1. INTRODUCTION.

Let k be a field and Y an algebraic scheme over k. By a deformation of Y we mean a flat morphism $q: V \to W$ of algebraic schemes V and W such that Y is isomorphic to the fiber of q at some rational point of W. The deformation is called non-singular if the fiber of q at each point in an open dense subset of W is non-singular.

We shall in the following work deal with the problem of constructing deformations of an affine scheme defined by vanishing of Pfaffians of an alternating matrix (see section 2 for definitions).

In section 3 we show that the generic schemes $P_{2s}(X)$ (the scheme of all alternating $m\times m$-matrices whose Pfaffians of order $2s$ vanish) have properties like generic determinantal schemes: they are reduced and irreducible and $P_{2s-2}(X)$ is the singular locus of $P_{2s}(X)$. It is proved by Room (see 7, 10.4.4, p. 200) that the scheme $P_{2s}(X)$ has dimension $[m(m-1)-(m-2s+1)(m-2s+2)]/2$. We give another proof of this dimension formula.

In section 4 we use results proved by D. Laksov in 5 to construct defomations of schemes defined by vanishing of Pfaffians. Especially, we show that a Gorenstein point Y in $\mathbb{A}^3 = \mathrm{Spec}(k[x_1,x_2,x_3])$ has a non-singular deformation. To obtain this we use a structure theorem for Gorenstein ideals I of height three in a regular local ring R. D.A. Buchsbaum and D. Eisenbud have in 1 showed that such ideals can be generated by Pfaffians of order $2n$ of an alternating $(2n+1)\times(2n+1)$-matrix with entries in R.

From this we obtain an element d in $k[x_1,x_2,x_3]$ such that Y is the closed subscheme of $\mathrm{Spec}(k[x_1,x_2,x_3]_d)$ ($k[x_1,x_2,x_3]_d$ is the quotient ring of $k[x_1,x_2,x_3]$ by the multiplicative set $\{1,d,d^2,\ldots\}$) where the Pfaffians of an alternating matrix with entries in $k[x_1,x_2,x_3]_d$ vanish. Uning this matrix we construct a deformation $q: V \to W$ of Y such that the fiber of q at all points in an open dense subset of W consists of distrinct points each of multiplicity one.

A. Iarrobino and J. Emsalem have in 4 proved that certain types of local Gorenstein algebras of length n in three variables have no deformations to $\mathrm{Spec}\,(k[x]/(x^n))$. Together with our result on non-singular deformations of Gorenstein points in $\mathbb{A}^3$, this gives the existence of points in $\mathbb{A}^3$ which have non-singular deformations although they have no deformations to $\mathrm{Spec}(k[x]/(x^n))$.

2. BASIC PROPERTIES OF PFAFFIANS.

Let R be a commutative ring. A square matrix with entries in R is called <u>alternating</u> if it is skew symmetric and if all its diagonal elements are zero.

Let M be an alternating $n \times n$-matrix with entries in R. If n is an odd number, then $\det(M) = 0$. For n even, $\det(M) = (\mathrm{Pf}(M))^2$, where $\mathrm{Pf}(M)$ is a polynomial function of the entries in M (see 1, Lemma 2.3 or 2, p. 82-84 or 6, Theorem 7, p. 373). The polynomial $\mathrm{Pf}(M)$ is called the <u>Pfaffian</u> of M.

Denote by $M_{i,j}$ the alternating $(n-2) \times (n-2)$-matrix obtained from M by deleting the i^{th} and j^{th} row and the i^{th} and j^{th} column. Then for any $1 \leq i \leq n$, the Pfaffian of M can be computed by the formula

$$\mathrm{Pf}(M) = \sum_{j \neq i} (-1)^j m_{i,j} \mathrm{Pf}(M_{i,j}) \tag{2.1}$$

where $m_{i,j}$ is the $(i,j)^{th}$ entry of M (see 1, Lemma 2.4).

If we delete the same $n-2s$ rows and columns from M, we get an alternating $2s \times 2s$-matrix. The Pfaffian of this matrix is called "a Pfaffian of M of order $2s$." Denote by $\mathrm{Pf}_{2s}(M)$ the ideal in R generated by all Pfaffians of M of order $2s$. By virtue of the formula (2.1) we get that

$$\mathrm{Pf}_{2s+2}(M) \subseteq \mathrm{Pf}_{2s}(M) \tag{2.2}$$

where $1 \leq s \leq n/2 - 1$.

