

FROBENIUS THEORY FOR POSITIVE MAPS
OF VON NEUMANN ALGEBRAS

by

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ABSTRACT

Frobenius theory about the cyclic structure of eigenvalues of irreducible non negative matrices is extended to the case of positive linear maps of von Neumann algebras. Semigroups of such maps and ergodic properties are also considered.

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1. Introduction

The spectral theory of positive maps has its origin in the classical work of Perron [1] and Frobenius [2], who considered the case of matrices with positive entries on finite dimensional vector spaces. For a compact exposition of Perron-Frobenius results see [3]. Let us distinguish two types of results in this theory. The first, due to Perron [1], is concerned with the existence and uniqueness of the maximal eigenvalue, the second, due to Frobenius [2], is concerned with the cyclic structure of the spectrum. Frobenius showed more particularly that a non negative irreducible matrix has always a simple eigenvalue r such that all other eigenvalues are contained in a circle of radius r around the origin. If the matrix is normalized such that $r = 1$ then the eigenvalues on the unit circle form a finite subgroup of the circle group which maps the system of all eigenvalues into itself.

In this paper we extend Frobenius results to the case of positive maps of von Neumann algebras. Let us first give some references to previous work. As the literature is quite extensive, especially concerning extensions of Perron's results, we shall mainly mention work related to Frobenius results (for additional references see [4]).

Frobenius type of results for compact operators on commutative C^* algebras and ordered vector spaces can be found in Krein and Rutman [5], who also extended Jentsch's work [6] on Perron type of results. For other extensions in the case of ordered vector spaces see e.g. [7] - [9].

Automorphisms of commutative C^* -algebras have been studied particularly in connection with ergodic theory, originating from classical work by Koopman [10], Carleman [11] and von Neumann [12], see [13].

Results of Frobenius type for groups of automorphisms in the general case of non commutative C^* -algebras have been obtained by Størmer [14].

For some particular spectral results which appeared in different contexts see the references in [14] and for recent related results see [15] - [17].

The extension of the entire Perron-Frobenius theory to the case of positive maps on finite-dimensional C^* -algebras has been obtained by Evans and Høegh-Krohn [4].

We shall now briefly discuss our results.

We consider a von Neumann algebra M and positive linear normalized maps ϕ of M into itself, satisfying the Kadison-Schwarz inequality $\phi(a^*a) \geq \phi(a)^*\phi(a)$ for any $a \in M$. Maps satisfying this inequality are well known (see e.g. [18] - [21]), in particular any 2-positive linear normalized map ϕ satisfies the inequality ([18], [21]). We recall that a map ϕ is called 2-positive if $\phi \otimes 1$ is positive on $M \otimes M_2$, where 1 is the unit matrix in the space M_2 of 2×2 matrices, so that in particular completely positive maps are 2-positive, hence satisfy the inequality. Such maps have found several applications recently, see e.g. [22] - [26].

Consider now a state invariant under ϕ and extend ϕ to the Hilbert space \mathcal{H} generated by applying M to the cyclic separating vector given by the state. Let ϕ be ergodic in the sense that no non-trivial projection is invariant under ϕ . Then we show that the set of eigenvalues on the unit circle for ϕ in M and for ϕ in \mathcal{H} is the same, it consists of simple eigenvalues ("roots") α which form a subgroup of the circle group acting by complex multiplication on the spectrum of ϕ as an operator in \mathcal{H} . The corresponding eigenvectors give unitary operators u_α and the map $\alpha \rightarrow u_\alpha$ is a

unitary multiplier representation of the group $\Gamma(\Phi)$ of roots. The restriction of Φ to the subalgebra M_Γ of M generated by the operators u_α is an ergodic automorphism and the restriction of the state to M_Γ is a trace. We give also more detailed results for the cases where $\Gamma(\Phi)$ is cyclic or finite.

We then extend (Th. 2.8-2.10) the considerations to the case of semigroups Φ_t , $t \geq 0$ obtaining Frobenius type of results for their infinitesimal generators. Ergodic properties are also discussed.

2. Dynamical Systems

Let M be a von Neumann algebra and ϕ a positive linear normalized map of M i.e. $\phi(M^+) \subseteq M^+$ where M^+ is the positive cone in M and $\phi(1) = 1$, such that ϕ satisfies the Schwarz inequality

$$\phi(a^*a) \geq \phi(a)^*\phi(a) \quad (2.1)$$

for any $a \in M$, which is the case (as remarked in Sect. 1) if ϕ is e.g. 2-positive.

Moreover, let ξ be a ϕ -invariant, $\xi \circ \phi = \xi$, cyclic and separating normal state on M . Then the triplet (M, ϕ, ξ) is called a dynamical system. Any $*$ -automorphism θ of M satisfies (2.1) so if θ is a $*$ -automorphism which leaves ξ invariant then (M, θ, ξ) is a dynamical system and we call this a closed dynamical system. By the GNS construction we may assume that ξ is a vector state

$$\xi(a) = (\Omega, a\Omega). \quad (2.2)$$

Let $\mathcal{H} = \overline{M\Omega}$ be the Hilbert space generated by M on the cyclic vector Ω . From (2.1) it follows that ϕ is a densely defined contraction (it is defined on $M\Omega$) hence it extends uniquely to a contraction on \mathcal{H} which we also denote by ϕ .

By the Tomita-Takesaki theory we have that the mapping $a\Omega \rightarrow a^*\Omega$ defined on $M\Omega$ extends uniquely to a closed antilinear map S of \mathcal{H} such that the modular operator of Tomita is given by $\Delta = S^*S$, where S^* is the adjoint of S . Moreover, if $S = J\Delta^{\frac{1}{2}}$ is the polar decomposition of S then J is an antiisometry of \mathcal{H} such that $a \rightarrow JaJ$ is an anti isomorphism of M with its commutant M' . Moreover $J^2 = 1$ and $J\Omega = \Omega$.

