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Introduction. The following pages contain the notes of a series of lectures given at the University of 0slo during the year 1974-75. Knowing that the alternative is not publishing, I have chosen to publish these notes in the present form even though, as a result of many suggestions and dissatisfied grunts from an otherwise very pleasant audience, I have become aware of the fact that the exposition is very far from being perfect.

The subject of these lectures were deformation theory, and in particular the existence and the structure of the hull of the various deformation functors in algebraic geometry.

These notes contain work done by the author over a long period of time. In fact some of the results date back to 1968-69. Since then, and in particular since 1971, when a first version of the theory presented in this paper was published (in the Preprint Series of the Institute of Mathematics at the University of Oslo), there has been done a lot of work on this subject.

I shall obviously not attempt to write a history of deformation theory, not even of this last period, but I think it may be proper to mention a few names and their relation to the results of these notes.

Inspired, I beleive, by results of Kodaira-Spencer and Grothendieck, Schlessinger and Lichtenbaum defined in (Li) a cotangent complex good enough to enable them to prove the first nontrivial theorems relating deformation theory to the cohomology of algebras. Later Andre (An) and Quillen (Qu) defined the correct cotangent complex, using quite different technics.

The approach of Quillen was then extended by Illusie (II) to yield
a global theory, working nicely for any topos.
The method used in (La4. was based upon the work of Andre and the study of the inductive and projective limit functors on small categories (see La 1, La 3).

The work of Illusie, having become standard, contains by far the most general results on the subject, thus suggesting that his methods might be the best suited for the purpose of deformation theory.

However, I have not resisted the temptation to continue the study of deformation theory along the lines of (La.4), and this paper presents the first results of this study.

Many of these results are therefore not entirely new. Some will, properly translated into the language of Illusie be found in his Springer Lecture Notes, others may be deduced from his general theorems.

This is particularly true for the following results: $(3.1 .12)$ (3.2.3) (4.1.14).

Now, the present study is based upon the following well known idea, that infinitesimal deformations of schemes (resp. morhisms of schemes) may be considered as infinitesimal deformations of the corresponding categories of algebras (resp. morphisms of algebras) associated to affine open coverings.

This way of looking at the problem of deforming schemes has many advantages. As will be shown we may, using well known functorial complexes, construct a global cohomology theory for any small category of algebras, having the same relation to the deformation theory
of the category as the Andre cohomology has to the deformation theory of a single algebra.

The main results of this paper are:
(3.1.12): Given any $S$-scheme $X$ and any quasicoherent $O_{X}$-Module M, there exist cohomology groups

$$
A^{n}(S, X ; M) \quad n \geq 0
$$

the abutment of a spectral sequence given by the term ${ }_{E_{2}^{p}}^{p}, q=H^{p}\left(X, \underline{A}^{q}(M)\right)$, where the sheaf $\underline{A}^{q}(M)$ is an $O_{X}-$ Module defined by $A^{q}(M)(U)=H^{q}(S, A ; M(U)) \quad$ whenever $U=\operatorname{Spec}(A)$ is an affine open subset of $X$, the last cohomology being that of Andre.
(3.1.14): Given any morphism of $S-s c h e m e s ~ f: X \rightarrow Y$, and any quasicoherent ${ }^{0}$-Module $M$, there exist cohomolog groups

$$
A^{n}(S, f ; M) \quad n \geq 0
$$

the abutment of a spectral sequence given by the term $E_{2}^{p, q}=H^{p}\left(Y, \underline{A}^{q}(f ; M)\right)$, where the sheaf $A^{q}(f ; M)$ is an $O_{Y}$ Module defined by $A^{q}(f ; M)(V)=A^{q}\left(B, f^{-1}(V) ; M\right)$ whenever $V=\operatorname{Spec}(B)$ is an open affine subset of $Y$.
(3.1.16): Let $Z$ be a locally closed subscheme of the s-scheme $X$, and let $M$ be any $O_{X}$-Module. Then there are cohomology groups

$$
A_{Z}^{n}(S, X ; M) \quad n \geq 0,
$$

the abutment of a spectral sequence given by the term $\mathrm{E}_{2}^{p, q}=A^{p}\left(S, X ; H_{Z}^{q}(N)\right)$. Moreover there is a long exact sequence
$\longrightarrow A_{Z}^{n-1}(S, X ; M) \rightarrow A^{n-1}(S, X ; M) \rightarrow A^{n-1}(S, X-Z ; M)$
$\longrightarrow A_{Z}^{n}(S, X ; M) \longrightarrow \cdots$
(3.2.3): Given any morphism of S-schemes $f: X \rightarrow Y$, and any $O_{X}$-Module $M$ there is a long exact sequence
$\longrightarrow A^{n-1}(S, f ; M) \longrightarrow A^{n-1}(S, X ; M) \longrightarrow A^{n-1}\left(S, Y ; R{ }^{\bullet} f \cdot M\right)$ $\longrightarrow A^{n}(S, f ; M) \longrightarrow \cdots$
(4.1.14): Let $\pi: R \rightarrow S$ be any surjective homomorphism of rings. Suppose $(\operatorname{ker} \pi)^{2}=0$ and consider a morphism of S-schemes $f: X \rightarrow Y$. Then there exists an obstruction element $O(f, \pi) \in A^{2}\left(S, f ; O_{X} S_{S}\right.$ ker $\left.\pi\right)$, such that $O(f, \pi)=0$ is a necessary and sufficient condition for the existence of a deformation of $f$ to $R$ (see definitions (4.1.)). The set of such deformations modulo an obvious equivalence relation, is a principal homogenous space over $A^{1}\left(S, f ; O_{X}{ }_{S} \operatorname{ker} \pi\right)$.
(4.2.5): Let $k$ be any field, and let $f: X \rightarrow Y$ be any morphism of algebraic k-schemes. Then the infinitesimal deformation functor of $£$ has a hull $H$ characterized in the following way: Let $T^{j}$ denote the completion of the symmetric k-algebra on the (topological) k-dual of $A^{i}\left(k, f ; O_{X}\right) \quad(\operatorname{see}(4.2)$ for definitions), then there exists a morphism of complete k-algebras

$$
o(f): T^{2} \rightarrow T^{1}
$$

with the following properties:

$$
\begin{equation*}
o(f)\left(\underline{m}_{T} 2\right) \subseteq\left(\underline{m}_{q} 1\right)^{2} \tag{i}
\end{equation*}
$$

(ii) $o(f)$ is unique up to automorphisms of $T^{1}$.
(iii) the leading term of $o(f)$ (the primary obstruction) is unique.
(iv) $H=T{ }_{T}^{1} \underset{T^{2}}{ } k$.
(5.1.2) : Severi-Kodaira-Spencer: Let $X$ be any closed subscheme of the algebraic k-scheme $Y$. Suppose $X$ is locally a complete intersection of $Y$, then if $f: X \rightarrow Y$ is the imbedding of $X$ in $Y$, we have: $A^{1}\left(k, f ; O_{X}\right)=H^{O}\left(X, N_{X / Y}\right)$ $A^{2}\left(k, f ; O_{X}\right)=H^{1}\left(X, N_{X / Y}\right)$ where $N_{X / Y}$ is the normal bundle of $X$ in $Y$.
(5.1.7): Let $Z$ be a locally closed subscheme of the algebraic $k$-scheme $X$. Suppose the $O_{X}$-Module $M$ has depth $\geq n+2$ at all points of $Z$, then the canonical morphism $A^{p}(k, X ; M) \rightarrow A^{p}(k, X-Z ; M)$ is an isomorphism for $p \leq n$.

Notations: Let $\vec{\varphi}>\vec{\psi}$. be two composable morphisms in some category. We shall denote by $\varphi \psi$ the composition of $\varphi$ and $\&$.
N.B. To avoid set theoretical difficulties we shall assume that all constructions involving categories, sets etc. take place in a fixed universe. No attempt is made to prove that the resulis emerging from these constructions are independent of the choice of this universe.

However, this seems rather obvious, see the corresponding discussions in (An).

Chapter 1. Sections of functors.
(1.1) Derivation functors associated to a functor.

Let $\pi: \underline{C} \rightarrow \underline{c}$ be a functor of small categories. We shall consider the category Mor c , for which

1. The objects are the morphisms of c .
2. If $\varphi, \varphi^{\prime}$ are objects in Mor $\underline{c}$ then the set of morphisms $\operatorname{Mor}\left(\varphi, \varphi^{\prime}\right)$ is the set of commutative diagrams


We write $\left(\psi, \psi^{\prime}\right): \varphi \rightarrow \varphi^{\prime}$ for such a morphism.
Let $\varphi \in$ Mor $c$ be an object (i.e. a morphism of $c$ ) and let $\pi^{-1}(\varphi)=\{\lambda \in$ Mor $\underline{C} \mid \pi(\lambda)=\varphi\}$.

If $\varphi_{1}$ and $\varphi_{2}$ are morphisms in $c$ which can be composed then we have a partially defined map:

$$
m: \pi^{-1}\left(\varphi_{1}\right) \times \pi^{-1}\left(\varphi_{2}\right)+\pi^{-1}\left(\varphi_{1} \circ \varphi_{2}\right)
$$

defined by composition of morphisms in $\underline{C}$.

We shall suppose that there exists a functor

$$
\text { Der: } \operatorname{Mox} \subset \rightarrow A b
$$

with properties:
(Der 1) There exists a map:

$$
\mu: \pi^{-1}(\varphi) \times \operatorname{Der}(\varphi) \rightarrow \pi^{-1}(\varphi)
$$

and a partially derined map

$$
\nu: \pi^{-1}(\varphi) \times \pi^{-1}(\varphi)+\operatorname{Der}(\varphi)
$$

defined on the subset of those pairs $\left(\lambda_{1}, \lambda_{2}\right)$ having same "source"
and same "aim". These maps satisfy the following relations

$$
\begin{aligned}
& \mu\left(\lambda_{,} \alpha+\beta\right)=\mu(\mu(\lambda, \alpha), \beta) \\
& \nu\left(\lambda_{1}, \lambda_{2}\right)=\alpha \text { is equivalent to } \lambda_{1}=\mu\left(\lambda_{2}, \alpha\right) .
\end{aligned}
$$

(i.e. the subsets of $\pi^{-1}(\varphi)$ with fixed source and aim are principal homogeneous spaces over $\operatorname{Der}(\varphi)$.)
(Der 2) Suppose $\varphi_{1}$ and $\varphi_{2}$ can be composed in $\underset{c}{ }$, then the diagram

$$
\begin{gathered}
\pi^{-1}\left(\varphi_{1}\right) \times \pi^{-1}\left(\varphi_{2}\right) \xrightarrow{m} \pi^{-1}\left(\varphi_{1} \circ \varphi_{2}\right) \\
\left(\pi^{-1}\left(\varphi_{1}\right) \times \operatorname{Der}\left(\varphi_{1}\right)\right) \times\left(\pi^{-1}\left(\varphi_{2}\right) \times \operatorname{Der}\left(\varphi_{2}\right)\right) \xrightarrow[\delta]{\mu} \pi^{-1}\left(\varphi_{1} \circ \varphi_{2}\right) \times \operatorname{Der}\left(\varphi_{1} \circ \varphi_{2}\right)
\end{gathered}
$$

commutes, with $\delta$ defined by:

$$
\delta\left(\left(\lambda_{1}, \alpha\right),\left(\lambda_{2}, \beta\right)\right)=\left(m\left(\lambda_{1}, \lambda_{2}\right), \operatorname{Der}\left(\operatorname{id}, \varphi_{2}\right)(\alpha)+\operatorname{Der}\left(\varphi_{1}, 1 \alpha\right)(\beta)\right)
$$

Note that $\left(1 d, \varphi_{2}\right): \varphi_{1} \circ \varphi_{2} * \varphi_{1}$ and $\left(\varphi_{1}, 1 d\right): \varphi_{1} \circ \varphi_{2} \leftarrow \varphi_{2}$ are morphisms in Mor $c$, since the diagrams

commute.

We shall from now on use the following notations:

$$
\begin{aligned}
\varphi_{1} \beta & =\operatorname{Der}\left(\varphi_{1}, i d\right)(\beta) \\
\alpha \varphi_{2} & =\operatorname{Der}\left(i \alpha_{,} \varphi_{2}\right)(\alpha) \\
\lambda_{1}-\lambda_{2} & =v\left(\lambda_{1}, \lambda_{2}\right)
\end{aligned}
$$

A functor with these properties will be called a derivation functor associated to $\pi$.

There are some obvious examples.

Ex.1. Let $\pi: R \rightarrow S$ be a surjective homomorphism of rings. Let $I=$ ker $\pi$ and suppose $I^{2}=0$. Consider the category $\underline{C}$ of flat R-algebras and the category $c$ of flat S-algebras. Tensorization with $S$ over $R$ defines a functor

$$
\pi: \underline{C} \rightarrow \underline{c}
$$

and the ordinary derivation functor

$$
\text { Der. Mor } c \rightarrow A b
$$

given by:

$$
\operatorname{Der}(\varphi)=\operatorname{Der}_{S}(A, B \underset{S}{X} I)
$$

where $\varphi: \mathrm{A} \rightarrow \mathrm{B}$ defines the A-module structure on $\mathrm{B} \underset{\mathrm{S}}{\otimes} \mathrm{I}$, is a derivation functor for $\pi$.

Ex.2. Let $\underline{C}_{0}$ be the full subcategory of $\underline{C}$ defined by the free R-algebras (i.e. the polynomial rings over $R$ in any set of variables), and let ${\underset{o}{o}}$ be the full subcategory of $\underline{c}$ defined by the free S-algebras. As above the ordinary derivation functor induces a derivation functor for the restriction $\pi_{0}$ of $\pi$ to $\mathrm{C}_{\mathrm{O}}$.

Ex.3. Let $\pi: R \rightarrow S$ be as before and let $\underline{C}$ be the category of R-flat affine group schemes over $R$ and $\underline{c}$ the category of $S$ flat affine group schemes over $S$.

Tensorization by $S$ over $R$ defines a functor

$$
\pi: \underline{C}+\underline{c}
$$

 homomorphism of $S-f l a t$ affine group schemes over $S$ ) and consider

$$
\left.\operatorname{Der}(\varphi)=\left\{\xi \in \operatorname{Der}_{S}(A, B \otimes \operatorname{ker} \pi) \mid \xi \circ \mu_{B}=\mu_{A} \circ(\varphi \otimes) \xi+\xi(\otimes) \varphi\right)\right\}
$$

where $\mu_{A}: A \rightarrow A \otimes A$ and $\mu_{\dot{B}} B \rightarrow B X B$ are the comultiplications defining the group scheme structure on $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ respectively.

Then Der is a derivation functor for $\pi$.
Remark. If $\pi^{-1}(\varphi)$ is empty then the conditions (Der 1) and and (Der 2) are vacuous.
(1.2) Obstructions for the existence of sections of functors.

Given a functor $\pi$ with a derivation functor Der: Nor $c \rightarrow \underline{A b}$, let us try to find conditions on $\underline{c}$ and $\pi$ under which there exists a section $\sigma$ for $\pi$, ie. a functor $\sigma: \underline{C} \rightarrow \underline{C}$ such that

$$
\sigma 0 \pi=1_{\mathrm{c}}
$$

We observe immediately that if such a o exists then certainly we must have

$$
\pi^{-1}(\varphi) \neq \emptyset \quad \text { for all } \varphi \in \underline{\text { Mors }} \subset
$$

and moreover there must exist a quasisection ide. a map
$\sigma^{\prime}:$ Mr $\underline{c} \rightarrow$ Nor $\underline{C}$ such that if $\varphi_{1}$ and $\varphi_{2}$ can be composed then $\sigma^{\prime}\left(\varphi_{1}\right)$ and $\sigma^{\prime}\left(\varphi_{2}\right)$ can be composed and $\sigma^{\prime}\left(\varphi_{1}\right) \circ \sigma^{\prime}\left(\varphi_{2}\right)$ have the same "source" and "aim" as $\sigma^{\prime}\left(\varphi_{1} \circ \varphi_{2}\right)$. Given such a quasisection $\sigma^{\prime}$ we deduce a map $\sigma_{0}: o b \underline{c}+o b \underline{C}$, which we shall call the stem of the quasisection $\sigma^{\prime}$.

Now, with all this we may prove:

Theorem (1.2.1) Suppose given a quasisection $\sigma^{\prime}$ of $\pi$, Then there exists an obstruction

$$
o\left(\sigma^{\prime}\right)=o\left(\sigma_{0}\right) \in \lim ^{(2)} \operatorname{Dor}
$$

such that $o\left(\sigma_{0}\right)=0$ if and only if there exists a section $\sigma$ of $\pi$ with the same stem $\sigma_{0}$ as $\sigma^{\prime}$. Moreover, if $o^{\prime}\left(\sigma^{\prime}\right)=0$ then the set of sections having the stem $\sigma_{0}$, modulo isomorphisms reducing to the identity, is a principal homogeneous space over

$$
\begin{aligned}
& \lim _{\stackrel{+}{c}}(1) \text { Der } \\
& \text { More } c
\end{aligned}
$$

Proof. Consider the complex $D^{\circ}=D^{*}$ (Der) of abelian groups defined by

$$
\begin{aligned}
& D^{\circ}(\operatorname{Der})=\prod_{c \in \operatorname{Dob} \underline{c}} \operatorname{Der}\left(1_{c}\right)
\end{aligned}
$$

where the indices are chains of morphisms in $\underline{c}$, and where

$$
d^{n}: D^{n} \rightarrow D^{n+1}
$$

is defined by:

$$
\begin{aligned}
& \left(d^{o} \xi\right)\left(\psi_{1}\right)=\psi_{1} \xi_{c_{1}}^{-\xi} c_{o} \psi_{1} \\
& \left(d^{n} \xi\right)\left(\psi_{1}, \cdots \psi_{n+1}\right)=\psi_{1} \xi\left(\psi_{2}, \cdots, \psi_{n+1}\right)+ \\
& \sum_{i=1}^{n}(-1)^{1} \xi\left(\psi_{1}, \cdots, \psi_{1} \circ \psi_{1+1}, \cdots, \psi_{n+1}\right)+(-1)^{n+1} \xi\left(\psi_{1}, \cdots, \psi_{n}\right) \psi_{n+1} \\
& \text { for } \quad n \geq 1 .
\end{aligned}
$$

One easily verifies that $d^{n} \circ d^{n+1}=0$ for all $n \geq 0$.

Lemma (1.2.2) $H^{n}\left(D^{\bullet}\right) \approx \lim (n)$ Der Mo r ${ }^{+}$©

The proof will be given in (1.3)

Now consider the quasisection $\sigma^{\prime}$ and define the element $\theta^{\prime}\left(\sigma^{\prime}\right)$ of $D^{2}$ by:

$$
\theta\left(\sigma^{\prime}\right)\left(\psi_{1}, \psi_{2}\right)=\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2}\right) .
$$

By assumption $\theta\left(\sigma^{\prime}\right)\left(\psi_{1}, \psi_{2}\right) \in \operatorname{Der}\left(\psi_{1} \circ \psi_{2}\right)$.
In fact $\theta\left(\sigma^{\circ}\right) \in$ ker $d^{2}$ since

$$
\begin{aligned}
& \left(d^{2} \theta\left(\sigma^{\prime}\right)\right)\left(\psi_{1}, \psi_{2}, \psi_{3}\right)=\psi_{1} \sigma\left(\sigma^{\prime}\right)\left(\psi_{2}, \psi_{3}\right)-\theta\left(\sigma^{\prime}\right)\left(\psi_{1} \circ \psi_{2}, \psi_{3}\right) \\
& +\theta\left(\sigma^{\prime}\right)\left(\psi_{1}, \psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\sigma^{\prime}\right)\left(\psi_{1}, \psi_{2}\right) \psi_{3} \\
& =\psi_{1}\left(\sigma^{\prime}\left(\psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\psi_{2}\right) \circ \sigma^{\prime}\left(\psi_{3}\right)\right)-\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right) \circ \sigma^{\prime}\left(\psi_{3}\right)\right) \\
& +\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2} \circ \psi_{3}\right)\right)-\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2}\right)\right) \psi_{3} \\
& =\left(\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2}\right) \circ \sigma^{\prime}\left(\psi_{3}\right)\right) \\
& -\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right) \sigma^{\prime}\left(\psi_{3}\right)\right) \\
& +\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2} \circ \psi_{3}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2} \circ \psi_{3}\right)\right) \\
& -\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right) \sigma^{\prime}\left(\psi_{3}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2}\right) \circ \sigma^{\prime}\left(\psi_{3}\right)\right) \\
& =0 .
\end{aligned}
$$

It follows that $\sigma\left(\sigma^{\prime}\right)$ defines an element $o\left(\sigma^{\circ}\right) \in H^{2}\left(D^{\circ}\right)$. Suppose $O\left(\sigma^{\prime}\right)=0$, then there is a $\xi \in D^{1}$ such that $d \xi=O\left(\sigma^{\prime}\right)$.

Now put

$$
\sigma(\varphi)=\sigma^{\prime}(\varphi)+\xi(\varphi)
$$

Then $\sigma\left(\psi_{1} \circ \psi_{2}\right)-\sigma\left(\psi_{1}\right) \circ \sigma\left(\psi_{2}\right)$

$$
\begin{aligned}
& =\left(\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right)+\xi\left(\psi_{1} \circ \psi_{2}\right)\right)-\left(\sigma^{\prime}\left(\psi_{1}\right)+\xi\left(\psi_{1}\right)\right) \circ\left(\sigma^{\prime}\left(\psi_{2}\right)+\xi\left(\psi_{2}\right)\right) \\
& =\sigma^{\prime}\left(\psi_{1} \circ \psi_{2}\right)-\sigma^{\prime}\left(\psi_{1}\right) \circ \sigma^{\prime}\left(\psi_{2}\right)-\left(\sigma^{\prime}\left(\psi_{1}\right) \xi\left(\psi_{2}\right)-\xi\left(\psi_{1} \circ \psi_{2}\right)\right. \\
& \left.+\xi\left(\psi_{1}\right) \sigma^{\prime}\left(\psi_{2}\right)\right)=\sigma^{\prime}\left(\sigma^{\prime}\right)\left(\psi_{1}, \psi_{2}\right)-(d \xi)\left(\psi_{1}, \psi_{2}\right)=0 .
\end{aligned}
$$

i.e. $\sigma$ is a functor, (we easily find that $\sigma\left(1_{c}\right)=\gamma_{\sigma_{o}}(c)$.

Obviously the stem of $\sigma$ is equal to the stem of $\sigma^{\circ}$ (i.e. $=\sigma_{0}$ ). Now let $\sigma_{1}$ and $\sigma_{2}$ be two sections of $\pi$ with the same stem $\sigma_{0}$. Then $\left(\sigma_{1}-\sigma_{2}\right)$ defines an element in $D^{1}$, by :

$$
\left(\sigma_{1}-\sigma_{2}\right)(\psi)=\sigma_{1}(\psi)-\sigma_{2}(\psi)
$$

Since $\sigma_{1}$ and $\sigma_{2}$ both are sections $\left(d^{1}\left(\sigma_{1}-\sigma_{2}\right)\right)\left(\psi_{1}, \psi_{2}\right)$ $=\psi_{1}\left(\sigma_{1}-\sigma_{2}\right)\left(\psi_{2}\right)-\left(\sigma_{1}-\sigma_{2}\right)\left(\psi_{1} \circ \psi_{2}\right)+\left(\sigma_{1}-\sigma_{2}\right)\left(\psi_{1}\right) \psi_{2}=0$, and therefore $\left(\sigma_{1}-\sigma_{2}\right)$ defines an element in $H^{1}\left(D^{*}\right)$.

Suppose this element is zero, then there exists an element $\zeta \in D^{\circ}$ such that

$$
\sigma_{1}(\psi)-\sigma_{2}(\psi)=\psi \zeta-\zeta \psi
$$

i.e.

$$
\sigma_{1}(\phi) \circ\left(1_{\sigma_{0}}\left(c_{1}\right)-\zeta c_{1}\right)=\left(1_{\sigma_{0}}\left(c_{0}\right)-\zeta c_{0}\right) \circ \sigma_{2}(\psi)
$$

for all

$$
\psi: c_{0} \rightarrow c_{1}
$$

Conversely, suppose $s \in H^{1}\left(D^{\circ}\right)$ is represented by $\xi \in D^{1}$ then given any section $\sigma$ of $\pi, \xi+\sigma$ is another section with the same stem as $\sigma$.

QED.
(1.3). Resolving functors for $\lim$,

Let $\underset{c}{c}$ be any small category ${ }^{+}$and denote $b y \underline{A b}^{0}$ the category of abelian functors on $\underline{c}^{\circ}$. Recall (see (La1)) the standard resolving complex

$$
\mathrm{C} \cdot: \underline{A b}^{\mathrm{c}^{0}} \rightarrow \text { CompI.ab} \cdot \underline{g r}
$$

defined by

$$
C^{p}(G)=c_{c_{1}+c_{1} \rightarrow o_{0}{ }_{\psi_{p} \rightarrow c_{p}}}^{\pi G\left(c_{p}\right)}
$$

with differential $d^{p}: C^{p}(G) \rightarrow C^{p+1}(G)$ given by :

$$
\begin{aligned}
& \left(d^{p}(\xi)\right)\left(\psi_{1}, \cdots, \psi_{p+1}\right)=G\left(\psi_{1}\right)\left(\xi\left(\psi_{2}, \cdots, \psi_{p+1}\right)\right) \\
& +\sum_{i=1}^{n}(-1)^{1} \xi\left(\psi_{1}, \cdots, \psi_{i} \circ \psi_{i+1}, \cdots, \psi_{p+1}\right)+(-1)^{n+1} \xi\left(\psi_{1}, \cdots, \psi_{p}\right)
\end{aligned}
$$

The basic properties of $c^{\circ}=C^{\circ}\left(\underline{c}^{0},-\right)$ are the following :

1) $C^{\circ}\left(c^{0},-\right)$ is an exact functor
2) $\mathrm{H}^{\mathrm{n}}\left(\mathrm{C}^{\bullet}\left(\underline{\mathrm{c}}^{0},-\right)\right)=\underset{{\underset{\mathrm{c}}{ }}^{0}}{\lim }(\mathrm{n})$ for $\mathrm{n} \geq 0$.

Now let $F$ be any abelian functor on Nor $c$ (ie. $F$ is an object of $\underline{A b}^{\text {Nor }} \xrightarrow{c}$ ) and put

Let $d^{p}$ be the homomorphism $D^{p}(F) \rightarrow D^{p+1}(F)$ defined by

$$
\begin{gathered}
\left(d^{p}(\xi)\right)\left(\psi_{1}, \cdots, \psi_{p+1}\right)=F\left(\psi_{1}, 1_{c_{p+1}}\right)\left(\xi\left(\psi_{2}, \cdots, \psi_{p+1}\right)\right) \\
+\sum_{i=}^{p}(-1)^{1} \xi\left(\psi_{1}, \cdots, \psi_{1}, \psi_{1+1}, \infty 0, \psi_{p+1}\right)+(-1)^{p+1} F\left(1_{c_{o}}, \psi_{p+1}\right)\left(\xi\left(\psi_{1}, \cdots, \psi_{p}\right)\right)
\end{gathered}
$$

(Remember that $\left(\psi_{1}, 1_{c_{p+1}}\right)$ is a morphism

$$
\psi_{2} \circ \cdots \circ \psi_{p+1} \rightarrow \psi_{1} \circ \cdots \circ \psi_{p+1}
$$

In Nor c and that $\left(1_{c_{o}}, \psi_{p+1}\right)$ is a morphism

$$
\psi_{1} \circ \cdots \circ \psi_{p} \rightarrow \psi_{1} \circ \cdots \circ \psi_{p+1}
$$

in More c).
It is easy to check that $\left(D^{p}(F), d^{p}\right)_{p \geq 0}$ is a complex of abelian
groups defining a functor

$$
\mathrm{D}^{\bullet}: \mathrm{Ab}^{\text {Mor }} \stackrel{C}{C}+\text { Compl } \cdot \underline{a b} \cdot \underline{\mathrm{gr}}
$$

Lemma (1.3.1). The functor $D^{*}=D^{*}(\underline{c},-)$ has the following properties:

1) $D^{\bullet}(\underline{c},-)$ is exact
2) $\quad H^{n}\left(D^{\bullet}(\underline{c},-)\right)=\underset{\sim}{\lim }(n)$ for $n \geq 0$.

Proof. Let $I$ be the constant functor on Mor c, i.e. $L(\varphi)=\mathbb{Z}$ for all $\varphi$.
We shall construct a projective resolution of $L$ in $A B$ Mor $C$. Let $\varphi: x \rightarrow y$ be any object of Mor $\subseteq$ and consider the sets
$\Delta^{o}(\varphi)=\left\{x^{\varepsilon} \xrightarrow{\varepsilon} c_{0} \stackrel{\rho}{\rightarrow} y \mid \varepsilon \circ \rho=\varphi\right\}$.
$\Delta^{n}(\varphi)=\left\{x \stackrel{\varepsilon}{+} c_{0} \psi_{1} c_{1} \rightarrow \cdots \rightarrow c_{n-1} \xrightarrow{\psi_{n}} c_{n} \stackrel{\rho}{+} y \mid \varepsilon \circ \psi_{1} \circ 00 \circ \psi_{n} \circ \rho=\varphi\right\}$
There exists maps:
$n_{1}^{n}: \Delta^{n}(\varphi) \rightarrow \Delta^{n+1}(\varphi)$
$\delta_{1}^{n}: \Delta^{n}(\varphi) \rightarrow \Delta^{n-1}(\varphi)$
defined by :


$$
\begin{aligned}
& \text { - } 15 \text { - } \\
& {\left[\begin{array}{c}
\varepsilon \circ \psi_{1} \psi_{2} \\
\left(x \rightarrow c_{1} \rightarrow c_{2}+\cdots \circ c_{n}+y\right), i=0 .
\end{array}\right.}
\end{aligned}
$$

$$
\begin{aligned}
& \left.\underset{\left(x \rightarrow c_{0}+0 \rightarrow c_{n-1}\right.}{\psi_{n} \rho \rho}+y\right), \quad i=n
\end{aligned}
$$

giving $\Delta^{n}(\varphi), n \geq 0$ the structure of a simplicial set. Moreover for each $n \geq 0, \Delta^{n}(\varphi)$ is functorial in $\varphi$ defining a functor

## $\Delta:$ Mor $c \rightarrow$ Simplicial sets

Composing $\Delta$ with the functor $C .(-, \mathbb{Z})$ we have constructed a complex of functors

$$
\text { C. : Mor } c+A b
$$

Now, by a standard argument we construct a contracting homotopy for $C$. thereby proving

$$
H_{i}\left(C_{0}\right)=\left\{\begin{array}{lll}
L & \text { for } & 1=0 \\
0 & \text { for } & 1 \neq 0
\end{array}\right.
$$

Moreover

$$
\begin{array}{rll}
C_{n}(\varphi)= & \left.\begin{array}{ll}
\mathbb{I} & \{\mathbb{I} \mathbb{Z} \\
(\varepsilon, \rho): \varphi^{\prime} \rightarrow \varphi & \psi_{1}, 00, \psi_{n}
\end{array}\right\} \\
& \text { in Mor } \subseteq & \psi_{1} \circ 00 \circ \psi_{n}=\varphi^{\beta}
\end{array}
$$

Using ( (Lrei), Prop.1.1.a) it follows that each $C_{n}$ is projective as an object of $A b \underline{M}$.

Therefore $C$. is a projective resolution of $L$ in AbMor $C$.

Since

$$
\begin{aligned}
& \text { Mor } \operatorname{Mor}_{\underline{c}}\left(\mathrm{C}_{\mathrm{n}}, F\right)=\pi F\left(\psi_{1} 00000 \psi_{n}\right) \\
& \mathrm{Ab} \\
& c_{0 \psi_{1}} c_{1} \rightarrow \cdots c_{n-1} \vec{\psi}_{n}^{c}{ }_{n}
\end{aligned}
$$

we find by a dull computation that

$$
D^{D^{0}(F) \simeq \operatorname{Mor}} \underset{\underline{M o r} C^{(C ., F)}}{ }
$$

thereby proving the lemma.
QED.

Chapter 2. Lifting of algebras and morphisms of algebras.

## (2.1) Leray spectral sequence for lim.

Let $c$ be any small category and let $c$ be an object of $c$. Consider the contravariant functor $c(\mathbb{Z}, c)$ defined by:

$$
c(\mathbb{Z}, c)\left(c^{\prime}\right)=\frac{11}{c^{\prime} \vec{\varphi} c} \mathbb{Z}
$$

We know (see (La1)) that these functors are projective objects in $A b^{\mathrm{C}^{\circ}}$.

Suppose $M$ is a full subcategory of $C$ and consider the restriction of $C(\mathbb{Z}, \mathrm{c})$ to $\mathbb{M}$. Let F be any contravariant functor on $M$ with values in $A B$, then we find,

$$
A b^{M^{\circ}}(C(\mathbb{Z}, c), F) \simeq \lim _{(M / C)^{\circ}} F
$$

Now, suppose $c_{o} \vec{\varphi} c$ in $c$ is an $M$ epinorphism, i.e. $c_{o} \in$ ob $M$ and the map

$$
\operatorname{Mor}\left(c^{\prime}, c_{0}\right) \rightarrow \operatorname{Mor}\left(c^{\prime}, c\right)
$$

is surjective for every $c^{\prime} \in$ ob $M$.
Suppose further that $C$ has fibered products and consider the system of morphisms

Put $c_{p}=\underbrace{c_{0} \times \ldots \times c_{c}}_{p+1}$.... c c denote by

$$
d_{p}^{i}: c_{p} \rightarrow c_{p-1} \quad i=0, \ldots, p
$$

the $p+1$ projection momphisms.

Consider for each $d_{p}^{i}$ the corresponding orphism
$\partial_{p}^{i}: C\left(\mathbb{Z}, c_{p}\right) \rightarrow C\left(\mathbb{Z}, c_{p-1}\right)$ and let $\partial_{p}=\sum_{j=0}^{p}(-1)^{i} \partial_{p}^{i}$. Then $\partial_{p} \partial_{p-1}=0$ for all $p \geq 1$.

