$$\exists x \in \mathcal{M}^{<\lambda,\nu>}_{<a,c>}(Q^{<\lambda,\nu>})\forall b \in I\ \exists y_b \in x \cap \mathcal{M}^{<\lambda,\nu>}_{<a,c,b>}\ (y_b \text{ has a notation}$$

$[<e,i>,<b,a>]$). But this is the same as

$$\exists x \in \mathcal{M}^{<\lambda,\nu>}_{<a,c>}(Q^{<\lambda,\nu>})\ (x \text{ has notation } [i,a]).$$

Now, $\lambda$ may be in $\mathcal{M}_{<a,c>'}$, but by $\Sigma_1$-reflection we find a $\sigma$ in $\mathcal{M}_{<a,c>}$ with the same properties.

Claim 5 If $a$ is minimal and not bad and $\langle i,a\rangle$ is a Q-notation for $x$, then $x \in \mathcal{M}_a$. If $a$ is bad, then $x \in \mathcal{M}_{a'}$.

Proof: Let $\nu$ be the associated position. Assume that $a$ is not bad. Then the last injury on this condition takes place at a stage $\langle\delta,\nu\rangle \in \mathcal{M}_a$. By claim 4 there is a $\sigma > \delta$, $\sigma \in \mathcal{M}_a$ such that at stage $\langle\sigma,\nu\rangle$ we find an $x \in \mathcal{M}_a^{\langle\sigma,\nu\rangle}$ having notation $[i,a]$. At this stage $\nu$ will be marked if it is not, and since this marker cannot be removed, we will add nothing to $Q$ to prevent $[i,a]$ from being a code for $x$.

If $a$ is bad, let $\delta = K^a_{n-1}$ and use claim 4 as above.

Claim 6 Let $a$ be minimal, $b$ recursive in $a$ and some tp(k-1)-element $i$. Then $Q_b \cap \mathcal{M}_a$ is not a-recursive in $Q_{-b} \cap \mathcal{M}_a$ via any subindividual.

Proof: Let $j \in tp(k-1)$. Let $\nu$ be the position of the condition

$$* \quad \mathcal{M}_a \setminus Q_{[i]^a} \neq [j,a]^{Q_{-[i]^a} \cap \mathcal{M}_a} .$$

Assume that all injuries of all conditions $\nu'$ for $\nu' \leq \nu$ has been done, and that $f(b,\xi,\nu)$ is constant for $\xi \geq \xi_o \in \mathcal{M}_a$.

Case 1 $\nu$ has a final marker, received at stage $\xi' \leq \xi_o$. Then $f(b,\xi',\nu) \subseteq Q_b$, but $\exists y \exists z \in \mathcal{M}_a R^{\xi'}_{\langle j,a\rangle}(f(b,\xi',\nu),y,z)$ & $y \subseteq Q^{\xi'}_{-b}$ & $z \cap Q^{\xi'}_{-b} = \emptyset$. Since the condition is not injured, we put no part of $z$ into $Q$ at any later stage. Thus

$$R_{\langle j,a\rangle}(f(b,\xi',\nu),y,z) \ \& \ y \subseteq Q_{-b} \ \& \ z \cap Q_{-b} = \emptyset .$$

But then $f(b,\xi',\nu)$ demonstrates that $*$ must hold.

<u>Case 2</u> $\nu$ has not a final marker. Then $f(b,\xi,\nu) = f(b,\xi_o,\nu) \in \mathcal{M}_a$ for all $\xi \geq \xi_o$. Moreover, we would never put any part of $f(b,\xi_o,\nu)$ into $Q_b$, and $f(b,\xi_o,\nu)$ is nonempty. Thus, if * fails we have

$$\exists y \exists z \in \mathcal{M}_a R_{<j,a>}(f(b,\xi_o,\nu),y,z) \;\&\; y \subset Q_{-b} \;\&\; z \cap Q_{-b} = \emptyset.$$

Since $Q$ is $\Sigma^*$-definable, we will by $\Sigma^*$-collection find $\xi' \in \mathcal{M}_a$ such that $y \subseteq Q^{\xi'}_{-b}$. But then at some stage $\xi > \max\{\xi_o,\xi'\}$, $\xi \in \mathcal{M}_a$, we would ask

$$\exists y \exists z \in \mathcal{M}_a R_{<j,a>}(f(b,\xi,\nu),y,z) \;\&\; y \subseteq Q_{-b} \;\&\; z \cap Q^{\xi}_{-b} = \emptyset \text{ ? }$$ and have the answer 'yes'. But then $\nu$ would receive a marker at stage $\xi$, contradicting the choice of $\xi_o$.

But if $Q_b$ is recursive in $Q_{-b}$, $a$ and some subindividual, there must be some $i$ $tp(k-1)$ for which * fails. This proves the claim.

Claim 5 and claim 6 give us the theorem. Note that the conclusion gives us that $Q_b$ is not recursive in $Q_{-b}$ and any $a$. Thus we have a family of incomparable strong degrees of cardinality $\aleph_k$. Moreover, all these degrees are r.e.-strong degrees.

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