Since ϕ is a positive map of M it must commute with the $*$ -map i.e. $\phi(a^*) = \phi(a)^*$ for any $a \in M$. Since $\xi \circ \phi = \xi$ we get

*) [27]

that ϕ leaves the domain of S in \mathcal{H} invariant and

$$\phi S = S \phi . \quad (2.3)$$

From this it follows that ϕ leaves the domain of Δ invariant and

$$\phi \Delta = \Delta \phi \quad (2.4)$$

as well as

$$\phi J = J \phi \quad (2.5)$$

since the polar decomposition of S is unique.

A consequence of (2.4) is that ϕ commutes with Tomita's modular automorphism σ_t on M .

Since ξ is separating we have that $M'\Omega$ is dense in \mathcal{H} , where M' is the commutant of M .

Let us now recall a construction from Tomita-Takesaki theory. To any $x \in \mathcal{H}$ we may associate a densely defined operator \hat{x} on \mathcal{H} with dense domain $M'\Omega$ defined for any $b' \in M'$ as

$$\hat{x}b'\Omega = b'x . \quad (2.6)$$

Let now $c \in M$ then with a' and b' in M' we have

$$(a'\Omega, b'c\Omega) = (a'c^*\Omega, b'\Omega) . \quad (2.7)$$

Since $M\Omega$ is dense in \mathcal{H} we therefore get by continuity for any $x \in D(S)$ that

$$(a'\Omega, b'x) = (a'Sx, b'\Omega) . \quad (2.8)$$

By (2.6) this gives us however that

$$(a'\Omega, \hat{x}b'\Omega) = (\widehat{Sx}a'\Omega, b'\Omega) . \quad (2.9)$$

Using now that $M'\Omega$ is dense in \mathcal{H} we get that, for any $x \in D(S) = D(\Delta^{\frac{1}{2}})$, \hat{x} has a densely defined adjoint the restriction of which to $M'\Omega$ is equal to \widehat{Sx} . Therefore both \hat{x} and \widehat{Sx} are

closable with the closure of \hat{x} equal to \widehat{Sx}^* and the closure of \widehat{Sx} equal to \hat{x}^* . For this reason we shall, for any $x \in D(\Delta^{\frac{1}{2}})$, let \hat{x} and \widehat{Sx} also denote the corresponding closed operators. In particular we have for any $x \in D(\Delta^{\frac{1}{2}})$ that

$$\hat{x}^* \hat{x} = \widehat{Sx} \hat{x} \quad (2.10)$$

is a positive self-adjoint operator affiliated with the von Neumann algebra M .

Let now $A \geq 0$ be a bounded positive symmetric operator in M . Then $\Phi(A) = a^2$ for some symmetric operator a in M and we have for $y = b'\Omega \in M'\Omega$

$$\begin{aligned} 0 \leq (y, \Phi(A)y) &= \xi(b'^* a^2 b') = \xi(a b'^* b' a) \leq \|b'^* b'\| \xi(a^2) = \\ &= \|b'\|^2 \xi(\Phi(A)). \end{aligned}$$

By the invariance of ξ under Φ we then get

$$0 \leq (y, \Phi(A)y) \leq \|b'\|^2 \xi(A) \quad (2.11)$$

and in particular

$$(y, \Phi(a^* a)y) \leq \|b'\|^2 \|a\Omega\|^2 \quad (2.12)$$

for any $a \in M$. Using now (2.1) we have from (2.12)

$$(y, \Phi(a)^* \Phi(a)y) \leq (y, \Phi(a^* a)y) \leq \|b'\|^2 \|a\Omega\|^2. \quad (2.13)$$

Thus for fixed $y = b'\Omega \in M'\Omega$ both sides of the inequality

$$(y, \Phi(a)^* \Phi(a)y) \leq (y, \Phi(a^* a)y) \quad (2.14)$$

define positive bilinear forms on \mathcal{H} by the identification $a \longleftrightarrow a\Omega$ of M with a dense subspace of \mathcal{H} .

Let now $A \geq 0$ be a positive self-adjoint (not necessarily bounded) operator affiliated with M , such that $\Omega \in D(A^{\frac{1}{2}})$. Let $f \in C(\mathbb{R})$ then we have from (2.11) that

$$0 \leq (y, \hat{\Phi}(f(A))y) \leq \|b'\|^2 \xi(f(A)) = \|b'\|^2 (\Omega, f(A)\Omega) \quad (2.15)$$

where $y = b'\Omega \in M'\Omega$. This shows that $f \rightarrow (y, \hat{\Phi}(f(A))y)$ is a continuous positive linear functional on $C(R)$ i.e. an integral on $C(R)$ with respect to a bounded measure with finite first moment, and denoting its first moment by $(y, \hat{\Phi}(A)y)$ we then have

$$0 \leq (y, \hat{\Phi}(A)y) \leq \|b'\|^2 \|A^{\frac{1}{2}}\Omega\|^2. \quad (2.16)$$

The first moment $(y, \hat{\Phi}(A)y)$ is then just the extension by positivity in A of the corresponding function defined on M^+ , the positive cone in M .

Since $\hat{a}\Omega = a$ for $a \in M$ we have from (2.13) that if $x \in M\Omega$ then for any $y = b'\Omega \in M'\Omega$

$$(y, \hat{\Phi}(x) * \hat{\Phi}(x)y) \leq (y, \hat{\Phi}(\hat{x} * \hat{x})y) \leq \|b'\|^2 \|x\|^2 \quad (2.17)$$

where $\hat{\Phi}(x) \equiv \widehat{\Phi(x)}$.

If $x \in D(\Delta^{\frac{1}{2}})$ we have that $\Omega \in D((\hat{x} * \hat{x})^{\frac{1}{2}})$, in fact

$$\|(\hat{x} * \hat{x})^{\frac{1}{2}}\Omega\| = \|x\|. \quad (2.18)$$

Hence the first moment $\mu_y(x, x)$ of the bounded positive measure corresponding to the integral

$$f \rightarrow (y, \hat{\Phi}(f(\hat{x} * \hat{x}))y) \quad (2.19)$$

exists. If $x = a\Omega$, $a \in M$ then obviously $\mu_y(x, x) = (y, \hat{\Phi}(a * a)y)$. So that $\mu_y(x, x)$ is an extension of the bilinear form $(y, \hat{\Phi}(a * a)y)$ defined on M to $D(\Delta^{\frac{1}{2}})$, where M is imbedded into $D(\Delta^{\frac{1}{2}})$ by $a \rightarrow a\Omega$. (2.17) shows that this first moment $\mu_y(x, x)$ is continuous in x in the strong topology in \mathcal{H} hence extends uniquely to a bounded bilinear form on \mathcal{H} . Furthermore, also by (2.17), we have

$$(y, \hat{\Phi}(x) * \hat{\Phi}(x)y) = \|\hat{\Phi}(x)y\|^2 = \|b' \hat{\Phi}(x)\|^2 \leq \mu_y(x, x) \leq \|b'\|^2 \|x\|^2. \quad (2.20)$$

We have thus proved the following theorem.