If $I_t(M)$ is the ideal in R generated by all minors of M of order t, then for each s with $1 \leq s \leq n/2$ we have that

$$I_{2s-1}(M) \subseteq \mathrm{Pf}_{2s}(M) \tag{2.3}$$

$$I_{2s}(M) \subseteq \mathrm{Pf}_{2s}(M) \subseteq \mathrm{Rad}(I_{2s}(M)) \tag{2.4}$$

where $\mathrm{Rad}(I_{2s}(M))$ denotes the redical of the ideal $I_{2s}(M)$ (see 1, Corollary 2.6).

3. GENERIC PFAFFIANS.

Let k be a field and let $x_{i,j}$, $1 \leq i < j \leq m$ be $m(m-1)/2$ algebraically independent elements over k (m is a number ≥ 2). If we put $x_{i,j} = 0$ and $x_{i,j} = -x_{j,i}$ for $i > j$, the $m \times m$-matrix $X = (x_{i,j})$ is alternating.

PROPOSITION 3.1.

Let Q be a minimal prime ideal in the polynomial ring

$P = k[x_{1,2},\ldots,x_{m-1,m}]$ containing the ideal $Pf_{2s}(X)$. Then the height of Q is $(m-2s+2)(m-2s+1)/2$.

Proof: We use induction on m . For $s=1$ the statement in the proposition is obvious.

Suppose $m \geq 4$ and $s \geq 2$.

Since $Pf_{2s}(X)$ is a homogeneous ideal we have that $\mathrm{Rad}\,(Pf_{2s}(X)) \subseteq (x_{1,2},\ldots,x_{m-1,m})$. But the closed subset $V(Pf_{2s}(X))$ of the $m(m-1)/2$-dimensional affine space contains points not in $V(x_{1,2},\ldots,x_{m-1,m})$, e.g. point $(1,0,\ldots,0)$. Hence $\mathrm{Rad}(Pf_{2s}(X)) \nsubseteq (x_{1,2},\ldots,x_{m-1,m})$ and we may suppose that $x_{1,2}$ is not in Q .

Considered as a matrix with elements in the localized ring $P_{x_{1,2}}$ we can operate on the rows and columns of X until X has the form

$$X' = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & X'' & \\ 0 & 0 & & & \end{array}\right)$$

where X'' is an alternating $(m-2)\times(m-2)$-matrix with entries $x_{i,j} + c_{i,j}$, $3 \leq i < j \leq m$, and $c_{i,j}$ consists of sums of elements from the first two rows of X .

Clearly the ideals $Pf_{2s}(X')$ and $Pf_{2s-2}(X'')$ in $P_{x_{1,2}}$ are equal. Using the formula (2.4) of section 2 we get that

$$\mathrm{Rad}\,(Pf_{2s}(X)) = \mathrm{Rad}(I_{2s}(X))$$
$$\mathrm{Rad}\,(I_{2s}(X')) = \mathrm{Rad}(Pf_{2s-2}(X'')) .$$

But, considered as ideals in $P_{x_{1,2}}$, Rad(I2s(X)) is equal to Rad $(I_{2s}(X)'))$. Consequently the ideal $QP_{x_{1,2}}$ will be a minimal prime ideal containing $Pf_{2s-2}(X'')$ in $P_{x_{1,2}}$. Thus, by induction, the height of $QP_{x_{1,2}}$ in $P_{x_{1,2}}$ (and hence also the height of Q in P) is equal to $(m-2s+2)(m-2s+1)/2$, as required.

Q.E.D.

PROPOSITION 3.2.

The affine scheme $P_{2s}(X) = \mathrm{Spec}(k[x_{1,2},\ldots,x_{m-1,m}]/Pf_2\ (X))$ $2 \leq 2s \leq m$, has the following properties:

(A) $P_{2s}(X)$ is a reduced and irreducible subscheme of codimension $(m-2s+2)(m-2s+1)/2$ in the affine $m(m-1)/2$-simensional space of all alternating $m \times m$-matrices.

(B) The scheme $P_{2s-2}(X)$ is the singular locus of the scheme $P_{2s}(X)$.