Lemma (2.1.1) The complex $C .=\left\{c\left(\mathbb{Z}, c_{p}\right), \partial_{p}\right\}_{p \geq 0}$ is a resolution of $C(\mathbb{Z}, \mathrm{c})$ in $\mathrm{Ab}^{\mathrm{M}}{ }^{\mathrm{O}}$.

Proof. See f. ex. (ix) p. 18.

Let $T^{\circ}$ be an infective resolution of $F$ in $A B^{M^{\circ}}$ and consider the double complex

$$
\operatorname{Mor}\left(C ., F^{\circ}\right)
$$

We shall compute the two associated spectral sequences. But first we have to establish the following lemma.

Lemma (2.1.2) Let $I: M / c \rightarrow M$ be the canonical forgetful functor and let $F$ be injective in $A M^{M^{\circ}}$, then the composed functor for $:(\mathbb{M} / c)^{\circ} \rightarrow A b$ is injective as an object of $A b{ }^{(M / c)^{0}:}$

Proof. The functor $f$ induces a functor

$$
f_{*}: A b^{M^{\circ}} \rightarrow A b(M / C)^{0}
$$

We want to prove that $\hat{I}_{*}$ takes injectives into injectives. To prove this we construct a left adjoint

$$
p: A b(M / c)^{0} \rightarrow A M^{\circ}
$$

Let $G$ be an object of $A b(M / c)^{\circ}$ and put

$$
\left.\rho(G)(m)=\frac{1}{\varphi \in \operatorname{Mor}(m, c)} \xrightarrow{G(m} \xrightarrow{\varphi} c\right)
$$

so that $\rho(G)$ is an object of $A b^{M^{\circ}}$.
One easily checks that there is a canonical isomorphism

$$
\operatorname{Mor}(\rho(G), F) \simeq \operatorname{Mor}\left(G, f_{:}(\mathbb{F})\right)
$$

proving that $\rho$ is left adjoint to $f_{*}$. Since $\rho$ is exact we know that $f_{*}$ takes injectives into injectives.

QED.

Going back to the double complex Mor(C., $\mathrm{F}^{*}$ ) we find the $\mathrm{F}_{2}$ terms of the two associated spectral sequences:
${ }^{1} \mathrm{H}^{p}, \mathrm{q}=\mathrm{H}^{\mathrm{p}}\left(\mathrm{H}^{\left.\left(\operatorname{Mor}\left(\mathrm{C} ., \mathrm{F}^{\bullet}\right)\right)\right)}\right.$
${ }_{H E}^{P}, q=H^{p}\left(\operatorname{Mor}\left(H_{q}\left(C_{0}\right), T^{0}\right)\right)$
We know already that
"祘 $q=0$ for $q \neq 0$
$\left.\mathrm{E}_{2}^{n, o}=H_{\left(M^{n} / \mathrm{c}\right)^{\circ}}^{\lim ^{\circ}}\left(\mathrm{F}^{\bullet}\right)\right)$
and by Lemma (2.1.2) we deduce
$" \mathrm{E}_{2}^{\mathrm{n}, o}=\underset{(\mathbb{M} / \mathrm{c})^{\circ}}{\lim }(\mathrm{n})_{\mathrm{F}}$.
Since
$\operatorname{Mor}\left(C_{p}, F^{\bullet}\right)=\underset{M / c_{p}}{\lim } \mathbb{F}^{\bullet}$,
we find, using Lemma (2.1.1) once more that

$$
\mathrm{E}_{2}^{p, q}=H_{M}^{p}\left(\lim _{\kappa}(q)_{\mathrm{F}}\right)
$$

We have proved the following theorem.

Theorem (2.1.3) Let $M \subseteq c$ and $\varphi: c_{o} \rightarrow c$ be given as above. Then there exists a Leray spectral sequence given by:

$$
\left.\mathbb{E}_{2}^{p, q}=F_{2}^{p, q}(M)=H_{(M / c \cdot)^{p}}^{\left(\lim _{N}\right.}(q)_{F}\right)
$$

converging to

$$
\lim _{\left(M^{\leftarrow} / \mathrm{c}\right)^{0}}(\cdot)_{\mathrm{F}}
$$

Remark 1. The spectral sequence above is nothing but the Leray spectral sequence associated to the "covering" $\varphi: c_{o} \rightarrow c$ in an appropriate Grothendieck topology.
2. Since $c_{0} \in$ ob M the category $M / c_{0}$ has a final object. Therefore $E_{2}^{0}, \underline{q}=0$ for all $q \geq 1$.

We deduce from this the formulas

$$
\begin{aligned}
& \lim _{(M / c)^{\circ}} \mathrm{F} \simeq \mathbb{E}_{2}^{0}, o \\
& \underset{(\mathbb{N} / \mathrm{c})^{\circ}}{\lim }(1)_{\mathrm{F}} \simeq \mathrm{E}_{2}^{1,0},
\end{aligned}
$$

and the exact sequence

$$
0 \rightarrow \mathrm{E}_{2}^{2,0} \rightarrow \lim _{(\underline{M} / c)^{\circ}}^{(2)_{\mathrm{F}}} \rightarrow \mathrm{E}_{2}^{1,1} \rightarrow \mathrm{E}_{2}^{3,0} \rightarrow \lim _{(M / c)^{\circ}}(3)_{\mathrm{F}}
$$

Corollary (2.1.4) Suppose that $\lim _{t}(i)_{F}=0$ for $i \geq 1$, $\left(M / c_{j}\right)^{0}$
$i+j=p$ and for $i+j=p-1$. Then

$$
\lim _{(\mathbb{M} /)^{\circ}}(p)_{F} \simeq \mathbb{E}_{2}^{p, o}
$$

Assume for a moment that there exists a functor i : $\mathrm{c} \rightarrow \mathrm{Ab}$ commuting with ribered products.

Corollary (2.1.5) Put $g=f \circ i$ and suppose

$$
\lim _{\underset{M}{M} / c_{p}} g=i\left(c_{p}\right) \quad \text { for all } p \geq 0
$$

Then

$$
\lim _{M / \mathrm{c}}^{\vec{M}(1)} \mathrm{g}=0 .
$$

Proof. Let $E$ be an injective abelian group and consider the functor

$$
F(-)=A b(g(-), E)
$$

We know that

$$
\begin{aligned}
& \frac{\mathrm{Ab}\left(\lim _{M / \mathrm{c}}(1), \mathrm{B}\right)}{\left(\lim _{(1)}(1)_{\mathrm{F}}\right.} \underset{(\mathrm{M})^{\circ}}{ } \\
& =\operatorname{ker}\left\{\lim _{\leftarrow} \mathrm{F} \rightarrow \lim \mathrm{~F}\right\} / \operatorname{in}\{\lim \mathrm{F} \rightarrow \lim \mathrm{~F}\} \\
& \left(M / c_{1}\right)^{\circ} \quad\left(M / c_{2}\right)^{0} \quad\left(M / c_{0}\right)^{\circ}\left(M / c_{1}\right)^{\circ} \\
& =A b\left(\operatorname{ker}\left\{i\left(c_{1}\right) \rightarrow i\left(c_{o}\right)\right\} / \operatorname{im}\left\{i\left(c_{2}\right) \rightarrow i\left(c_{1}\right)\right\}, E\right)
\end{aligned}
$$

But since $i\left(c_{p}\right)=\underbrace{i\left(c_{0}\right) \times \ldots \times i\left(c_{0}\right)}_{p+1}$
this last group is zero.
Since this holds fox all injective abelian groups $E$ we have proved that $\lim _{M / \mathrm{c}}^{\mathrm{M}}(1) \mathrm{g}=0$.

Remark. The last corollary and the next one are important in our development of the lifting theory for algebras.

Corollary (2.1.6) Let $M_{0} \subseteq M$ be two full subcategories of $c$. Suppose c has fibered products and let $\mathrm{c} \in \mathrm{ob} \mathrm{c}$.

Assume that ( $c, M_{0}, M$ ) satisfies the following conditions:
$\left(c_{1}\right)$ There exists an object $c_{0}$ of $M_{0}$ and an M-epimorphism $\varphi: c_{0} \rightarrow c$.
( $c_{2}$ ) For any M-epimorphism $\psi: d_{0} \rightarrow d$ in $c$ with $d_{0} \in M_{0}$ there exist objects $\theta_{p} \in M_{0}$ and M-epimorphisms

$$
\psi_{p}: e_{p} \rightarrow \underbrace{d_{o}^{o} \times \ldots \times d_{d}}_{p+1}) \quad p \geq 2
$$

Then we may conclude

$$
\lim _{(\mathbb{M} / c)^{0}}(0) \simeq \lim _{\left(M_{0}^{+} / c\right)^{0}}
$$

Proof. We first observe that $\left(c_{1}\right)$ and $\left(c_{2}\right)$ together with (2.1.1) imply that there are canonical isomorphisms

$$
\begin{equation*}
\lim _{\left(M / c_{p}\right)^{0^{\prime}}} \underset{\left(\mathcal{N}_{0}^{\sim}\right.}{\lim _{p}^{*}} \tag{1}
\end{equation*}
$$

where


Now the canonical momphism

$$
t^{n}: \lim _{(\mathbb{M} / c)^{0}}(n) \rightarrow \lim _{\left(\mathcal{M}_{-0}^{+} / c\right)^{0}}(n)
$$

induces morphisms of spectral sequences

$$
\left.\operatorname{ti}_{2}^{p}, q: \mathbb{E}_{2}^{p}, q_{(\mathbb{M}}\right) \rightarrow \mathbb{E}_{2}^{p}, q_{\left(\mathbb{N}_{0}\right)}
$$

Using (1) we find isomorphisms

Thereby proving that $t_{2}^{1}$ is an isomorphism. By an easy induction argument we may assume that $t \frac{p}{2}, q$ are isomorphisns for all $p, q$ with $p+q \leq n$ or $q<n$. This implies that
are isomorphisms for all $p, q$ with $p+q=n$, thereby proving that $t^{n}$ is an isomorphism.

QED.

## (2.2) Lifting of algebras

Let $S$ be any commutative ring with unit. Let $S-a l g$ denote the category of $S$ algebras and let S-free denote the category of free S-algebras (i.e. the cateogry of polynomial algebras, in any set of variables, over $S$ ).

Let $A$ be any object of $S-a l g$ and consider the subcategories $M_{0}$ and $M$ of $S-a l g / A$ where $M=S-$ free $/ A$ and $M_{0}=(S-f r e e / A)^{e p i}$ is the full subcategory of $M$ defined by the epimorphisms $F \rightarrow A$ 。 Thus we hove $M_{0} \subseteq M \subseteq S-a l g / A$. We observe that we have isomorphisms of categories:

$$
\begin{aligned}
& M_{0} \simeq M_{0} /\left(A \overrightarrow{1}_{A} A\right) \\
& M \simeq M /\left(A \overrightarrow{1}_{A}^{A}\right) \\
& M-a l g \\
& \simeq M-a l g / A) /\left(A_{A} A\right)
\end{aligned}
$$

Let $\mathcal{I}$ (resp. $f_{o}$ ) be the forgetful functor $M \rightarrow s-a l g$ (resp. $\left.M_{0} \rightarrow S-a l g\right)$. By straight forward verification we find that $M_{0} \subseteq M \subseteq S-a l g / A$ and the object $\left(A^{1} A A\right)$ satisfy the conditions of Corollary $(2.1 .6)$. We therefore conclude

Lemma (2.2.1) There are canonical isomorphisms of functors

$$
\underset{(S-\operatorname{free} / A)^{\circ}(n)}{\lim _{\leftarrow}(S-\operatorname{free} / A)^{\text {epi,o }} \quad n \geq 0}
$$

Let $i: S-a l g \rightarrow A B$ be the forgetful functor, then $i$ commutes with fibered products. Thus Corollary (2.1.5) implies

Lemma (2.2.2) Let $g=f i$ (resp. $g_{0}=f_{0} i$ ) be the composed func.. tor, then

$$
\begin{aligned}
& \lim _{\rightarrow \rightarrow} \mathrm{G}=\mathrm{A}, \lim _{\rightarrow \rightarrow}(1)^{\mathrm{g}}=0 \\
& \text { S-free/A } \quad \text { S-free/A } \\
& \text { (resp. } \lim _{\rightarrow} g=A, \quad \lim _{\rightarrow}(1)^{g}=0 \text { ) } \\
& (s-\text { free } / A)^{\text {epi }}(s-\text { sree } / A)^{\text {epi }}
\end{aligned}
$$

Remark. The isomorphism of (2.2.1) is obviously induced by the natural homomorphism of complexes

$$
C \cdot\left(S-\text { free } / A^{0},-\right) \rightarrow C \cdot\left((S \text { free } / A)^{\text {epi }, o},-\right)
$$

Now recall (see (An)) that given any A-module il the algebra cohomeology $H^{\circ}(S, A ; M)$ is defined by:

$$
H^{n}(S, A ; M)=\underset{(S-\text { free } / A}{\lim _{\sim}^{\circ}}(n) \quad \operatorname{Der}_{S}(-, M)
$$

where

$$
\operatorname{Der}_{S}(-, M):(S-f r e e / A)^{\circ} \rightarrow A b
$$

is the functor defined by:

$$
\operatorname{Der}_{S}(\varphi \downarrow, M)=\operatorname{Der}_{S}(F, M)
$$

where it is understood that $M$ is considered as an $F$-module via $\varphi$ 。

Lemma (2.2.1) therefore tells us that we may compute $H^{n}(x, A ; M)$ using only the subcategory (S-free/A) epi of (S-free/A), or stated in a form we shall need later on: the homomorphism of complexes

$$
C^{\bullet}\left((\text { S-free } / A)^{\circ}, \operatorname{Der}_{S}(-, M)\right) \rightarrow C^{\circ}\left((S-\text { free } / A)^{\text {epi }, o}, \operatorname{Der}_{S}(\ldots, M)\right)
$$

is a quasiisomorphism (i.e. induces isomorphisms in cohomology).

Consider any $s$-module $I$ and let

$$
\operatorname{Der}_{S}\left(-,-Q_{S} I\right): \operatorname{Mor}(S-f r e e / A) \rightarrow A b
$$

be the functor defined by:

$$
\operatorname{Der}_{S}\left(-, \cdots \otimes_{S}^{\otimes} I\right)\left(F_{0} \xrightarrow[\delta_{0} \pm A^{L} \delta_{1}]{\alpha_{1}} F_{1}\right)=\operatorname{Der}_{S}\left(F_{0}, F_{1} \otimes_{S} I\right)
$$

where $\mathrm{F}_{1} \underset{\mathrm{~S}}{\otimes} \mathrm{I}$ is considered as an $\mathrm{F}_{\mathrm{o}}$-module via the morphism $\alpha_{1}$.
Let

$$
\operatorname{Der}_{S}(-, A \otimes I): \operatorname{Mor}(S-\text { free } / A) \rightarrow A b
$$

be the functor defined by

$$
\operatorname{Der}_{S}(\ldots, A \otimes I)\left(F_{0} \xrightarrow{\alpha_{1}} \xrightarrow{\delta_{0} \searrow_{A}} I_{1}\right)=\operatorname{Der}_{S}\left(F_{0}, A \otimes I\right)
$$

where $A \underset{S}{\otimes} I$ is considered as an $F_{o}$-module via the morphism $\delta_{0}\left(=\alpha_{1} \delta_{1}\right)$.
Obviously there is a morphism of functors

$$
\operatorname{Der}_{\mathrm{S}}(-,-\otimes I) \rightarrow \operatorname{Der}_{\mathrm{S}}(-, A \otimes I)
$$

The restriction of this morphism to the subcategory $\operatorname{Mor}(\mathrm{S}-\mathrm{free} / \mathrm{A})^{\mathrm{epi}}$ of $\operatorname{Mor}($ S-free/A $)$ is moreover surjective. Notice that by construction

$$
\begin{aligned}
& D^{\bullet}\left(\left(S-\text { free }_{A}\right), \operatorname{Der}_{S}(-, A \otimes I)\right)=C_{S}^{\bullet}\left((S-\text { free } / A)^{0}, \operatorname{Der}_{S}(-, A \otimes I)\right) \\
& D^{\bullet}\left(\left(S-\text { free }_{S}\right)^{e p i}, \operatorname{Der}_{S}(-, A \otimes I)\right)=C^{0}\left(\left(S-\text { free }_{A}\right)^{\text {epi,o }}, \operatorname{Der}_{S}(-, A \otimes I)\right)
\end{aligned}
$$

Thus there is a commutativ diagram of complexes
$D^{\bullet}\left((S-\operatorname{sree} / A), \operatorname{Der}_{S}(-,-\underset{S}{\otimes} I)\right) \stackrel{1}{\rightarrow} C^{\bullet}\left(\left(S-\operatorname{free}^{(A)}\right)^{\circ}, \operatorname{Der}_{S}(-, A \otimes I)\right)$ $\downarrow k \quad i \downarrow$

in which $i$ is a quasiisomorphism and $j$ is a surjection. Put:

$$
K^{\bullet}=\operatorname{ker} j
$$

Now let

$$
\pi: R \rightarrow S
$$

be any surjective homomorphism of commutative rings and consider the diagram

$$
\mathrm{e}=\{R \vec{\pi} S \rightarrow A\}
$$

Definition (2.2.3) A lifting of $e$, or a lifting of $A$ to $R$, is a commutative diagrom of commutative rings

such that:
(1) $\quad \begin{array}{rl}A^{\prime} \\ R & S \\ \sim\end{array}$
(2) $\operatorname{Tor}_{1}^{R}\left(A^{\prime}, S\right)=0$

Abusing the language we shall usually call A' a lifting of $A$ to $R$.

Definition (2.2.4) Two liftings, $A^{\prime}$ and $A^{\prime \prime}$, of $A$ to $P$ are equivalent (written $A^{\prime} \sim A^{\prime \prime}$ ), if there exists an isomorphism of rings

$$
\theta: A^{\prime} \rightarrow A^{\prime \prime}
$$

such that the following diagram commutes


The set of liftings of $A$ to $R$ modulo this equivalence relation is denoted

$$
\operatorname{Def}(\mathrm{e})=\operatorname{Def}(R \rightarrow S \rightarrow A) .
$$

The purpose of this paragraph is to answer the following two questions

1) When does there exist liftings of $A$ to $R$ ?
2) If there do exist some, how many are there?

As usual the answers given will be rather formal and only partial.

In fact we shall have to assume that

$$
(\operatorname{ker} \pi)^{2}=0,
$$

implying that $I=k e r \pi$ has a natural structure of $S$-modul. Notice that in this case we already know ((1.1) Ex. 2) that the functor

$$
\operatorname{Der}_{S}(-,--\otimes I): \operatorname{Mor}(S-f r e e) \rightarrow A b
$$

is a derivation functor for the functor

$$
-\underset{R}{-S}: R \text {-iree } \rightarrow \text { S-free },
$$

the restriction of $-\underset{R}{S}$ to the subcategory $R$ free of $\mathrm{R}-\mathrm{Alg}$ 。

Suppose there exist a section $\sigma$ of this last functor, then an easy argument shows that the R-algebra

$$
A^{\prime}=\lim _{S \rightarrow \vec{f}+e}(f \circ \sigma)
$$

where

$$
\text { I:S-free/A } \rightarrow \text { S-free }
$$

is the forgetfull functor, defines a lifting of $A$ to $R$. Now, clearly, the existence of a section $\sigma$ of $-\underset{R}{\otimes} S$ is too much to hope for, but the idea, properly modified, is still good.

In fact there axe lots of quasisections $\sigma^{\prime}$ of $-\underset{R}{\otimes} S: R$-free $\rightarrow$ S-free (but only one stem ) Picking one we find an obstruction cocycle $O\left(\sigma^{\prime}\right)$ in $D^{2}\left(S-f_{r e e}, \operatorname{Der}_{S}(-,-\underset{S}{\otimes} I)\right.$ ) (see (1.2)). Obviously the forgetfull functor $f$ defines a morphism of complexes

$$
D^{\bullet}\left(\text { S-free }, \operatorname{Der}_{S}(-,--\underset{S}{\otimes} I)\right) \rightarrow D^{\circ}\left(S-\text { free }_{A}, \operatorname{Der}_{S}(-,-{\underset{S}{S}} I)\right)
$$

Thus $O\left(\sigma^{\prime}\right)$ defines a 2-cocycle $O^{\prime}\left(\sigma^{\prime}, A\right)$ of $D^{0}(S-$ iree $\left./ A), \operatorname{Der}_{S}(-,-\underset{S}{\otimes} I)\right)$, which maps to a 2-cocycle

$$
O\left(\sigma^{\prime}, A\right)=I\left(O^{\prime}\left(\sigma^{\prime}, A\right)\right) \in C^{2}\left(\left(S-\text { free }_{A}\right)^{O}, \operatorname{Der}_{S}(-, A \otimes I)\right)
$$

under the morphism 1 (see diagram above).

We already know that the corresponding cohomology class

$$
o(\pi, A) \in H^{2}(S, A ; A \otimes I)
$$

does not depend upon the choice or quasisection $\sigma^{\prime}$. Moreover we shall prove the following

Theorem (2.2.5) There exists an obstruction

$$
o(\pi, A) \in H^{2}(S, A ; A \otimes I)
$$

such that $o(\pi, A)=0$ if and only if there exists a lifting of $A$ to $R$. In that case $\operatorname{Def}(R \rightarrow S \rightarrow A)$ is a principal homogeneous space over $H^{1}(S, A ; A \underset{S}{S} I)$.

Proof. Consider the diagrams of functor

$$
\begin{array}{cc}
\left.\begin{array}{ll}
\text { R-free } \\
\downarrow-\otimes S \\
(\text { SHIre } / A) \\
f
\end{array}\right)
\end{array}
$$

Definition (2.2.6) A map

$$
\sigma^{\prime}: \operatorname{mor}(S-f r e e / A) \rightarrow \operatorname{mor}(R-f r e e)
$$

(resp. $\left.\sigma_{0}^{\prime}: \operatorname{mor}\left(S-f_{r e e}^{A}\right)^{\text {ep }} \rightarrow \operatorname{mor}(R-f r e e)\right)$ respecting the objects (ie. objects are mapped onto objects) will be called an $\hat{i}$ (resp. $f_{o}$ ) - quasisection provided

$$
\sigma^{\prime} u=f \quad\left(x \operatorname{esp} \cdot \sigma_{o}^{\prime} u_{0}=f_{0}\right)
$$

Let $\sigma^{\prime}$ (resp. $\sigma_{o}^{\prime}$ ) be any $f$ (resp. $f_{o}$ )- quasisection and consider the cochin $O\left(\sigma^{\prime}\right)$ (resp. $O\left(\sigma_{0}^{1}\right)$ ) of
$C^{2}\left((\text { S-free } / A)^{\circ}, \operatorname{Der}_{S}(-, A \otimes I)\right) \quad(r e s p$.
$C^{2}\left(\left(S-\operatorname{free}_{A}\right)\right.$ epi, $\left.\left.0, \operatorname{Der}_{S}(-, A \otimes I)\right)\right)$ defined by

$$
O_{0}\left(\sigma^{\prime}\right)\left(\begin{array}{llll} 
& \alpha_{1} & & \alpha_{2} \\
F_{0} & F_{1} & \rightarrow \\
\delta_{0} & F_{0} & \psi_{1}
\end{array}\right)=\left(\sigma_{2}^{\prime}\left(\alpha_{1} \alpha_{2}\right)-\sigma^{\prime}\left(\alpha_{1}\right) \sigma^{\prime}\left(\alpha_{2}\right)\right)\left(\delta_{2} \otimes 1_{I}\right)
$$



One proves as in (1.2) that $O_{0}\left(\sigma^{\prime}\right)$ (resp. $O_{0}\left(\sigma_{0}^{\prime}\right)$ ) is a cocycle, and that the corresponding cohomology class coincides with the cohomology class $o(\pi, A)$ constructed above. Now suppose there exists a lifting $A^{\prime}$ of $A$ to $R$. Then we may, fox every object ( $\pi_{0} \stackrel{\delta}{o}^{\circ} \mathrm{A}$ ) of S -free $/ \mathrm{A}$, pick an object ( $\mathrm{F}_{0}^{\prime} \stackrel{\delta_{0}^{\prime}}{o} \Lambda^{\prime}$ ) of R-free/ $A^{\prime}$ such that $\delta_{0}^{\prime}{\underset{R}{R}}_{\otimes} S=\delta_{0}$. Obviously $\sigma^{\prime}\left(\delta_{0}\right)=F_{0}^{\prime}$, and let us put

$$
\sigma_{A^{\prime}}^{\prime}\left(\delta_{0}\right)=\delta_{0}^{\prime} .
$$

With these notations let $Q_{0}=Q_{o}\left(\sigma^{\prime}, A^{\prime}\right)$ be the 1. cochain of $C^{\cdot}\left((S-\text { free } / A)^{\circ}, \operatorname{Der}_{S}(-, A \otimes I)\right)$ defined by

$$
Q_{0}\left(\begin{array}{ccc}
F_{0} & \alpha_{1}^{\prime} & F_{1} \\
\delta_{0} & \delta_{A} \delta_{1}
\end{array}\right)=\sigma^{\prime}\left(\alpha_{1}\right) \sigma_{A^{\prime}}^{\prime}\left(\delta_{1}\right)-\sigma_{A^{\prime}}^{\prime}\left(\delta_{0}\right)
$$

We find

$$
\begin{aligned}
& -\left(\sigma^{\prime}\left(\alpha_{1} \alpha_{2}\right) \sigma_{A^{\prime}}^{\prime}\left(\delta_{2}\right)-\sigma_{A^{\prime}}^{\prime}\left(\delta_{0}\right)\right)+\left(\sigma^{\prime}\left(\alpha_{1}\right) \sigma_{A^{\prime}}^{\prime}\left(\delta_{1}\right)-\sigma_{\Lambda^{\prime}}^{\prime}\left(\delta_{0}\right)\right) \\
& =\sigma^{\prime}\left(\alpha_{1}\right)\left(\sigma^{\prime}\left(\alpha_{2}\right) \sigma_{A^{\prime}}^{\prime}\left(\delta_{2}\right)-\sigma_{A^{\prime}}^{\prime}\left(\delta_{1}\right)\right)-\left(\sigma^{\prime}\left(\alpha_{1} \alpha_{2}\right) \sigma_{A^{\prime}}^{\prime}\left(\delta_{2}\right)-\sigma_{A^{\prime}}^{\prime}\left(\delta_{0}\right)\right) \\
& +\left(\sigma^{\prime}\left(\alpha_{1}\right) \sigma_{A^{\prime}}^{\prime}\left(\delta_{1}\right)-\sigma_{A^{\prime}}^{\prime}\left(\delta_{0}\right)\right)=\left(\sigma^{\prime}\left(\alpha_{1}\right) \sigma^{\prime}\left(\alpha_{2}\right)-\sigma^{\prime}\left(\alpha_{1} \alpha_{2}\right)\right) \sigma_{\Lambda^{\prime}}^{\prime}\left(\delta_{2}\right) \\
& =-0\left(\sigma^{\prime}\right)\left(\alpha_{1}, \alpha_{2}\right) .
\end{aligned}
$$

Thus $O\left(\sigma^{\prime}\right)=-d \otimes_{0}\left(\sigma^{\prime}, A^{\prime}\right)$ and $o(\pi, A)=0$, proving the "if" part of the theorem.

Suppose $o(\pi, A)=0$, then there exists a 1.-cochain 5 of $C^{\circ}\left((S-f r e e / A)^{\circ}, \operatorname{Der}_{S}(-, A \otimes I)\right)$ such that $O\left(\sigma^{\prime}, A\right)=d \zeta$.

Since $j: D^{\bullet} \rightarrow C^{\bullet}$ is surjective there exists a 1-cochain $\xi$ of $D^{*}$ such that $j(\xi)=i \zeta$. Let $\sigma_{1}$ be given by

$$
\sigma_{1}(\alpha)=\sigma^{\prime}\left(f_{0}(\alpha)\right)+\xi(\alpha) .
$$

Then $\sigma_{1}$ is a $f_{0}$-quasisection.
One checks that the 2. cochain $\omega$ of $D^{*}$ defined by

$$
\omega\left(\begin{array}{c}
F_{0} \rightarrow F_{1} \xrightarrow{\alpha_{2}} F_{2} \\
\delta_{0} \rightarrow \delta_{1} \downarrow \\
\underset{A}{ }
\end{array}\right)=\sigma_{2}\left(\alpha_{1} \alpha_{2}\right)-\sigma_{1}\left(\alpha_{1}\right) \sigma_{1}\left(\alpha_{2}\right)
$$

is mapped to zero by $j$, thus sits in $K^{2}$. Now

$$
\begin{aligned}
& A^{\prime}=\lim \quad \sigma_{1}=\operatorname{coker}\left(\mu \sigma_{1}\left(\delta_{0}\right) \vec{\mu} \sigma_{1}\left(\delta_{0}\right)\right) \\
& (S-f r e e / A)^{\mathrm{epi}}
\end{aligned}
$$

exists as an R-modul. We shall show that $A^{\prime}$ is a lifting of A , thus justifying our claim of "good idea" above. Consider the resolving complex $C .=C .\left((S-f r e e / A)^{\text {epic }},-\right)$ of


Since $\sigma_{1}$ is not a functor $C_{0}\left(\sigma_{1}\right)$ will not necessarily be a complex, but nevertheless we may consider the commutative diagran

in which all sequences of morphisms marked with solid arrows are exact.

In fact we have $C .\left(\sigma_{1}\right) \underset{R}{\otimes} I=C_{0}\left(\mathrm{~g}_{0}\right) \underset{S}{\otimes} I$ and $C .\left(\sigma_{1}\right) \underset{R}{Q}=C .\left(g_{0}\right)$
where, we recall, $g_{0}=f_{o}^{i}(s e e(2.2 .2))$. The vertical sequences are exact since all $C_{p}\left(\sigma_{1}\right)$ are R-free, the lower horizontal sequence is exact due to Corollary (2.2.2), and finally, part of the middle horizontal sequence is exact by the definition of $\Lambda^{\prime}$.

Remember that we do not know that $\delta^{\circ} \gamma=0$. In fact it may well be that $\delta^{\circ} \gamma \neq 0$. However $\operatorname{in}\left(\delta^{\circ} \gamma\right) \subseteq C_{o}\left(\sigma^{\prime \prime}\right) \underset{R}{\otimes I}$ and fortunately we have arranged the situation such that

$$
\beta(i m(\delta \circ \gamma))=0 .
$$

This follows by observing that the image of $\delta^{\circ} \mathrm{Y}$ consists of sums of elements of the form

$$
\left(\sigma_{1}\left(\alpha_{1} \alpha_{2}\right)-\sigma_{1}\left(\alpha_{1}\right) \sigma_{1}\left(\alpha_{2}\right)\right)(\zeta)=\omega\left(\alpha_{1}, \alpha_{2}\right)(\zeta)
$$

where

is an object of $(\text { s-free } / A)^{\text {ep }}$ and

$$
\xi \in \sigma_{1}\left(\delta_{0}\right)=\xi_{0}^{\prime}
$$

Since $\omega \in \mathbb{K}^{2}$ we conclude

$$
\beta\left(\omega\left(\alpha_{1}, \alpha_{2}\right)(\xi)\right)=0 .
$$

Using this we may easily see that $\alpha$ is injective.
But $\alpha$ is infective if and only if

$$
\operatorname{Tor}_{1}^{R_{1}}\left(A^{\prime}, S\right)=0
$$

We have to show that $A^{\prime}$ is an R-algebra. Consider a system of homomorphisms

$$
F_{1} \stackrel{d}{\rightarrow} F_{0} \underset{\Delta^{\prime}}{\times F_{0} \stackrel{p_{1}^{\prime}}{\underset{\sim}{p_{2}^{\prime}}}} \vec{F}_{0} \xrightarrow{\rho} A
$$

in which $F_{0}$ and $F_{1}$ are free $S$ algebras, $\rho$ and $d$ are surjective, $p_{1}^{\prime}$ and $p_{2}^{\prime}$ are the projections and $\Delta^{\prime}$ is the diagonal. Let $\Delta: F_{0} \rightarrow F_{1}$ be a homomorphism such that $\Delta^{\circ} d=\Delta^{\prime}$, and put $p_{i}=d^{\circ} p_{i}^{\prime}$.