Theorem 2.1

Let (M, ϕ, ξ) be a dynamical system. Let Δ be the modular operator given by ξ . Then for any $x \in D(\Delta^{\frac{1}{2}})$ we have that \hat{x} defined on $M'\Omega$ by $\hat{x}b'\Omega = b'x$ extends uniquely to a closed operator affiliated with M with densely defined adjoint $\hat{x}^* = \widehat{Sx}$.

The extension of ϕ to \mathcal{H} is a contraction which commutes with the modular automorphism, thus leaving $D(\Delta^{\frac{1}{2}})$ invariant. For a fixed $y = b'\Omega \in M'\Omega$ the bilinear form $(y, \hat{\phi}(x)^* \hat{\phi}(x)y) = \|\hat{\phi}(x)y\|^2$ is, as a function of $x \in D(\Delta^{\frac{1}{2}})$, strongly continuous on \mathcal{H} and extends to a bounded form on \mathcal{H} . For $y = b'\Omega \in M'\Omega$ let $\mu_y(x, x)$ be the first moment of the bounded positive measure given by the integral $f \rightarrow (y, \phi(f(\hat{x}^* \cdot \hat{x})y)$. Then for any $x \in D(\Delta^{\frac{1}{2}})$ the first moment $\mu_y(x, x)$ is bounded and defines a positive bilinear form which is bounded on \mathcal{H} hence extends to a positive bilinear form on \mathcal{H} .

Moreover we have the following inequalities for $x \in D(\Delta^{\frac{1}{2}})$ and $y = b'\Omega \in M'\Omega$:

$$\|\hat{\phi}(x)y\|^2 \leq \mu_y(x, x) \leq \|b'\|^2 \|x\|^2.$$

For fixed $x \in D(\Delta^{\frac{1}{2}})$ we see that $\mu_y(x, x)$ is a positive quadratic form in y hence we write for $y \in M'\Omega$

$$\mu^x(y, y) = \mu_y(x, x). \tag{2.21}$$

It follows then from the definition that $\mu^x(y, b'z) = \mu^x(b'^*y, z)$ for any $b' \in M'$ and $y, z \in M'\Omega$. This gives that if $y = b'\Omega$ then

$$\mu^x(y, y) = w_x(b'^*b') \tag{2.22}$$

where $w_x(b') = \mu^x(1, y)$ is a positive bounded function on M' i.e. a state (not normalized) on M' , in fact we have by the invariance of ξ under ϕ that

$$w_x(1) = \|x\|^2. \quad (2.23)$$

Now

$$\|\hat{\phi}(x)y\|^2 = \|b'\phi(x)\|^2 = (\phi(x), b'^*b'\phi(x)) \quad (2.24)$$

where $y = b'\Omega$, $b' \in M'$. Hence by the inequality of theorem 2.1 we have that for any $b' \in M'$

$$\omega_x(b'^*b') \equiv w_x(b'^*b') - (\phi(x), b'^*b'\phi(x)) \geq 0. \quad (2.25)$$

Hence ω_x defines a state on M' which obviously is dominated by w_x so that $\omega_x(1) \leq \|x\|^2$, and more precisely we have

$$\omega_x(1) = \|x\|^2 - \|\phi(x)\|^2. \quad (2.26)$$

Let us now assume that there is an eigenvalue α with $|\alpha| = 1$ of the mapping ϕ of \mathcal{H} , i.e. there is an $x_\alpha \in \mathcal{H}$ such that

$$\phi(x_\alpha) = \alpha x_\alpha. \quad (2.27)$$

Since ϕ commutes with the modular automorphism we have that the eigenspace E_α of ϕ corresponding to the eigenvalue α is invariant under Δ , and since Δ is self-adjoint its restriction to the invariant subspace E_α is also self-adjoint. From this it follows that $D(\Delta^{\frac{1}{2}}) \cap E_\alpha$ is dense in E_α , since $D(\Delta^{\frac{1}{2}}) \cap E_\alpha$ is the domain of the restriction of $\Delta^{\frac{1}{2}}$ to E_α . Hence we may take the eigenvector x_α in (2.27) to be in $D(\Delta^{\frac{1}{2}})$. With this x_α we get from (2.26) that $\omega_{x_\alpha}(1) = 0$ and since ω_{x_α} is a state, we have that $\omega_{x_\alpha} \equiv 0$. But this is to say that

$$\omega_{x_\alpha}(b'^*b') = \mu_y(x_\alpha, x_\alpha) - (y, \hat{\phi}(x_\alpha)^*\hat{\phi}(x_\alpha)y) = 0 \quad (2.28)$$

with $y = b'\Omega$, $b' \in M'$. By the inequality of theorem 2.1 we have that

$$\mu_y(x, x) - (y, \hat{\phi}(x)^*\hat{\phi}(x)y) \geq 0 \quad (2.29)$$

for all $x \in D(\Delta^{\frac{1}{2}})$, i.e. (2.29) is a positive form on $D(\Delta^{\frac{1}{2}})$ which is zero for $x = x_\alpha$. By Schwarz inequality we then have for any $x \in D(\Delta^{\frac{1}{2}})$ that

$$\mu_y(x, x_\alpha) - (y, \hat{\Phi}(x) * \hat{\Phi}(x_\alpha)y) = 0 \quad (2.30)$$

i.e.

$$\mu_y(x, x_\alpha) = (y, \hat{\Phi}(x) * \alpha \hat{x}_\alpha y). \quad (2.31)$$

From the definition of μ_y and (2.28) we get that the first moment of the integral

$$f \rightarrow (y, \hat{\Phi}(f(\hat{x}_\alpha * \hat{x}_\alpha))y) \quad (2.32)$$

is equal to $(y, \hat{\Phi}(x_\alpha) * \hat{\Phi}(x_\alpha)y)$. But by (2.24) we have

$$(y, \hat{\Phi}(x_\alpha) * \hat{\Phi}(x_\alpha)y) = \|b' \hat{\Phi}(x_\alpha)\|^2 = \|b' x_\alpha\|^2 = (y, \hat{x}_\alpha * \hat{x}_\alpha y) \quad (2.33)$$

where in the second equality we have used (2.27).