Proof: The codimension formula of (A) follows at once from Proposition 3.1. Moreover if $s=1$ both (A) and (B) is obvious. Suppose $s \geq 2$ and let $b_1,\ldots,b_l$ denote the Pfaffians of X of order $2s-2$. Let $P_{2s}(X)_{b_i}$, $1 \leq i \leq l$, be the affine open subscheme of $P_{2s}(X)$ defined by

$$P_{2s}(X)_{b_i} = \mathrm{Spec}(k[x_{1,2},\ldots,x_{m-1,m}]_{b_i}/Pf_{2s}(X)k[x_{1,2},\ldots,x_{m-1,m}]_{b_i})$$

LEMMA 3.3.

(i) $P_{2s}(X)_{b_i}$ is regular and irreducible.

(ii) $b_1,\ldots,b_l$ can be arranged in a sequence such that for each $2 \leq k \leq l$, $P_2\ (X)_{b_k} \cap P_{2s}(X)_{b_t}$ is non-empty for at least one t, $1 \leq t < k$.

(iii) The union of the schemes $P_{2s}(X)_{b_i}$ is dense in $P_{2s}(X)$.

Proof of Lemma 3.3: Let b be the Pfaffian of order $2s-2$ obtained by deleting the first $m-2s+2$ rows and the first $m-2s+2$ columns from X . Using the formula for expansion of Pfaffians along a row (see(2.1) in section 2) we get that $(m-2s+2)(m-2s+1)/2$ of the generators of $Pf_{2s}(X)$ can be written

$$bx_{i,j} + A_{i,j} \quad , \quad 1 \leq i < j \leq m-2s+2$$

where $A_{i,j}$ is a polynomial in the variables $x_{u,v}$, $v \geq m-2s+3$. Indeed, $bx_{i,j} + A_{i,j}$ is the Pfaffian of X of order $2s$ obtained by deleting the first $m-2s+2$ rows expect the i^{th} and j^{th} row and the first $m-2s+2$ columns except the i^{th} and j^{th} .

Let I be the ideal in $k[x_{1,2},\dots,x_{m-1,m}]$ generated by $bx_{i,j} + A_{i,j}$, $1 \leq i < j \leq m-2s+2$. The scheme $P'_b = \mathrm{Spec}\,(k[x_{1,2},\dots,x_{m-1,m}]_b/Ik[x_{1,2},\dots,x_{m-1,m}]_b)$ is isomorphic to $\mathrm{Spec}\,(k[x_{1,2},\dots,x_{m-1,m}]_b/J$ where J is the ideal in $k[x_{1,2},\dots,x_{m-1,m}]_b$ generated by $x_{i,j}$, $1 \leq i < j \leq m-2s+2$. This gives that P'_b is a regular and irreducible affine scheme of dimension $[m(m-1)-(m-2s+2)(m-2s+1)]/2$.

But $P_{2s}(X)_b$ is a closed subscheme of P'_b of the same simension as P'_b . Hence P'_b and $P_{2s}(X)_b$ are equal, and $P_{2s}(X)_b$ is regular and irreducible.

To prove (ii) we must arrange $b_1,\dots,b_l$ in a sequence such that for every k, $2 \leq k \leq l$, $b_k b_t$ is not in $\mathrm{Rad}(Pf_{2s}(X))$ for at least one t , $1 \leq t < k$. But, suppose b_k and b_t are Pfaffians of two submatrices of X of size $2s-2$ which has $(2s-3)(2s-4)/2$ common entries.

Then any Pfaffian of X of order $2s$ consists of sums of monomials such that each term in this sum contains a variable which is not in b_k and b_t . Hence $b_k b_t$ is not in $\mathrm{Rad}\,(\mathrm{Pf}_{2s}(X))$.

On the other hand we can list $b_1,\ldots,b_l$ in a sequence such that for each $k \geq 2$, the matrix defining b_k has $(2s-3)(2s-4)/2$ common entries with at least one of the matrices defining $b_1,\ldots,b_{k-1}$. This gives a proof of (ii).

Let Q be a minimal prime ideal in $P_{2s}(X)$. From Propostion 3.1. we conclude that Q is not in $P_{2s-2}(X)$. But the complement of $P_{2s-2}(X)$ in $P_{2s}(X)$ is equal to $\bigcup_{i=1}^{l} P_{2s}(X)_{b_i}$, so this union is dense in $P_{2s}(X)$. Thus the last part of the lemma is shown.