Then A is the inductive limit of the system


Apply the $f_{0}-q u a s i s e c t i o n ~ \sigma_{1}$ on the corresponding morphisms of (Sheree $/ \mathrm{A})^{\text {epic }}$. Then we get a diagram of R -algebras

$$
\underset{\sigma_{1}(\Delta)}{\stackrel{\sigma_{1}\left(p_{1}\right)}{\sigma_{1}\left(p_{2}\right)} F_{o}^{\prime}}
$$

Since we have the commutative diagram

in which $\alpha$ is injective and all sequences involving morn-phisms marked with solid arrows are exact we conclude that $\beta$ is an isomorphism. We therefore are reduced to prove that the $R$-module kex $\rho^{\prime}=\operatorname{im}\left(\sigma_{1}\left(p_{1}\right)-\sigma_{1}\left(p_{2}\right)\right)$ is on ideal of $F_{o}^{\prime}$. Suppose $x \in \operatorname{im}\left(\sigma_{1}\left(p_{1}\right)-\sigma_{1}\left(p_{2}\right)\right)$ and $y \in F_{0}^{\prime}$. We have to prove that $y x \in \operatorname{im}\left(\sigma\left(p_{1}\right)-\sigma_{1}\left(p_{2}\right)\right)$. First, assume $x \in F_{0} \otimes I$, then $\rho^{\prime}(y x)=(\rho \otimes 1)(\vec{y} \cdot x)=\rho(\vec{y}) \cdot(\rho \otimes 1)(x)=0$ where $\bar{y}$ iss
the image of $y$ in $F_{0}$. Thus $y x \in k e r p^{\prime}$ 。
Since $\sigma_{1}$ is an $f_{0}$-quasisection we have

$$
\sigma_{1}\left(F_{o}^{F_{0}} \xrightarrow{1 F_{0}} F_{0}\right)=F_{o}^{\prime} \xrightarrow{1_{F_{0}^{\prime}}} F_{o}^{\prime} .
$$

Therefore

$$
\sigma_{1}(\Delta) \sigma_{1}\left(p_{i}\right)=1_{P_{c}^{\prime}}-w\left(\Delta, p_{i}\right) \quad i=1,2 .
$$

We have already seen that for all $y \in T_{0}^{\prime}$

$$
\begin{aligned}
& \omega\left(\Delta, p_{i}\right)(y) \in F_{0} \otimes I \\
& \omega\left(\Delta, p_{i}\right)(y) \in \operatorname{ker} \rho^{\prime}
\end{aligned}
$$

Now

$$
y=\sigma_{1}\left(p_{i}\right)\left(\sigma_{1}(\Delta)(y)\right)+w\left(\Delta, p_{i}\right)(y) \quad i=1,2
$$

and since $x \in \operatorname{im}\left(\sigma_{1}\left(p_{1}\right)-\sigma_{1}\left(p_{2}\right)\right)$ there is a $u \in F_{1}^{\prime}$ such that $x=\sigma_{1}\left(p_{1}\right)(u)-\sigma_{1}\left(p_{2}\right)(u)$
therefore

$$
\begin{aligned}
\mathrm{yx} & =\left(\sigma_{1}\left(p_{1}\right)\left(\sigma_{1}(\Delta)(y)\right)+w\left(\Delta, p_{1}\right)(y)\right)\left(\sigma_{1}\left(p_{1}\right)(u)\right) \\
& \cdots\left(\sigma_{1}\left(p_{2}\right)\left(\sigma_{1}(\Delta)(y)\right)+w\left(\Delta, p_{2}\right)(y)\right)\left(\sigma_{1}\left(p_{2}\right)(u)\right) \\
& =\sigma_{1}\left(p_{1}\right)\left(\sigma_{1}(\Delta)(y) \cdot u\right)+\sigma_{1}\left(p_{1}\right)(u) \cdot w\left(\Delta, p_{1}\right)(y) \\
& -\sigma_{1}\left(p_{2}\right)\left(\sigma_{1}(\Delta)(y) \cdot u\right)+\sigma_{1}\left(p_{2}\right)(u) \cdot w\left(\Delta, p_{2}\right)(y) .
\end{aligned}
$$

But, since we already know that

$$
\sigma_{1}\left(p_{i}\right)(u) \cdot w\left(\Delta, p_{i}\right)(y) \in \operatorname{ker} \rho^{\prime}
$$

this shows that $y x \in \operatorname{ker} \rho^{\prime}=\operatorname{ira}\left(\sigma_{1}\left(p_{1}\right)-\sigma_{1}\left(p_{2}\right)\right)$.
Therefore $A^{\prime}$ is an algebra and we have proved that it is a lifting of $A$ to $R$.

Fixing the quasisection $\sigma^{\prime}$, let $A^{\prime}$ be any lifting of $A$, and consider the 1.cochain $Q_{0}\left(\sigma^{\prime}, A^{\prime}\right)$ constructed above. Remember that $O\left(\sigma^{1}\right)=-d Q_{0}$. The corresponding $\sigma_{1}$ in the construction above, which is unique up to elements of $K^{2}$, will be denoted $\sigma^{\prime}\left(\Lambda^{\prime}\right)$. For any object ( $\mathrm{T}_{\mathrm{O}}{ }^{\delta_{0}} \mathrm{~A}$ ) of (Swiree/A) let us put $\sigma^{\prime}\left(A^{\prime}\right)_{A^{\prime}}\left(\delta_{0}\right)=\sigma_{A^{\prime}}^{\prime}\left(\delta_{0}\right)$. For any morphism

of $(S-\text { free } / A)^{\text {epi }}$ we have, by definition of $Q_{0}\left(\sigma^{\prime}, A^{\prime}\right)$ a commutative diagrom

$$
{ }_{\alpha_{A^{\prime}}^{\prime}\left(\delta_{0}\right)}^{F_{0}^{\prime}} \xrightarrow[A^{\prime}]{\sigma^{\prime}\left(A^{\prime}\right)\left(\alpha_{1}\right)} F_{1}^{\prime}
$$

which implies

$$
\lim _{\left(\mathrm{S} \rightarrow \text { free } / R^{\text {epi }}\right.} \sigma^{\prime}\left(\mathrm{A}^{\prime}\right)=A^{\prime} .
$$

Given a lifting $A^{\prime}$ there is thus a unique, up to elements of $K^{2}, f_{o}$-quasisection $\sigma^{\prime}\left(A^{\prime}\right)$ such that

$$
\begin{aligned}
& \lim _{(\mathrm{m}} \sigma^{\prime}\left(A^{\prime}\right)=A^{\prime} . \\
& (\mathrm{Seee} / A)^{\text {epi }}
\end{aligned}
$$

Let $A^{\prime}$ and $A^{\prime \prime}$ be two liftings of $A$, then the corresponding cochain

$$
Q_{0}\left(A^{\prime}\right)-Q_{0}\left(A^{\prime \prime}\right) \in C^{1}\left((S-\text { free } / A)^{\circ}, \operatorname{Der}_{S}(-, A \otimes I)\right)
$$

is a cocycle defininga cohomology class

$$
\lambda\left(A^{\prime}, A^{H}\right) \in H^{1}(S, A ; A \otimes I)
$$

One easily checks that this class does not depend upon the choices made.
On the other hand in $\lambda$ is an element of $H^{1}(S, A ; A \otimes I)$, let $S_{0}$ be a 1.cocycle of $C^{\circ}\left(\left(S-\text { free }_{A}\right)^{\circ}, \operatorname{Der}_{S}(-, A \otimes I)\right)$ reprosenting $\lambda$, then we consider the 1 cochain

$$
\zeta=Q_{0}\left(\sigma^{\prime}, A^{\prime}\right)-\zeta_{O}
$$

Obviously $d \zeta=-O\left(\sigma^{\prime}\right)$ and so there correspond $\xi_{0}, \xi \in D^{1}$ such that $j\left(\xi_{0}\right)=i\left(\zeta_{0}\right) \quad j(\xi)=i(\zeta)$. The quasisection $\sigma_{1}=\sigma^{\prime}+\xi$ defines a lifting $A^{\prime \prime}$ of $A$. One easily checks that

$$
\begin{aligned}
& i\left(Q_{0}\left(A^{\prime}\right)-Q_{0}\left(A^{\prime \prime}\right)\right)=i\left(\zeta_{0}\right) \\
& \sigma\left(A^{\prime \prime}\right)=\sigma\left(A^{\prime}\right)-\xi_{0} \quad\left(\text { modulo } K^{2}\right)
\end{aligned}
$$

Suppose $\lambda=0$ then $\zeta_{0}$ is a coboundary. We may assume $\xi_{0}=d \eta$ with $\eta \in D^{0}$ 。
For every orphism

of $(S-\text { free } / A)^{\text {ep i }}$ consider the diagram


Since $i\left(\zeta_{0}\right)=d \eta$ we find that the diagrams of morphisms represented by solid arrows commute. But this implies that there exist a morphism $A^{\prime} \rightarrow A^{\prime \prime}$ which joined to the solid diagram will not distroy the commutativity. Obviously then $A^{\prime \prime} \sim A^{\prime}$. This proves the theorem.

> Q.E.D.

## (2.3) Obstructions for lifting morphisms of algebras

Let $\pi_{1}: R_{1} \rightarrow S_{1}$ and $\pi_{2}: R_{2} \rightarrow S_{2}$ be two surjective homomorphisms of commutative rings.
Let $A_{1}$ be an $S_{1}-a l g e b r a$ and $A_{2}$ be an $S_{2}$-algebra and suppose given morphisms of rings $\beta_{0}, \beta_{1}$ and $\beta_{2}$ making the following diangram commutative:


Suppose given a lifting $A_{1}^{\prime}$ of $A_{1}$ to $R_{1}$ and a lifting $A_{2}^{\prime}$ of $A_{2}$ to $R_{2}$.

Definition (2.3.1) A homomorphism of rings $\beta_{2}^{\prime}: A_{1}^{\prime} \rightarrow A_{2}^{\prime}$ is a lifeting of $\beta_{2}$ to $\beta_{0}$ with respect to $A_{1}^{\prime}$ and $A_{2}^{\prime}$ if the following diagram comminutes


Definition (2.3.2) Iwo lifting $\beta_{2}^{\prime}$ and $\beta_{2}^{\prime \prime}$ of $\beta_{2}$ to $\beta_{0}$ with respect to $A_{1}^{\prime}$ and $A_{2}^{\prime}$ are equivalent (written $\beta_{2}^{\prime} \sim \beta_{2}^{\prime \prime}$ ) if
there exist automorphisms of R-algebras $\theta_{1}: A_{1}^{\prime} \rightarrow A_{1}^{\prime}$ and $\theta_{2}: A_{2}^{\prime} \rightarrow A_{2}^{\prime}$ such that the following diagram commutes


The set of lifting of $\beta_{2}$ to $\beta_{0}$ w.r.t. $A_{1}^{\prime}$ and $A_{2}^{\prime}$ modulo this equivalence relation is called

$$
\operatorname{Def}\left(\left(B_{0}, \beta_{1}, \beta_{2}\right), A_{1}^{\prime}, A_{2}^{\prime}\right)
$$

Now suppose $\operatorname{ker} \pi_{1}^{2}=\operatorname{ker} \pi_{2}^{2}=0$. Then we may prove the following:

Theorem (2.3.3) Given lifting $A_{1}^{\prime}$ and $A_{2}^{\prime}$ of $A_{1}$ and $A_{2}$ respectively to $R_{1}$ and $R_{2}$ there exists an obstruction

$$
o\left(\beta_{2}\right)=o\left(\beta_{2}, A_{1}^{\prime}, A_{2}^{\prime}\right) \in H^{\prime}\left(S_{1}, A_{1} ; A_{2} \otimes \operatorname{ser} \pi_{2}\right)
$$

such that $o\left(\beta_{2}\right)=0$ if and only if there exists a lifting of $\beta_{2}$ to $\beta_{0}$ with respect to $A_{1}^{\prime}$ and $A_{2}^{\prime}$. In this case $\operatorname{Def}\left(\left(\beta_{0}, \beta_{1}, \beta_{2}\right), A_{1}^{1}, A_{2}^{\prime}\right)$ is a principal homogeneous space over

$$
H^{O}\left(S_{1}, A_{1} ; A_{2} \otimes \operatorname{ker} \pi_{2}\right)
$$

Proof. As in the proof of (2.2.5) pick any quasisection $\sigma_{i}^{1}$ of
$-S_{i}: R_{i}-$ free $\rightarrow S_{i}$-free, $i=1,2$.
Consider the corresponding 1.cochain $Q_{0}\left(\sigma_{i}^{1}, A_{i}^{\prime}\right)$ of
$C^{\circ}\left(\left(S_{i}-\text { free }_{A_{i}}\right)^{\circ}, \operatorname{Der}_{S_{i}}\left(-, A_{i} S_{S_{i}} \operatorname{ker} \pi_{i}\right)\right) \quad i=1,2$.
Let $O\left(\sigma_{1}^{1}, \sigma_{2}^{1} ; \mathrm{A}_{1}^{1}, \mathrm{~A}_{2}^{\prime}\right)$ be the 1.cochain of

$$
C^{\circ}\left(\left(S_{1}-\text { iree } / A_{1}\right)^{\circ}, \operatorname{Der}_{S_{1}}\left(-, A_{2} \otimes \operatorname{ser} \pi_{2}\right)\right)
$$

defined by

$$
\begin{aligned}
& \left.-\sigma_{1}^{\prime}\left(\alpha_{1}\right)_{R_{1}} 1_{R_{2}}\right)\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}\left(\delta_{1}\right)+Q_{0}\left(\sigma_{1}^{\prime}, A_{1}^{\prime}\right)\left(\alpha_{1}\right)\left(\beta_{2} \otimes \beta_{0}\right) \\
& -\left(1_{F_{0} S_{1}}^{\otimes \beta_{1}}\right) Q_{0}\left(\sigma_{2}^{\prime}, A_{2}^{\prime}\right)\left(\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}\left(\alpha_{1}\right)\right)
\end{aligned}
$$

where

$$
\left(\beta_{0}, \beta_{1}, \beta_{2}\right) *:\left(S_{1}-\text { free } / A_{1}\right) \rightarrow\left(S_{2}-\text { free } / A_{2}\right)
$$

is the functor defined by

$$
\left(\beta_{0}, \beta_{1}, \beta_{2}\right) *\left(\begin{array}{c}
\mathrm{F}_{0} \\
\downarrow \delta_{0} \\
A_{1}
\end{array}\right)=\begin{gathered}
\mathrm{F}_{0} \otimes S_{2} \\
S_{1} S_{2} \\
\downarrow \delta_{o}^{\prime} \\
A_{2}
\end{gathered},
$$

$\delta_{0}^{\prime}$ being the composition:

$$
F_{\mathrm{o}_{\mathrm{S}_{1}}^{\otimes} \mathrm{S}_{2}} \rightarrow \mathrm{~A}_{1} \mathrm{~S}_{1} \mathrm{~S}_{2} \rightarrow \mathrm{~A}_{2}
$$

One checks that $0\left(\sigma_{1}^{1}, \sigma_{2}^{1} ; A_{1}^{1}, A_{2}^{\prime}\right)$ is a 1.cocycle, and that the corresponding cohomology class $o\left(A_{1}^{\prime}, \Lambda_{2}^{\prime}\right) \in H^{1}\left(S_{1}, A_{1} \otimes S_{2}\right.$ ker $\left.\pi_{2}\right)$ does not depend upon the choices made. Consider for every morphism of $\left(S_{1}\right.$-free/ $\left.A_{1}\right)$,

the diagram:

where $\delta_{i}^{i}=\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}\left(\delta_{i}\right), \quad i=1,2$ and $\alpha_{1}^{\prime}=\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}\left(\alpha_{1}\right)$. Put

$$
V^{\prime}\left(\delta_{i}\right)=\left(1_{F_{i}^{\prime}} \otimes \beta_{0}\right) \sigma_{2 A_{2}^{\prime}}^{\prime}\left(\delta_{i}^{\prime}\right) \quad i=0,1 .
$$

Then this diagram induces the following diagram

and we find that

$$
\begin{aligned}
& O\left(\sigma_{1}^{\prime}, \sigma_{2}^{\prime} ; A_{1}^{\prime}, A_{2}^{\prime}\right)\left(\begin{array}{c}
\mathrm{F}_{0} \xrightarrow{\alpha_{1}} \mathrm{~F}_{1} \\
\delta_{0} \searrow_{1} / \delta_{1} \\
A_{1}
\end{array}\right)=v^{\prime}\left(\delta_{0}\right)-\sigma_{1}^{\prime}\left(\alpha_{1}\right) \nu^{\prime}\left(\delta_{1}\right) \\
& +Q_{0}\left(\sigma_{1}^{\prime}, A_{1}^{\prime}\right)\left(\alpha_{1}\right)\left(\beta_{2}{ }_{S_{1}}^{\otimes} \beta_{0}\right)
\end{aligned}
$$

Suppose now that there exists a lifting $\beta_{2}^{1}$ of $\beta_{2}$ to $\beta_{0}$ w.r.t. $A_{1}^{\prime}$ and $A_{2}^{\prime}$ then let $Q_{1}\left(\beta_{2}^{\prime}\right)$ be the 0 . cochain of $C^{\cdot}\left(\left(S_{1}-f \text { free } / A_{1}\right)^{0}, \operatorname{Der}_{S_{1}}\left(-, A_{2}{\underset{S}{2}}^{\otimes}\right.\right.$ kex $\left.\left.\pi_{2}\right)\right)$ defined by

$$
Q_{1}\left(\beta_{2}^{\prime}\right)\left(\begin{array}{l}
F_{o}^{o} \\
\downarrow_{\mathrm{o}} \delta_{o} \\
A
\end{array}\right)=\nu^{\prime}\left(\delta_{o}\right)-\sigma_{1 A^{\prime}}^{\prime}\left(\delta_{o}\right) \beta_{2}^{\prime} .
$$

We find

$$
O\left(\sigma_{1}^{\prime}, \sigma_{2}^{\prime} ; A_{1}^{\prime}, A_{2}^{\prime}\right)=-d Q_{1}
$$

Thus proving the "if" part of the theorem.
Let us consider the image of $O\left(\sigma^{\prime}, A_{1}^{1}, A_{2}^{1}\right)$ in $C^{\circ}$ By definition of $\sigma_{1}^{\prime}\left(A_{1}^{\prime}\right)$ we find

$$
\begin{aligned}
& \nu^{\prime}\left(\delta_{0}\right)-\sigma_{1}^{\prime}\left(\alpha_{1}\right) \nu^{\prime}\left(\delta_{1}\right)+Q_{0}\left(\sigma_{1}^{\prime}, A_{1}^{\prime}\right)\left(\alpha_{1}\right)\left(\beta_{2} \underset{S_{1}}{\otimes} \beta_{0}\right) \\
= & \nu^{\prime}\left(\delta_{0}\right)-\sigma_{1}^{\prime}\left(A_{1}^{\prime}\right)\left(\alpha_{1}\right) \nu^{\prime}\left(\delta_{1}\right) .
\end{aligned}
$$

whenever

in an object of $(\text { S-free } / A)^{e p i}$.

With this done, suppose $o\left(A_{1}^{\prime}, A_{2}^{\prime}\right)=0$, then there exists a O. cochain $x$ of

$$
C^{\bullet}\left(\left(S_{1}-\operatorname{free}_{A_{1}}\right)^{\circ}, \operatorname{Der}_{S_{1}}\left(-, A_{2}{\left.\left.\underset{S_{2}}{\otimes} \operatorname{ker} \pi_{2}\right)\right)}^{\prime}\right.\right.
$$

such that

$$
\mathrm{d} x=O\left(\sigma_{1}^{\prime}, \sigma_{2}^{\prime} ; \mathrm{A}_{1}^{\prime}, A_{2}^{\prime}\right) .
$$

Put

$$
\nu\left(\delta_{0}\right)=\nu^{\prime}\left(\delta_{0}\right)-x\left(\delta_{0}\right)
$$

for every object

$$
\left(\begin{array}{l}
F_{0} \\
\downarrow \delta_{0} \\
A_{1}
\end{array}\right)
$$

of (Scree/ $A_{1}$ ), then for every morphism

of $\left(S_{1}-f x e e / A_{1}\right)$ epis we find a commutative diagram

which proves that there exists a lifting $\beta_{2}^{\prime}$ of $\beta_{2}$ to $\beta_{0}$ w.r.t. $A_{1}^{\prime}$ and $A_{2}^{\prime}$. In fact we know that $A_{1}^{\prime}=\left(\lim _{1}^{-}-\text {free }_{A_{1}}\right)^{\operatorname{epi}} \sigma_{1}^{\prime}\left(A_{1}^{\prime}\right)$. This ends the proof of (2.3.3)

Remark (2.3.4) By construction we have an equality

$$
\begin{aligned}
& \nu\left(\delta_{0}\right)=\sigma_{1 A_{i}^{\prime}}^{\prime}\left(\delta_{0}\right) \beta_{2}^{\prime} \text { which implies that } \\
& Q_{1}\left(\beta_{2}^{\prime}\right)\left(\delta_{0}\right)=\nu^{\prime}\left(\delta_{0}\right)-\sigma_{1 A_{1}^{\prime}}^{\prime}\left(\delta_{0}\right) \beta_{2}^{\prime}=n\left(\delta_{0}\right) \\
& \text { for all objects }\left(S_{1}-\frac{f r e e}{} / A_{1}\right) .
\end{aligned}
$$

Remark (2.3.5) Consider any diagram of commutative rings

$$
\begin{aligned}
& \underset{\pi_{1} \downarrow}{R_{1}} \xrightarrow{\beta_{0}} \underset{\pi_{2} \downarrow}{R_{2}} \xrightarrow{\gamma_{0}}{ }_{\pi_{3} \downarrow} R_{3} \\
& S_{1} \xrightarrow{\beta_{1}} S_{2} \xrightarrow{\gamma_{1}} S_{3} \\
& \mu_{1} \downarrow \quad \beta_{2} \mu_{2} \downarrow \gamma_{2} \mu_{3} \downarrow \\
& A_{1} \xrightarrow{\beta_{2}} A_{2} \xrightarrow{\gamma_{2}} A_{3}
\end{aligned}
$$

Assume $\pi_{1}, \pi_{2}, \pi_{3}$ are surjective and, moreover, that $\operatorname{ker} \pi_{1}^{2}=\operatorname{ker} \pi_{2}^{2}=\operatorname{ker} \pi_{3}^{2}=0$. Suppose given lifting $A_{1}^{\prime}, A_{2}^{\prime}$ and $A_{3}^{\prime}$ of $A_{1}, A_{2}$ and $A_{3}$ respectively, and suppose we have found lifting $\beta_{2}^{\prime}, \gamma_{2}^{\prime}$, and $\left(\beta_{2} \gamma_{2}\right)$ of $\beta_{2}, \gamma_{2}$ and $\beta_{2} \gamma_{2}$ respectively, w.r.t. $A_{1}^{\prime}$ and $A_{2}^{\prime}, A_{2}^{\prime}$ and $A_{3}^{\prime}$, and $A_{1}^{\prime}$ and $A_{3}^{\prime}$ respectively.
Pick any object ( $F_{0} \rightarrow{ }^{\delta} A_{1}$ ) of $S_{1}-f x e e / A_{1}$ and consider the diagram


An easy computation shows that

$$
\begin{aligned}
& \sigma_{1 A_{1}^{\prime}}^{\prime}\left(\delta_{0}\right)\left(\beta_{2} \gamma_{2}\right)^{\prime}=\beta_{2}^{\prime} \gamma_{2}^{\prime}-\left(Q_{1}\left(\gamma_{2}^{\prime}\right)\left(\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}\left(\delta_{0}\right)\right)\right. \\
& \left.-Q_{1}\left(\left(\beta_{2} \gamma_{2}\right)^{\prime}\right)\left(\delta_{0}\right)+Q_{1}\left(\beta_{2}^{\prime}\right)\left(\delta_{0}\right)\left(\gamma_{2} \otimes \gamma_{0}\right)\right) .
\end{aligned}
$$

## Chapter 3. Global cohomology.

## (3.1) Definitions and some spectral sequences

Let us first recall some fundamental constructions. If e is any small category, one may consider the category of functors (covariant) on $e$ with values in the category of abelian groups Ab. We shall denote by $A b$ - this category, which we know is abelian having enough injectives and projectives.
Jet $C^{\cdot}(\underline{e})=C^{\cdot}(\underline{e},-): A b^{e} \rightarrow$ Compl.ab。gr. be the following functor: Let $F$ be any object of $A B=\underline{E}$ and put:
where the indices mun through all strings of $p$ composable nor. phisms in e. Let the differential $d^{p}: C^{p}(e, F) \rightarrow C^{p+1}(\underline{e}, F)$ be defined by: For $\beta=\left(\beta_{\mu_{1}, \mu_{2}, \ldots, \mu_{p}}\right) \in C^{p}(\underline{e}, F)$ let $d^{p}(\beta)=$ $\left(d^{p}(\beta)_{\mu_{1}, \mu_{2}, \infty,, \mu_{p+1}}\right)$ be given by the formula

$$
\begin{aligned}
& d^{p}(\beta)_{\mu_{1}, \mu_{2}, 000, \mu_{p+1}}=F\left(\mu_{1}\right)\left(\beta_{\mu_{2}}, \mu_{3}, 000, \mu_{p+1}\right)+ \\
+ & \sum_{i=1}^{p}(-1)^{i} \beta_{\mu_{1}, 000, \mu_{i} \mu_{i+1}, \infty 00, \mu_{p+1}}+(-1)^{p+1} \beta_{\mu_{1}, \mu_{2}, 000, \mu_{p}}
\end{aligned}
$$

It is easy to show that $d^{p} d^{p+1}=0$ thereby proving that $C^{\cdot}(\underline{e}, F)$ $=\left\{C^{p}(e, T), d^{p}\right\}$ is a complex. Moreover, we observe that $C^{\circ}(\underline{e}, \cdots)$ is an exact functor, and, almost by constmuction, we have (see Appendix (1.3) or (Ia 1)):

$$
H^{n}\left(C^{\circ}(\underline{e},--)\right) \simeq \lim _{\underline{e}}(n)
$$

Given the category e we define (see (1.1)) the category Mor e for which the objects are the morphisms of $e$ and for which the
morphisms $(\alpha, \beta): \mu \rightarrow \epsilon$ are commutative diagrams of the form:


It turned out that for this special category there exists another functorial complex $D^{\bullet}(\underline{e},-): \underline{A b}^{\text {Mor }} \underline{e} \rightarrow$ Compl.ab。gr. with the same property as $C^{\bullet}$ (More, --) but better suited for our purpose (see (1.3)). If $G$ is an object of $A b^{M o r e}$ e then $D^{\bullet}(e, G)$ is given ber:

$$
D^{p}(\underline{e}, G)=e_{o \vec{\mu}_{1}}^{T} e_{1 \vec{\mu}_{2}} \cdots \vec{\mu}_{p} e_{p} G\left(\mu_{p}, \mu_{2}, \ldots o, \mu_{p}\right)
$$

and

$$
d^{p}: D^{p}(e, G) \rightarrow D^{p+1}(\underline{e}, G)
$$

where

$$
\begin{aligned}
& d^{p}(\beta)_{\mu_{1}, \mu_{2}, \ldots,, \mu_{p+1}}=G\left(\left(\mu_{1}, 1_{e_{p+1}}\right)\right)\left(\beta_{\mu_{2}, \ldots, \mu_{p+1}}\right) \\
& +\sum_{i=1}^{p}(-1)^{i_{i}} \beta_{\mu_{1}, \infty,, \mu_{i} \mu_{i+1}, \ldots,, \mu_{p+1}}+(-1)^{p+1} G\left(\left(1_{e_{0}}, \mu_{p+1}\right)\right)\left(\beta_{\mu_{p}, \infty, \mu_{p}}\right) .
\end{aligned}
$$

$D^{*}(e,-)$ is an exact functor, and we proved in (1.3) that

$$
H^{n}\left(D^{\bullet}(e,-)\right) \simeq \lim _{\operatorname{Mor}^{2}}(n)
$$

Definition (3.1.1) Let $G$ be an object of $A b^{\text {Mor }}$ - The n.th cohomology of $e$ with values in $G$ is the abelian group:

$$
H^{n}(\underline{e}, G)=H^{n}\left(D^{\cdot}(\underline{e}, G)\right)=\lim _{\operatorname{Mo} \tilde{t}^{(n)}}^{(n} \cdot
$$

(Why not? )
Now, with these generalities done, we shall start the constmuction
of the global algebra cohomology which eventually will lead to the cohomology groups $A^{i l}(S, X ; M)$ refered to in the Introduction.

Let $s$ be any commutative ring with unit element, and let us make the following definition:

Definition (3.1.2) A 2.S-algebra is a morphism of S-algebras. If $\mu: A \rightarrow B$ and $\mu^{\prime}: A^{\prime} \rightarrow B^{\prime}$ are 2.S-algebxas then a morphism $(\alpha, \beta): \mu \rightarrow \mu^{\prime}$ is a commutative diagram of the form:


Let 2.S-alg denote the category of 2.S-algebras, and consider a small subcategory $d$ of $2.5-a l g$.

Obviously the functor $\mathrm{A} \rightarrow(\mathrm{S} \rightarrow \mathrm{A})$ defines an imbedding of $\mathrm{S}-\mathrm{al} \mathrm{g}$ in $2 . S-a l g$. Ve shall therefore identify any small subcategory of s-alg with the corresponding subcategory of 2.s-alg -

Examples (3.1.3) (I) Let $Y$ be any $S$-scheme and let $U$ be any affine open Zariski covering of $Y$. Then $W$ as a subset of the topology of $Y$ is an ordered set, therefore a category the morphisins being the inclusions $U \subseteq V$. The dual category $C_{T D}$ is a category of $S$-algebras. (II) Let $\mathrm{I}: X \rightarrow Y$ be a morphism of $S$-schemes, and let $U$ (resp. W) be an affine open covering of $Y$ (xesp. X) . Then the set $f(W, W)=\left\{(U, V) \mid U \in W, V \in W, V \subseteq f^{-1}(U)\right\}$ is an ordered set and the dual category $d_{W} W$ is a category of 2.S-algebras. In fact, if $(U, V) \in f(W, W)$ then $U=S p e c(A)$,
$V=\operatorname{Spec}(B)$ and $f \mid V: V \rightarrow U$ corresponds to an S-algebra morphism $A \rightarrow B$ 。

Definition (3.1.4) A d-Module $M$ is a functor $M: a \rightarrow A b$ such that for any object $\mu: A \rightarrow B$ of $d M(\mu)$ is a BModule, and such that for every morphism $(\alpha, \beta): \mu \rightarrow \mu^{\prime}$ of $\underline{d}$ with $\mu^{\prime}: A^{\prime} \rightarrow B^{\prime}$, the corresponding homomorphism $M((\alpha, \beta)): M(\mu) \rightarrow M\left(\mu^{\prime}\right)$ is $\beta: B \rightarrow B^{\prime}$ linear.

Example (3.1.5) In the situation of (3.1.3), (II) any $O_{X}$-Module will, in an obvious way, induce a $\underset{-}{d}, W^{-M o d u l e}$.

Now, consider a moxphism $(\alpha, \beta)$ of $\alpha$, i.e. a commutative diagram


Let $M^{\prime}$ be any $B^{\prime}$-module and consider the functor

$$
(\alpha, \beta)_{*}: A-f r e e / B \rightarrow A^{\prime}-\text { free } / B^{\prime}
$$

(see (2.2) for definitions) defined by:

$$
(\alpha, \beta)_{*}(\delta)=\text { composition of } \quad \delta \otimes \wedge_{A} \text { and } \underset{A}{B \otimes A^{\prime}} \rightarrow B^{\prime} \text {. }
$$

Here $\delta: A[x] \rightarrow B$ denotes any object of $A$-free $B$. The functor $(\alpha, \beta)_{*}$ induces a morphism of complexes

$$
\begin{aligned}
C^{\bullet}\left((\alpha, \beta), M^{\prime}\right): C^{0}\left(A^{\prime}-f r e e / B^{\prime}, \operatorname{Der}_{A^{\prime}}\left(-, M^{\prime}\right)\right) \rightarrow \\
C^{\cdot}\left(A-f r e e / B^{\circ}, \operatorname{Der}_{A}\left(-, M^{\prime}\right)\right) .
\end{aligned}
$$

If $T: M^{\prime} \rightarrow M^{\prime \prime}$ is any homomoxphism of $B^{\prime}$-modules, then the corresponding homomorphisms of abelian groups

$$
\tau_{\delta^{\prime}}: \operatorname{Der}_{A^{\prime}}\left(A^{\prime}[\underline{x}], M^{\prime}\right) \rightarrow \operatorname{Der}_{A^{\prime}}\left(A^{\prime}[x], M^{\prime \prime}\right)
$$

where $\delta^{\prime}$ runs through $A^{\prime}-$ free/ $B^{\prime}$ defines a morphism

$$
\tau: \operatorname{Der}_{A^{\prime}}\left(-, M^{\prime}\right) \rightarrow \operatorname{Der}_{A^{\prime}}\left(-, M^{\prime \prime}\right)
$$

which in turn induces a morphism of complexes

$$
\begin{aligned}
& C^{\circ}\left(A^{\prime}-\underline{f r e e} / B^{\prime}, T\right): C^{\bullet}\left(A^{\prime}-\underline{f r e e} / B^{\prime}, \operatorname{Der}_{A^{\prime}}\left(-, M^{\prime}\right)\right) \rightarrow \\
& C^{\circ}\left(A^{\prime}-\text { free } / B^{\prime}, \operatorname{Der}_{A^{\prime}}\left(-, M^{\prime \prime}\right)\right) .
\end{aligned}
$$

Let $M$ be any d-Module then it follows from what has been said above that the map

$$
(\alpha, \beta) \rightarrow C^{\bullet}\left(A-f r e e / B^{0}, \operatorname{Der}_{A}\left(-, M\left(\mu^{\prime}\right)\right)\right)
$$

induces a functor

$$
\left.C^{\circ}(-, \text { Der_( }-, M)\right): \text { Mor } \underset{\sim}{d} \rightarrow \text { Compl_ab_gr. }
$$

We may therefore consider the double-complex

$$
K_{\underline{d}}^{\circ}(M)=D^{\circ}\left(\underline{d}, C^{\circ}\left(-, \operatorname{Der}_{-}(-, M)\right)\right)
$$

Definition (3.1.6) The global algebraic cohomology of $\alpha$ with values in $M$, denoted by

$$
A^{\mathrm{n}}(\mathrm{~S}, \underline{\alpha} ; \mathrm{M}) \quad \mathrm{n} \geq 0,
$$

is the cohomology of the simple complex associated to the double complex $\mathrm{K}_{\underline{\mathrm{d}}}{ }^{\circ}(\mathrm{M})$.

For $q \geq 0$ let $A^{q}(\mathbb{M})$ denote the $q^{\text {th }}$ cohomology of the functor $C^{\circ}(-, \operatorname{Der} . .(-, M))$, then $A^{q}(M)$ is a functor on Mor d with values in Ab

Lemma (3.1.2) $A^{n}(S, d ; M)$ is the abutment of a spectral sequence given by

$$
\mathrm{E}_{2}^{p, q}=H^{p}\left(\underline{d}, \underline{A}^{q}(M)\right)
$$

Proof. This is just the first spectral sequence of $K_{\underline{d}}^{\circ}$. Q.E.D.

Let ${\underset{o}{o}}$ be any subcategory of the category $d$. Given a d-Module $M$ we may consider the restriction of $M$ to Mor ${\underset{o}{o}}^{0}$ (usually denoted $M$, thus abusing the language).

There is a canonical surjective morphism of double complexes

$$
K_{\underline{d}}^{\bullet}(M) \rightarrow K_{\dot{d}}^{\bullet}(M)
$$

Let $K_{\underline{d} /{\underset{\sim}{-}}_{0}}(\mathbb{M})$ denote the kernel of this morphism, and put:

Definition (3.1.8) The global algebraic cohomology of a relative to $d_{0}$, with values in $M$, denoted by

$$
A_{-0}^{n}(\underline{\alpha}, M) \quad n \geq 0
$$

is the cohomology of the simple complex associated to the double complex $\mathbb{K}_{\underline{d} / d_{0}^{\circ}}^{\circ}(M)$.