Consider now the self-adjoint operator $A = \hat{x}_\alpha * \hat{x}_\alpha$ with spectral resolution

$$A = \int_0^\infty \lambda dE_\lambda.$$

We have

$$\mu_y(x_\alpha, x_\alpha) = \int_0^\infty \lambda d(y, \hat{\Phi}(E_\lambda)y). \quad (2.34)$$

By (2.33) we then get

$$\int_0^\infty \lambda d(y, \hat{\Phi}(E_\lambda)y) = \int_0^\infty \lambda d(y, E_\lambda y). \quad (2.35)$$

This implies however that

$$\int_0^N \lambda d\hat{\Phi}(E_\lambda) \leq A \quad (2.36)$$

and

$$\lim_{N \rightarrow \infty} \int_0^N \lambda d\hat{\Phi}(E_\lambda) = A \quad (2.37)$$

in the sense of positive bilinear forms on $M'\Omega$, and hence by uniform boundedness from (2.36) as bilinear form on $D(A^{\frac{1}{2}})$.

Let us call the eigenvalues of Φ on \mathcal{H} on the unit circle the roots of Φ and denote the set of all roots by $\Gamma(\Phi)$. We shall call the corresponding eigenvectors and eigenspaces root vectors and root spaces respectively.

Now (2.37) is a consequence of (2.31) in the special case where $x = x_\alpha$. It follows more generally from (2.31) that if $z \in M'\Omega \subset D(\hat{x}_\alpha)$ then

$$\Phi(\hat{x}_\alpha z) = \alpha \hat{x}_\alpha \Phi(z). \quad (2.38)$$

Remark that since $\Phi : M\Omega \rightarrow M\Omega$ and commutes with J it also maps $M'\Omega$ into $M'\Omega$. Φ is therefore a bounded operator that intertwines between the closed operators \hat{x}_α and $\alpha \hat{x}_\alpha$. This follows from (2.38) since $M'\Omega$ is dense in the graph norm of the closed operator \hat{x}_α .

Since Φ commutes with S and S is antilinear, we get that Sx_α is a root vector for Φ corresponding to the root $\bar{\alpha}$. Especially we get that $\Gamma(\Phi)$ is invariant under complex conjugation

$$\overline{\Gamma(\Phi)} = \Gamma(\Phi). \quad (2.39)$$

But we get also that, for $z \in M'\Omega$,

$$\Phi(\hat{x}_\alpha^* z) = \bar{\alpha} \hat{x}_\alpha^* \Phi(z) \quad (2.40)$$

since $Sx_\alpha = \hat{x}_\alpha^*$. Hence we have that

$$\Phi \hat{x}_\alpha = \alpha \hat{x}_\alpha \Phi \quad \text{and} \quad \Phi \hat{x}_\alpha^* = (\alpha \hat{x}_\alpha)^* \Phi \quad (2.41)$$

in the sense that the contraction Φ is an intertwining operator for the two pairs $\{\hat{x}_\alpha, \alpha \hat{x}_\alpha\}$ and $\{\hat{x}_\alpha^*, (\alpha \hat{x}_\alpha)^*\}$ of closed operators. From this it follows that Φ intertwines the self-adjoint operator

$A = \hat{x}_\alpha^* \hat{x}_\alpha = (\alpha \hat{x}_\alpha)^* \hat{x}_\alpha$, since $\bar{\alpha}\alpha = 1$, i.e.

$$A\phi = \phi A. \quad (2.42)$$

Hence ϕ is a bounded operator on \mathcal{H} commuting with A , and it is then well known that this implies that ϕ commutes with its spectral projections

$$\phi E_\lambda = E_\lambda \phi. \quad (2.43)$$

Applying now both sides of (2.43) to Ω we get

$$\phi(E_\lambda) = E_\lambda \quad (2.44)$$

for the action of ϕ in M .

We shall say that ϕ is ergodic if there is no projection in M different from 0 or 1 which is invariant under ϕ . If ϕ is ergodic, we shall also say that the dynamical system (M, ϕ, ξ) is ergodic. Let now $\alpha \in \Gamma(\phi)$ and x_α a normalized root vector, i.e. $\|x_\alpha\| = 1$, and let us assume that the dynamical system (M, ϕ, ξ) is ergodic. In this case from (2.44) we have that $A = \hat{x}_\alpha^* x_\alpha = 1$, and if we consider the root vector Sx_α for the root $\bar{\alpha}$ we get in the same manner $\widehat{Sx}_\alpha^* \widehat{Sx}_\alpha = \hat{x}_\alpha^* \hat{x}_\alpha = 1$ so that

$$\hat{x}_\alpha^* \hat{x}_\alpha = \hat{x}_\alpha \hat{x}_\alpha^* = 1 \quad (2.45)$$

for any normalized root vector x_α , i.e. \hat{x}_α is a unitary element of M . Hence the eigenvalues on the unit circle are the same for ϕ in \mathcal{H} as for ϕ in M . Using now that \hat{x}_α is unitary, (2.41) may be written as

$$\hat{x}_\alpha^* \phi \hat{x}_\alpha = \alpha \phi \quad (2.46)$$

so that for $\alpha \in \Gamma(\phi)$ we have that ϕ and $\alpha \phi$ are unitarily equivalent. This gives us that $\Gamma(\phi)$ is a subgroup of the unit circle