We now complete the proof of Propostion 3.2.

Let S be the singular locus of $P_{2s}(X)$ and denote by $f_1,\ldots,f_r$ the Pfaffians of X of order $2s$. Using the expansion formula for Pfaffians (see (2.1) of section 2) we get that all entries in the Jacobian matrix $(\frac{\partial f_i}{\partial x_{u,v}})$ are Pfaffians of X of order $2s-2$ or zero. It follows at once that $P_{2s-2}(X) \subseteq S$. But the complement of $P_{2s-2}(X)$ in $P_{2s}(X)$ is equal to the union $\bigcup_{i=1}^{l} P_{2s}(X)_{b_i}$ and each of the schemes $P_{2s}(X)_{b_i}$ are regular (see (i) of Lemma 3.3). Therefore $S = P_{2s-2}(X)$ and (B) is proved.

Let R be a noetherian ring and look at the following conditions about R for $k = 0,1,2\ldots$:

(S_k) it holds that depth $(R_p) \geq \inf(k,\mathrm{ht}(p))$ for all $p \in \mathrm{Spec}(R)$.

(R_k) if $p \in \mathrm{Spec}\,(R)$ and $\mathrm{ht}(p) \leq k$, then R_p is regular.

It is proved in EGA (see 3, Proposition 5.8.5, p. 108) that R is reduced if and only if (R_o) and (S_1) are satisfied.

Put $R = k[x_{1,2},\dots,x_{m-1,m}]/Pf_{2s}(X)$ and take a prime ideal Q in R with $ht(Q) \leq 1$. Then by Proposition 3.1. Q is not in $P_{2s-2}(X)$ and it follows from statement (B) of the proposition that R_Q is regular. Hence both (R_o) and (S_1) holds for R and we have shown that $P_{2s}(X)$ is reduced.

It remains to prove that $P_{2s}(X)$ is irreducible.

Suppose $P_{2s}(X) = Z_1 \cup Z_2$ and suppose we have proved, that $P_{2s}(X)_{b_i} = Z_1 \cap P_{2s}(X)_{b_i}$, $1 \leq i \leq k-1$, $2 \leq k \leq l$. We have that $P_{2s}(X)_{b_k} = [P_{2s}(X)_{b_k} \cap Z_1] \cup [P_{2s}(X)_{b_k} \cap Z_2]$. But $P_{2s}(X)_{b_k}$ is irreducible (see Lemma 3.3, (i)) and therefore equal to $P_{2s}(X)_{b_k} \cap Z_1$ or $P_{2s}(X)_{b_k} \cap Z_2$. Using that $P_{2s}(X)_{b_k}$ intersects one of the schemes $P_{2s}(X)_{b_i}$, $1 \leq i \leq k-1$ (see Lemma 3.3, (ii)) and that $P_{2s}(X)_{b_k}$ is non-singular (see Lemma 3.3, (i)) we conclude that $P_{2s}(X)_{b_k} = P_{2s}(X)_{b_k} \cap Z_1$.

Thus Z_1 contains the union of the schemes $P_2\ (X)_{b_i}$, $1 \leq i \leq l$, and since this union is dense in $P_{2s}(X)$ (see Lemma 3.3, (iii)) we have that $P_{2s}(X)$ is irreducible.

Q.E.D.

REMARK 3.4.

If $m = 2s+1$ the scheme $P_{2s}(X)$ is Cohen-Macanlay, i.e. the ring $k[x_{1,2},\dots,x_{2s,2s+1}]/Pf_{2s}(X)$ is Cohen-Macanlay (see 1, Proposition 6.1).

For other values of s (except the trivial cases $s = 1$

or $2s=m$) it is not known if $P_{2s}(X)$ is Cohen-Macanlay or not.

4. CONSTRUCTION OF DEFORMATIONS OF SCHEMES DEFINED BY VANISHING OF PFAFFIANS.

Let $Z = \mathrm{Spec}(A)$ be an affine open subset of the p-deimensional affine space $\mathbb{A}^p = \mathrm{Spec}(k[Z_1,\dots,Z_p])$. Put $\mathbb{A}^q = \mathrm{Spec}(k[Y_1,\dots,Y_q])$ and let $f: Z \to \mathbb{A}^q$ be a morphism of affine schemes. Denote by $f_j(Z)$ the image of Y_j by the homomorphism $k[Y_1,\dots,Y_q] \to A$ corresponding to the morphism f. Moreover, denote by $G = \mathrm{Spec}(k[U_{1,1},U_{1,2},\dots,U_{p,q},V_1,\dots,V_q])$ the affine space of $(p+1)\mathrm{x}q$-matrices and by e the rational point of G corresponding to the matrix with all entries equal to zero.