There is a long exact sequence of cohomology

$$
\cdots \rightarrow A_{\alpha_{0}}^{n}(\underset{\alpha}{\alpha}, M) \rightarrow A^{n}(\underset{\alpha}{\alpha}, M) \rightarrow A^{n}\left(\underline{d}_{0}, M\right) \rightarrow A_{\underline{d}_{0}}^{n+1}(\underline{d}, M) \rightarrow \cdots
$$

Lemma ( 3.1 .8 ) Let $e$ be any small category, and consider the functor

$$
\epsilon: \text { Mor } \underline{e} \rightarrow \underline{e}
$$

defined by:

$$
\epsilon\left(e_{1} \rightarrow e_{2}\right)=e_{2} .
$$

Let $F: e \rightarrow A b$ be any functor, then there are natural isomorphisms

$$
\lim _{\lim e}^{(n)} \in F \underset{\underline{e}}{\underset{\lim }{\lim ^{( }}(n)_{F},} \quad n \geq 0
$$

Proof. This is trivial, due to the fact already observed in (2.2) that

$$
D^{0}(\underline{e}, \varepsilon \mathbb{F})=C^{0}(\underline{e}, \mathbb{F}) .
$$

Q.E.D.

Corollary (3.1.9) Let $e_{0}$ be any subcategory of the category e. Let $\underline{n}$ be the full subcategory of More the objects of which are the moxphisms $e_{0} \rightarrow e_{1}$ with $e_{0} \in$ ob。 $e_{0}$.

Let $F:{\underset{\sim}{e}}_{0} \rightarrow A b$ be any functor, then there are natural isomorphisms

$$
\underset{\underset{n}{\lim }}{ }(n) \in F \underset{{\underset{\sim}{e}}_{-}^{\lim }}{ }(n)_{F}, \quad n \geq 0
$$

Proof. This is an easy consequence of (3.1.8). In fact More $e_{-}$is a cofinal subcategory of $n$ (see Appendix (1.3))
Q.J.D.

Now let us apply some of these generalities to algebraic geometry. Let $X$ be an $S$-scheme, and let $\mathbb{Z}_{X}$ be the open covering of $X$ consisting of all affine open subsets. Let ${\underset{\mathrm{c}}{\mathrm{X}}}={\underset{-}{\mathbb{C}_{X}}}$ (see (3.1.3) I.) be the dual category of S-algebras.

Let $F$ be any $O_{X}$-Module then $F$ is a $C_{X}$-Module which, via the functor $\varepsilon: M o r{\underset{X}{X}} \rightarrow \underline{C}_{X}$ may be considered a functor on Mor ${\underset{X}{X}}$.

Theorem (3.1.10) Suppose $F$ is quasicoherent, then there are natural isomorphisms

$$
H^{n}\left(\underline{c}_{X}, F\right) \simeq H^{n}(X, F), \quad n \geq 0 .
$$

Proof. By (3.1.8) there are natural isomorphisms

$$
H^{n}\left(c_{X}, F\right)=\lim _{\operatorname{Mor}^{-} \underline{c}_{X}}^{(n)} \varepsilon F \simeq \underset{\underline{c}_{X}}{\lim _{X}}(n)_{F} .
$$

Moreover we have isomorphisms
Q.E.D.

Definition (3.1.11) The global algebrajc cohomology of $X$ with values in $F$ are the groups

$$
A^{n}(S, X ; F)=A^{n}\left(S, \underline{C}_{X} ; \mathbb{F}\right), \quad n \geq 0 .
$$

If $\mu: A \rightarrow B$ is any object of Mor ${\underset{X}{X}}$, then

$$
\begin{aligned}
A^{q}(F)(\mu) & =H^{q}\left(C^{\circ}\left(S-\text { free } / A^{\circ}, \operatorname{Der}_{S}(-, F(\operatorname{Spec}(B)))\right)\right) \\
& =H^{q}(S, A ; F(\operatorname{Spec}(B)))
\end{aligned}
$$

By (An)p. 85 we find that

$$
H^{\mathrm{q}}(S, A ; F(\operatorname{Spec}(B)))=H^{\mathrm{q}}(S, B ; F(\operatorname{Spec}(B)))
$$

and, in fact, $A^{q}(\mathbb{F})$ is the composition of $\varepsilon$ with a sheaf on $X$. This sheaf, which we shall still denote by $A^{q}(F)$ is quasicoherent whenever $F$ is.

Theorem (3.1.12) Suppose $F$ is quasicoherent, then the global algebraic cohomology $A^{*}(S, X ; F)$ is the abutment of a spectral
sequence given by the term

$$
E_{2}^{p}, q=H^{p}\left(X, A^{q}(F)\right)
$$

Proof. This is a trivial consequence of (3.1.10).
Q.E.D.

Consider any morphism of $S-$ schemes $f: X \rightarrow Y$, and let $\mathbb{Z}_{X}$, and $\mathbb{Z}_{Y}$ be the ordered sets of affine open subsets of $X$ and $Y$ respectively. Put ${\underset{d}{f}}^{f}=f\left(\mathbb{Z}_{X}, \mathbb{Z}_{Y}\right)^{0}$ see (3.1.2).

Let $F$ be any $O_{X}$-Module. In an obvious way we may consider $F$ a. $\quad \underline{a}_{f}-$ Module .

Definition (3.1.13) The global algebraic cohomology of $f$ with values in $F$ are the groups

$$
A^{n}(f ; F)=A^{n}\left(S,{\underset{d}{f}}^{f} F\right), \quad n \geq 0
$$

Theorem (3.1.14) $A^{\bullet}(f ; F)$ is the abutment of a spectral sequence with

$$
\mathrm{E}_{2}^{p}, \mathrm{q}=\mathrm{H}^{\mathrm{p}}\left(\underline{\mathrm{c}}_{\mathrm{Y}} ;{\underset{\mathrm{A}}{\mathrm{f}}}_{\mathrm{q}}^{(F))}\right.
$$



$$
A_{-}^{q}(F)\left(A_{1} \rightarrow A_{2}\right)=A^{q}\left(A_{1}, f^{-1}\left(\operatorname{Spec} A_{2}\right) ; F\right)
$$

Proof. Let $\tau: \operatorname{Mor}_{f} \rightarrow$ Mor $\mathcal{c}_{\mathrm{X}}$ be the functor defined by

$$
r\left(\begin{array}{ll}
A & \rightarrow \\
\downarrow & \\
\downarrow \\
A^{\prime} \rightarrow & B^{\prime}
\end{array}\right)=\begin{aligned}
& A \\
& \downarrow \\
& A^{\prime}
\end{aligned}
$$

Let $\pi: \operatorname{Mor}_{Y} \rightarrow \mathbb{P} \operatorname{Mor}{\underset{\sim}{f}}_{f}$ be the map given by

$$
\pi(x)=\tau^{-1}(\hat{x})
$$

where $\left.\hat{x}=\left\{x^{\prime} \in \operatorname{Mor}{\underset{Y}{Y}}\right\rfloor x \rightarrow x^{\prime}\right\}$. Using (La 3)(1.3) we find a homomorphism of compleces

$$
C^{\bullet}\left(\operatorname{Mor} \underline{d}_{f^{\prime}}-\infty\right) \rightarrow C^{\bullet}\left(\operatorname{Mor}{\underset{Y}{Y}}, C^{\bullet}(\pi,--)\right)
$$

inducing isomorphisms in cohomologyr.
For $x$ an object of Mors $c_{Y}$ put $x=A \rightarrow A^{\prime}$ and put

$$
\pi_{0}(x)= \begin{cases}A \rightarrow B \\ \downarrow & \downarrow \\ A^{\prime} \rightarrow \pi(x) \mid B^{\prime} & \rightarrow B \quad \text { making the two triangles } \\ \text { commutative },\end{cases}
$$

then $\pi_{0}(x)$ is cofinal in $\pi(x)$ (see ( La 3 ) (1.2.4)).
In fact, given any object

$$
w=\left(\begin{array}{lll}
A_{1} & & B_{1} \\
\downarrow & & \downarrow \\
A_{1}^{\prime} & & \rightarrow \\
B_{1}^{\prime}
\end{array}\right)
$$

of $\pi(x)$, there exists by definition of $\pi(x)$ a commutative diagra gram


Put

$$
u=\left(\begin{array}{ccc}
A & \longrightarrow & A^{\prime} \otimes_{1}^{\otimes_{1}} B_{1} \\
\downarrow & \downarrow \\
A^{\prime} & \downarrow & B_{1}^{\prime}
\end{array}\right)
$$

then $u \in \pi_{0}(x)$ and there exists a unique orphism $u \rightarrow W$ in $\pi(x)$. Thus the canonical homomorphism of complexes

$$
C^{\bullet}(\pi,-) \rightarrow C^{\circ}\left(\pi_{0},-\right)
$$

induces isomorphisms in cohomology.

Now, observe that $\pi_{0}(x)$ is isomorphic to $\operatorname{Mor}{\underset{f}{f}}^{-1}\left(\operatorname{Spec}\left(A^{\prime}\right)\right)$ where $f^{-1}\left(\operatorname{Spec}\left(A^{\prime}\right)\right)$ is considered as an A-scherne. This implies that the cohomology of the double complex

$$
C^{\circ}\left(\Pi_{0}(x), C^{\circ}\left(-, \operatorname{Der}_{-}(-, F)\right)\right)
$$

being isomorphic to the cohomology of the double complex

$$
C^{\circ}\left(\operatorname{Mor}_{f^{-1}}^{-1}\left(\operatorname{Spec}\left(A^{\prime}\right)\right), C^{\circ}\left(-, \operatorname{Der}_{-}(-, F)\right)\right)
$$

is equal to $A \cdot(F)(x)$.
Since we already know that the homomorphism of double complexes

$$
\begin{aligned}
& C^{\bullet}\left(\operatorname{Mor}_{f}{\underset{f}{f}}, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right) \rightarrow \\
& C^{\bullet}\left(\operatorname{Mor}{\underset{Y}{Y}}, C^{\bullet}\left(\Pi, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right)\right)
\end{aligned}
$$

induces isomorphisms in cohomology, the first spectral sequence of the double complex

$$
C^{\bullet}\left(\operatorname{Mor} C_{Y}, C^{\bullet}\left(\pi, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right)\right)
$$

being given by

$$
\mathrm{E}_{2}^{p}, \underline{q}=H^{p}\left(\underline{c}_{Y} ;{\underset{\mathrm{A}}{\mathrm{~F}}}_{q}^{q}(F)\right)
$$

converges to the cohomology of $C^{\circ}\left(\operatorname{Mor}{\underset{f}{f}}^{f}, C^{\bullet}(-, \operatorname{Der}(-, F))\right)$ which is the same as the cohomology of the complex

$$
D^{\bullet}\left(d_{f}, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right)
$$

This proves the theorem.
Q.E.D.

Consider a closed subscheme $Z$ of the S-scheme X . The category $\underline{-x}_{X-Z}$ is a full subcategory of $\underline{c}_{X}$. Let $F$ be any $\mathrm{O}_{\mathrm{X}}$ Module, and let's make the following definition.

Definition (3.1.15) The global algebraic cohomology of $X$ with values in $F$ and support in $Z$, are the groups

$$
A_{Z}^{n}(S, X ; F)=A_{C(X-Z)}^{n}\left(S, c_{X} ; F\right) \quad n \geq 0
$$

By construction we have a long exact sequence

$$
\rightarrow A_{Z}^{n}(S, X ; F) \rightarrow A^{n}(S, X ; F) \rightarrow A^{n}(S, X-Z ; F) \rightarrow A_{Z}^{n+1}(S, X ; F) \rightarrow
$$

Let for any subset ${\underset{\sim}{o}}$ of the ordered set $e$,

$$
c^{\bullet}\left(e / e_{0},-\right)
$$

denote the kernel of the canonical moxphism

$$
C^{\bullet}(\underline{e},-) \rightarrow C^{\circ}(\underline{e},-\infty)
$$

Recall that we denote by $\hat{e}_{0}$ the subset of $e$ defined by

$$
\hat{\underline{e}}_{0}=\left\{x \in \underline{e} \mid f x^{\prime} \in \underline{e}_{0}, x \leq x^{\prime}\right\}
$$

By definition we have an exact sequence of double complexes

$$
\begin{gathered}
0 \rightarrow C^{\bullet}\left(\operatorname{Mor}_{C_{X}} / \operatorname{mor}{\left.\underset{X X-Z}{ }, C^{*}\left(-, \operatorname{Der}_{-}(-, F)\right)\right) \rightarrow}_{C^{\bullet}\left(\operatorname{Mor}_{X}, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right) \rightarrow}^{C^{\bullet}\left(\operatorname{Mor}_{X-Z}, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right) \rightarrow 0,}\right.
\end{gathered}
$$

inducing the long exact sequence above.
Using Corollary (3.1.9) we may prove that the canonical morphism of double complexes

$$
\begin{aligned}
& C^{\bullet}\left(\operatorname{Mor}{\underset{X X}{X}}^{\operatorname{Mor}_{C_{X-Z}}}, C^{\bullet}\left(-, \operatorname{Der}_{-}(-, F)\right)\right) \rightarrow
\end{aligned}
$$

induces isomorphisms in cohomology (use the short exact sequence of complexes and the first spectral sequence of the third member).

Let the maps

$$
\pi_{i}: \operatorname{Mor}{\underset{\sim}{X}}^{\operatorname{Pr}} \operatorname{Mor}{\underset{X}{X}}^{i}=1,2
$$

be given by

$$
\pi_{1}(x)=\hat{x} \text { and } \pi_{2}(x)=\hat{x} \cap \widehat{\operatorname{Mor} C_{X}-Z}
$$

By (La 3) (1.3) there is a canonical morphism of double complexes

$$
\begin{aligned}
& C^{\cdot}\left(\operatorname{Mor}{\underset{X}{X}}, C^{\bullet}\left(\pi_{1} / \pi_{2}, C^{\bullet}\left(S-\text { free }^{\prime} /-^{\circ}, \operatorname{Der}_{S}(-, F)\right)\right)\right)
\end{aligned}
$$

inducing isomorphisms in cohomology.
Let $x=A \rightarrow B$ be an object of Morc $A_{X}$ then for any object $x^{\prime}=$ $\left(A^{\prime} \rightarrow B^{\prime}\right)$ of $\pi_{1}(x)$ there is a unique commutative diagram


Corresponding to this diagram there is a functor

$$
\text { S-free } / A^{\prime} \rightarrow \text { S-free } / A
$$

inducing a morphism of functors on $\pi_{1}(x)$

$$
C^{\circ}\left(S-\text { free } / A^{\circ}, \operatorname{Der}_{S}(-, F)\right) \rightarrow C^{\circ}\left(S-\text { free } /-^{\circ}, \operatorname{Der}_{S}(-, \mathbb{F})\right)
$$

We already know, see (An) p. 83 , that this moxphism induces isomorphisms in cohomology. Therefore the canonical morphism of double complexes

$$
\begin{aligned}
& C^{\circ}\left(\pi_{1}(x) / \pi_{2}(x), C^{0}\left(\text { S-iree } / A^{0}, \operatorname{Der}_{S}(-, F)\right)\right) \rightarrow \\
& C^{0}\left(\pi_{1}(x) / \pi_{2}(x), C^{0}\left(\text { S-free } /-0, \operatorname{Der}_{S}(-, F)\right)\right)
\end{aligned}
$$

induces isomorphisms in cohomology.

Now there is a canonical isomorphism

$$
\begin{aligned}
& C^{\bullet}\left(\pi_{1}(x) / \pi_{2}(x), C^{\bullet}\left(S-f_{r e e} / A^{\circ}, \operatorname{Der}_{S}(-, F)\right)\right) \leadsto \\
& C^{\bullet}\left(S-f r e e / A^{\circ}, \operatorname{Der}_{S}\left(-, C^{\bullet}\left(\pi_{1}(x) / \pi_{2}(x), F\right)\right)\right)
\end{aligned}
$$

Putting things together we find a morphism of complexes

$$
\begin{aligned}
& C^{\circ}\left(\text { Mr } \mathrm{c}_{\mathrm{X}} / \text { Mr } \mathrm{c}_{\mathrm{X} \ldots \mathrm{Z}}, \mathrm{C}^{\bullet}\left(\mathrm{S}-\text { free }^{\left(\sim^{\circ}\right.}, \operatorname{Der}_{S}(-, F)\right)\right) \rightarrow \\
& C^{\bullet}\left(\operatorname{Mor}{\underset{X}{X}}, C^{\bullet}\left(S-f r e e /-^{\circ}, \operatorname{Der}_{S}\left(-, C^{\bullet}\left(\pi_{1} / \pi_{2}, F\right)\right)\right)\right)
\end{aligned}
$$

inducing isomorphisms in cohomology.
Consider the exact sequence of complexes

$$
0 \rightarrow C^{\bullet}\left(\pi_{1}(x) / \pi_{2}(x), F\right) \rightarrow C^{\bullet}\left(\pi_{1}(x), F\right) \rightarrow C^{\bullet}\left(\pi_{2}(x), F\right) \rightarrow 0
$$

Suppose $F$ is quasicoherent, then by (3.1.9) and (3.1.10) we find
thus

$$
\begin{aligned}
& H^{q}\left(C^{\bullet}\left(\pi_{1}(x)\right)\right)=H^{q}(\operatorname{Spec}(B), F) \\
& H^{q}\left(C^{\bullet}\left(\pi_{2}(x)\right)\right)=H^{q}(\operatorname{Spec}(B)-Z, F) \\
& H^{q}\left(C^{\bullet}\left(\pi_{1}(x) / \pi_{2}(x), J^{F}\right)\right)=H_{Z}^{q}(F)(\operatorname{Spec}(B))
\end{aligned}
$$

Theorem (3.1.16) $A_{Z}^{\circ}(S, X ; F)$ is the abutment of a spectral sequence given by

$$
E_{2}^{p, q}=A^{p}\left(S, X ; H_{Z}^{q}(F)\right)
$$

Proof. Take the first spectral sequence of the double complex

$$
C^{\bullet}\left(\operatorname{Mor} \underline{C}_{X}, C^{\bullet}\left(S-\operatorname{free}^{\circ}, \operatorname{Der}_{S}\left(\cdots, C^{\bullet}\left(\pi_{1} / \pi_{2} \pi^{I}\right)\right)\right)\right.
$$

Consider a morrphism of $\mathrm{S}-$ schemes

$$
f: X \rightarrow Y
$$

and let $X_{0} \subseteq X$ and $Y_{0} \subseteq Y$ be two closed subschemes satisfying $f^{-1}\left(Y_{0}\right) \subseteq X_{0}$ 。 Then $f$ induces a morphism of $S$-schemes

$$
f_{0}: X-X_{0} \rightarrow Y-Y_{0}
$$

The corresponding category ${\underset{d}{f}}$ of $2 . S$-algebras is, in a natural way, identifyed with a subcategory of $\alpha_{f}$ 。

Definition (3.1.17) The global algebraic cohomology of $f$ relative to $f_{o}$ (or with support in $\left(X_{0}, Y_{0}\right)$ ) with values in $F$ are the groups

$$
A_{f}^{n}(f ; F)=A_{\underline{d}_{f}}^{n}\left(\underline{d}_{f} ; F\right) \quad n \geq 0
$$

Theorem ( 3.1 .18 ) $A_{f}^{\circ}(f ; F)$ is the abutment of a spectral sequence given by the term

$$
\mathrm{E}_{2}^{p}, \underline{q}=\mathrm{H}^{p}\left(c_{Y} ; \mathrm{A}_{\mathrm{f}}^{\mathrm{q}} / \mathrm{f}_{\mathrm{O}}(\mathrm{~F})\right)
$$

where $A_{-f}^{q} / f_{0}(F)$ is the functor on Mor $c_{Y}$ defined by

$$
A_{f / f_{0}}^{q}(F)\left(A_{1} \rightarrow A_{2}\right)=A_{f^{-1}}^{1}\left(\operatorname{Spec}\left(A_{2}\right)\right) n x_{0}\left(A_{1}, f^{-1}\left(\operatorname{Spec}\left(A_{2}\right) ; F\right)\right.
$$

Proof. Virtually the same as for (3.1.14).

$$
Q . E \cdot D
$$

## (3.2) Iong exact sequence associated to a morphism of schemes

We shall need a technical lemma.
Let $g: \underline{e}_{1} \rightarrow \underline{e}_{2}$ be a functor of small categories. Suppose $\underline{e}_{1}$ has an initial object $S$. Consider the category $\underset{\sim}{c} \mathrm{~g}$ for which the objects are the triples ( $A, B, \rho$ ) where $A$ is an object of $\underline{e}_{1}$, $B$ is an object of $e_{2}$ and $\rho$ is a morphism $g(A) \rightarrow B$.

A morphism $(A, B, \rho) \rightarrow\left(A^{\prime} \cdot B^{\prime}, \rho^{\prime}\right)$ in $C / g$ is by definition a pair of morphisms $\left(\psi_{1}, \psi_{2}\right), \psi_{1}: A \rightarrow A^{\prime}, \psi_{2}: B \rightarrow B^{\prime}$ in $\Theta_{1}$ respectively $e_{2}$, making the diagram

$$
\begin{array}{cll}
\mathrm{g}(\mathrm{~A}) & \rho & \mathrm{B} \\
\mathrm{~g}\left(\psi_{1}\right) \downarrow & & \downarrow \psi_{2} \\
\mathrm{~g}\left(\mathrm{~A}^{\prime}\right) & \overrightarrow{\rho^{\prime}} & \mathrm{B}^{\prime}
\end{array}
$$

commutative. Let $\Phi_{1}: \underline{C} / g \rightarrow \underline{e}_{1}$ and $\Phi_{2}: \underline{C} / g \rightarrow \underline{e}_{2}$ be the functors defined by $\Phi_{1}(A, B, \rho)=A, \Phi_{2}(A, B, \rho)=B$. Consider functors $\mathrm{F}: \underline{e}_{1}^{0} \rightarrow \mathrm{Ab}, G: \underline{e}_{2}^{0} \rightarrow \mathrm{Ab}$.

Lemma (3.2.1) Then there are natural isomorphisms
(1) $\quad \lim ^{(n)}{ }_{1} 1^{\circ} \mathrm{F} \simeq \lim ^{(n)}{ }_{F}$
$\mathrm{C} / \mathrm{g}^{\circ}$
$\mathrm{e}_{1}^{0}$
(2) $\quad \lim _{i=}(n)_{\Phi_{2}} G^{G} \simeq \lim ^{(n)} G$
$\underline{g} g^{\circ} \quad e_{2}^{o}$

Proof. For $n=0$ there is nothing to prove. Let for evexy object $A^{\prime}$ of $e_{1} \quad I_{A^{\prime}}$ be an injective abelian group. The functor $C: \underline{e}_{-1}^{0} \rightarrow A b$ given by $C(A)=\Pi I_{\Lambda^{\prime}}$ is injective as an object of $\mathrm{A}^{\prime} \rightarrow \mathrm{A}$
the category of abelion functors on $\underline{e}_{1}^{0}$. Let for every ( $A^{\prime}, B^{\prime}, P^{\prime}$ ) of $\mathrm{C} / \mathrm{g}$

$$
I\left(A^{\prime}, B^{\prime}, \rho^{\prime}\right)= \begin{cases}0 & \text { if } \rho^{\prime} \neq 1_{g}\left(A^{\prime}\right): g\left(A^{\prime}\right) \rightarrow g\left(A^{\prime}\right) \\ I_{A^{\prime}} & \text { if } \rho^{\prime}=1_{g\left(A^{\prime}\right)}: g\left(A^{\prime}\right) \rightarrow g\left(A^{\prime}\right)\end{cases}
$$

then the functor $C^{\prime}: C / g^{9} \rightarrow A b$ defined by

$$
\begin{gathered}
C^{\prime}(A, B, \rho)=\prod_{\left(A^{\prime}, B^{\prime}, \rho^{\prime}\right) \rightarrow(A, B, \rho)}\left(A^{\prime}, B^{\prime}, \rho^{\prime}\right)
\end{gathered}
$$

is injective as an object of the category of abelian functors on C/g. Moreover

$$
C^{\prime}=\Phi_{1} C .
$$

Since the functor

$$
F \rightarrow \Phi_{1} F
$$

is
exact and takes enarg injectives into injectives (1) follows from the spectral sequence associated to a composition of functors.

Consider for every object $B^{\prime}$ of $\underline{e}_{2}$ an injective abelian group $J_{B^{\prime}}$. The functor $D:{\underset{-}{e}}_{0}^{o} \rightarrow \underline{A b}$ given by $D(B)=\prod_{B^{\prime} \rightarrow B} J_{B^{\prime}}$ is injective as an object of the category of abelian functors on ${\underset{-}{e}}_{o}^{o}$. Let for every object

$$
\left(A^{\prime}, B^{\prime}, \rho^{\prime}\right) \text { of } C / g \quad J\left(A^{\prime}, B^{\prime}, \rho^{\prime}\right)= \begin{cases}0 & \text { if } A^{\prime} \neq S \\ J_{B^{\prime}} & \text { if } A^{\prime}=S\end{cases}
$$

then the functor $D^{\prime}: C / g^{\circ} \rightarrow \mathrm{Ab}$ defined by

$$
\begin{gathered}
D^{\prime}(A, B, \rho)=\Pi J\left(A^{\prime}, B^{\prime}, P^{\prime}\right) \quad \text { is injective } \\
\left(A^{\prime}, B^{\prime}, P^{\prime}\right) \rightarrow(A, B, P)
\end{gathered}
$$

as an object of the category of abelian functors on $\mathrm{c} / \mathrm{g}^{\circ}$.

Moreover

$$
D^{\prime}=\Phi_{2} D
$$

Since the functor

$$
G \rightarrow \Phi_{2} G
$$

is exact and takes enough injectives into injectives (2) follows. Q.E.D.

Let $\psi: A \rightarrow B$ be a morphisms of S-algebras. Then $\psi$ induces a functor $\psi_{*}: S-$ free $/ A \rightarrow$ S-free/ $B$.

Let $e_{1}=S-\underline{f r e e}^{\prime} / A, \underline{e}_{2}=S-\underbrace{}_{\text {free }} / B$ and put $g=\psi_{*}$. Then the category $C / g=C / \mathbb{X}$ of $(3.2 .1)$ is the category whose objects are commutative diagrams of the form:

$$
\begin{aligned}
F_{1} & \rightarrow F_{2} \\
\delta_{1} \downarrow & \delta_{2} \downarrow \\
\mathrm{~S} \rightarrow \mathrm{~A} & \rightarrow \mathrm{~B}
\end{aligned}
$$

with $\delta_{1} \in \mathrm{ob}$ S-free $/ \mathrm{A}$ and $\delta_{2} \in \mathrm{ob}$ S-free $/ \mathrm{B}$
A morphism of

is a pair of morphisms of S-algebras

$$
\left(F_{1} \rightarrow F_{1}^{\prime}, F_{2} \rightarrow F_{2}^{\prime}\right)
$$

making all diagrams commutative.
The functors $\Phi_{i}: \underline{C} / \psi \rightarrow \underline{e}_{\mathbf{i}} \quad i=1,2$ are in this case defined by:

$$
\left.\begin{array}{l}
\Phi_{1}\left[\begin{array}{rrr}
F_{1} \rightarrow F_{2} \\
\delta_{\downarrow} \downarrow & \delta_{2} \downarrow \\
S \rightarrow A & \rightarrow & B
\end{array}\right]={ }_{1} \downarrow \\
F_{1} \\
A
\end{array}\right]\left[\begin{array}{rr}
F_{1} & F_{2} \\
\delta_{\downarrow} \downarrow & \delta_{2} \downarrow \\
S \rightarrow A & \rightarrow
\end{array}\right]=\delta_{2} \downarrow \begin{array}{r}
F_{2} \\
B
\end{array}
$$

Let $M$ be any B-module and define three functors

$$
D_{i}(M): C / \psi_{*}^{\circ} \rightarrow A b \quad i=1,2,3
$$

By:

$$
\begin{aligned}
& D_{1}(M)\left[\begin{array}{ll}
F_{1} \rightarrow & F_{2} \\
W^{2} & \downarrow^{2} \\
B
\end{array}\right]=\operatorname{Der}_{S}\left(F_{1}, M\right)=\operatorname{Der}_{S}\left(\Phi_{1}(-), M\right) \\
& D_{2}(M)\left[\begin{array}{ll}
W_{1} \rightarrow F_{2} \\
S \rightarrow A \rightarrow & { }^{2}
\end{array}\right]=\operatorname{Der}_{S}\left(F_{2}, M\right)=\operatorname{Der}_{S}\left(\Phi_{2}(-), M\right) \\
& D_{3}(M)\left[\begin{array}{lll}
W_{1} \rightarrow & F_{1} \\
\vdots & \downarrow & \downarrow \\
S & A
\end{array}\right]=\operatorname{Der}_{F_{1}}\left(F_{2}, M\right)=\operatorname{Der}_{A}\left(F_{2} \otimes A, M\right)
\end{aligned}
$$

We may consider the subcategory ${\underset{\sim}{O}}^{C} / \Psi_{*}$ of $\underline{C} / \psi_{*}$ whose objects are diagrams of the form

with $\left(F_{2} \rightarrow B\right) \in$ ob $F_{1}-$ free /B,$F_{1} \rightarrow F_{2}$ being the structure morphism。 There is a natural functor $\Phi: \underline{C}_{0} / \Psi_{*} \rightarrow A-f r e e / B$ which takes
$\mathrm{F}_{1} \rightarrow \mathrm{~F}_{2}$
$\downarrow$
$\mathrm{~A} \rightarrow \mathrm{~B}$
into
A

inducing a morphism of complexes

$$
C^{\bullet}\left(\left(A-\text { free }^{2}\right)^{\circ}, \operatorname{Der}_{A}(-, M)\right) \rightarrow C^{\bullet}\left(\underline{C}_{0} / \psi_{*}^{\circ}, D_{3}(\mathbb{M})\right) .
$$

Obviously we get an exact sequence of functors, restricted to $C_{o} / \psi$

$$
0 \rightarrow D_{3}(M) \rightarrow D_{2}(M) \rightarrow D_{1}(M) \rightarrow 0
$$

The corresponding exact sequence of complexes

$$
0 \rightarrow C^{\bullet}\left(\underline{C}_{0} / \psi_{*}, D_{3}(\mathbb{M})\right) \rightarrow C^{\bullet}\left(\underline{C}_{0} / \psi_{*}, D_{2}(M)\right) \rightarrow C^{\circ}\left(\underline{C}_{0} / \psi_{*}, D_{1}(M)\right) \rightarrow 0
$$

will induce a long exact sequence in cohomology

$$
\begin{aligned}
& \rightarrow H^{n}(A, B ; M) \rightarrow H^{n}(S, B ; M) \rightarrow H^{n}(S, A ; M) \\
& \rightarrow H^{n+1}(A, B ; M) \rightarrow \ldots
\end{aligned}
$$

This is a consequence of Lemma (3.2.1) and the following result:

Lemma (3.2.2) The canonical morphism of complexes

$$
C^{\bullet} \cdot\left(\left(A-\text { free }_{B}\right)^{0}, \operatorname{Der}_{A}(-, M)\right) \xrightarrow{r_{0}^{\bullet}} C^{\bullet}\left(\underline{C}_{0} / 4, D_{3}(M)\right)
$$

induces isomorphisms in cohomology.

Proof. Consider the functor

$$
C_{i}^{0}:\left(A-a l_{g} / B\right)^{0} \rightarrow \text { Comploloab。g. }_{0} \quad i=1,2
$$

defined by

$$
\begin{aligned}
& \mathrm{C}_{1}^{0}(\mathrm{~A} \underset{\varphi}{ } C \rightarrow B)=C^{0}\left(\left(\mathrm{~A}-\mathrm{free} / C^{\circ}, \operatorname{Der}_{A}(-, M)\right)\right. \\
& C_{2}^{0}\left(\mathrm{~A} \vec{\varphi}^{C} \rightarrow B\right)=C^{0}\left(\underline{C}_{o} / \varphi_{*}^{\circ}, D_{3}(M)\right)
\end{aligned}
$$

Obviously there exists a canonical morphism of functors

$$
C_{1} \xrightarrow{x_{0}^{0}} c_{2}^{0}
$$

which evaluated on the initial object ( $A \vec{\psi}^{B}{\overrightarrow{\imath_{B}}}_{B}$ ) of $\left(A-a l_{g} / B\right)^{0}$
is the morphism $r_{0}^{\circ}$.
Now consider an object $F_{0} \rightarrow B$ of $(A-f r e e / B)^{\text {epi }}$ and let $\ddot{F}_{p}=$ $\overrightarrow{\mathrm{F}}_{\mathrm{O}} \times_{\mathrm{B}} \ldots{\underset{\mathrm{B}}{\mathrm{F}}}^{\mathrm{O}} \quad(\mathrm{p}+1$ factors).
There are $p+1$ projection morphisms

$$
d_{p}^{j}: \vec{F}_{p} \rightarrow \bar{F}_{p-1} \quad p \geq 1, j=0, \ldots, p
$$

Let

$$
\delta_{p i}^{j}: C_{i}^{0}\left(A \rightarrow \bar{F}_{p-1} \rightarrow B\right) \rightarrow C_{i}^{0}\left(A \rightarrow \overline{\mathbb{F}}_{p} \rightarrow B\right) \quad i=1,2
$$

be the induced morphisms of complexes.
Put

$$
\delta_{p, i}=\sum_{j=0}^{p+1}(-1)^{j} \delta_{p, i}^{j}
$$

Then we prove as in (2.1.1) that the family

$$
\left\{C_{i}^{p q}\right\}_{p, q>0}=\left\{C_{\dot{i}}^{q}\left(A \rightarrow \overline{\mathrm{~F}}_{p} \rightarrow B\right), \delta_{p, i}\right\}_{p, q>0} \quad i=1,2
$$

is a double complex and fixing $q$ the complex

$$
\left\{C_{i}^{q}\left(A \rightarrow \bar{i}_{p} \rightarrow B\right), \delta_{p, i}\right\}_{p \geq 0} \quad i=1,2
$$

is a resolution of

$$
C_{i}^{q}\left(A \rightarrow B \overrightarrow{1}_{B} B\right) \quad i=1,2
$$

Moreover we find that $r^{*}$ induces a morphism of double complexes

$$
\mathrm{C}_{1}^{0 \cdot} \xrightarrow{\mathrm{~m}_{2}^{00}} \mathrm{C}_{2}^{0}
$$

Now we easily check that

$$
H^{O}\left(r^{*}\right)(A \underset{\varphi}{\varphi} C \rightarrow B): H^{\circ}\left(C_{1}^{0}(A \rightarrow C \rightarrow B)\right) \rightarrow H^{O}\left(C_{2}^{0}\left(A \vec{\varphi}_{\varphi}^{C} \rightarrow B\right)\right)
$$

is an isomorphism. Since $\bar{F}$ is A-free we find that both $H^{1}\left(C_{1}^{\circ}(A \rightarrow \bar{F} \rightarrow B)\right)$ and $H^{1}\left(C_{2}^{\circ}(A \rightarrow \overline{\mathrm{~F}} \rightarrow B)\right)$ are zero. Using the
morphism induced by $r^{\bullet}$ on the first spectral sequences of $C_{1}^{00}$ and $C_{2}^{\circ}$, we obtain what we want.
Q.E.D.