(looked upon as a group, the circle group) and that the group $\Gamma(\Phi)$ acts by complex multiplication on $\text{Sp}(\Phi)$, the spectrum of Φ . Moreover if α and β are two roots, then $\hat{x}_\alpha^* x_\beta$ is a root vector corresponding to the root $\bar{\alpha}\beta$. By the ergodicity we then have that if x_α and x_α' are two normalized root vectors for the root α , then $\hat{x}_\alpha^* \widehat{x_\alpha'} = 1$. We also observe that if $\alpha, \beta \in \Gamma(\Phi)$ with root operators u_α and u_β , then both $u_\alpha u_\beta$ and $u_\beta u_\alpha$ are root operators for the root $\alpha\beta$ and u_α^* is a root operator of $\bar{\alpha}$. This gives us by the ergodicity that $u_\alpha u_\beta = \gamma(\alpha, \beta) u_\beta u_\alpha$, where γ is a multiplier, so that $\alpha \rightarrow u_\alpha$ is a multiplier unitary representation of the group $\Gamma(\Phi)$, with multiplier $\gamma(\alpha, \beta)$. If $\Gamma(\Phi)$ is cyclic, i.e. has a simple generator, then the multiplier is trivial, hence $\alpha \rightarrow u_\alpha$ is a unitary representation of the abelian group $\Gamma(\Phi)$, hence in this case the algebra generated by the root operators is abelian. We have thus proven the following theorem.

Theorem 2.2.

Let (M, Φ, ξ) be an ergodic dynamical system, where ξ is a cyclic separating vector state for M invariant under Φ . Let \mathcal{H} be the corresponding Hilbert space. Then the discrete eigenvalues on the unit circle for Φ as an operator in \mathcal{H} coincide with the discrete eigenvalues on the unit circle for Φ in M . Let $\Gamma(\Phi)$ be the set of all roots of Φ , i.e. the discrete eigenvalues on the unit circle. $\Gamma(\Phi)$ is a subgroup of the circle group which acts by complex multiplication on the spectrum $\text{Sp}(\Phi)$ of Φ in \mathcal{H} . If $\alpha \in \Gamma(\Phi)$ then α is a simple eigenvalue of Φ and the corresponding root operator u_α in M is proportional to a unitary operator in M and $x_\alpha = u_\alpha \Omega$ is the corresponding root vector in \mathcal{H} , where Ω is the vector corresponding to the vector state ξ . The

invariance of $\text{Sp}(\Phi)$ under multiplication by the root α is given by the unitary equivalence

$$u_\alpha^* \Phi u_\alpha = \alpha \Phi,$$

if the root operator u_α is normalized so that it is unitary. If α and β are in $\Gamma(\Phi)$ with root operators u_α and u_β , then $u_\alpha u_\beta$ is a root operator for the root $\alpha\beta$ and u_α^* is a root operator for $\bar{\alpha}$. Hence if we select for each $\alpha \in \Gamma(\Phi)$ a unitary operator u_α then $u_\alpha u_\beta = \gamma(\alpha, \beta) u_\beta u_\alpha$, where $\gamma(\alpha, \beta)$ is a multiplier for the group $\Gamma(\Phi)$ and $\alpha \rightarrow u_\alpha$ is a unitary multiplier representation of the group $\Gamma(\Phi)$ with multiplier $\gamma(\alpha, \beta)$. If $\Gamma(\Phi)$ is cyclic, i.e. has a single generator, then $\alpha \rightarrow u_\alpha$ is a unitary representation of the abelian group $\Gamma(\Phi)$ and therefore the algebra generated by the root operators is abelian. \blacksquare

Remark: Results of this type were proven by Frobenius [2] for commutative, finite-dimensional von Neumann algebras. For the commutative infinite dimensional case with Φ compact, results were given by Krein and Rutman [5] and for the commutative infinite dimensional case with Φ an automorphism by Koopman [10] and von Neumann [12]. In the infinite dimensional non-commutative case with Φ an automorphism results of this type were obtained by Størmer [14] and in the finite dimensional non-commutative case with general Φ by Evans and Høegh-Krohn [4].

If Φ is compact in \mathcal{H} , $\Gamma(\Phi)$ must be a finite subgroup of the unit circle and since any such group has the form

$$\Gamma_m = \{e^{\frac{2\pi i k}{m}}, k=0,1,\dots,m-1\} \quad (2.47)$$

we have that $\Gamma(\Phi) = \Gamma_m$ where $m = |\Gamma(\Phi)|$ is the order of $\Gamma(\Phi)$.

We shall say that Φ is primitive if $|\Gamma(\Phi)| = 1$ i.e. $\Gamma(\Phi) = \{1\}$ and imprimitive if not, and following Frobenius we call $|\Gamma(\Phi)|$ the imprimitivity of Φ . Especially we have that if Φ is compact in \mathcal{H} then it has finite imprimitivity. If Φ is of trace class in \mathcal{H} then the Fredholm determinant $|1 - z\Phi|$ of $1 - z\Phi$ exists and defines an entire function

$$f_{\Phi}(z) = |1 - z\Phi| \quad (2.48)$$

such that $f_{\Phi}(z_0) = 0$ if and only if z_0^{-1} is an eigenvalue for Φ . Especially we get that the set of zeros of f on the unit circle is $\Gamma(\Phi)$. Recalling now that for $\alpha \in \Gamma(\Phi)$

$$u_{\alpha}^* \Phi u_{\alpha} = \alpha \Phi \quad (2.49)$$

by the unitary equivalence of $\alpha \Phi$ and Φ , we get then

$$f_{\Phi}(\alpha z) = f_{\Phi}(z) \quad (2.50)$$

because the Fredholm determinant is a unitary invariant. Since α in (2.50) is any m -th root of the unit and f is entire, we have that there exists an entire function $g(z)$ such that $f_{\Phi}(z) = g(z^m)$. Let us also remark that since $\Gamma(\Phi) = \Gamma_m$ is cyclic, we have that the algebra generated by the root operators is commutative. Let now $\gamma = e^{\frac{2\pi i}{m}}$ and u' be a root operator corresponding to γ then $u'^m = \bar{c} \cdot 1$ where $|\bar{c}| = 1$. Let now $u = c^{1/m} u'$ then $u^m = 1$. Since u is unitary and $u^m = 1$ we have the spectral decomposition

$$u = \sum_{k=0}^{m-1} \gamma^k P_k \quad (2.51)$$

where P_k are the spectral projections for u .