Define a homomorphism of rings

$$\psi: k[Y_1,\dots,Y_q] \to A[U_{1,1},U_{1,2},\dots,U_{p,q},V_1,\dots,V_q]$$

by $\psi(Y_j) = \sum_{i=1}^{p} U_{i,j}\, Z_i + V_j + f_j(Z)$. Let $F: G\times Z \to \mathbb{A}^q$ be the morphism of affine schemes corresponding to ψ.

Let $\Phi = D_o \subseteq D_1 \subseteq \dots \subseteq D_c = D$ be a sequence of irreducible subschemes of $\mathbb{A}^q = M$ and suppose D is Cohen-Macaulay. Moreover, assume that D_{i-1} is the singular locus of D_i, $i=1,\dots,c$.

Denote by V the open subscheme of the scheme $F^{-1}(D) = (G\times Z)\mathrm{x}_M D$ where the morphism

$$q_D : F^{-1}(D) \to G$$

induced by the projection of $G\times Z$ onto the first factor, is flat (see 3, IV_3, (11.1.1)).

For each rational point g of the scheme G we denote by f_g the restriction of the morphism F to the scheme $(g \times Z) \simeq Z$. Note that by the associativity formula, the fiber $q_D^{-1}(g) = g \times_G (G \times Z) \times_M D$ is isomorphic to the inverse image $f_g^{-1}(D) = (g \times Z) \times_M D$ of D by f_g.

D. Laksov has in 5 proved that q_D and f_g have the following properties (see 5, Theorem 2 of section 3 and the proposition of section 4):

PROPOSITION 4.1. (D. Laksov)

If $f_e^{-1}(D)$ is a subscheme of Z of pure codimension codim $(D'M)$, then the following conditions hold:

(a) The fiber $q_D^{-1}(e)$ is contained in V.

(b) There exists an open dense subset U of G such that for each point g of U the following assertions holds:

(i) The fiber $q_D^{-1}(g) \simeq f_g^{-1}(D)$ is contained in V.

(ii) Each scheme $f_g^{-1}(D_i)$ in the sequence

$$\Phi = f_g^{-1}(D_o) \subseteq f_g^{-1}(D_1) \subseteq \dots \subseteq f_g^{-1}(D_c) = f_g^{-1}(D)$$

is of pure codimension $\operatorname{codim}(D_i, M)$ in Z (empty if $\operatorname{codim}(D_i, M)$ is greater than $\dim M$).

(iii) $f_g^{-1}(D_{i-1})$ is the singular locus of the scheme $f_g^{-1}(D_i)$ for $i = 1, \dots, c$.

We are interested in the following special case: Let Y be a closed subscheme of pure codimension three in $Z = \operatorname{Spec}(A)$ defined by vanishing of Pfaffians of order $2n$ of an alternating $(2n+1)\times(2n+1)$-matrix $F = (f_{i,j})$ with entries in A .

Let $M=\mathrm{Spec}(k[x_{1,2},\ldots,x_{2n,2n+1}])$ be the affine $n(2n+1)$-dimensional space of alternating $(2n+1)\times(2n+1)$-matrices. Denote by P_{2s} the scheme of all alternating $(2n+1)\times(2n+1)$-matrices whose Pfaffians of order 2s vanish $0 \leq s \leq n$.

In section 3 we have proved the following:

$$\Phi = P_0 \subseteq P_2 \subseteq \ldots\ldots \subseteq P_{2n}$$

is a sequence of irreducible subschemes of M and P_{2s-2} is the singular locus of P_{2s} , $s=1,\ldots,n$ (see Proposition 3.2). Moreover, $P_{2n} = P$ is Cohen-Macanlay (see Remark 3.4).

Now, define a homomorphism of rings

$$\Phi : k[x_{1,2},\ldots,x_{2n,2n+1}] \to A$$

by sendign $x_{i,j}$ to $f_{i,j}$, $1 \leq i < j \leq 2n+1$. Then Y is the scheme theoretic inverse image of P by the morphism of affine schemes

$$f : Z \to M$$

corresponding to φ .