Let $f: X \rightarrow Y$ be any morphism of $S$-schemes and consider any $O_{X}$ Module $F$. $F$ induces a ${\underset{-f}{f}}^{-M o d u l e}$, still denoted $F$. Consider the functor
defined by

$$
\left\{\begin{array}{lllll}
S & \rightarrow & A & \Psi & B \\
\| & & \downarrow & & \downarrow \\
S & \rightarrow & A & \Psi^{\prime} & B
\end{array}\right\} \rightarrow C^{\circ}\left(C, \circ, D_{i}\left(F\left(\psi^{\prime}\right)\right)\right) \quad, \quad i=1,2,3
$$

Remember $F\left(\psi^{\prime}\right)=F(B)=F(\operatorname{Spec}(B))$. The short exact sequence * above will induce a short exact sequence of double complexes:

$$
\begin{aligned}
& 0 \rightarrow C^{\bullet}\left(\operatorname{Mox}{\underset{f}{f}}, C^{\circ}\left(C_{0} / \cdots, D_{3}(T(-))\right)\right) \\
& \rightarrow C^{*}\left(\operatorname{Mor}{\underset{\sim}{f}}, C^{\bullet}\left(\underline{C}_{0} /-, D_{2}(F(-))\right)\right) \\
& \rightarrow C^{\bullet}\left(\operatorname{Mor} \underline{d}_{f}, C^{\circ}\left(\underline{C}_{0} /-, D_{1}(F(-))\right)\right) \rightarrow 0
\end{aligned}
$$

With all this done, we shall state the main result of this paragraph:

Theorem (3.2.3) Let $f: X \rightarrow Y$ be any morphism of S-schemes, and consider any $O_{X}$-Module $F$. Then there is a long exact sequence
$\ldots \rightarrow A^{n}(f ; F) \rightarrow A^{n}(S, X ; F) \rightarrow A^{n}\left(S, Y ; R^{\circ} f_{*} F\right) \rightarrow A^{n+1}(f ; F) \rightarrow \ldots$
where $A^{n}\left(S, Y ; R^{0} \mathbb{I}_{*} F\right)$ denotes hypercohomology of the complex $\mathrm{R}^{*} \mathrm{f}_{*}{ }^{F}$ 。

Proof. Consider the subcategory $\underset{-}{C_{0} / \psi}$ of $\underline{C} \psi$. It is easy to see that for any diagram of $C / \psi$ of the form

with, $y_{1}, y_{2}$ and $y$ objects of $\underline{C}_{0} / \psi$, there exists an object $z$ of ${\underset{\sim}{-}}^{\psi_{\psi}}$ and morphisms $\psi_{1}, \psi_{2}$ and $\psi_{0}$ such that $\varphi_{1} \psi_{1}=\varphi_{2} \psi_{2}$, $\varphi_{1} \psi_{0}=\varphi_{2} \psi_{0}$. This implies that $\underset{\sim}{C} / \psi$ is cofinal in $\underset{\psi}{ } \quad$ (see the Appendix (1.3))

By an easy spectral sequence argument, using this and (3.2.1), we find that the following morphisms of double complexes induce isomoxphisms in cohomology.

$$
\begin{aligned}
& C^{\circ}\left(\text { Mor }{\underset{f}{f}}, C^{\bullet}\left(- \text { Pree }_{-}^{\circ}, \operatorname{Der}_{-}(-, F(-))\right)\right) \\
& \downarrow r^{\circ} \\
& C^{\bullet}\left(\operatorname{Mor}_{-f}, C^{*}\left(\underline{C}_{-0} /{ }_{-}^{0}, D_{3}(F(-))\right)\right) \\
& C^{\circ}\left(\operatorname{Mor}{\underset{f}{f}}^{f}, C^{\bullet}\left(S-f^{f r e e} / \psi_{2}^{\circ}(-), \operatorname{Der}_{S}\left(-, T\left(\psi_{2}^{1}(-)\right)\right)\right)\right. \\
& \downarrow \\
& C^{\bullet}\left(\operatorname{Mor} \operatorname{di}_{f}, C \cdot\left(\underset{-}{C}, D_{2}(F(-))\right)\right) \\
& \downarrow \\
& C^{\circ}\left(\operatorname{Mor}{\underset{\sim}{f}}, C \cdot\left(C_{-0} /{ }_{-}^{0}, D_{2}(F(-))\right)\right) \\
& C^{\circ}\left(\operatorname{Mor}{\underset{f}{f}}, C^{\bullet}\left(\operatorname{sineee} / \psi_{1}^{\circ}(-), \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right)\right. \\
& \downarrow \\
& \left.C^{\bullet}\left(\operatorname{Mor}_{-\mathrm{f}}, \mathrm{C}^{0}\left(\underline{C} /{ }^{\circ}, \mathrm{D}_{1}(\mathrm{~F}-)\right)\right)\right) \\
& \downarrow \\
& C^{\bullet}\left(\operatorname{Mor}{\underset{f}{f}}, C^{\bullet}\left(\underset{\sim}{C} /{ }_{-\infty}^{0}, D_{1}\left(F_{-}\right)\right)\right)
\end{aligned}
$$

where

$$
\psi_{i}^{j}: \operatorname{Mor}{\underset{\mathrm{d}}{\mathrm{f}}}^{\mathrm{j}} \rightarrow \mathrm{~S}-\mathrm{alg} \quad i=1,2, j=0,1
$$

are the functors defined by:

$$
\begin{aligned}
& \psi\left(\begin{array}{rrr}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A^{\prime} & \rightarrow & B^{\prime}
\end{array}\right)=A \\
& \psi_{1}^{1}\left(\begin{array}{rrr}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A^{\prime} & \rightarrow & B^{\prime}
\end{array}\right)=A^{\prime} \\
& \psi_{2}^{0}\left(\begin{array}{rrr}
A & -B \\
\downarrow & \downarrow \\
A^{\prime} & \rightarrow B^{\prime}
\end{array}\right)=B \\
& \psi_{2}^{1}\left(\begin{array}{rrr}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A^{\prime} & \rightarrow & B^{\prime}
\end{array}\right)=B^{\prime}
\end{aligned}
$$

Notice that $\psi_{i}^{0}$ is contravariant and $\psi_{i}^{1}$ is covariant, $i=1,2$. Consider the functor

$$
\psi: \operatorname{Mor} \underline{d}_{f} \rightarrow \operatorname{Mor} \underline{c}_{\mathrm{X}}
$$

defined by

$$
\psi\left(\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A^{\prime} & \rightarrow & B^{\prime}
\end{array}\right)=\begin{gathered}
B \\
\downarrow \\
B^{\prime}
\end{gathered}
$$

Then we have an equality of functors of complexes on Mord ${ }_{f}$

$$
\begin{aligned}
& C^{\bullet}\left(\text { S-free } / \psi(-\infty)^{o}, \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right. \\
= & \psi C^{\circ}\left(\text { S-free } / 0, \operatorname{Der}_{S}(-, F(-))\right)
\end{aligned}
$$

This shows that there is a morphism of double conplexes

$$
\begin{aligned}
& \left.C^{\bullet}\left(\operatorname{Mor}{\underset{X}{X}}, C^{\bullet}\left(\text { S-free } /-{ }^{\circ}, \operatorname{Der}_{S}(-, F)\right)\right)\right) \\
& \quad \downarrow I^{\cdots} \\
& C^{\bullet}\left(\operatorname{Mor}{\underset{f}{f}}, C^{\bullet}\left(\text { S-free } / \psi_{2}^{\circ}(-), \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right)\right)
\end{aligned}
$$

This moxphism induces a morphism of the corresponding $E_{2}$-terms of the first spectral sequences

$$
\begin{aligned}
& \downarrow \\
& H^{p}\left(C^{\bullet}\left(\operatorname{Mor}{\underset{\sim}{f}}, \psi A^{q}(F)\right)\right)=\lim _{\underset{\leftarrow}{ }(p)}^{\operatorname{Mor}{\underset{d}{f}}^{q}}
\end{aligned}
$$

(see (3.1.7)). Now the functor $A^{q}(F)$ on Mor ${\underset{C}{X}}$ is a sheaf on $X$, therefore (see (3.1.8))

$$
\lim _{\operatorname{Mor}_{\underline{c}}^{c}}(p) A^{q}(F)={\underset{-}{c}}_{\lim _{x}(p)} A^{q}(F)
$$

$$
\lim _{\operatorname{Mor}{\underset{\sim}{f}}^{d}}(p) A^{q}(F)=\lim _{\dot{a}_{f}}(p) A^{q}(F)
$$

To show that these groups are isomorphic under the given morphism is now nothing but a simple Leray spectrol sequence argument for the morphism $f: X \rightarrow Y$. In fact, let $G$ be any presheaf on $X$, let $W$ (resp. W) be any open covering of $X$ (resp. $\Psi$ ) then consider the ordered set

$$
f(W, W)=\left\{(U, V) \mid U \in W, V \in W, V \subseteq I^{-1}(U)\right\}
$$

and the order-preserving map

$$
\psi: f(W, W) \rightarrow W
$$

defined by $\psi(U, V)=V$. The image of $\psi$ is an open covering of $X$, the intersection of $W$ and $f^{-1}(W)$, let us call it IT . Given $U \in W$, let $u(U)$ be the subset of $f(W, W)$ defined by:

$$
x(U)=\left\{\left(U^{\prime}, V^{\prime}\right) \mid U^{\prime} \subseteq U, V^{\prime} \subseteq f^{-1}\left(U^{\prime}\right)\right\}
$$

It is easy to see that the subset

$$
x_{o}(U)=\left\{\left(U, V^{\prime}\right) \mid V^{\prime} \subseteq f^{-1}(U)\right\} \simeq\left\{V^{\prime} \in W \mid V^{\prime} \subset f^{-1}(U)\right\}
$$

is cofinal in $x(U)$ (see (L a3(1,3)). Moreover

$$
\underset{U \in \mathbb{W}}{U} x(U)=\mathbb{T}
$$

It then follows from ((La 3) ( )) that the morphisms of complexes

induce isomorphisms in cohomology.
We have proved that $1^{.0}$ induces isomorphisms in cohomology. To complete the proof of the theorem we shall compare the doublecomplex
and the following triple complex

$$
C^{\bullet}\left(\operatorname{Mor}{\underset{Y}{Y}}, C^{\bullet}\left(S-f^{\operatorname{free}} /-^{\circ}, \operatorname{Der}_{S}\left(-, C^{\bullet}\left(\operatorname{Mor}{\underset{Y}{ }}_{-1}^{(\operatorname{Spec}(-))}, T\right)\right)\right)\right)
$$

In fact let us consider the obvious functor

$$
h: \operatorname{Mor}{\underset{f}{f}} \rightarrow \operatorname{Mor}_{\mathrm{Y}}
$$

Given an object $A \rightarrow A^{\prime}$ of Mors $c_{Y}$, let $x\left(A \rightarrow A^{\prime}\right)$ be the full subcategory of Nor ${\underset{f}{f}}$ the objects of which are those diagrams

$$
x=\left(\begin{array}{ccc}
A_{0} & - & B_{0} \\
\downarrow & & \downarrow \\
A_{0}^{\prime} & \rightarrow & B_{0}^{\prime}
\end{array}\right)
$$

such that there exists a morphism

$$
\left(A \rightarrow A^{\prime}\right) \rightarrow h(x)=\left(A_{0} \rightarrow A_{0}^{\prime}\right)
$$

of $\operatorname{Mor} c_{Y}$, i.e. such that there is a commutative diagram

$$
\begin{aligned}
& A_{0} \rightarrow A \\
& \downarrow^{\prime} \rightarrow{ }_{0}^{\prime} \\
& A_{0}^{\prime}-A^{\prime} .
\end{aligned}
$$

Given such an $x$ we find a commutative diagram


It is not too difficult to see that this implies that the full subcategory

$$
\mu_{0}\left(A \rightarrow A^{\prime}\right)
$$

of $x\left(A \rightarrow A^{\prime}\right)$, the objects of which are the diagrams of the form

$$
\begin{aligned}
& A \rightarrow B \\
& \downarrow \rightarrow{ }^{A} \\
& A^{\prime} \rightarrow B^{\prime}
\end{aligned}
$$

is cofinal.
Remember that both Mor ${\underset{d}{f}}$ and Mor ${\underset{Y}{Y}}$ are oxdered sets and $h$ being a functor, is order preserving.
Since $C^{\bullet}\left(S-f r e e / \psi_{1}^{0}(-), \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right)$ is a functor on Mor $\underline{X}_{f}$, we find (see (ta 3) (1.3) a morphism of double complexes

$$
\begin{gathered}
C^{\bullet}\left(\operatorname{Mox}{\underset{d}{f}}, C^{\bullet}\left(\text { S-free } / \psi_{1}^{0}(-), \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right)\right. \\
\downarrow \\
C^{\bullet}\left(\operatorname{Mor}{\underset{S}{Y}}, C^{\bullet}\left(\mu(-), C^{\bullet}\left(\text { S-free } / \psi_{1}^{\circ}(-), \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right)\right)\right.
\end{gathered}
$$

which induces isomorphisms in cohomology.

The morphism of double complexes

$$
\begin{aligned}
& C^{\bullet}\left(\mu\left(A \rightarrow A^{\prime}\right), C^{0}\left(S-\operatorname{rree} / \psi_{1}^{0}(-), \operatorname{Der}_{S}\left(-, T\left(\psi_{2}^{1}(-)\right)\right)\right)\right) \\
& \downarrow \\
& C^{\bullet}\left(\mu_{0}\left(A \rightarrow A^{\prime}\right), C^{\bullet}\left(S-\text { free } / \psi_{1}^{0}(-), \operatorname{Der}_{S}\left(-, H\left(\psi_{2}^{1}(-)\right)\right)\right)\right) \\
& \| \\
& C^{\bullet}\left(S-f r e e / A^{0}, D_{S}\left(-, C^{\bullet}\left(\mu_{0}\left(A \rightarrow A^{\prime}\right), F\left(\psi_{2}^{1}(-)\right)\right)\right)\right)
\end{aligned}
$$

induced by the inclusion $x_{0}\left(A \rightarrow A^{\prime}\right) \subseteq x\left(A \rightarrow A^{\prime}\right)$ induces isomorphisms in cohomology. Moreover it follows from the description of the objects of $x_{0}\left(A \rightarrow A^{\prime}\right)$ that $h$ maps $x_{o}\left(A \rightarrow A^{\prime}\right)$ isomorphically (as ordered set, or category) onto

$$
\operatorname{Mor}{\underset{f}{f}}^{-1}\left(\operatorname{Spec}\left(A^{\prime}\right)\right)
$$

Thus, composing, we find a morphism of double complexes inducing isomorphisms in cohomology

$$
\begin{aligned}
& C^{\circ}\left(\operatorname{Mor}{\underset{-1}{f}}, C \cdot\left(S-\text { free }^{*} / \psi_{1}^{\circ}(-), \operatorname{Der}_{S}\left(-, F\left(\psi_{2}^{1}(-)\right)\right)\right)\right) \\
& \downarrow \\
& C^{\bullet}\left(\operatorname{Mor}{\underset{Y}{Y}}, C^{\bullet}\left(S-\operatorname{free} /-^{0}, \operatorname{Der}_{S}\left(-, C^{\bullet}\left(\operatorname{Mor}{\underset{f}{f}}^{-1}(\operatorname{Spec}(-))\right), F\right)\right)\right)
\end{aligned}
$$

The conclusion of the theorem then follows from the exact sequence **。

## Chapter 4. Global obstruction theory

(4.1) Definitions and the main theorem

We shall define the notion of deformation of categories of 2.s-algebras in such a way that it generalizes the classical notion of infinitesimal deformations (liftings) of algebras and schemes, and, moreover takes care of the case of morphisms of schemes.

The applications we have in mind are many. We shall deduce results on moduli spaces and, in particular on the local structure of the Hilbert scheme. In a later chapter we shall also need the resulws of this chapter in the study of (possibly non-flat) descent.

This last application is responsible for the seemingly hopeless generalities that now follow.

Let $3 . S-a l g$ be the category in which the objects are the pairs of composable morphisms of S-alg, i.e. diagrams of the form $R \xrightarrow[\rightarrow]{\pi} A \xrightarrow{\mu} B$ in $S-a l g$. If ( $\pi, \mu$ ) and ( $\pi^{\prime}, \mu^{\prime}$ ) are two objects of $3 . S$-alg then a morphism $(\pi, \mu) \rightarrow\left(\pi^{\prime}, \mu^{\prime}\right)$ of $3 . S-a l g$ is a tripple $\left(\rho_{0}, \beta_{1}, 8_{2}\right)$ of morphisms of $S$ alg making the following diagram commutative

$$
R \xrightarrow{\beta_{0}} R^{\prime}
$$



Let

$$
\Phi=\hat{q}_{1,3}: 3.5-\underline{l_{g}} \rightarrow 2.5-\underline{a l g}
$$

be the functor defined by composition, i.e.

Chapter 4. Global obstruction theory
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This last application is responsible for the seemingly hopeless generalities that now follow.

Let 3.Swalg be the category in which the objects are the pairs of composable morphisms of $S-a l g$, i.e. diagrams of the form $R \xrightarrow{\pi} A \xrightarrow[H]{\mu} B$ in $S-a l g$. If ( $\pi, \mu$ ) and ( $\pi^{\prime}, \mu^{\prime}$ ) are two objects of $3 . S$ alg then a morphism $(\pi, \mu) \rightarrow\left(\pi^{\prime}, \mu^{\prime}\right)$ of $3 . S-a l g$ is a tripple $\left(\theta_{0}, \beta_{1}, 8_{2}\right)$ of morphisms of S-alg making the following diagram commutative

$$
\begin{aligned}
& R \xrightarrow{\beta_{0}} R^{\prime} \\
& \stackrel{\pi}{A} \underset{\beta_{1}}{ } \vec{A}^{i} \\
& \mu \downarrow \quad \downarrow{ }^{\prime} \\
& B \underset{\beta_{2}}{ } \mathrm{~B}^{\prime}
\end{aligned}
$$

Let

$$
\Phi=\bar{o}_{1,3}: 3 \cdot S-a l g \rightarrow 2 . S-a 1 g
$$

be the functor defined by composition, i.e.

$$
\Phi(R \rightarrow A \rightarrow B)=R \rightarrow B
$$

and let

$$
{ }_{1,2}: 3.5-81 g+2 . S-81 g
$$

be the functor defined by $\quad{ }_{1,2}(R \rightarrow A \rightarrow B)=R \rightarrow A$.

Definition (4.1.1) A deformation of $e$ is a functor $\sigma$ making the following diagram commutative.

$$
\begin{aligned}
& \mathrm{e} \xrightarrow{\sigma} 3 . \mathrm{S}-\mathrm{alg} \\
& \text { sle } \leq \quad \downarrow \\
& \text { 2.s-alg }
\end{aligned}
$$

such that for every object $(R \rightarrow A \rightarrow B)$ of $e$ writing $\sigma(R \rightarrow A \rightarrow B)=R \rightarrow \sigma(B) \rightarrow B$ the following two conditions hold:
(1)

$$
\sigma(B) \underset{R}{\otimes} A=B
$$

(2)

$$
\operatorname{Tor}_{1}^{\mathrm{R}}(\sigma(\mathrm{~B}), \mathrm{A})=0
$$

Defjnition (4.1.2) Two deformations $\sigma$ and $\sigma^{\prime}$ of $e$ are equivalent (written $\sigma \sim \sigma^{\prime}$ ) if there is an isomorphism of functors

$$
\theta: \sigma \rightarrow \sigma^{\prime}
$$

such that $\delta(\theta)$ is the identity on $\delta$ e.

Remark (4.1.3) It is easy to see that $\sim$ defines an equivalence relation in the set of deformations of e (N.B. we shall prefer not to enter into any set theoretical considerations at this point. See the Introduction.)

Definition (4.1.4) Let $e$ be any small subcategory of $3.5-a l g$, then we shall denote by

$$
\operatorname{Def}(\mathrm{e})
$$

the set of deformations of $e$ modulo the equivalence relation $\sim$ defined above.

Remark (4.1.5) Abusing the language we shall sometimes use the notation $\sigma$ both for a deformation of $e$ and for its equivalence class, hoping that this will simplify the exposition without introducing too much confusion.

Let $\underline{\theta}_{o}$ be any subcategory of $e$, thus the inclusion $e_{0} \subseteq$ e induces a canonical map

$$
\operatorname{Def}(\underline{e}) \rightarrow \operatorname{Def}\left(\underline{e}_{0}\right) .
$$

In fact we may consider Def as a functor on the ordered set (category) of small subcategories of 3.S-alg.

Remark (4.1.6) Let $e$ be such that for every object $R \rightarrow A \rightarrow B$ of $e$ the morphism $\pi$ is surjective, then a deformation of e will be refered to as a lifting of e.

Example (4.1.7) Let $A$ be any $S$ algebra, and let $\pi: R \rightarrow S$ be any surjective morphism of commutative rings. Let $e$ be the subcategory of 3.0 忽 -lg consisting of the single object $(R \rightarrow S \rightarrow A)$ and the identity morphism. A deformation $\sigma$ of e is then an R-algebra $\sigma(R)$ together with a morphism $\sigma(\mathrm{A}) \rightarrow \mathrm{A}$ such that the following conditions hold:

1. The diagram

is commutative.
2. $\quad \sigma(A) \otimes \mathbb{R} \underset{R}{\otimes}$.
3. $\operatorname{Tor}_{1}^{\mathrm{R}}(\sigma(\mathrm{A}), \mathrm{S})=0$.

Thus a deformation (lifting) of $e$ is simply a lifting of the $S$-algebra $A$ to $R$.

Examole (4.1.8) Let $X$ be an $S-s c h e m e, ~ a n d ~ c o n s i d e r ~ t h e ~ c a t e-~$ tory of $S$-algebras $C_{X}$ (see (3.1)). Let $\pi: R \rightarrow S$ be any homomorphism of commutative rings and consider the subcategory $e$ of 3 . $\mathbb{Z}-a l g$ the objects of which are the pairs of morphisms of $\mathbb{Z}$-algebras

$$
R \rightarrow S \rightarrow A
$$

where $S \rightarrow A$ is the structure morphism of an object of $\mathcal{C}_{X}$, the morphisms of $e$ being the morphisms of ${\underset{\mathrm{c}}{X}}$ extended in the obvious way.

If $\pi$ is surjective then a deformation (lifting) of $e$ is a section $\sigma$ of the functor

$$
-\underset{\mathrm{R}}{\otimes \mathrm{~S}}: \mathrm{R}-\underline{a l g} \rightarrow \mathrm{~S}-\underline{\mathrm{al} g} \mathrm{~g}
$$

defined on the subcategory ${ }_{-}{ }_{X}$ of S-alg, such that for any object $A$ of $\frac{\mathrm{c}}{\mathrm{X}} \quad \sigma(\mathrm{A})$ is a lifting of A to R . Suppose $\pi$ has nilpotent kernel, i.e. that for some $n_{9}(\operatorname{ker} \pi)^{n}=0$, then $\sigma$ corresponds to an $R$-scheme $X^{\prime}$
which is a deformation of $X$ to $R$. In fact for each affine open subset $\operatorname{Spec}(A)$ of $X$ we take $\operatorname{Spec}(\sigma(A))$ and we glue. This sets up a one-to-one correspondence between the set of deformations of $e$ and the set of deformations of the scheme $X$ to $R$.

Exampel (4.1.9) Let $f: X \rightarrow Y$ be any morphism of $S$-schemes, and consider the category ${\underset{d}{f}}$ of $2 . S-a l g e b r a s$ (see (3.1)). Let $\pi: R \rightarrow S$ be any morphism of S-algebras, and consider the subcategory e of 3.S-alg the objects of which are the pairs of morphisms

$$
A \underset{S}{\otimes} R \xrightarrow[T_{A}^{\otimes \pi}]{ } A \underset{S}{\otimes} S=A \longrightarrow B
$$

$\mu$ running through the set of objects of $\underline{d}_{f}$, and the morphisms being the morphisms of ${\underset{\sim}{f}}$ extended in the obvious way.

Suppose $\pi$ is surjective and has nilpotent kernel, then a deformation $\sigma$ of $e$ corresponds to morphisms of $S-s c h e m e s$ $\epsilon$ and $f^{\prime}$ making the following diagram cartesian

and satisfying the following condition

$$
\operatorname{Tor}_{1}{ }^{O_{S} \otimes_{S}^{R}}\left(O_{X}, O_{Y}\right)=0
$$

which reduces to

$$
\operatorname{Tor}_{1}^{R}\left(O_{X}, S\right)=0
$$

In fact, let $\operatorname{spec}(A)$ be any open affine subset of $Y$, let

Spec(B) be any open affine subset of $X$ contained in $f^{-1}(\operatorname{Spec}(A))$, then $\sigma\left(A \otimes_{S} R \rightarrow A H B\right)=A \otimes_{S} R \mu_{\mu}^{\prime} \sigma(B) \rightarrow B$ 。 We may glue the $\operatorname{Spec}(\sigma(B))^{\prime}$ is together to form a scheme $X^{i}$. The morhisms $\varepsilon$ and $f^{\prime}$ correspond to the morphisms $\xi: \sigma(B) \rightarrow B$, and the morphisms $\mu$ ' respectively. This sets up a one-to-one correspondence between the set of deformations of $e$ and the set of deformations of $f$ to $R$.

Now, suppose $e$ is any small subcategory of $3 . S-a l g$ such that for any object $R \xrightarrow{\Pi} A \underset{H}{\mu}$ of $e$ the morphism $\pi$ is surjective and $(\text { ker } \pi)^{2}=0$.

Then ker $\pi$ is an A-module, and the correspondence

$$
(R \rightarrow A \rightarrow B) \rightarrow B \underset{A}{\otimes} \operatorname{ker} \pi
$$

defines a functor

$$
0 \otimes \operatorname{ker} \Phi_{1,2}: e \rightarrow \underline{A b}
$$

which is an e-Module.
We shall construct a functor

$$
C^{\cdot}\left(-, \operatorname{Der}\left(-, 0 \otimes \operatorname{ker} \bar{z}_{1}, 2\right)\right): \text { Mor } e \rightarrow \text { Compl,ab.gr. }
$$

analogous to the functor

$$
C \cdot\left(-, \text { Der_ }_{-}(-, M)\right): \text { Mor } \alpha \rightarrow \text { Compl, } \mathrm{ab}_{2} g r .
$$

Studied in (3.1).
Let $\left(\beta_{o}, \beta_{1}, \beta_{2}\right)$ be an object of Mor e, i.e. a morphism $(\pi, \mu) \rightarrow\left(\pi^{\prime}, \mu^{\prime}\right)$ of $e$. We then have a commutative diagram

$$
\begin{array}{rll}
R & \overrightarrow{\beta_{0}} & R^{\prime} \\
\pi \downarrow & & \downarrow^{\prime} \\
\mathrm{A} & \overrightarrow{\beta_{1}} & \mathrm{~A}^{\prime} \\
\mu \downarrow & & \downarrow \mu^{\prime} \\
\mathrm{B} & \overrightarrow{\beta_{2}} & \mathrm{~B}^{\prime}
\end{array}
$$

Exactely as before we may convince ourselves that the correspondence

$$
\left(\beta_{0}, B_{1}, B_{2}\right) \rightarrow C \cdot\left(A-\operatorname{free}_{B^{\prime}}{ }^{\circ}, \operatorname{Der}_{A}\left(-, B_{A^{\prime}}^{\prime} \otimes \operatorname{ker} \pi^{\prime}\right)\right)
$$

defines a functor Mor e $\rightarrow$ Complab.gr . This is the functor $C \cdot\left(-, \operatorname{Dex}_{-}\left(-, 0 \otimes \operatorname{ker} \mathbf{1}_{1,2}\right)\right)$.
Consider the double complex

$$
K_{\underline{e}}^{\cdot}=D^{\cdot}\left(\underline{e}, C^{\cdot}\left(-, \operatorname{Der}_{-}\left(-, 0 \otimes \operatorname{ker} \Phi_{1,2}\right)\right)\right)
$$

Definition (4.1.10) We shall denote by

$$
A^{n}(e, 0) \quad n \geq 0
$$

the cohomology of the simple complex associated to the double complex $\mathrm{K}_{\mathrm{e}}^{\circ}$.

Examples (4.1.11) In the situation of (4.1.7) there are canonical isomorphisms

$$
A^{n}(e, 0) \simeq H^{n}(S, A ; A \underset{S}{\otimes} \operatorname{ker} \pi) \quad n \geq 0,
$$

provided $(\text { ker } \pi)^{2}=0$.
In the situation of (4.1.8) there are canonical isomorphisms

$$
A^{n}(\underline{e}, 0) \simeq A^{n}\left(S, X ; O_{X} \otimes \operatorname{ker} \pi\right) \quad n \geq 0
$$

provided $(\text { ker } \pi)^{2}=0$.

In the situation of (4.1.9) there are canonical isomorphisms

$$
A^{n}(\underline{e}, 0) \simeq A^{n}\left(f ; O_{X} \underset{S}{\otimes \operatorname{ker} \pi)} \quad n \geq 0\right.
$$

In fact, the category $e$ in these three cases is isomorphic to $\{S \rightarrow A\}, \quad \underline{c}_{d}$ and $\underline{d}_{f}$ respectively.

Let ${\underset{-}{e}}_{e}$ be any subcategory of the category $e$. Then there is a canonical morphism of double complexes

$$
\underline{K}_{\underline{e}}^{\bullet \cdot} \rightarrow \underline{K}_{\underline{e}_{0}^{0}}^{0}
$$

Let $\mathrm{K}_{\mathrm{e}}^{\bullet \cdot 0}$ eo be the kernel of this morphism.

Definition (4.1.12) We shall denote by

$$
A_{e_{0}}^{n}(e, 0) \quad n \geq 0
$$

the cohomology of the simple complex associated to the double complex $\mathrm{K}_{\underline{e} / \ddot{e}_{o}}$.

Thus, by definition, there is a long exact sequence of cohomology

$$
\cdots \rightarrow A_{\underline{e}_{0}}^{n}(\underline{e}, 0) \rightarrow A^{n}(\underline{e}, 0) \rightarrow A^{n}\left(\underline{e}_{0}, 0\right) \rightarrow{\underset{A_{0}}{n}}_{n+1}^{(\underline{e}, 0) \rightarrow \ldots}
$$

Example (4.1.13) In the situation of (4.1.8) let $X_{o}$ be a closed subscheme of $X$. Consider the subcategory ${\underset{-}{c}}_{X}-X_{o}$ of ${\underset{\mathrm{c}}{X}}$ and the corresponding subcategory $\underline{e}_{o}$ of $\underline{e}$, then there are canonical isomorphisms

$$
A_{e_{0}}^{n}(\underline{e}, 0) \simeq A_{X_{0}}^{n}\left(S, X, O_{X} \otimes \operatorname{ker} \pi\right) \quad n \geq 0 .
$$

In the situation of (4.1.9) let $X_{o}$ be a closed subscheme of $X$ and $Y_{o}$ a closed subscheme of $Y$ such that
$f^{-1}\left(Y_{0}\right) \subseteq X_{0}$. Consider the restriction of $f$ to $X-X_{0}$,

$$
f_{0}: X-X_{0} \rightarrow Y-Y_{0}
$$

and the corresponding subcategory $\underline{d}_{f}$ of $\underline{d}_{f}$. Obviously ${ }_{-}^{d} f_{o}$ corresponds to a subcategory ${\underset{\sim}{e}}$ of $\underline{e}$, and there are canonical isomorphisms

$$
A_{e_{0}}^{n}(e, 0) \simeq A_{f_{0}}^{n}\left(f, O_{x} \underset{S}{\otimes k e r} \pi\right) \quad n \geq 0
$$

Theorem (4.1.14) There is an obstruction

$$
o(\underline{e}) \in A^{2}(\underline{e}, 0)
$$

such that $o(e)=0$ is a necessary and sufficient condition for the existence of a deformation of $\underline{e}$. If $O(\underline{e})=0$ then Def(e) is a principal homogenous space over $A^{1}(\underline{e}, 0)$.