Since $\Phi(u) = \gamma u$ we see that

$$\Phi(P_k) = P_{k-1} \quad \text{and} \quad \Phi(P_0) = P_{m-1}, \quad k = 1, \dots, m-1. \quad (2.52)$$

Especially we have that

$$\phi^m(P_k) = P_k \tag{2.53}$$

so that ϕ^m is not ergodic. It is easy to see that the restriction of ϕ^m to the algebra $M_k = P_k M P_k$ is ergodic and in fact primitive. These results depend obviously only on the fact that $\Gamma(\phi)$ is of finite order. We have thus the following theorem

Theorem 2.3

Let (M, ϕ, ξ) be as in theorem 2.2. Then if ϕ has finite imprimitivity we have $\Gamma(\phi) = \Gamma_m$, where Γ_m is the group of m -th roots of the unit. Let $\gamma = e^{\frac{2\pi i}{m}}$ then a root operator u corresponding to $\bar{\gamma}$ may be normalized so that $u^m = 1$. For this u we have that $u = \sum_{k=0}^{m-1} \gamma^k P_k$ is the spectral resolution of the unitary operator u . Hence $\{P_k\}$ is a resolution of the identity in M and the algebra generated by the root operators is the abelian algebra generated by $\{P_k\}$. Moreover $\phi(P_k) = P_{k-1}$ and $\phi(P_0) = P_{m-1}$. Especially $\phi^m(P_k) = P_k$, so that ϕ^m is not ergodic. However the restriction of ϕ^m to the algebra $M_k = P_k M P_k$ is ergodic and primitive. In fact $|\Gamma(\phi)| = m$ if and only if ϕ^m is not ergodic. If ϕ is compact, then ϕ has finite imprimitivity. If in addition ϕ is of trace class in $B(\mathcal{H})$, then there is an entire function $g(z)$ such that

$$|1-z\phi| = g(z^m)$$

where $|1-z\phi|$ is the Fredholm determinant of ϕ . ■

Let now $\Gamma(\phi)$ be cyclic but not finite. Then for any root $\gamma \in \Gamma(\phi)$ we have that $\gamma/2\pi$ is irrational and that γ generates $\Gamma(\phi)$, i.e.

$$\Gamma(\phi) = \{\gamma^n; n=0, \pm 1, \dots\}. \tag{2.54}$$

Let now u be a root operator corresponding to $\bar{\gamma}$, normalized so that u is unitary. A root operator corresponding to γ^n is then given by u^{-n} . Let ν be the spectral measure on the unit circle for the unitary operator u . Since obviously ϕ restricted to the subalgebra generated by u is an automorphism, we have that ϕ induces a transformation of the spectrum of u , and since $\phi(u^n) = \gamma^n u^n$ it follows that this transformation coincides with the restriction to the spectrum of u of the transformation $z \rightarrow \gamma z$. Hence if

$$u = \int_{|z|=1} z dE_z \quad (2.55)$$

is the spectral resolution of u , we must have that

$$\phi(E_z) = E_{\gamma z} \quad (2.56)$$

for ν -almost all z in the unit circle. Since there are no other root operators than the u^n , $n = 0, \pm 1, \dots$, it follows that ν is ergodic with respect to the transformation $z \rightarrow \gamma z$ of the unit circle. That ν is invariant under this transformation follows from $\xi = \xi \circ \phi$ and $\phi(u^n) = \gamma^n u^n$ for all $n \in \mathbb{Z}$. Hence we have proved the following theorem.

Theorem 2.4

Let (M, ϕ, ξ) be an ergodic dynamical system, such that $\Gamma(\phi)$ is cyclic but not finite. Then for any $\gamma \in \Gamma(\phi)$ we have that $\gamma/2\pi$ is irrational and that γ generates $\Gamma(\phi)$, i.e. $\Gamma(\phi) = \{\gamma^n; n = 0, \pm 1, \pm 2, \dots\}$. Let now u be the root operator corresponding to $\bar{\gamma}$ normalized so that u is unitary.

Let ν be the spectral measure on the unit circle for the unitary operator u corresponding to the state ξ , i.e.

$\xi(u^n) = \int_{|z|=1} z^n d\nu(z)$, and let $u = \int_{|z|=1} z dE_z$ be the spectral resolution of u . Then the projection valued measure dE_z is absolutely continuous with respect to ν . ν is an invariant ergodic measure with respect to the transformation $z = \gamma z$ of the unit circle and

$$\phi(E_z) = E_{\gamma z}$$

for ν -almost all z on the unit circle. ■

Let now α and β be two roots of the ergodic dynamical system (M, ϕ, ξ) with corresponding root operators u_α and u_β . Since $u_\alpha \cdot u_\beta$ then is a root operator for the root $\alpha \cdot \beta$ we have

$$\phi(u_\alpha u_\beta) = \alpha\beta u_\alpha u_\beta = \phi(u_\alpha) \cdot \phi(u_\beta). \quad (2.57)$$