Proof. We shall start by constructing a 2-cocycle of the simple complex associated to the double complex $\mathrm{K}_{\mathrm{O}}^{\cdot 0}$, defining the cohomology class o. Using results of Chapter 2, we shall then prove that this cohomology class has the required property. The rest will be rather straightiorward. The component of dimension 2 of the single complex associated to $\mathrm{K}_{\underline{e}} \cdot$ has the form

$$
\begin{aligned}
& \left(K_{\underline{e}} \cdot \cdot\right)^{2}=K_{\underline{e}}^{0,2} \oplus K_{\underline{e}}^{1}, 1_{\oplus K_{\underline{e}}}^{2, O}=D^{0}\left(\underline{e}, C^{2}\left(-, \operatorname{Der} \underset{-}{ }\left(-, 0 \otimes \operatorname{ker} 1_{1,2}\right)\right)\right)
\end{aligned}
$$



Let for every object $R \xrightarrow{\pi} A \stackrel{!}{\leftrightharpoons} B$ of $\quad$ e $\sigma_{\pi \mu}^{\prime}$ be an $f$-quasisectron (see (1.2)) of the functor

$$
-{ }_{R}^{\otimes} A^{R} \text {-free } \rightarrow \text { A-free }
$$

and consider the 2-cochain $0_{o}$ of $K_{e}^{\cdot \cdot}$ defined by
 By construction $0_{0}$ is an element of the component $K_{e}^{o, 2}$. Let $d_{1}$ and $d_{2}$ denote the two differentials of the double comflex $K_{e}^{\cdot}$. We already know (see (1.2)) that $d_{2}\left(0_{0}\right)=0$. Let us compute $d_{1}\left(0_{0}\right)$. We find

$$
\alpha_{1}\left(0_{0}\right)=-d_{2}\left(O_{1}\right)
$$

where $0_{1} \in K_{e}^{1,1}$ is given by

1) See the Appendix for the calculations.

$$
\left.\sigma_{\pi \mu}^{\prime}\left(\alpha_{1}\right) \otimes{ }_{R}^{1} 1_{R}\right)\left(\delta_{1} \otimes 1_{A} 1_{\text {ker } \pi^{\prime}}\right) \tau
$$



Moreover we observe that

$$
\alpha_{1}\left(0_{1}\right)=0
$$

Let $0=0_{0}+O_{1}$ and let $d$ be the differential of the simple complex associated to $K_{e}^{\cdot}$, then $d(0)=0$. Thus 0 defines a cohomology class $o \in A^{2}(\underline{e}, 0)$.
Now $\underline{o}=0$ is equivalent to the existence of an element $Q=Q_{o}+Q_{1} \in\left(K_{e}^{\cdot}\right)^{1}=K_{e}^{0,1} \oplus K_{e}^{1,0}$ such that

1. $0_{0}=d_{2}\left(-Q_{0}\right), \quad 2 \cdot \quad 0_{1}=\alpha_{1}\left(Q_{0}\right)-d_{2}\left(Q_{1}\right), \quad$ 3. $\quad 0=d_{1}\left(Q_{1}\right)$.

By the proof of (2.2.5) 1. is equivalent to the following statem ment: For all objects ( $\mathrm{R} \xrightarrow[\rightarrow]{\mathrm{H}} \mathrm{A} \xrightarrow[\rightarrow]{\mu} \mathrm{B}$ ) of e there exists a lifting $\sigma_{0}(B)$ of $B$ as A-algebra to $R$, i.e. there exists a commutative diagram

such that $\sigma_{0}(B) \underset{R}{\otimes A} \leadsto B$ and $\operatorname{Tor}_{1}^{R}\left(\sigma_{0}(B), A\right)=0$.
The set of such diagrams corresponds to the set of $Q_{o}$ 's with the property 1. . Given a $Q_{0}$ with the property 1. then by the proof of (2.3.3) 2. is equivalent to the following statement: For every morphism $\left(\beta_{0}, \beta_{1}, \beta_{2}\right)$,

1) See the Appendix for the caldulations.

of $e$, there exists a morphism of rings $\sigma_{0}\left(\beta_{2}\right)$ making the following diagram commutative

$$
\begin{aligned}
& \xrightarrow{R} \xrightarrow[\beta_{O}]{ } R^{\prime} \\
& \sigma_{0}(\mathrm{~B})_{\sigma_{0}} \overrightarrow{\left(\beta_{2}\right)} \sigma_{0}\left(\mathrm{~B}^{\prime}\right) \\
& \varepsilon \downarrow \xrightarrow[\beta_{2}]{ } \downarrow^{\prime}
\end{aligned}
$$

The set of such morphisms corresponds to the set of $Q_{1}$ 's with this property.

Finally 3. is equivalent to the following statement:

$$
\sigma_{0}: \underline{e} \rightarrow 3 . S-a 1 g
$$

defined by

$$
\sigma_{0}(R \rightarrow A \rightarrow B)=R \rightarrow \sigma_{0}(B) \rightarrow B
$$

is a functor. This follows from an inspection of the proof of (2.3.3) and from (2.3.5).

The rest is straightforward.
Q.E.D.

Corollary (4.1.15) Let $e_{o}$ be any subcategory of $e$. Then the homomorphism

$$
A^{2}(\underline{e}, 0) \rightarrow A^{2}\left(e_{0}, 0\right)
$$

maps $o(\underline{e})$ onto $o\left(\underline{e}_{0}\right)$. Moreover, if both are zero, the map

$$
\operatorname{Def}(\underline{e}) \rightarrow \operatorname{Def}\left(\underline{\mathrm{e}}_{0}\right)
$$

is a morphism of principal homogeneous spaces via the homomorphism

$$
A^{1}(\underline{e}, 0) \rightarrow A^{1}\left(\underline{e}_{0}, 0\right) .
$$

Proof. This follows immediately from the proof of (5.1.14). Q.E.D.

Let $\underline{e}_{0}$ be any subcategory of the category of 3.S-algebras e (no conditions on e are needed), and suppose given a deformation $\sigma_{0}$ of $e_{0}$.

Definition (4.1.16) We shall denote by

$$
\operatorname{Def}\left(\underline{e} / e_{0} ; \sigma_{0}\right)
$$

the subset of $\operatorname{Def}(\underline{e})$ which maps to $\sigma_{o}$ under the map $\operatorname{Def}(\underline{e}) \rightarrow \operatorname{Def}\left(\underline{e}_{0}\right)$.

Remark (4.1.17) Let $\sigma$ be any deformation of e, then the cochains $Q_{0}(\sigma(B))$ and $Q_{1}\left(\sigma\left(\beta_{2}\right)\right)$ (denoted $Q_{0}\left(A^{\prime}\right)$ and $\left.Q_{1}\left(\beta_{2}\right)\right)$ defined in the proofs of (2.2.5) and (2.3.3) respectively fuse to define cochains $Q_{0}(\sigma)$ and $Q_{1}(\sigma)$ of $\underline{K}_{\underline{e}}^{\bullet}$ characterizeing the deformation $\sigma$.

Consider now a subcategory ${\underset{\sim}{e}}^{e}$ of $e$ and suppose we are given a deformation $\sigma_{0}$ of ${\underset{\sim}{e}}_{0}$. Then $\sigma_{o}$ is character-
 respectively. And we know that the obstruction cocycle $0\left(\underline{e}_{0}\right)=o_{0}\left(\underline{e}_{0}\right)+o_{1}\left(\underline{e}_{0}\right)$ of $K_{e_{0}}$ is a coboundary, and in fact we have:

$$
\begin{aligned}
0_{0}\left(\underline{e}_{0}\right) & =d_{2}\left(-Q_{0}\left(\sigma_{0}\right)\right) \\
0_{1}\left(\underline{e}_{0}\right) & =d_{1}\left(Q_{0}\left(\sigma_{0}\right)\right)-d_{2}\left(Q_{1}\left(\sigma_{0}\right)\right) \\
0 & =d_{1}\left(Q_{1}\left(\sigma_{0}\right)\right) .
\end{aligned}
$$

Considering the short exact sequence of double complexes

$$
0 \rightarrow K_{\underline{e} / \underline{e}_{0}}^{\ddot{e_{0}}} \rightarrow K_{\underline{e}}^{\ddot{ }} \xrightarrow{p} K_{\underline{e}_{0}}^{\cdots} \rightarrow 0
$$

we find 1 -cochains $Q_{o}^{:} \in K_{\underline{e}}^{0,1}$ and $Q_{1}^{1} \in K_{\underline{e}}^{1,0}$ such that $p\left(Q_{0}^{1}\right)=Q_{0}\left(\sigma_{0}\right), p\left(Q_{1}\right)=Q_{1}\left(\sigma_{0}\right)$.
Let $Q^{\prime}=Q_{0}^{\prime}+Q_{1}^{\prime}$.
Since $\rho\left(O(\underline{e})-d Q^{i}\right)=0$ we find that

$$
O\left(\underline{e} / \underline{e}_{0}\right)=O(\underline{e})-\alpha Q^{2}
$$

sits in $\mathrm{K}_{\mathrm{e}}^{e} / \mathrm{e}_{\mathrm{o}}$. The corresponding cohomology class

$$
o\left(e / \underline{e}_{0}\right) \in{\underset{A_{0}}{2}}_{2}^{(e, 0)}
$$

depends only on the choice of $\sigma_{0}$.
Suppose there exists a deformation $\sigma$ of $e$ such that $\sigma$ maps onto $\sigma_{0}$ under the map

$$
\operatorname{Def}(\underline{e}) \rightarrow \operatorname{Def}\left(\underline{e}_{0}\right)
$$

then we have:

$$
\begin{aligned}
0_{o}(\underline{e}) & =d_{2}\left(-Q_{o}(\sigma)\right) \\
0_{1}(\underline{e}) & =d_{1}\left(Q_{o}(\sigma)\right)-d_{2}\left(Q_{1}(\sigma)\right) \\
0 & =d_{1}\left(Q_{1}(\sigma)\right)
\end{aligned}
$$

and, by construction, there exists a $\xi \in \mathrm{K}_{\underline{e}_{0}^{0}}^{0} 0$, such that:

$$
\begin{aligned}
& o\left(Q_{0}(\sigma)\right)-Q_{0}\left(\sigma_{0}\right)=\alpha_{2}(\xi) \\
& \rho\left(Q_{1}(\sigma)\right)-Q_{1}\left(\sigma_{0}\right)=\alpha_{1}(\xi)
\end{aligned}
$$

Pick a cochain $\xi^{\prime} \in{\underset{\underline{e}}{\mathrm{e}}}_{0,0}$ with $\rho\left(\xi^{\prime}\right)=\xi$, and put:

$$
\begin{aligned}
& Q_{0}^{\prime}=Q_{0}(\sigma)-d_{2}\left(\xi^{\prime}\right) \\
& Q_{1}^{\prime}=Q_{1}(\sigma)-d_{1}\left(\xi^{\prime}\right)
\end{aligned}
$$

Then $\rho\left(Q_{0}^{\prime}\right)=Q_{0}\left(\sigma_{0}\right), \quad \rho\left(Q_{f}\right)=Q_{1}\left(\sigma_{0}\right)$ and the corresponding

$$
O\left(\underline{e} / \underline{e}_{0}\right)=O(\underline{e})-d Q^{\prime}=0,
$$

thus $o\left(e / e_{0}\right)=0$.
On the other hand suppose $o\left(e / e_{0}\right)=0$, then

$$
O\left(\underline{e} / \underline{e}_{0}\right)=O(\underline{e})-d Q^{2}=d R
$$

With $R=R_{o}+R_{1} \in K_{\underline{e}}^{\bullet} / \underline{e}_{0}$.
In particular there exists a deformation $\sigma$ of $\underline{e}$, and one with

$$
Q_{0}(\sigma)=Q_{0}^{1}+R_{0}, Q_{1}(\sigma)=Q_{1}+R_{1}
$$

Since

$$
\begin{aligned}
& \rho\left(Q_{0}(\sigma)\right)=o\left(Q_{0}^{\prime}+R_{0}\right)=Q_{0}\left(\sigma_{0}\right) \\
& \rho\left(Q_{1}(\sigma)\right)=\rho\left(Q_{1}+R_{1}\right)=Q_{1}\left(\sigma_{0}\right)
\end{aligned}
$$

we find that the map

$$
\operatorname{Def}(\underline{e}) \rightarrow \operatorname{Def}\left(\underline{e}_{0}\right)
$$

maps $\sigma$ onto $\sigma_{0}$.
We have thus proved the following result:

Theorem (4.1.17) Given a deformation $\sigma_{0}$ of $\underline{e}_{0}$, then there is an obstruction

$$
o\left(\underline{e} / \underline{e}_{o} ; \sigma_{0}\right) \in \underline{A}_{\underline{e}_{0}}^{2}(\underline{e}, 0)
$$

such that $o\left(\underline{e} / \underline{e}_{0} ; \sigma_{0}\right)=0$ if and only if $\operatorname{Def}\left(\underline{e} / \underline{e}_{0} ; \sigma_{0}\right)$
is nonempty.
In this case $\operatorname{Def}\left(\underline{e} / \underline{e}_{0} ; \sigma_{0}\right)$ is a principal homogeneous space over

$$
A_{\underline{\theta}_{0}}^{1}(\underline{e}, 0)
$$

## (4.2) Formal moduli

Let $V$ be any local ring with maximal ideal ${\underset{m}{m}}_{V}$ and residue field $k=V / \underline{m}_{V}$, and consider the category 1 of local $V$-algebras of finite length with residue field $k$.
Given an object $R$ of $\underline{I}$ we shall denote by $\underline{m}_{R}$ the maximal ideal of $R$. Thus $k=R /{\underset{m}{m}}_{R}$.

There is a filtration of the category 1 , the $n^{\text {th }}$ member of which is the full subcategory $I_{n}$ of $I$ defined by the objects $R$ with $\underline{m}_{R}^{n}=0$. Moreover there are functors

$$
\lambda_{n}^{n+1}: 1_{n+1} \rightarrow \underline{1}_{n} \quad n \geq 1
$$

defined by

$$
\lambda_{n}^{n+1}(R)=R / m_{R}^{n}
$$

Consider any pair of subcategories $\underset{-}{d} \subseteq \underline{d}$ of $2 . \mathrm{k}-\mathrm{alg}$. We shall have to divide the further discussion into two cases.

Case 1. $V$ is in this case supposed to be a k-algebra.

Case 2. $V$ is in this case arbitrary, but we shall require d (and therefore $d_{0}$ ) to be a subcategory of k-alg, usually denoted c 。

Let $R$ be any object of 1 . We shall consiader the following subcategories

$$
{\underset{\mathrm{e}}{\mathrm{oR}}}^{\subseteq_{\mathrm{e}}} \underline{-}_{\mathrm{R}}
$$

of 3.V-alg.
In case 1. the objects of $e_{R}$ (resp. $e_{o R}$ ) are the diagrams of the form

$$
\underset{k}{R} \otimes A \rightarrow A \rightarrow B
$$

where $(A \rightarrow B)$ is an object of $d$ (resp. ${\underset{-}{0}}_{0}$ ).
The morphisms of. $\underline{e}_{R}$ (resp. $e_{o R}$ ) are those induced by the morphisms of $d$ (resp. $d_{o}$ ).

In Case 2. the objects of $e_{R}$ resp. $e_{o R}$ ) are the diagrams of the form

$$
\mathrm{R} \rightarrow \mathrm{k} \rightarrow \mathrm{~B}
$$

where $(k \rightarrow B)$ is an object of $d$ (resp. $d_{0}$ ) (i.e. $B$ is $n n$ object of $\underset{\sim}{c}\left(r e s p .{\underset{\sim}{0}}^{0}\right.$ )). The morphisms are those induced by the morphisms of $\underset{\sim}{d}$ (resp. ${\underset{o}{0}}^{0}$ ).

With these notations, we shall define the functors

$$
\begin{aligned}
& \operatorname{Def}(\underline{d}): 1 \rightarrow \underline{S e t s} \\
& \operatorname{Def}\left(\underline{d}_{0}\right): 1 \rightarrow \underline{S e t s}
\end{aligned}
$$

by:

$$
\begin{aligned}
& \operatorname{Def}(\underline{d})(R)=\operatorname{Def}\left(\underline{e}_{R}\right) \\
& \operatorname{Def}({\underset{d}{0}})(R)=\operatorname{Def}\left(\underline{e}_{0 R}\right) .
\end{aligned}
$$

(We shall leave as an exercise the verifications needed to show that these objects are functors.)

In Case 1. both functors are pointed, in fact $R$ given there is a canonical trivial deformation of $e_{R}$ (resp. $e_{o R}$ ) given by the diagrams


In Case 2. we shall assume that the functor $\operatorname{Def}\left(d_{-}\right)$is pointed. In both cases we shall denote the point of $\operatorname{Def}({\underset{\mathrm{d}}{0}})$ by * .

Let

$$
\operatorname{Def}\left(d / \alpha_{0}\right): 1 \rightarrow \text { Sets }
$$

be the functor defined by

$$
\operatorname{Def}\left(d / d_{0}\right)(R)=\operatorname{Def}\left(\underline{e}_{R} / \underline{e}_{0 R} ; *\right)
$$

The purpose of this paragraph is to prove that this functor has a hull, and moreover, to give the structure of this hull.

We need some preparations. Let $\rho: R \rightarrow R^{\prime}$ be any morphism of 1 such that

$$
\underline{m}_{R} \cdot \operatorname{ker} \rho=0
$$

Notice that in this case there are canonical isomorphisms of 1

$$
\underset{R^{\prime}}{R} \times R=\underset{R^{\prime}}{\times} \times R^{\prime}[\text { ker } \rho]=R \underset{k}{x} \underset{k}{ }[\text { ker } \rho]
$$

making the following diagrams commutative

| $R \underset{R^{\prime}}{R}$ | $=$ | $\mathrm{R} \times \mathrm{k} \times[\operatorname{ker} \rho]$ | $R \underset{R^{\prime}}{\times R}$ |  | $\underset{k}{R} \times k[\operatorname{ker} \rho]$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\downarrow \mathrm{pr}_{1}$ |  | $\downarrow^{p r} r_{1}$ | $\downarrow \mathrm{pr}_{2}$ |  | $\downarrow^{\mu}$ |
| R | $=$ | R | R | $=$ | R |

where $\mu$ is defined by

$$
\mu(x,(\alpha, x))=x+x .
$$

In this situation we shall prove the following souped up version of (4.1.17).

Theorem (4.2.1) Given an element $\vec{\sigma} \in \operatorname{Def}\left(\underline{d} / d_{o}\right)\left(R^{\prime}\right)$ there is an obstruction

$$
o(\bar{\sigma}, \rho) \in A_{A_{-}^{d}}^{2}\left(\underline{d}, o_{\underline{d}} \otimes \operatorname{ker} \rho\right)
$$

such that $o(\bar{\sigma}, \rho)=0$ if and only if

$$
\bar{\sigma} \in \operatorname{im} \operatorname{Def}\left(d / \alpha_{0}\right)(p) .
$$

In any case the diagram above induces a commutative diagram

$$
\begin{aligned}
& \operatorname{Def}\left(\underline{d} /{\underset{-}{0}}^{0}\right)\left(\underset{R^{\prime}}{R} R\right)=\operatorname{Def}\left(d / d_{0}\right)\left(R \times R^{\prime}[\operatorname{ker} \rho]\right) \\
& \downarrow \mathrm{p}
\end{aligned}
$$

$$
\begin{aligned}
& \operatorname{Def}\left(d / \alpha_{0}\right)\left(R^{\prime}\right) \\
& \downarrow^{p r} 1 \\
& \operatorname{Def}\left(\alpha / d_{0}\right)(R)=\operatorname{Def}\left(d / d_{0}\right)(R)
\end{aligned}
$$

in which the maps $p$ and $\mu^{1}$ are surjections.

Proof. Let $\bar{\sigma} \in \operatorname{Def}\left(\alpha / d_{0}\right)\left(R^{\prime}\right)$ and pick a representative $\sigma$ of $\bar{\sigma}$. Consider the following subcategory e of 3.V-alg。

In Case 1. the objects of $e$ are the diagrams

$$
\left.\underset{k}{\mathrm{R} \otimes \mathrm{~A}} \rightarrow \mathrm{R}^{\prime} \otimes \mathrm{A} \rightarrow \underset{\mathrm{k}}{\otimes\left(\mathrm{R}^{\prime} \otimes \mathrm{A}\right.} \underset{\mathrm{k}}{\otimes \mathrm{~A}} \rightarrow \mathrm{~B}\right)
$$

where $A \rightarrow B$ is an object of $d$. The morphisms of $e$ are those induced by the morphisms of $d$. Obviously ${\underset{o}{o}}$ corresponds to a subcategory $e_{o}$ of e.

In Case 2. the objects of $e$ are the diagrams

$$
R \rightarrow R^{\prime} \rightarrow \sigma\left(R^{\prime} \rightarrow S \rightarrow B\right)
$$

where $S \rightarrow B$ is an object of $d$ (joe. Bis an object of $c$ ). The morphisms of $e$ are those induced by the morphisms of $d$ (ie. $c$ ). Obviously $\underset{-}{d}$ corresponds to a subcategory $\underline{e}_{o}$ of $\underline{e}$. By (4.1.17) there is an obstruction

$$
o\left(\underline{e} / e_{0} ; *\right) \in A_{e_{0}^{2}}^{2}(0,0)
$$

such that $o\left(e / e_{0} ; *\right)=0$ if and only if there is a deformation of e reducing to $*$ on $\underline{e}_{0}$. The first part of the theorem then
follows from the existence of canonical isomorphisms:

$$
A_{\underline{Q}_{0}}^{n}(\underline{e}, 0)=A_{\underline{d}_{0}}^{n}\left(\underline{a}_{\underline{a}_{k}}^{\infty} \otimes \operatorname{ker} \rho\right), \quad n \geq 0
$$

The cohomology on the left side is given by the double complex

In Case 1. let

be an object of More, then

$$
\begin{aligned}
& C^{\cdot}\left(-, \operatorname{Der}_{-}\left(-, O \otimes \operatorname{ker} \Phi_{1,2}\right)\right)\left(\beta_{0}, \beta_{1}, \beta_{2}\right) \\
& C^{\cdot}\left(\left(R^{\prime} \otimes A_{1}\right)-\operatorname{free}^{\prime} / \sigma\left(R^{\prime} \otimes A_{1} \rightarrow A_{1} \rightarrow B_{1}\right)^{o},\right. \\
& \operatorname{Der}_{R^{\prime} \otimes A_{1}}\left(-, \sigma\left(R^{\prime} \otimes A_{2} \rightarrow A_{2} \rightarrow B_{2}\right)_{R^{\prime}}^{\otimes \otimes_{k} A_{2}}{\left.\left.\operatorname{ker} \pi_{2}\right)\right)}\right.
\end{aligned}
$$

Now kex $\Pi_{2}=A_{2}{ }_{k}^{\otimes k e r} \rho$, therefore

$$
\sigma\left(R^{\prime} \otimes A_{2} \rightarrow A_{2} \rightarrow B_{2}\right) R_{R^{\prime} \otimes A_{2}}^{\otimes} \operatorname{ker} \pi_{2}=B_{2} \otimes \operatorname{ker} \rho .
$$

There exists a canonical functor

$$
\left(R^{\prime} \otimes A_{1}\right)-\text { free } / \sigma\left(R^{\prime} \otimes A_{1} \rightarrow A_{1} \rightarrow B_{1}\right) \rightarrow A_{1}-\text { free } / B_{1}
$$

defined by tensorization with $A_{1}$ over $R^{\prime} \otimes A_{1}$. This functor incuces a morphism of complexes:

$$
\begin{aligned}
& C^{\cdot}\left(A_{1}-\text { free }^{\prime} B_{1}^{0}, \operatorname{Der}_{A_{1}}\left(-, B_{2} \otimes \operatorname{ker} \rho\right)\right) \\
& \downarrow \\
& C^{\cdot}\left(-, \text { Der_ }\left(-, 0 \otimes \operatorname{ker} \phi_{1,2}\right)\right)\left(\beta_{0}, \beta_{1}, \beta_{2}\right)
\end{aligned}
$$

Notice that $e\left(r e s p . e_{o}\right)$ is, in a natural way, isomorphic to $\underset{\sim}{a}$ (resp. ${\underset{o}{o}}$ ). Thus the morphism above induces a morphism of double complexes

Due to a result of André (see (An)p. ) the corresopnding morphisin of the first spectral sequences is an isomorphism. This ends the proof of the first part of the theorem.

The only remaining difficulty is the following. Suppose ( $\bar{\sigma}_{1}, \bar{\sigma}_{2}$ ) is an elenent of

$$
\begin{gathered}
\operatorname{Def}\left(\underline{\alpha} / \alpha_{0}\right)(R) \times \operatorname{Def}\left(\underline{\alpha} / \alpha_{0}\right)(R) \\
\operatorname{Def}\left(\alpha / \underline{\alpha}_{0}\right)\left(R^{\prime}\right)
\end{gathered}
$$

then by definition the map

$$
\operatorname{Def}\left(d / \underline{\alpha}_{0}\right)(\rho): \operatorname{Def}\left(\underline{d} / \alpha_{0}\right)(R) \rightarrow \operatorname{Def}\left(\alpha / \alpha_{0}\right)\left(R^{\prime}\right)
$$

maps $\vec{\sigma}_{1}$ and $\vec{\sigma}_{2}$ onto the same elenent. But remember that we are talking about classes of deformations. This implies that if $\sigma_{1}$ and $\sigma_{2}^{\prime}$ are representatives of $\bar{\sigma}_{1}$ and $\bar{\sigma}_{2}$ respectively, then $\sigma_{1} \underset{R}{\otimes} R^{\prime}$ and $\sigma_{2}^{\prime} \underset{R}{\otimes} R^{\prime}$ are equivalent deformations of $\underline{e}_{R^{\prime}}$, but they need not be equal.

However, if

$$
\theta^{\prime}: \sigma_{1} \underset{R}{\otimes} R^{\prime} \simeq \sigma_{2}^{\prime}{ }_{R}^{\otimes} R^{\prime}
$$

is an equivalence we easily prove (see the Appendix ( 2.1 ) that there is a third deformation $\sigma_{2}$ of $e_{R}$ and an equivalence

$$
\theta: \sigma_{2} \leadsto \sigma_{2}^{\prime}
$$

such that:

$$
\theta \underset{R}{\otimes} R^{\prime}=\theta^{\prime}
$$

Therefore $\sigma_{2}$ is another representative of $\quad \bar{\sigma}_{2}$ and this time we have

$$
\sigma_{1} \underset{R}{\otimes} \mathrm{R}^{\prime}=\sigma_{2} \underset{\mathrm{R}}{\otimes} \mathrm{R}^{\prime}
$$

With these notations we have to construct a deformation $\sigma_{0}$ of $\mathrm{e}_{\mathrm{R}} \underset{\mathrm{R}^{1}}{ } \mathrm{R}$ such that

$$
\operatorname{Def}\left(a / \alpha_{0}\right)\left(p r_{i}\right)\left(\sigma_{0}\right)=\sigma_{i} \quad i=1,2
$$

This construction is, in full generality, both lengthy and dull. The point will be equally well understood restricting our situation to the following simple one:

$$
\begin{aligned}
& \mathrm{d}=\{S \rightarrow A\} \quad{\underset{o}{0}}=\varnothing \\
& \mathrm{R}^{\prime}=\mathrm{S} .
\end{aligned}
$$

Then the lifting $\sigma_{1}$ and $\sigma_{2}$ corresponds to the commutative diagram


We know that

$$
A_{i}=\lim _{\rightarrow i} \sigma^{\prime}\left(A_{i}\right) \quad i=1,2
$$

(see the proof of (2.2.5)) where $\sigma^{\prime}\left(A_{i}\right)$ are the $f$-quasisections corresponding to the listings $A_{i} \quad i=1,2$.

Let for any ring $T, T[X]$ denote the polynomial algebra on one variable. Then since

$$
\underset{R^{\prime}[X]}{R[X]} \times R[X]=\underset{R^{\prime}}{R}=R[X]
$$

the two f-quasisections $\sigma^{\prime}\left(A_{1}\right)$ and $\sigma^{\prime}\left(A_{2}\right)$ fuse to give an $f$-quasisection

$$
\sigma^{\prime}\left(A_{1}\right) \underset{R^{\prime}}{\times \sigma^{\prime}}\left(A_{2}\right)
$$

of the functor

$$
\left(R \underset{R^{\prime}}{\times} R\right)-\text { free } \rightarrow S \text {-free } .
$$

Since the obstruction cocycles of $\sigma^{\prime}\left(A_{1}\right)$ and $\sigma^{\prime}\left(A_{2}\right)$ both are zero, the obstruction cocycle of $\sigma^{\prime}\left(A_{1}\right) \underset{R^{\prime}}{\times}\left(A_{2}\right)$ is also zero, therefore
is a lifting of $A$ to $R \underset{R^{\prime}}{\times} R$, which, by construction has the properties required.

Moreover, via the canonical isomorphisms

$$
\begin{aligned}
& \operatorname{Def}\left(\underline{d} / \underline{a}_{0}\right)(R \times R) \simeq \operatorname{Def}\left(\underline{d} / \underline{\alpha}_{0}\right)(R) \times \operatorname{Def}\left(\underline{d} / \underline{-}_{0}\right)(\operatorname{s}[\operatorname{ker} \rho]) \\
& \operatorname{Def}\left(\underset{\alpha}{\alpha} / \alpha_{0}\right)(R \times R) \simeq \operatorname{Def}\left(\underset{\sim}{d} \alpha_{0}\right)(R) \times A_{\alpha_{0}}^{1}\left(\alpha, O_{\alpha} \otimes \operatorname{ker} \rho\right)
\end{aligned}
$$

induced by the canonical isomorphisms

$$
R \underset{S}{\times R} \simeq R \underset{S}{x} S[\text { ker } \rho]
$$

This lifting $A_{0}$ corresponds to the pairs $\left(A_{1}, A_{21}\right)$ and $\left(A_{1}, \lambda\left(A_{2}, A_{1}\right)\right)$ respectively, where

$$
A_{21}=\underset{\substack{\rightarrow \text { free }}}{\lim _{\text {Sim }}} \sigma_{21}
$$

with the f-quasisection $\sigma_{21}$ of

$$
S[\text { kex } \rho] \text {-free } \rightarrow S \text {-free }
$$

defined by
where for $x \in F_{o}$,

$$
\sigma_{21}(\psi)(x)=\psi(x)+\left(\sigma^{\prime}\left(A_{2}\right)(\psi)-\sigma^{\prime}\left(A_{1}\right)(\psi)(x)\right.
$$

Which is meaningfull since all coefficients of the polynomial

$$
\sigma^{\prime}\left(A_{2}\right)(\psi)(x)-\sigma^{\prime}\left(A_{1}\right)(\psi)(x)
$$

sits in kero $\rho$.
This ends the proof of (4.2.1).

> Q.E.D.

Remark (4.2.2) We have tacitely used the fact that a deformation of a deformation is a deformation. This follows from elementtarry diagram chasing.

Corollary (4.2.3) Consider a commutative diagram of 1 of the form

and suppose

$$
\underline{m}_{R_{1}} \cdot \operatorname{ker} \rho_{1}=\underline{m}_{R_{2}} \cdot \operatorname{ker} \rho_{2}=0
$$

Let $\bar{\sigma}_{1}$ be an element of $\operatorname{Def}\left(\underline{\alpha} / \alpha_{0}\right)\left(R_{1}^{1}\right)$ and put
$\ddot{\sigma}_{2}=\operatorname{Def}\left(\mathrm{d} / \alpha_{0}\right)\left(\rho^{\prime}\right)\left(\bar{\sigma}_{1}\right)$.
Consider the homomorphisms induced by $\rho$,

$$
\rho_{*}^{n}: A_{d_{0}}^{n}\left(\underline{d}, o_{d} \otimes \operatorname{ker} \rho_{1}\right) \rightarrow A_{-0}^{n}\left(\underline{d}, o_{d} \otimes \operatorname{ker} \rho_{2}\right) .
$$

Then

$$
\rho_{*}^{2}\left(o\left(\bar{\sigma}_{1}, \rho_{1}\right)\right)=o\left(\bar{\sigma}_{2}, \rho_{2}\right)
$$

Moreover there is a commutative diagram

$$
\begin{aligned}
& \operatorname{Def}\left(d / d_{0}\right)\left(R_{1}\right) \times A_{d_{0}}^{1}\left(d, o_{d} \otimes \operatorname{ker} \rho_{1}\right) \rightarrow \operatorname{Def}\left(d / d_{0}\right)\left(R_{2}\right) \times A_{d}^{1}\left(d, o_{d} \otimes \operatorname{ker} \rho_{2}\right) \\
& \downarrow \mu^{1}
\end{aligned}
$$

Proof. This follows immediately from the definitions.
Q.E.D.

Consider the category of $k$-vector spaces, k-mod. Let $V$ be any object of k-mod . Pick a basis $\left\{v_{i}\right\}_{i \in I}$ for $V$ and put the topology on $V$ in which a basis for the neighbourhoods of the neutral element consists of the subspaces containing all but a finite number of the elements $v_{i}$. Consider the corresponding category of topological k-vector spaces, k-top.mod. Let Hom ${ }_{k}^{c}$ denote the Hom functor in this category. Obviously all finite dimensional vectorspaces will be discrete. Moreover, there is a natural topology, defined by the dual basis, on the topological dual $V^{*}=H o m_{l k}^{c}(V, k)$ of any object $V$ of k-top.mod. . And one easily checks that there is a canonical isomorphism

$$
V \simeq V^{* *} .
$$

We shall now use these generalities in the construction of a hull for $\operatorname{Def}\left(\alpha / \alpha_{0}\right)$.
Fix a basis for $A_{\underline{d}}^{i}\left(\underline{d}, O_{\underline{d}}\right) \quad i=1,2$,
and consider the symmetric $V$-algebra

$$
\operatorname{sym}_{V}\left(A_{\underline{d}}^{i}\left(\underline{d}, O_{\underline{d}}\right)^{*}\right) \quad i=1,2
$$

on the topological dual of $A_{\underline{d}_{o}}^{i}\left(\underline{d}, O_{\underline{d}}\right)$.
Let

$$
T^{i} \quad i=1,2
$$

be the completion of $\operatorname{Sym}_{V}\left(\mathrm{~A}_{\underline{d}_{-}}^{\dot{1}}\left(\underline{d}, \mathrm{O}_{\underline{\alpha}}\right)^{*}\right)$ in the topology defined by the topology on $A_{{\underset{O}{O}}^{i}}^{\left(d, O_{d}\right)}{ }^{*}$. i.e. the topology in which a basi.s for the neighbourhoods of the neutral element consists of the ideals containing some power of the ideal defined by $A_{d_{-}^{d}}^{i}\left(\underline{\alpha}, O_{d}\right)^{*}$ and intersecting $A_{d}^{i}\left(\alpha, O_{d}\right)^{*}$ in an open subspace.
Notice, that if $A_{d}^{i}\left(d, O_{d}\right)$ has finite dimension, then the topology of $A_{d}^{i}\left(\underset{d}{d}, O_{d}\right) *$ is discrete and $T^{i}, i=1,2$ is a convergent power series algebra on $\hat{\theta}$.
Moreover, if $A_{d}^{i}\left(d, O_{d}\right)$ has a countable basis, then in the corresponding topology on $T^{i}$ there is a countable basis for the system of neighbourhoods of the neutral element.

Theorem (4.2.4) Suppose $A_{\underline{\alpha}}^{1}\left(\underline{\alpha}, 0_{\underline{\alpha}}\right)$ has a countable basis as a $k$-vector space. Pick such a basis, then there exists a morphism of complete local rings

$$
0: T^{2} \rightarrow T^{1}
$$

such that

$$
H\left(d / \alpha_{0}\right)=T^{1} \hat{\mathbb{Q}}^{2} k
$$

is a hull for the functor $\operatorname{Def}\left(d / d_{0}\right)$.

Proof. For each $n \in \mathbb{N}$ let's put $T_{n}^{i}=T^{i} / m_{n} n_{i} \cdot T_{n}^{i}$ has a natural topology, the quotient of the topology of $\mathrm{m}^{i}$.

Our first step will be to prove that in Case 1. $\mathbb{T}_{2}^{1}$ prorepresent the functor Def( $\alpha / \alpha_{0}$ ) restricted to $\underline{1}_{2}$. This is rather easy. In sact let $R$ be any object of $I_{2}$. By definition $\underline{m}_{R}^{2}=0$ imply. ing that $R$ as a commutative ring (k-algebra) is equal to $k\left[m_{R}\right]$, the Nagata ring of the k-vectorspace $\underline{m}_{R}$ 。

Now let $M_{V}^{c} x_{V}^{c}$ denote the set of continuous morphisms in the category of topological local rings and consider the canonical isomorphisins:

$$
\begin{aligned}
& \operatorname{Mor}_{V}^{c}\left(\mathbb{T}_{2}^{1}, R\right)=\operatorname{Mor}_{V}^{c}\left(\left(V / \underline{m}_{V}^{2}\right)\left[A_{\underline{d}}^{1}\left(\underline{\alpha}, O_{\underline{d}}\right) *\right], R\right) \\
= & \operatorname{Hom}_{\underline{k}}^{c}\left(A_{\underline{d}}^{1}\left(\underline{\alpha}, O_{\underline{d}}\right) *, \underline{m}_{R}\right)=A_{\underline{d}_{0}^{\prime}}^{1}\left(\underline{d}, o_{\underline{d}}\right) \otimes \underline{m}_{R} \\
= & A_{d_{0}}^{1}\left(\underline{d}, O_{d} \otimes \underline{m}_{R}\right) .
\end{aligned}
$$

Since in Case 1 every object $R$ of 1 is a k-algebra and all norphisms are k-algebra morphisms there is a canonical element $\sigma_{0}$ in $\operatorname{Def}\left(\mathrm{d} / \mathrm{d}_{0}\right)(\mathrm{R})$ namely the trivial deformation $\left(*=\operatorname{Def}\left(\mathrm{d} / \subseteq \operatorname{Def}\left(\mathrm{d} / \mathrm{a}_{0}\right)\right.\right.$ By (4.2.1) this implies that there is a canonical isomorphism

$$
A_{d_{0}}^{\prime}\left(\underline{d}, O_{d} \otimes \underline{m}_{R}\right)=\operatorname{Def}\left(\underline{d} /{\underset{o}{0}}^{0}\right)(R)
$$

proving what we wanted to prove.
In Case 2. there is, as in Case 1 , an initial object of $1_{2}$, namely $\mathrm{V} /{\underset{m}{m}}_{2}^{2}$ and by (4.2.1) there is an obstruction

$$
o_{1} \in A^{2}\left(\underline{\alpha}_{,}, o_{\underset{k}{\alpha}}^{\otimes}{\underset{m}{m}}_{V} / \underline{m}_{V}^{2}\right)
$$

which is zero if and only if $\operatorname{Def}\left(\underset{d}{(1)}\left(\operatorname{m}_{V}^{2}\right)\right.$ is non empty.

Now we have canonical isomorphisms and a canonical inclusion

$$
\begin{aligned}
& A^{2}\left(\underline{\alpha}, \underline{O}_{\underline{d}}^{\otimes} \underset{V}{m_{V}} / \underline{m}_{V}^{2}\right)=\operatorname{Hom}_{V}^{c}\left(A^{2}\left(\underline{\alpha}, \underline{O}_{\underline{\alpha}}\right) *, \underline{m}_{V} / \underline{m}_{V}^{2}\right) \\
& =\operatorname{Mor}_{V}^{c}\left(\left(V / \underline{\underline{m}}_{V}^{2}\right)\left[A^{2}\left(\underline{\alpha}, \underline{0}_{\underline{d}}\right)^{*}\right], V / \underline{m}_{V}^{2}\right)=\operatorname{Mor}_{V}^{c}\left(\mathbb{T}_{2}^{2}, V / \underline{m}_{V}^{2}\right) \\
& \subseteq \operatorname{Mor}_{V}^{c}\left(T_{2}^{2}, T_{2}^{1}\right) \text {. }
\end{aligned}
$$

Let $R$ be any object of $I_{2}$ and consider the canonical morphism

$$
v: V / \underline{m}_{V}^{2} \rightarrow R .
$$

We know (see (4.2.3)) that the image of $o_{1}$ considered as an lemont of $\operatorname{Mor}_{V}^{c}\left(T_{2}^{2}, V / m_{V}^{2}\right)$ in $\operatorname{Mor}_{V}^{c}\left(T_{2}^{2}, R\right)=A^{2}\left(\underline{d}, O_{d} \otimes{\underset{m}{R}}\right)$ under the map induced by $v$ is zero if and only $\operatorname{Def}(d)(R)$ is non empty. Let

$$
V_{2}=V / \underline{m}_{V}^{2} \underset{\mathbb{T}_{2}^{2}}{\otimes} k, \quad H_{2}=\mathbb{T}_{2}^{1} \underset{\mathbb{T}_{2}^{2}}{\otimes} k=V_{2}\left[A^{1}\left(\underline{\alpha}, O_{\underline{\alpha}}\right) *\right]
$$

$V / m_{V}^{2}$ and $T_{2}^{1}$ being considered as $T_{2}^{2}$-modules via the morphism $o_{1} \in \operatorname{Mor}_{V}^{c}\left(T_{2}^{2}, V / \underline{m}_{V}^{2}\right) \subseteq \operatorname{Mor}_{V}^{c}\left(T_{2}^{2}, T_{2}^{1}\right) . V_{2}$ is the largest quotient of $\mathrm{V} / \underline{m}_{V}^{2}$ to which $\alpha$ may be lifted.
Since we know that $\operatorname{Def}(d)\left(V_{2}\right)$ is nonempty we may pick an element $\sigma_{1}$ in $\operatorname{Def}(\underset{\sim}{d})\left(V_{2}\right)$. This element will take the place of the trivial deformation in Case 1.

For any object $R$ of $l_{2}$ we have canonical isomorphisms

$$
\begin{aligned}
& \operatorname{Mor}_{V}^{c}\left(H_{2}, R\right)=\operatorname{Mor}_{V}^{c}\left(V_{2}\left[A^{1}\left(\underline{\alpha}, o_{\underline{d}}\right)^{*}\right], R\right) \\
= & \left\{\begin{array}{l}
\phi \text { if } \nu: V / \underline{m}_{V}^{2} \rightarrow R \text { does not factor through } V_{2} \\
\operatorname{Hom}_{k}^{c}\left(A^{1}\left(\underline{\alpha}, \underline{o}_{\underline{d}}\right)^{*}, \underline{m}_{R}\right)=A^{1}\left(\underline{\alpha}, 0_{\underline{d}} \otimes \underline{m}_{R}\right) \text { if } v \text { does factor through } \\
V_{2} .
\end{array}\right.
\end{aligned}
$$

Using the element $\sigma_{1} \in \operatorname{Def}(\underline{d})\left(V_{2}\right)$ we find functorial isomorphisms

$$
A^{1}\left(\underline{d}, O_{d} \otimes \underline{m}_{R}\right) \simeq \operatorname{Def}(\underline{d})(R)
$$

whenever the latter is non empty, thus proving the existence of a natural isomorphism

$$
\operatorname{Mox}_{V}^{c}\left(\mathrm{H}_{2},-\right) \simeq \operatorname{Def}(\mathrm{d})
$$

on the category $\underline{1}_{2}$.
Let in Case 1. $o_{1}: \mathbb{T}_{2}^{2} \rightarrow \mathbb{T}_{2}^{1}$ be the trivial morphism (i.e. the composition of $\mathbb{T}_{2}^{2} \rightarrow k \rightarrow T_{2}^{1}$ ) . We have then, in both cases, proved the following statement:

There exists a continuous morphism

$$
o_{1}: T_{2}^{2} \rightarrow T_{2}^{1}
$$

such that the corresponding closed fiber,

$$
H_{2}=\mathrm{T}_{2}^{1} \otimes{ }_{2}^{2}{ }_{2}^{2} k
$$

prorepresents the functor $\operatorname{Def}\left(\underline{\alpha}_{\alpha}\right)$ restricted to $\underline{1}_{2}$.
In order to extend this result to all subcategories $I_{n}$ of $I$ we shall have to make the isomorphism

$$
\operatorname{Mor}_{V}^{c}\left(H_{2},-\right) \simeq \operatorname{Def}\left(\underset{\sim}{\alpha} / \alpha_{0}\right)
$$

more explicite.
This can be done in the following way. Let $I_{2}$ denote the set of open ideals of $H_{2}$ and consider the following subcategory $\underline{e}_{2}$ of 3.V-Alg . An object of $\underline{e}_{2}$ is in Case 1. a diagram

$$
A \otimes H_{2} / o r \rightarrow A \rightarrow B
$$

where $A \rightarrow B$ is an object of $d$ and $\alpha \in I_{2}$. The momphisms of $e_{2}$ are those induced by the morphisms of $d$ and by the morphisms of the form $\mathrm{H}_{2} / \sigma \rightarrow \mathrm{H}_{2} / \neq$ with $\alpha, \dot{d} \in I_{2}, \alpha \subseteq \psi_{0}$. Obviously the subcategory ${\underset{\sim}{-}}_{0}$ of $\underset{d}{d}$ corresponds to a subcategory
$\underbrace{}_{02}$ of $\underline{e n}_{2}$.
In Case 2. the objects of $e_{2}$ are the diagrams

$$
\mathrm{H}_{2} / \alpha \rightarrow \mathrm{k} \rightarrow \mathrm{~B}
$$

where $k \rightarrow B$ is an object of $d$ (i.e. $A$ is an object of $c$ ) and. $a \in I_{2}$. The morphisms of $\underline{e}_{2}$ being those induced by the morphisms of $d$ (i.e. ©) and the morphisms

$$
\mathrm{H}_{2} / \rightarrow \mathrm{H}_{2} /
$$

above. Obviously ${\underset{\mathrm{d}}{0}}$ corresponds to a subcategory $\underline{e}_{02}$ of $\underline{e}_{2}$. If we consider $I_{2}$ as an ordered set (by inclusion), therefore as a category, we find

$$
\underline{e}_{2} \simeq \underline{\alpha} \times I_{2}, \quad \underline{e}_{02} \simeq \underline{a}_{0} \times I_{2}
$$

Now we apply (4.2.1). There is an obstruction

$$
o \in A_{e_{-22}}^{2}\left(e_{2}, 0\right)
$$

such that $\underline{o}=0$ if and only if $\underline{e}_{2}$ admits a deformation trivial on $e_{02}$. In that case the set of such deformations modulo isomorphisms is a principal homogenous space over $A_{e_{-02}}^{1}\left(e_{2}, 0\right)$. Using (Lia1)(5.3)) and the isomorphisms $e_{2} \simeq d \times I_{2} \quad e_{02} \simeq{\underset{o}{0}} \times I_{2}$, we find a spectral sequence given by the term
converging to $A_{e_{-2}}^{p+q}\left(e_{2}, 0\right)$.
Using the fact that $I_{2}$ contains a countable cofinal subset, we find isomorphisms:

$$
\begin{aligned}
& A_{e_{02}}^{2}\left(\underline{e}_{2}, 0\right)=\lim _{\bar{\Sigma}_{2}} A_{\underline{d}_{0}}^{2}\left(\underline{\alpha}, O_{\underline{\alpha}}\right) \otimes\left(\underline{m}_{H_{2}} / o c\right) .
\end{aligned}
$$

Notice that for each or $I_{2}$ the category $e\left(H_{2} / o c\right)$ is a subcategory of $\mathrm{e}_{2}$. The obstruction for deforming $e_{\left(\mathrm{H}_{2} / \sigma\right) \text { rela- }}$ tive to $\underline{e}_{0}\left(\mathrm{H}_{2} / \sigma\right)$ sits in $\mathrm{A}_{\mathrm{d}_{\mathrm{O}}^{2}}^{2}\left(\underset{d}{ }, \mathrm{O}_{\mathrm{d}}\right) \underset{k}{\otimes}\left(\underline{m}_{\mathrm{H}_{2}} / \sigma\right)$.

We already know that this obstruction is zero. Using this we find $\underline{o}=0$ 。
Moreover, there is a nice canonical element of $A_{e_{-2}}^{1}\left(e_{2}, 0\right)$. In fact we observe that

$$
\left.A_{\underline{e}_{o 2}}^{1}\left(e_{2}, 0\right)=\lim _{\underset{\sim}{\leftarrow} \in I_{2}} A_{\underline{d}}^{1}\left(\underline{\alpha}, o_{\underline{\alpha}}\right) \otimes\left({\underset{\sim}{v}}_{2} \oplus{\underset{A}{a}}_{0}^{1}\left(\underline{\alpha}, o_{\underline{d}}\right) *\right) / \sigma\right)
$$

The converging sum

$$
\sigma_{2}=\sum_{k} \mathrm{e}_{\mathrm{k}} \otimes\left(\mathrm{o}, \mathrm{e}_{\mathrm{k}}^{*}\right)
$$

then defines an element of $A_{\underline{e}_{02}}^{1}\left(\underline{e}_{2}, 0\right)$.
Since we may, exactly as above, identify $A_{e_{2}}^{1}\left(e_{2}, 0\right)$ with the set of isomorphism classes of deformations of $e_{2}$ relative to $e_{o 2}$, $\sigma_{2}$ correspond to an isomorphism class of deformations of $\underline{e}_{2}$ relative to eq . We shall pick a representative of this class and, abusing the language, we shall let $\sigma_{2}$ denote this representative. Thus we find an element

$$
\sigma_{2} \in \lim _{I_{2}} \operatorname{Def}\left(d_{-0}\right)\left(\mathrm{H}_{2} / \sigma\right)
$$

and we may convince ourselves about the fact that $\sigma_{2}$ determines the isomorphism of functors on $I_{2}$ :

$$
\psi_{2}: \operatorname{Mor}^{c}\left(H_{2},-\right) \propto \operatorname{Def}\left(d / d_{0}\right)
$$

Now let $H_{m}$ for any $m$ be a topological quotient of $T_{m}^{1}$ and let $I_{m}$ denote the ordered set of open ideals of $H_{m}$.
Let

$$
\mathrm{e}_{-\mathrm{m}}\left(\begin{array}{ll}
\text { resp. } & \left.\underline{e}_{\mathrm{om}}\right) \quad \mathrm{m} \leq \mathrm{n}
\end{array}\right.
$$

denote the following subcategory of $3 . V-a l g$. An object of $e_{-m}$ (resp. $e_{o m}$ ) is in Case 1. a diagram of the form

$$
\left(H_{m} / \sigma\right){\underset{k}{2}}_{\otimes A \rightarrow A \rightarrow B} \rightarrow A
$$

where $A \rightarrow B$ is an object of $\underline{d}$ (resp. $d_{0}$ ) and $\sigma \in I_{m}$. The morphisms of $e_{m}$ (resp. $e_{-m}$ ) are those induced by the morphisms of $d$ (resp. $a_{0}$ ) and by inclusions among the or's. In Case 2. the objects of $e_{-m}$ (resp. $e_{o m}$ ) are the diagrams of the form

$$
\mathrm{H}_{\mathrm{m}} / \sigma \rightarrow \mathrm{k} \rightarrow \mathrm{~B}
$$

where $k \rightarrow B$ is an object of $d$ (resp. $d_{0}$ ). The morphisms are those induced by the morphisms of $\underline{d}$ (resp. $d_{0}$ ) and by the inclusions among the ou's.
By induction on $m$ we shall prove that there exists for every $m$ a topological quotient $H_{m}$ of $T_{m}^{1}$ and a deformation $\sigma_{m}$ of $e_{m}$ such that the restriction of $\sigma_{r n}$ to $e_{-m}$ is $*$, satisfying the following conditions:

1) The canonical morphism of topological rings

$$
t_{m}: T_{m}^{1} \rightarrow T_{m-1}^{1}
$$

induces a morphism of topological rings

$$
h_{m}: H_{m} \rightarrow H_{m-1}
$$

2) The corresponding map

$$
\lim _{I_{m}} \operatorname{Def}\left(\underline{\alpha} / \alpha_{0}\right)\left(H_{m} / \sigma\right) \rightarrow \lim _{I_{m-1}} \operatorname{Def}\left(\alpha_{m} / \underline{\alpha}_{0}\right)\left(H_{m-1} / \sigma\right)
$$

maps $\sigma_{\mathrm{m}}$ onto $\sigma_{\mathrm{m}-1}$.
3) $\sigma_{m}$ determines a smooth morphism of functors on $I_{m}$

$$
\psi_{\mathrm{n}}: \operatorname{Mox}^{c}\left(\mathrm{H}_{\mathrm{m}},-\right) \rightarrow \operatorname{Def}\left(\underset{\alpha_{0}}{d_{0}}\right)
$$

Assume such $H_{m}$ and $\sigma_{m}$ exist for $m \leq n$, and put

$$
\begin{aligned}
& C_{m}=\operatorname{ker}\left(T_{m}^{1} \rightarrow H_{m}\right), \quad m \leq n, \\
& C_{m}^{1}=m_{m+1}^{1} \cdot \operatorname{ker}\left(T_{m+1}^{1} \rightarrow H_{m}\right), \quad m \leq n, \\
& H_{m}^{\prime}=T_{m+1}^{1} / C_{m}^{\prime}, \quad m \leq n .
\end{aligned}
$$

Let

$$
h_{m+1}^{\prime}: H_{m}^{\prime} \rightarrow H_{m}, m \leq n
$$

be the canonical morphism of topological rings. Obviously

$$
{\underset{m}{H}}^{H_{m}^{\prime}} \cdot \operatorname{ker} h_{\mathrm{m}+1}^{\prime}=0, \quad m \leq n .
$$

Let $I_{m}^{\prime}$ be the ordered set of open ideals of $H_{m}^{\prime}$, and let

$$
e_{n}^{\prime}\left(r e s p \cdot e_{o n}^{\prime}\right)
$$

be the following subcategory of $3 . V-2 l \mathrm{~g}$. In Case 1. the objects of $e_{n}^{\prime}$ (resp. $e_{o n}^{\prime}$ ) are the diagrams of the form

$$
\left(H_{n}^{\prime} / \sigma\right) \underset{k}{\otimes} A \rightarrow\left(H_{n} / h_{n+1}^{\prime}(O \alpha)\right) \otimes A \rightarrow \sigma_{k}\left(\left(H_{n} / h_{n+1}^{\prime}(O C)\right) \otimes \underset{k}{\otimes} A \rightarrow A \rightarrow B\right)
$$

with ot $\in I_{n}^{\prime}$ and $A \rightarrow B$ an object of $d$ (resp. ${\underset{-}{0}}^{0}$ ). In Case 2. the objects are the diagrams of the form

$$
\left(H_{n}^{\prime} / \sigma\right) \rightarrow H_{n} / h_{n+1}^{\prime}(\sigma) \rightarrow \sigma_{n}\left(H_{n} / h_{n+1}^{\prime}(\alpha) \rightarrow k \rightarrow B\right)
$$

where $\alpha \in I_{n}^{\prime}$, and $k \rightarrow B$ is an object of $d$ (resp. ${\underset{o}{0}}^{d}$ ). The morphisms being defined accordingly.

We observe that

$$
\underline{e}_{n}^{\prime} \simeq \underset{d}{d} \times I_{n}^{\prime} \quad\left(r e s p \cdot \quad e_{0}^{\prime} \simeq{\underset{-0}{d}}_{0} \times I_{n}^{\prime}\right)
$$

 that the restriction to ${\underset{-}{o n}}_{1}^{\prime}$ is $*$, is an element

$$
\begin{aligned}
& =\lim _{\underset{I_{n}^{\prime}}{\prime}} \operatorname{Hom}_{l k}^{c}\left(A_{\underline{d}}^{2}\left(\underline{\alpha}, O_{\underline{d}}\right)^{*},\left(\left(k \operatorname{er} h_{n+1}^{\prime}+\sigma\right) / \sigma\right)\right)
\end{aligned}
$$

Given any $\sigma \in I_{n}^{\prime}$, let $o_{n}(\sigma)$ be the projection of $o_{n}$ on

$$
\left.\operatorname{Hom}_{k}^{c}\left(A_{d}^{2}\left(d, o_{d}\right)^{*},\left(\operatorname{ker~h}_{n+1}+o r\right) / o r\right)\right)
$$

and let $\alpha^{\prime}$ be the ideal of $H_{n}^{\prime}$ containing $\sigma$ and such that as an ideal of $H_{n}^{\prime} / \sigma$, $\sigma c^{\prime} / \sigma$ is generated by the image of $o_{n}(o c)$
 Put

$$
\begin{aligned}
& H_{n+1}=\lim _{\underset{i}{\prime}}^{I_{n}^{\prime}} H_{n}^{\prime} / \sigma^{\prime} \\
& h_{n+1}: H_{n+1} \rightarrow H_{n}
\end{aligned}
$$

Let

$$
\left.\mathrm{e}_{\mathrm{n}}^{\mathrm{n}+1} \text { (xesp} \cdot{e^{n+1}}_{o n}\right)
$$

denote the subcategory of 3.V-alg. analoguous to the subcategory ${\underset{-}{e}}_{-1}^{(r e s p}$ ( ${\underset{\sim}{e}}_{\prime}^{\prime}$ ) defined above, with $H_{m}^{\prime}$ and $h_{m+1}^{\prime}$ replaced by $H_{n+1}$ and $h_{n+1}$ respectively.

By construction the obstruction for deforming $e_{n}^{n+1}$ relative to $e_{o n}^{n+1}$ vanish, therefore there exists a deformation $\sigma_{n+1}$ of $e_{n}^{n+1}$ relative to $\frac{e_{o n}^{n+1}}{-1}$. Obviously $\sigma_{n+1}$ is a deformation of $e_{n+1}$ relative to $e_{o n+1}$. Moreover, by construction $H_{m}$ and $\sigma_{m}$ have the properties 1) and 2) for $m \leq n+1$.

Before we prove 3) for $m \leq n+1$, we shall study the ideals $C_{m}$ and $C_{m}^{\prime}$, and their relations to the family of homomorphism $o_{m}\left(\sigma_{\mathrm{m}}\right)$, $\quad \pi \in I_{m}$ 。
By construction we have:

$$
\begin{gathered}
C_{n}^{\prime}=t_{n+1}^{-1}\left(C_{n}\right) \cdot m_{T_{n+1}}^{1} \\
\operatorname{ker} h_{n+1}^{\prime}=t_{n+1}^{-1}\left(C_{n}\right) / C_{n}^{\prime} \\
C_{n+1} / C_{n}^{\prime} \subseteq t_{n+1}^{-1}\left(C_{n}\right) / C_{n}^{\prime}
\end{gathered}
$$

These relations imply:

$$
m_{n+1} n_{n+1} \subseteq c_{n}^{\prime}
$$

which again implies that the morphism

$$
t_{n+2}: T_{n+2}^{1} \rightarrow T_{n+1}^{1}
$$

maps $C_{n+1}^{\prime}$ into $C_{n}^{\prime}$.
There follows a commutative diagram
(

Denote by $J_{m}$ the ordered set of open ideals of $T_{m}^{1}$. Using $(4,2,3)$ on the parallellograms with two vertical edges, we find that the homomorphism

$$
\begin{aligned}
& \lim _{J_{n+1}}^{J_{n}} \operatorname{Hom}_{k}^{c}\left(A_{d_{0}}^{2}\left(\alpha, o_{d}\right) *,\left(t_{n+1}^{-1}\left(c_{n}\right)+\alpha c\right) /\left(c_{n}^{\prime}+\alpha\right)\right)
\end{aligned}
$$

maps $o_{n}$ onto $o_{n-1}$ •
Notice that we know already that $C_{2}$ is generated as ideal by the image of $O_{1}$ considered as an element of

$$
\begin{aligned}
& \lim _{\mathcal{J}_{2}} \operatorname{Hom}_{k}^{c}\left(A_{d_{0}}^{2}\left(\underline{d}, o_{\underline{d}}\right)^{*},{\underset{T}{T}}_{2} 1 / o r\right) \\
& =\operatorname{Hom}_{k}^{c}\left(A_{d_{0}}^{2}\left(\underline{d}, o_{\underline{d}}\right)^{*}, m_{T_{2}} 1\right) .
\end{aligned}
$$

Put

$$
{\underset{T}{T}}^{i}={\underset{m}{m}}^{i} /{\underset{m}{p}}^{2} i=A_{\underline{d}_{0}}^{i}\left(\underline{\alpha}, 0_{\underline{\alpha}}\right)^{*}, \quad i \geq 0
$$

and consider the commutative diagram:


All sequences in this diagram are exact. In fact, since $J_{n+1}$ and $J_{n}$ contain countable cofinal subsets, the only point to prove is that the upper horizontal sequence is exact. Now this is a consequence of a lemma of Mittag-Leffler type (see (La2) (1.)) and the following equality:

$$
t_{n+1}\left(C_{n}^{\prime}\right)=C_{n-1}^{\prime},
$$

which easily is seen to follow from the corresponding equality

$$
t_{n}\left(C_{n}\right)=C_{n-1}
$$

We shall prove this by induction on $n$, knowing, of course, that it is true for $n=2$.

Suppose there exists elements

$$
\begin{aligned}
0_{m} & \left.\in{\underset{\Im}{J_{m}}}_{\lim _{m}}^{\operatorname{Hom}_{k}^{c}\left(t_{T} 2\right.},\left(t_{m}^{-1}\left(c_{m-1}\right)+\sigma\right) / \sigma\right) \\
& \subseteq \operatorname{Hom}_{k}^{c}\left(\underline{t}_{T} 2, T_{m}^{1}\right) \quad m \leq n
\end{aligned}
$$

such that

$$
\begin{array}{ll}
k_{m}\left(O_{m}\right)=O_{m-1}, & m \leq n \\
r_{n}\left(O_{m}\right)=O_{m-1}, & m \leq n
\end{array}
$$

and such that $C_{m}$ is generated by the image of $O_{m}$ for $m \leq n$, then obviously

$$
t_{m}\left(C_{m}\right)=C_{m-1}, m \leq n
$$

In particular, therefore, the upper horizontal sequence in the diagram above is exact.
But then, by elementary diagram chasing, we find an element

$$
o_{n+1} \in \lim _{J_{n+1}^{-}} \operatorname{Hom}_{k}^{c}\left(t_{T}^{2},\left(t_{n+1}^{-1}\left(c_{n}\right)+\sigma\right) / \sigma\right)
$$

such that:

$$
\begin{aligned}
& k_{n+1}\left(o_{n+1}\right)=o_{n} \\
& r_{n+1}\left(o_{n+1}\right)=o_{n}
\end{aligned}
$$

Knowing, as we do, that $C_{n+1} / C_{n}^{\prime}$ is generated by the image of $o_{-n}$ in $t_{n+1}^{-1}\left(C_{n}\right) / C_{n}^{\prime}$, and recalling that

$$
c_{n}^{\prime}=t_{n+1}^{-1}\left(c_{n}\right) \cdot m_{T_{n+1}}^{1}
$$

we conclude that $C_{n+1}$ is generated by the image of $0_{n+1}$.
Then go on. We have proved that there exists an element

$$
\begin{aligned}
0=\left\{0_{n}\right\}_{n \geq 1} & \in \lim _{\stackrel{-}{m}} \operatorname{Hom}_{k}^{c}\left(t_{T} 2, T_{n}^{1}\right) \\
& =\operatorname{Hom}_{k}^{c}\left(\underline{t}_{T}, T^{1}\right) \\
& =\operatorname{Mor}^{c}\left(T^{2}, T^{1}\right)
\end{aligned}
$$

such that

$$
H=\lim _{\hat{n}} H_{n}=T \hat{T}^{1} \hat{Q}^{2} k
$$

Moreover, in the process, we have proved that

$$
H_{n+1} / m_{H_{n+1}}^{n}=H_{n}, \quad n \geq 1
$$

To complete the proof of the theorem, we have to prove that $\left(H_{m}, \sigma_{m}\right)$ has the property 3 ).

Take any object $R$ of $I_{n+1}$ and consider the commutative diagram

$$
\begin{array}{cc}
\operatorname{Mor}^{c}\left(H_{n+1}, R\right) & \xrightarrow{\psi_{n+1}(R)} \\
\downarrow & \operatorname{Def}\left(\alpha / \underline{\alpha}_{0}\right)(R) \\
& \downarrow \\
\text { Mor }\left(\operatorname{Hom}, R / m_{R}^{n}\right) \xrightarrow{\psi_{n}\left(R / m_{R}^{n}\right)} \xrightarrow{D} \operatorname{Def}\left(\underline{\alpha} / \underline{\alpha}_{0}\right)\left(R / m_{R}^{n}\right)
\end{array}
$$

which exists since we have proved that $H_{n+1} /{\underset{m}{n}}_{n+1}^{n}=H_{n}$. Now, use (4.2.1) to see that this diagram may be completed by the following commutative diagram:

$$
\begin{aligned}
& !!\quad! \\
& \operatorname{Mor}^{c}\left(H_{n+1}, R\right) \times \operatorname{Mor}^{c}\left(H_{n+1}, k\left[\underline{m}_{R}^{n}\right]\right) \rightarrow \operatorname{Def}\left(\underline{\alpha} / \underline{\alpha}_{0}\right)(R) \times \operatorname{Def}\left(\underline{\alpha}_{\alpha}^{\alpha}\right)\left(k\left[\underline{m}_{R}^{n}\right]\right) \\
& \|^{\mu_{1}} \\
& \downarrow^{\mu 2} \\
& \operatorname{Mor}^{C}\left(H_{n+1}, R\right) \times \operatorname{Mor}^{C}\left(H_{n+1}, R\right) \rightarrow \operatorname{Def}\left(\alpha / d_{o}\right)(R) \times \operatorname{Def}\left(\underline{\alpha}_{\alpha_{0}}\right)(R) \\
& \operatorname{Mor}^{c}\left(H_{n+1}, R / \underline{m}_{R}^{n}\right) \quad \operatorname{Def}\left(\underline{\alpha} / \underline{\alpha}_{0}\right)\left(R / \underline{m}_{R}^{n}\right)
\end{aligned}
$$

in which the dotted arrow is an isomorphism, since $H_{2}$ prorepresente $\operatorname{Def}\left(\underline{d} \underline{\alpha}_{0}\right)$ on $\underline{I}_{2}$.

We know by the induction hypotheses that ${ }_{~_{n}}\left(R / m_{R}^{n}\right)$ is surjective. Let us prove that this implies that $\psi_{n+1}(R)$ is surjective. Let, to that end, $\bar{\sigma}_{R}$ be any element of $D\left(\underline{\alpha}_{\alpha}^{\alpha} \underline{\alpha}_{0}\right)(R)$ and let

$$
\bar{\sigma}_{\mathrm{R} / \underline{m}_{\mathrm{R}}^{\mathrm{n}}}=\bar{\sigma}
$$

be the image of $\bar{\sigma}_{R}$ in $\operatorname{Def}\left(\underline{d} / \underline{d}_{0}\right)\left(R / m_{R}^{n}\right)$. There exists a morphism of topological rings

$$
\varphi: H_{n} \rightarrow R / m_{R}^{n}
$$

such that $\psi_{n}\left(R / m_{R}^{n}\right)(\varphi)=\bar{\sigma}$. This of course means that

$$
\bar{\sigma}=\operatorname{Def}\left(\underline{d}{\underset{-}{0}}^{0}\right)(\varphi)\left(\sigma_{n}\right)
$$

Consider the diagram


We may clearly find a morphism $\bar{\varphi}$ making the completed diagram commutative. Since $\bar{\varphi}$ maps $\operatorname{ker}\left\{T_{n+1}^{1} \rightarrow H_{n}\right\} \cdot \underline{m}_{T} 1$ onto zero, $\bar{\varphi}$ factors through $H_{n}^{\prime}$. Finally, since by (4.2.3) the induced moxphism in cohomology maps the obstruction

$$
o_{n}=o\left(\sigma_{n}, h_{n}^{\prime}\right)
$$

to zero as $\bar{\sigma}=\bar{\sigma}_{R} / \underline{m}_{R}^{n}$ may be lifted to $R, \tilde{\varphi}$ factors through $H_{n+1}$, and we obtain a commutative diagram

$$
\begin{array}{lll}
\mathrm{H}_{\mathrm{n}+1} & \overrightarrow{\varphi^{\prime}} & \mathrm{R} \\
\downarrow & & \downarrow \\
\mathrm{H}_{\mathrm{n}} & & \vec{\varphi} \\
\mathrm{R} / \mathrm{m}_{R}^{n}
\end{array} .
$$

This together with the nice diagram above, in which $\mu_{1}$ and $\mu_{2}$ are surjective, completes the proof of the theorem.
Q.E.D.