Hence if M_Γ is the strongly closed subalgebra of M generated by the root operators u_α , $\alpha \in \Gamma \equiv \Gamma(\phi)$, then ϕ maps M_Γ and the restriction of ϕ to M_Γ is an automorphism. Let $\mathcal{H}_\Gamma = \overline{M_\Gamma \Omega}$ then \mathcal{H}_Γ is a ϕ invariant subspace of \mathcal{H} and the restriction of ϕ to \mathcal{H}_Γ is obviously unitary with discrete spectrum equal to Γ , and $u_\alpha \Omega$, $\alpha \in \Gamma$ is a complete set of orthogonal eigenvectors for ϕ in \mathcal{H}_Γ . Hence Ω is the only invariant eigenvector and from this we also get that the restriction of ϕ to M_Γ is ergodic. From the orthogonality of Ω and $u_\alpha \Omega$ for $\alpha \neq 1$ we have that $\xi(u_\alpha) = 0$ for $\alpha \neq 1$. But then $\xi(u_\alpha u_\beta) = \xi(u_\beta u_\alpha) = 0$ for $\alpha \neq \bar{\beta}$ and if $\beta = \bar{\alpha}$ then $u_\beta = c u_\alpha^*$ where c is an element in the unit circle, and since u_α is unitary we have that if $\beta = \bar{\alpha}$ then $u_\alpha u_\beta = u_\beta u_\alpha$ so that $\xi(u_\alpha u_\beta) = \xi(u_\beta u_\alpha)$ in any case. This shows that for a and b in M_Γ then $\xi(ab) = \xi(ba)$ i.e. the restriction of ξ to M_Γ is a trace. That the restriction of an ergodic state to the root algebra M_Γ is a trace was observed by

Størmer [14] in the case where Φ is an automorphism. We have thus proven the following theorem.

Theorem 2.5

Let (M, Φ, ξ) be an ergodic dynamical system with root system Γ . Let M_Γ be the root algebra, i.e. the strongly closed subalgebra of M generated by the root operators and let Φ_Γ be the restriction of Φ to M_Γ . Then Φ_Γ is an automorphism of M_Γ and $(M_\Gamma, \Phi_\Gamma, \xi_\Gamma)$, where ξ_Γ is the restriction of ξ to M_Γ , is an ergodic dynamical system. Moreover ξ_Γ is a trace on M_Γ . ■

One could now ask if it is so that M_Γ is always commutative for an ergodic dynamical system. The following example shows that this is not the case.

Example 2.6

Let $\mathcal{H} = L_2(\mathbb{R})$ and set $(V(x)f)(y) = f(y-x)$ and $(U(x)f)(y) = e^{ixy} f(y)$. Then V and U are both strongly continuous unitary representations of the abelian group \mathbb{R} on $L_2(\mathbb{R})$. Moreover

$$U(x)V(y) = e^{ixy} V(y)U(x).$$

Let $\lambda > 0$ and u and m in \mathbb{Z} then

$$U(\lambda n)V(\lambda m) = e^{i\lambda^2 nm} V(\lambda m)U(\lambda n).$$

Let M be the strongly closed subalgebra of $B(\mathcal{H})$ generated by $U(\lambda n)$ and $V(\lambda m)$ for n and m in \mathbb{Z} . Then M is noncommutative if and only if λ^2 is not an integral multiple of 2π .

Define a state ξ on M by $\xi(U(\lambda n)) = \xi(V(\lambda n)) = 0$ for $n \neq 0$ and $\xi(1) = 1$. Let now α and β be two real numbers and set $W = U(\alpha)V(\beta)$. Then

$$W^* U(\lambda n) W = V^*(\beta) U(\lambda n) V(\beta) = e^{i\lambda\beta n} U(\lambda n)$$

and

$$W^* V(\lambda n) W = V^*(\beta) U^*(\alpha) V(\lambda n) U(\alpha) V(\beta) = e^{i\lambda\alpha n} V(\lambda n).$$

Set now for $a \in M$, $\theta(a) = W^* a W$, then θ is an automorphism of M and (M, θ, ξ) is a dynamical system. Moreover it follows from the above equations that if α, β and $2\pi/\lambda$ are independent over \mathbb{Z} (the ring of integers) then (M, θ, ξ) is an ergodic dynamical system. We have from the equation above that the root system $\Gamma = \Gamma(\theta)$ is given by

$$\Gamma = \{e^{i\lambda(\alpha m + \beta n)}; (m, n) \in \mathbb{Z} \times \mathbb{Z}\}$$

and a root operator corresponding to $e^{i\lambda(\alpha m + \beta n)}$ is given by $U(\lambda m) V(\lambda n)$. M is noncommutative if λ^2 is not an integral multiple of 2π , and $M = M_\Gamma$.

If ϕ is primitive, then 1 is the only eigenvalue on the unit circle and it is also a simple eigenvalue. This gives us that ϕ^n converges weakly in \mathcal{H} to the projection with range the subspace generated by 1 . Hence for any a and b in M we have

$$\lim_{n \rightarrow \infty} \xi(a \phi^n(b)) = \xi(a) \xi(b) \quad (2.58)$$

i.e. the dynamical system is strongly mixing. Conversely strong mixing implies that 1 is the only eigenvalue of ϕ , as seen by taking $a = b$ to be an eigenvector in (2.58). Moreover we observe that if ϕ is ergodic but not necessarily primitive, then we still have, 1 being a simple eigenvalue, that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi(a \phi^k(b)) = \xi(a) \xi(b) \quad (2.59)$$

i.e. that the mean ergodic theorem holds.

We have thus proven the following theorem.

Theorem 2.7

If the dynamical system (M, Φ, ξ) is ergodic, then the mean ergodic theorem (2.59) holds. Moreover Φ is primitive, i.e. $\Gamma(\Phi) = \{1\}$, if and only if the dynamical system (M, Φ, ξ) is strongly mixing i.e. (2.58) holds for arbitrary a and b in M . ■

We shall now consider the case of semigroups of positive maps, instead of the iterates of a single positive map Φ .