Chapter 5. Some applications
(5.1) Tocal structure of moduli schemes

Let $k$ be any field and consider a morphism of algebraic $k$-schemes

$$
f: X \rightarrow Y .
$$

Denote by $\mathrm{k} / \mathrm{sch} / \mathrm{k}$ the category of pointed k -schemes, and let the functor

$$
D_{\mathrm{f}}: \mathrm{k} / \mathrm{sch} / \mathrm{k} \rightarrow \text { Sets }
$$

be defined by

$$
\begin{aligned}
D_{f}(\operatorname{Spec}(k) \vec{\varphi} T) & =\left\{\left.\begin{array}{llll}
X^{\prime} & f^{\prime} & Y \times T \\
\uparrow & & \Upsilon^{T} \\
X & \rightarrow & Y
\end{array} \right\rvert\, \quad X^{\prime} \quad \text { flat over } T\right. \\
X & \text { and the diagram being cartesian }\} / \sim
\end{aligned}
$$

here $\sim$ denotes the equivalence relation defined by
if and only if there is an isomorphism $X^{\prime} \rightarrow X^{\prime \prime}$ making $2 I l$ diagrams commute.

Suppose $D_{f}$ is representable, and let

$$
\text { Spec (k) } \stackrel{h}{\rightarrow} H
$$

be the representing object of $\mathrm{k} / \mathrm{sch} / \mathrm{k}$.
Then for every object $\operatorname{Spec}(k) \rightarrow I$ of $k / s c h / k$,

$$
D_{f}(\operatorname{Spec}(k) \rightarrow T)=\operatorname{Mor}_{p t}(T, H)
$$

where Morpt denotes morphisms respecting the base points.

Obviously $\hat{i} \in D_{f}(\operatorname{Spec}(k) \stackrel{1}{\rightarrow} \operatorname{Spec}(k))$ corresponds to $h \in \operatorname{Mor}_{p t}(\operatorname{Spec}(k), H)$, i.e. to the base point of $H$. Let $1^{0}$ be the subcategory of $k / \operatorname{sch} / k$ whose objects $\operatorname{Spec}(k) \rightarrow T$ are such that $\llbracket=\operatorname{Spec}(R)$ with $R$ a local artinian k-algebra with residue field $k$.

The restriction of $D_{f}$ to $\underline{1}^{0}$ is given by

$$
\begin{aligned}
& D_{f}(\operatorname{Spec}(k) \rightarrow \operatorname{Spec}(R))=\operatorname{Mor}_{p t}(\operatorname{Spec}(R), H) \\
= & \operatorname{Mor}\left(O_{H, h}, R\right)=\operatorname{Mor}\left(\hat{O}_{H, h}, R\right)
\end{aligned}
$$

where the last morphism set is the set of homomorphisms of local k--algebras.
It follows that $\hat{O}_{H, h}$ prorepresents the functor $D_{f}$ restricted to $1^{0}$.

Thus we have the following result:

Theorem (5.1.1) With the above notations,

$$
\hat{\mathrm{O}}_{\mathrm{H}, \mathrm{~h}}=\operatorname{Sym}\left(\mathrm{A}^{1}\left(\mathrm{f} ; \mathrm{O}_{\mathrm{X}}\right)^{*}\right)^{\wedge} \hat{Q}_{\operatorname{Sym}\left(\mathrm{A}^{2}\left(\mathrm{f}_{;} \mathrm{O}_{\mathrm{X}}\right)^{*}\right)^{\wedge}}
$$

In particular the imbedding dimension of $O_{H, h}$ is equal to $\operatorname{dim}_{k} A^{1}\left(f ; O_{X}\right)$, and $O_{H, h}$ is regular if and only if

$$
0: \operatorname{Sym}\left(A^{2}\left(f ; O_{X}\right)^{*}\right)^{\wedge} \rightarrow \operatorname{Sym}\left(A^{1}\left(f, O_{X}\right)^{*}\right)^{\wedge}
$$

is trivial

Proof. This follows immediately from (4.2.4), and (4.1.9).

Let us consider the special case where $f: X \rightarrow Y$ is a closed embedding. Using (3.1.14) we find a spectral sequence with

$$
\mathrm{E}_{2}^{p, q}=H^{p}\left(Y, A_{f}^{q}\left(O_{X}\right)\right)
$$

where ${ }_{-}^{A}\left(O_{X}\right)$ is the quasicoherent sheaf on $Y$ defined by

$$
A_{f}^{q}\left(O_{X}\right)(\operatorname{Spec}(A))=A^{q}\left(A, f^{-1}(\operatorname{Spec}(A)) ; O_{X}\right)
$$

converging to $A^{\circ}\left(f ; O_{X}\right)$.
Let $J$ be the ideal of $O_{Y}$ vanishing on $X$. Then $I^{-1}(\operatorname{Spec}(A))$
$=\operatorname{Spec}(B)$ with $B=A / J(\operatorname{Spec}(A))$ 。 Thus

$$
A_{f}^{q}\left(O_{X}\right)(\operatorname{Spec}(A))=H^{q}(A, B ; A)
$$

In particular $A_{-1}^{O}\left(O_{X}\right)=0$ and :

$$
A_{P}^{1}\left(O_{X}\right)=\operatorname{Hom}\left(J / J^{2}, O_{X}\right)=N_{X / Y}
$$

is the normal bundle of $X$ in $Y$.

Theorem (5.1.2) (Severi--Kodairamspencer) Let $X$ be any closed subscheme of the algebraic $k$ scheme $Y$. Suppose $X$ is locally a complete intersection of $Y$, then if $f: X \rightarrow Y$ is the imbedding of $X$ in $Y$, we have

$$
A^{n}\left(x, O_{X}\right)=H^{n-1}\left(X, N_{X / Y}\right) \quad n \geq 0
$$

where $N_{X / Y}$ is the normal bundle of $X$ in $Y$.

Proof. This follows from the fact that $H^{n}(A, B ;-)=0$ for $n \geq 2$ whenever $B$ is a complete intersection of $A$,
Q.E.D.

Suppose the Hilbert scheme $\mathrm{Hilb}_{Y}$ of $Y$ exists. Let $\{X\}$ be the point of $\mathrm{Hilb}_{Y}$ corresponding to the imbedding f. Then

Theorem (5.1.3) With the assumptions of (5.1.2) there is a morphism of complete local rings

$$
0: \operatorname{Sym}\left(\mathrm{H}^{\wedge}\left(\mathrm{X}, \mathrm{~N}_{\mathrm{X} / \mathrm{Y}}\right)^{*}\right)^{\wedge} \rightarrow \operatorname{Sym}\left(\mathrm{H}^{\mathrm{O}}\left(\mathrm{X}, \mathrm{~N}_{\mathrm{X} / \mathrm{Y}}\right)^{*}\right)^{\wedge}
$$

such that

$$
\begin{array}{r}
\hat{\mathrm{O}}_{\mathrm{Hilb}}^{\mathrm{Y}},\{\mathrm{X}\} \simeq \operatorname{Sym}\left(\mathrm{H}^{\circ}\left(\mathrm{X}, \mathrm{~N}_{\mathrm{X} / \mathrm{Y}}\right)^{*}\right)^{\wedge} \hat{\otimes} \mathrm{k} \\
\operatorname{Sym}\left(\mathrm{H}^{1}\left(\mathrm{X}, \mathrm{~N}_{\mathrm{X} / \mathrm{Y}^{*}}\right)^{*}\right)^{\wedge}
\end{array}
$$

In particular $H_{i l b}$ is nonsingular at the point $\{X\}$ if and only if 0 is trivial.

Proof. This is a simple consequence of (5.1.1) and (5.1.2).
Q.E.D.

Remark (5.1.4) The above theorem generalizes a theorem of Kodaira and Spencer, see (Ko1,2) and ( Mu ), p. 157.). In ( the theorem is stated in the following form: Let $X$ be a curve on the surface $Y$. Say that $X$ is semi-regular if $H^{1}\left(Y, O_{Y}(X)\right) \rightarrow H^{1}\left(X, N_{X / Y}\right)$ is the zero map. Then if $\operatorname{Char}(k)=0$ and $X$ is semi-regular the scheme classifying all curves on $X$ is nonsingular at the point $X$.

This follows froin (5.1.1) and (5.1.2) since an easy computation shows that in this case the morphism 0 restricted to $H^{1}\left(X, N_{X / Y}\right)^{*}$ factors via $H^{1}\left(Y, O_{Y}(X)\right)^{*}$.

Another special case to consider is the case where $f: X \rightarrow \operatorname{Spec}(\mathbb{I})$ is the structure morphism of an algebraic scheme $X$ 。 Using (3.1.12) we find a spectral sequence given by

$$
E_{2}^{p, q}=H^{p}\left(X, A^{q}\left(O_{X}\right)\right)
$$

where $A^{q}\left(O_{X}\right)$ is the quasicoherent sheaf given by

$$
\underline{A}^{q}\left(O_{X}\right)(\operatorname{Spec}(A))=H^{q}(k, A ; A)
$$

whenever $\operatorname{Spec}(A)$ is an open subset of $X$, converging to $A^{*}\left(k, X ; O_{X}\right)=A^{\bullet}\left(f ; O_{X}\right)$.

Put

$$
\theta_{X}=A^{0}\left(O_{X}\right)
$$

then we have the following result:

Theorem (5.1.5) Suppose $X$ is nonsingular, then there exists a morphism of complete local rings

$$
0: \operatorname{Sym}\left(\mathrm{H}^{2}\left(\mathrm{X}, \theta_{\mathrm{X}}\right)^{*}\right)^{\wedge} \rightarrow \operatorname{Sym}\left(\mathrm{H}^{1}\left(\mathrm{X}, \theta_{\mathrm{X}}\right)^{*}\right)^{\wedge}
$$

such that

$$
\begin{aligned}
& H=\operatorname{Sym}\left(H^{1}\left(X, \theta_{X}\right)^{*}\right)^{\wedge} \hat{\otimes} k \\
& \operatorname{Sym}\left(H^{2}\left(X, \theta_{X}\right)^{*}\right)^{\wedge}
\end{aligned}
$$

is the hull of the deformation functor of X .

Proof. Since $X$ is nonsingular, $A^{q}\left(O_{X}\right)=0$ for $q \geq 1$. Q.E.D.

Remark $(5.1 .6)$ With the assumptions of $(5.1 .5)$ the obstruction morphism 0 has a first approximation given by a homomorphism

$$
H^{1}\left(X, \theta_{X}\right) \operatorname{Sym}_{\otimes}^{H^{1}}\left(X, \theta_{X}\right) \rightarrow H^{2}\left(X, \theta_{X}\right)
$$

This is a graded Lie product and must coincide with the product of Kodaira-Spencer (Ko 1).

The later approximations of 0 will give rise to cohomology operations on $H^{1}\left(X, \theta_{X}\right)$, the nature of which we do not fully understand.

The computation of 0 and the corresponding study of these cohomology operations will be treated in a later paper.

Let us end this paragraph by writing up a couple of easy consequences of (3.1.16.).

Theorem (5.1.7) Let $Z$ be a closed subscheme of the S-scheme X. Consider any $\mathrm{O}_{\mathrm{X}}$-Module M . Then the canonical morphism

$$
A^{p}(S, X ; M) \rightarrow A^{p}(S, X-Z ; M)
$$

is injective for $p \leq \inf _{x \in Z}$ depth $M_{x}+1$ and bijective for $p \leq \inf _{x \in z} \operatorname{depth} \cdot M_{x}+2$.

Proof. This follows from the fact that

$$
H_{Z}^{p}(M)=0 \quad \text { for } p \leq \inf _{x \in Z} \operatorname{depth} M_{x}+1
$$

and from the exact sequence following (3.1.15).

Corollary (5.1.8) (Schlessingers comparison theorem) If A is any $S$-algebra, $M$ an A-module and $J$ an ideal of $A$, then the canonical homomorphism

$$
H^{p}(S, A ; M) \rightarrow A^{p}(S, \operatorname{Spec}(A)-V(J) ; \tilde{M})
$$

is injective for $p \leq \operatorname{depth}_{1} M+1$ and an isomorphism for $\mathrm{p} \leq \operatorname{depth}_{\mathrm{I}^{M+2}}$.

Remark(5.1.9) Schlessinger proved this theorem for the case $p=1$, see (Sch), Svaenæs (Sv) has proved a theorem related to (5.1.8) in the general case.

There are many applications of the general theory. In this chapter we have mentioned only a few. As an easy consequence
of (5.1.1) one may prove that the Hilbert scheme is nonsingular at all points corresponding to zero dimensional subschemes $X$ of $\mathbb{P}^{n}$ provided $X$ is locally a complete intersection (a sufficient condition is local.liftability, thus codimension 2 Cohen Macaulay will do). This is a theorem of Fogarty (Fo). For results refining $(5.1 .1)$ see the paper of Ellingsrud (El). The graded theory has been studied by Kleppe (Kl). His results generalize results of Pinkham (Pi) and others.

We shall, in a later paper study subjects like secant bundles, dual schemes and equisingular deformations. Winally we hope to be able to use the above machinery in the study of non-flat descent.

Appendix (1.3) Let ${\underset{c}{o}}^{0}$ be any subcategory of the category $c$, and let

$$
F: c^{0} \rightarrow A b
$$

be a functor.
Given any object $c$ of $c$ we may consider the simplicial set

$$
\Lambda_{c}\left(\underline{c}_{0}\right)=\left\{\Lambda_{c}^{p}\left(\underline{c}_{0}\right)\right\}_{p>0}
$$

defined by:

$$
\begin{aligned}
\Lambda_{c}^{p}\left({\underset{-}{c}}_{c}\right)= & \left\{\left(c_{\vec{\rho}} c_{0} \vec{\psi}_{T} c_{1} \rightarrow \cdots \vec{\psi}_{p} c_{p}\right) / \psi_{i} \text { morphisms of }{\underset{o}{o}}\right. \\
& \text { for } p \geq i \geq 1\},
\end{aligned}
$$

the face morphisms

$$
\partial_{i}: \Lambda_{c}^{p}\left(\underline{c}_{0}\right) \rightarrow \Lambda_{c}^{p-1}\left(\underline{c}_{0}\right) \quad i=0, \ldots, p
$$

being defined by

Let $C_{0}(c)=C_{0}\left(\Lambda_{c}\left({\underset{-}{o}}_{0}\right) ; \mathbb{Z}\right)$ be the simplicial chain complex associated to $\Lambda_{c}\left({\underset{c}{o}}^{0}\right.$ ). One checks that $c \rightarrow C$ (c) is a functor

$$
\text { C. }: c^{0} \rightarrow \text { Compl.ab.gr. }
$$

Moreover if all $\Lambda_{c}\left(\underline{-}_{o}\right)$ are nonempty there is an augmentation morphism

$$
\text { C. } \rightarrow \mathbb{Z}
$$

where $\mathbb{Z}$ is the constant functor.
Now it follows from (La.1) that $C^{p}$ is a projective object of the category of abelian functors on $c^{\circ}$. Therefore if for every object $c$ of $c, \Lambda_{c}\left({\underset{\sim}{c}}^{c}\right)$ is ascyclic,

$$
\text { C. } \rightarrow \mathbb{Z}
$$

is a projective resolution of $\mathbb{Z}$ in the category of abelian fundtors on $c^{\circ}$. This is obviously the case when ${\underset{\sim}{c}}^{c}=\underset{\sim}{c}$.

Theorem (see (La 1), (L aL)). Consider any functor $F: \underline{c}^{\circ} \rightarrow \mathrm{Ab}$. Suppose for every object $c$ of $c, \Lambda_{c}\left(c_{o}\right)$ is ascyclic, then the canonical morphism of complexes

$$
C^{\bullet}\left(\underline{c}^{0}, \vec{F}\right) \rightarrow C^{\cdot}\left(\underline{c}_{o}^{0}, F\right)
$$

induces isomorphisms in cohomology

Proof. This is a simple consequence of the identities:
Q.E.D.

Corollary Suppose $c_{0}$ is a subcategory of $c$ satisfying the following conditions,
(1) given any object $c$ of $c$ there is a orphism $\rho: c \rightarrow c_{0}$ of $c$ with $c_{o}$ an object of $c_{-}$,
(2) given any diagram

of $c$ with $c_{1}$ and $c_{2}$ (resp. $c_{0}$ ) objects of $c_{0}$ there exist $\varphi_{1}, \varphi_{2}$ (resp. $\varphi_{0}$ ) of ${\underset{-}{o}}$ such that

$$
\rho_{1} \varphi_{1}=\rho_{2} \varphi_{2} \quad\left(\text { resp } \quad \rho_{1} \varphi_{0}=\rho_{2} \varphi_{0}\right)
$$

Then for every object $c$ of $c, \Lambda_{c}({\underset{\sim}{c}})$ is ascyclic.

Proof. Let $\Lambda_{o}$ be any finite simplicial subset of $\Lambda_{c}\left({\underset{\sim}{o}}^{o}\right)$. Then the condition (2) implies that $\Lambda_{o}$ is contractible in $\Lambda_{c}\left(\underline{c}_{0}\right)$. Q.E.D.

Appendix (2.1) If $S[X] \vec{j} A$ is a surjective homomorphism of S-algebras then we know that

$$
H^{1}(S, A ; B \otimes I) \simeq H_{S[X]}(\text { ker } j, B \otimes I) / \text { Derivations }
$$

Let $R[X] \vec{j}, A^{\prime}$ be a lifting of $j$ to $R$ and observe that

$$
\operatorname{Ker} j^{\prime} \underset{R}{\otimes A} \simeq \operatorname{ker} j
$$

since $\operatorname{Tor}_{1}^{\mathrm{R}}\left(\mathrm{A}^{\prime}, \mathrm{S}\right)=0$.
Let $\nu^{\prime}: R[X] \rightarrow B^{\prime}$ be a lifting of $j^{\circ} \psi$, then $\nu^{\prime}$ defines an $R[X]$-module homomorphism

$$
\text { Ker } j^{\prime} \rightarrow B^{\prime} \underset{R}{\otimes} I
$$

vanishing on ker $j^{\prime} \underset{R}{\otimes}$.
Therefore $\nu^{\prime}$ induces a homomorphism

$$
\nu: \operatorname{ker} j \rightarrow B^{\prime} \underset{R}{\otimes} I \simeq B \underset{S}{\otimes} I .
$$

One may check that $\nu$ represents the class

$$
o_{\pi}\left(\forall ; A^{\prime}, B^{\prime}\right) .
$$

Let $A^{\prime}$ and $B^{\prime}$ be liftings of $A$ and $B$ respectively and consider the map

$$
\Psi_{*}: H^{1}(S, A ; A \otimes I) \rightarrow H^{1}(S, A ; A \otimes I)
$$

defined by

$$
\Psi_{*}(\lambda)=o_{\pi}\left(\psi^{\prime} ; A^{\prime}, B^{\prime}\right)-o_{\pi}\left(\psi A^{\prime \prime}, B^{\prime}\right)
$$

where $\lambda$ corresponds to the difference $A^{\prime}-A^{\prime \prime}$.

Theorem $\Psi_{*}$ is induced by $\psi \otimes 1_{I}: A \otimes I \rightarrow B \otimes I$.

Corollary Suppose $A$ and $B$ can be lifted to $R$ and suppose
$o\left(\psi: A^{\prime}, B^{\prime}\right) \in \operatorname{im} \psi_{*}$ for some $A^{\prime}$ and $B^{\prime}$ lifting $A$ and $B r e-$ spectively. Then there exists on $A^{\prime \prime}$ lifting $A$ and $a$ $\psi^{\prime \prime}: A^{\prime \prime} \rightarrow B^{\prime}$ lifting $\psi$ 。

Corollary Let $\zeta: A \rightarrow B$ be an isomorphism and suppose $A$ and $B$ can be lifted to $R$. Then there exists for every lifting $B^{\prime}$ of $B$ a unique lifting $A^{\prime}$ of $A$ and a morphism

$$
\zeta^{\prime}: A^{\prime} \rightarrow B^{\prime}
$$

lifting $\zeta$.

Consider the map

$$
\Psi^{*}: H^{1}(S, B ; B \otimes I) \rightarrow H^{1}(S, A ; \otimes I)
$$

defined by

$$
\Psi^{*}(\mu)=o_{\pi}\left(\psi ; A^{\prime}, B^{\prime \prime}\right)=o_{\pi}\left(\psi, A^{\prime}, B^{\prime}\right)
$$

where $\mu$ corresponds to the difference $B^{\prime \prime}-B^{\prime}$.

Theorem $\Psi^{*}$ is induced by $\psi: A \rightarrow B$.

Proof. See (La 4) (3.1.6).

Corollary Suppose $A$ and $B$ can be lifted to $R$ and suppose $o\left(\psi ; A^{\prime}, B^{\prime}\right) \in \operatorname{im} \psi^{*}$ for some $A^{\prime}, B^{\prime}$ lifting $A$ and $B$ respectively. Then there exists an $B^{\prime \prime}$ lifting $B$ and $a \psi^{\prime \prime}: A^{\prime} \rightarrow B^{\prime \prime}$ lifting $\psi$ 。

Corollary Let $\psi: A \rightarrow B$ be an isomorphism and suppose $A$ and $B$ can be lifted to $R$. Then there exists for every lifting $A^{\prime}$ and $A$ a unique lifting $B^{\prime}$ of $B$ and a morphism $\psi^{\prime}: A^{\prime} \rightarrow B^{\prime}$ lifting $\psi$.

Corollary Let $\mu \in H^{\prime}(S, A ; A \otimes I)$ correspond to $A^{\prime}-A^{\prime \prime}$ where $A^{\prime}$ and $A^{\prime \prime}$ are two liftings of $A$ to $R$. Then $\mu=o_{\pi}\left(1_{A}, A^{\prime}, A^{\prime \prime}\right)$.

Appendix (4.1) Calculations of $d_{1}\left(O_{0}\right), d_{2}\left(O_{2}\right)$ and $d_{1}\left(O_{1}\right)$. By definition of $K^{*}$ we find,

$$
\begin{aligned}
& \alpha_{1}\left(O_{0}\right)\left(\begin{array}{ccc}
R & \longrightarrow & R^{\prime} \\
\Pi \downarrow & \beta_{0} & \downarrow^{\prime \prime} \\
A & & \beta_{1} \\
A^{\prime} \\
\mu & & \downarrow^{\prime \prime} \\
B & \overrightarrow{\beta_{2}} & B^{\prime}
\end{array}\right)\left(\begin{array}{ccc}
F_{0} \xrightarrow[\alpha_{1}]{ } & F_{1} \xrightarrow[\alpha_{2}]{ } / F_{2} \\
\delta_{0} & \downarrow_{1}^{\delta_{1}} & \delta_{2}
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\left(O_{0}\left(\begin{array}{c}
R \\
\psi^{\pi} \\
A \\
\psi^{\mu} \\
B
\end{array}\right)\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}\right)\left(\begin{array}{ccc}
F_{0} \xrightarrow{O_{1}} & \alpha_{1} & \alpha_{1} \\
\delta_{0} & \downarrow_{2} & \sigma_{2} \\
0 & & \delta_{2}
\end{array}\right)=U,
\end{aligned}
$$

where $\left(\beta_{0}, \beta_{1}, \beta_{2}\right)^{*}: C^{0}\left(A^{\prime}-\right.$ free $/ B^{\prime}{ }^{\circ}, \operatorname{Der}_{A^{\prime}}\left(-, B_{A^{\prime}} \otimes\right.$ ken $\left.\left.\pi^{\prime}\right)\right)$ $\rightarrow C^{*}\left(A-\right.$ free $\left./ B, \operatorname{Der}_{A^{\prime}}\left(-, B_{A^{\prime}} \otimes \operatorname{ker} \pi^{\prime}\right)\right)$ is the canonical morphism induced by

$$
\left(\beta_{1}, \beta_{2}\right)_{*}: A-\text { free } / B \rightarrow A^{\prime}-\text { free } / B^{\prime} \quad(\text { see }(4.1))
$$

and

$$
\left(\beta_{1} \otimes 1_{F}\right): \operatorname{Der}_{A^{\prime}}\left(A_{A}^{\prime} \otimes F, B_{A^{\prime}}^{\prime} \otimes \operatorname{ker} \pi^{\prime}\right) \rightarrow \operatorname{Der}_{A}\left(E, B_{A^{\prime}}^{\prime} \otimes \operatorname{ker} \pi^{\prime}\right),
$$

and where $\left(\beta_{0}, \beta_{1}, \beta_{2}\right)_{*}: C^{\circ}\left(A-\right.$ free $\left./ B, \operatorname{Der}_{A}(-, B \otimes \operatorname{ker} \pi)\right)$
$\rightarrow C^{\bullet}\left(A-f r e e / B, \operatorname{Der}_{A}\left(-, B_{A^{\prime}}^{\prime} \otimes \operatorname{ker} \pi^{\prime}\right)\right)$ is the morphism induced by

$$
\beta_{2} \underset{\beta_{1}}{\otimes} \beta_{0} \mid \operatorname{ker} \pi: B \otimes \operatorname{ker} \pi \rightarrow B_{A}^{\prime} \otimes \operatorname{ker} \pi^{\prime}
$$

We find,

$$
\begin{aligned}
& U=\left(\beta_{1} \otimes 1_{F_{0}}\right) 0_{0}\left(\begin{array}{c}
R^{\prime} \\
\downarrow^{\Pi^{\prime}} \\
A^{\prime} \\
\downarrow^{\mu^{\prime}} \\
B^{\prime}
\end{array}\right)\left(\begin{array}{ccc}
A^{\prime} \otimes F_{0} & A^{\prime} \otimes F_{1} & A^{\prime} \otimes F_{2} \\
A & & \downarrow \\
& \delta_{0}^{\prime} & A^{\prime} \otimes B \\
& & \downarrow^{\prime}
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\beta_{1} \otimes 1_{F_{0}}\right)\left(\sigma_{\pi^{\prime}}^{\prime}\left(\left(1_{A^{\prime}} \otimes \alpha_{1}\right)\left(1_{A^{\prime}} \otimes \alpha_{2}\right)\right)-\sigma_{\pi^{\prime}}^{\prime}\left(1_{A^{\prime}} \otimes \alpha_{1}\right) \sigma_{\pi^{\prime}}^{\prime}\left(1_{A^{\prime}} \otimes \alpha_{2}\right)\right)\left(\delta_{2}^{\prime} \otimes 1_{\text {ker } \pi}\right) \\
& -\left(\sigma_{\pi}^{\prime}\left(\alpha_{1} \alpha_{2}\right)-\sigma_{\pi}^{\prime}\left(\alpha_{1}\right) \sigma_{\pi}^{\prime}\left(\alpha_{2}\right)\right)\left(\delta_{2}^{\otimes} 1_{\text {ker } \pi}\right)\left(\beta_{2} \otimes_{\beta_{1}}^{\beta_{0}} \mid \operatorname{kex} \pi\right) \text {. }
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& +O_{1}\left(\begin{array}{ccc}
R & \beta_{0} & R^{\prime} \\
\pi \downarrow & & \pi^{\prime} \\
A & \overrightarrow{\beta_{1}} & A^{\prime} \\
\mu \downarrow & \downarrow^{\prime}
\end{array}\right)\left(\begin{array}{cc}
\mu_{0} & \alpha_{1} \\
F_{1} \\
\left.\delta_{0}\right\rangle_{-} & \swarrow \\
\delta_{1}
\end{array}\right)=V
\end{aligned}
$$

where $\alpha_{1}^{*}$ is the map induced by

$$
\begin{aligned}
& \alpha_{1}: \operatorname{Der}_{A}\left(F_{1}, B_{\Lambda^{\prime}}^{\prime} \otimes \operatorname{ker} \pi^{\prime}\right) \rightarrow \operatorname{Der}_{A}\left(F_{o}, B^{\prime} \otimes \operatorname{ker} \pi^{\prime}\right) . \\
& V=\alpha_{1}\left(\beta_{1}^{\otimes} 1_{A}{\underset{1}{1}}\right)\left(\sigma_{\pi^{\prime}}^{\prime}\left(\alpha_{2}^{\otimes} 1_{A} A^{\prime}\right)-\sigma_{\pi}^{\prime}\left(\alpha_{2}\right) \otimes_{R} 1_{R^{\prime}}\right)\left(\delta_{2}^{\prime} \underset{A^{\prime}}{\otimes} 1_{\text {ker } \pi^{\prime}}\right)
\end{aligned}
$$

Now

$$
\alpha_{1}\left(1_{\mathbb{F}_{1}} \otimes \beta_{1}\right)=\left(1_{\mathrm{F}_{0}} \otimes \beta_{1}\right)\left(\alpha_{1} \hat{\mathrm{~A}}^{\otimes 1_{\mathrm{A}^{\prime}}}\right)
$$

and
therefore

$$
\begin{aligned}
& -\sigma_{\pi^{\prime}}^{\prime}\left(\alpha_{1} \alpha_{2}^{\otimes} 1_{A} A^{\prime}\right)+\sigma_{\pi}^{\prime}\left(\alpha_{1} \alpha_{2}\right){\underset{R}{ }}_{\otimes}^{1_{R^{\prime}}} \\
& \left.+\left(\sigma_{\pi^{\prime}}^{\prime}\left(\alpha_{1}{\underset{A}{A}}_{\otimes 1_{A^{\prime}}}\right)-\sigma_{\pi^{\prime}}^{\prime}\left(\alpha_{1}\right) \underset{R}{\otimes} 1_{R^{\prime}}\right)\left(\sigma_{\Pi^{\prime}}^{\prime}\left(\alpha_{2}\right) \underset{R}{\otimes 1_{R^{\prime}}}\right)\right)\left(\delta_{2}^{\prime} \underset{A^{\prime}}{\otimes} 1_{\text {ker } \pi^{\prime}}\right) \\
& =-\mathrm{U} \text {, }
\end{aligned}
$$

thus proving

$$
d_{1}\left(O_{0}\right)=-d_{2}\left(O_{1}\right)
$$

Now, in the same way we find,

$$
\begin{aligned}
& \alpha_{1}\left(O_{1}\right)\left(\begin{array}{ccccc}
R & \longrightarrow & R^{\prime} & & R^{\prime \prime} \\
\pi & \beta_{0} & \downarrow^{\pi^{\prime}} & \beta_{0}^{\top} & \downarrow^{\pi^{\prime \prime}} \\
A & \cdots & \bar{\beta}_{1} & A^{\prime} & \\
\mu^{\prime} & \beta_{1}^{\prime} & A^{\prime \prime} \\
\downarrow & & \downarrow^{\prime \prime} & & \downarrow^{\prime \prime} \\
B & \overrightarrow{\beta_{2}} & B^{\prime} & \overrightarrow{\beta_{2}^{\prime}} & B^{\prime \prime}
\end{array}\right)\left(\begin{array}{ccc}
F_{0} & \alpha_{1} & F_{1} \\
\delta_{0} & & \delta_{1}
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =0 \text { 。 }
\end{aligned}
$$

## Bibliography

(An) André, M., Méthode Simpliciale en Algébre Homologique et Algébre Commutative, Springer Lecture Note nr. 32 (1967).
(Ar) Artin, M., Grothendiick Topologies, Department of Mathematics, Harvard University (1962).
(巴l) Ellingsrud, G., Sur le schema de Hillbert des variétés de codimension 2 à cõne de Cohen-Miacaulay (to appear in Annales Scientiques de l'Ecole Normale Sup.).
(Il) Illusie, $L_{0}$, Complexe Cotangent et Deformationes I \& II, Springer Lecture Notes no. 239 (1971) et no. 283 (1972).
(Kl.) Kleppe, Jan, Deformation of Graded Algebras, Preprint Series. Department of Math. University of Oslo, nr. 14 (1975).
(Ko1) Kodaira, K. and Spencer, D.C., On deformations of complex analytic structures I and II, Annals of Math., Vol 67 (68) pp. 328-466.
(Ko 2) Kodaira, K. and Spencer, D.C., A theorem of completeness of charactristic systems of complete continuous systems, Am.J. Mathe 1979 (81) p. 477.
(La1) Laudal, O.A., Sur la limite projective et la theorie de la dimension I et II. Seminaire C. Ehresmann, Paxis 1961.
(La 2) Laudal, O.A., Cohomologie locale. Applications Math.Scand. 12 (1963) pp. 147-162.
(La 3) Laudal, O.A., Sur la théorie des limites projectives et inductives, Am. Sci. l'Ecole Normale Sup. 82 (1965) pp. 241296.
(La.4) Laudal, O.A., Sections of functors and the problem of lifting algebraic structures. Preprint Series, Dept. of Math., University of Oslo, nr. 12 (1971).
(Ji) Lichtenbaum, S. and Schlessinger, M., The cotangent complex of a morphism. Trans. Amer. Math. Soc., Vol 128(1967)pp.41-70.
(Mu) Mumford, D., Iectures on curves on an algebraic surface, Annals of Math. Studies, No.59, Princeton University, 1966.
(Pi) Pinkham, H.C., Deformations of Cones with negative Grading, Journal of Algebra, Vol 30 (1-3), June 1974, pp. 92-102.
(Qu1) Quillen, D., Homotopical algebra. Lecture Notes in Mathematics. Springer, Berlin (1967).
(Qu2) Quillen, D., On the (co-) homology of commutative rings, Proceedings of Symposia in Pure Mathematics, Vol XVII (1970) pp. 65-87.
(Sch1) Schlessinger, M., Infinitesimal deformations of singularities Ph. D. Thesis, Harvard University, Cambridge, Mass. 1964.
(Sch 2) Schlessinger, M., Functors of Artin Rings. Transactions of the American Math. Soc., Vol 130 (1968) pp. 208-22.2.
(Sch 3) Schlessinger, M., On rigid singularities. Proc. of the Rice University Conferance 1972 (to appear).
(Sch 4.) Schlessinger, M., Rigidity of quotient singularities. Invent. Math. 14 (1971) pp. 17-26.
(Sv 1) Svenes, $\mathrm{I}_{\mathrm{o}}$, Coherent cohomology on flag manifolds and rigigity. Ph. D. Theses, M.I.T.,Cambridge, Mass. (1972).
(Sv2) Svanes, T., Arithmetic Normality for projective embeddings of flag manifolds. Math. Scand. 33 (1973) pp. 55-68.
(Sv 3) Svanes, T., Some Criteria for rigidity of noetherian Rings. Preprints.series 1973/74, no. 15. Aarhus University.
(Wa) Wahl, J.M., Equi singulax deformations of plane algebroid curves. Mronsaction Amer. Math. Soc., Vol 193 (1974) pp. 143-170.