Let M be a von Neumann algebra and Φ_t , $t \geq 0$ a semigroup of positive normalized maps of M i.e. $\Phi_0 = 1$, $\Phi_t \circ \Phi_s = \Phi_{t+s}$, $\Phi_t(M^+) \subseteq M^+$ and $\Phi_t(1) = 1$ such that the Φ_t satisfy the Schwarz inequality

$$\Phi_t(a^* a) \geq \Phi_t(a)^* \Phi_t(a) \quad (2.60)$$

for any $a \in M$ and all t . Moreover if ξ is a cyclic and separating normal state on M such that $\xi(a \Phi_t(b))$ is measurable as a function of t , and ξ is invariant under Φ_t i.e. $\xi \circ \Phi_t = \xi$, we say that (M, Φ_t, ξ) is a dynamical system with continuous time or a dynamical flow. We say that the dynamical flow is ergodic iff $\Phi_t(a) = a$ for all t implies that $a = \lambda 1$. As for the discrete dynamical systems (M, Φ, ξ) considered before, (2.58) implies that Φ_t extends to a measurable, hence strongly continuous, contraction semigroup on \mathcal{H} , where \mathcal{H} is the Hilbert space obtained by the GNS construction from the state ξ . We denote the continuous extension to \mathcal{H} also by Φ_t , and we let iA be the infinitesimal generator of Φ_t in \mathcal{H} i.e.

$$\Phi_t = e^{itA} \quad t \geq 0. \quad (2.61)$$

Since Φ_t is a contraction, we have that $i(A - A^*) \geq 0$ so that the spectrum of A is confined to the closed upper half plane. Let Γ

be the discrete part of the spectrum of A on the real line. Then of course for any $t \geq 0$ we have that $e^{it\Gamma}$ is the discrete spectrum of Φ_t on the unit circle. Let now $\alpha \in \Gamma$ and x_α be a corresponding normalized eigenvector. As in the proof of theorem 2.2 we may choose $x_\alpha \in D(\Delta^{\frac{1}{2}})$ and then we find that (2.44) holds with Φ_t replacing Φ , for t arbitrary positive. This then implies (2.45) by the ergodicity of the flow, and then also (2.46) for all $t \geq 0$. In this way we prove the following theorem.

Theorem 2.8

Let (M, Φ_t, ξ) be an ergodic dynamical flow. Then the discrete eigenvalues on the real line for the infinitesimal generator of Φ_t in \mathcal{H} coincide with the discrete eigenvalues on the real line for the infinitesimal generator of Φ_t in M . Let the set of these discrete eigenvalues on the real line be denoted by Γ , the root system of the flow, then Γ is a subgroup of the additive group of the real line. Moreover the spectrum of the semigroup Φ_t in \mathcal{H} is invariant under this additive group. Moreover, for any $\alpha \in \Gamma$, $e^{2\pi i \alpha}$ is a simple eigenvalue of the semigroup Φ_t and a corresponding root operator $u_\alpha \in M$ is proportional to a unitary operator in M . The invariance of the spectrum of the semigroup Φ_t is given by the unitary equivalence

$$u_\alpha^* \Phi_t u_\alpha = e^{2\pi i \alpha} \Phi_t$$

where u_α is a normalized root operator corresponding to $\alpha \in \Gamma$. If α and β are in Γ with root operators u_α and u_β then $u_\alpha u_\beta$ is a root operator for $\alpha + \beta$ and u_α^* is a root operator for $-\alpha$. Hence if we select for each $\alpha \in \Gamma$ a unitary root operator u_α then $u_\alpha u_\beta = \gamma(\alpha, \beta) u_\beta u_\alpha$, where $\gamma(\alpha, \beta)$ is a multiplier for Γ , and $\alpha \rightarrow u_\alpha$ is a unitary multiplier representation with

multiplier γ . Γ is either a dense subgroup of \mathbb{R} or discrete i.e. $\Gamma = \{n\alpha, n \in \mathbb{Z}\}$. If Γ is discrete, then the strongly closed subalgebra M_Γ generated by the root operators is abelian. \blacksquare

The restriction of Φ_t to M_Γ is obviously an automorphism and as in the discrete case we get that the restriction of ξ to M_Γ is a trace. In the special case where Γ is discrete, so that $\Gamma = \{n\alpha, n \in \mathbb{Z}\}$, M_Γ is abelian and generated by the root operator u corresponding to α . Let u be normalized to be unitary, then M_Γ is simply the von Neumann algebra generated by u . Since Φ_t restricted to M_Γ is a one parameter group of automorphism, it is induced by a one parameter flow on the spectrum of u . Since $\Phi_t(u^n) = e^{i t n \alpha} u^n$ this flow on the spectrum of u must coincide with the flow $e^{i\varphi} \rightarrow e^{i(\varphi + \alpha t)}$ on the spectrum of u . From the fact that 1 is an eigenvalue of multiplicity one for the semigroup Φ_t restricted to M_Γ it follows that (M_Γ, Φ_t, ξ) is an ergodic dynamical flow so that the flow $e^{i\varphi} \rightarrow e^{i(\varphi + \alpha t)}$ is ergodic with respect to the spectral measure μ for u in ξ , i.e. the measure μ such that

$$\xi(f(u)) = \int_{|z|=1} f(z) d\mu(z). \quad (2.62)$$

Hence μ is an invariant and ergodic measure with respect to the flow induced by the rotation of the unit circle. Hence since Φ_t is also strongly continuous, we have that $d\mu$ is the Haar measure on the unit circle, and that u has constant spectral multiplicity. We have thus the following theorem.

Theorem 2.9

Let (M, Φ_t, ξ) be an ergodic dynamical flow, and let Γ be its root system. Then the restriction of ξ to the von Neumann algebra

M_Γ generated by the root operators is a trace, ϕ_t leaves M_Γ invariant and the restriction of ϕ_t to M_Γ is a one parameter group of automorphisms. Moreover (M_Γ, ϕ_t, ξ) is an ergodic dynamical flow. Γ consists either of one point, or is discrete or is dense. In the first case (M, ϕ_t, ξ) is strongly mixing. In the discrete case we have $\Gamma = \{n\alpha; n \in \mathbb{Z}\}$. Let in this case u be a normalized root operator corresponding to α . Then u has Lebesgue spectrum and in fact the spectral measure for u in the state ξ is the Haar measure on the unit circle and u has constant spectral multiplicity. Moreover the flow (M_Γ, ϕ_t, ξ) is induced by rotating the spectrum of u at the constant speed α . ■

From the spectrum of the strongly continuous contraction semigroup ϕ_t in \mathcal{H} we also have the following theorem.

Theorem 2.10

If the dynamical flow (M, ϕ_t, ξ) is ergodic, then the mean ergodic theorem holds i.e.

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \xi(a \phi_t(b)) = \xi(a) \xi(b)$$

for all a and b in M . Moreover if $\Gamma = \{1\}$ then the dynamical flow (M, ϕ_t, ξ) is strongly mixing.

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