Remember that $\mathrm{codim}(P,M)$ is three, and since V is supposed to have pure codimension three in Z we can use Propostion 4.1 to obtain the following result:

THEOREM 4.2.

Let $Z = \mathrm{Spec}(A)$ be an affine open subset of the pdimensional affine space $\mathbb{A}^p$, $p \geq 3$. Suppose Y is the closed subscheme of Z where the Pfaffians of order 2n of an alternating $(2n+1)\times(2n+1)$ -matrix F with entries in A vanish. Moreover, suppose Y has pure codimension three in Z .

Then there exists a flat morphism

$$q : V \to W$$

from an algebraic scheme V to a regular, irreducible algebraic scheme W and an open dense subset U of W such that:

(a) There exists a rational point e in W such that the scheme Y is isomorphic to the fiber of q at e .

(b) For each rational point g of U there exists an alternating $(2n+1)\times(2n+1)$ matrix F(g) with entries in A with the following properties:

(i) The fiber $q^{-1}(g)$ is isomorhic to $P_{2n}(F(g))$ (the closed subscheme of Z where the Pfaffians of F(g) of order 2n vanish).

(ii) Each scheme $P_{2s}(F(g))$ in the sequence

$$\Phi = P_0(F(g)) \subseteq P_2(F(g)) \subseteq \dots \subseteq P_{2n}(F(g))$$

is empty or of pure codimension $(2n-2s+3)(n-s+1)$ in Z .

(iii) $P_{2s-2}(F(g))$ is the singular locus of the scheme $P_{2s}(F(g))$, $1 \leq s \leq n$.

COROLLARY 4.3.

Let Y be a Gorenstein point in $\mathbb{A}^3$, i.e. $Y = \mathrm{Spec}(k[x_1,x_2,x_3]/I)$ where $k[x_1,x_2,x_3]/I$ is a local Gorenstein ring of dimension zero. Then Y has non-singular deformations.

Proof of the corollary: We will show that there exists an element d in $k[x_1,x_2,x_3]$ such that $k[x_1,x_2,x_3]/I$ is ssomorphic to $k[x_1,x_2,x_3]_d/Ik[x_1,x_2,x_3]_d$ and such that the ideal $Ik[x_1,x_2,x_3]_d$ is generated by Pfaffians of an alternating matrix with entries in $k[x_1,x_2,x_3]_d$.

First, localizing in the maximal ideal Q containing I , we can write $k[x_1,x_2,x_3]/I$ as a quotient of the local ring $k[x_1,x_2,x_3]_Q$ by the ideal $Ik[x_1,x_2,x_3]_Q$. We then use the Pfaffian structure of Gorenstein ideals of height three in regular local rings (see 1, Theorem 2.1): If R is a regular local ring and J is a Gorenstein ideal in R of height three (i.e. R/J is a Gorenstein ring of demension $\dim R-3$) then there exists an alternating $(2n+1)\times(2n+1)$-matrix N with entries in R such that J is equal to $\mathrm{Pf}_{2n}(N)$. Thus we get that the ideal $Ik[x_1,x_2,x_3]_Q$ is generated by the Pfaffians of order $2n$ of an alternating, matrix F' with entries in $k[x_1,x_2,x_3]_Q$. If we multiply each entry in F' by the procuct of the denominators of the entries in F' we get an altenating matrix F with entries in $k[x_1,x_2,x_3]$ such that $\mathrm{Pf}_{2n}(F') = \mathrm{Pf}_{2n}(F)k[x_1,x_2,x_3]_Q$.

Since $Ik[x_1,x_2,x_3]_Q = \mathrm{Pf}_{2n}(F)k[x_1,x_2,x_3]_Q$ we can find an element d in $k[x_1,x_2,x_3]$, d not in Q , such that $Ik[x_1,x_2,x_3]_d = \mathrm{Pf}_{2n}(F)k[x_1,x_2,x_3]_d$.

By Theorem 4.2 with $Z = \mathrm{Spec}(k[x_1,x_2,x_3]_d)$ we can construct a deformation

$$q : V \to W$$

where the fiber of q at all points g in an open dense subset of W has a stratification

$$\Phi = P_o(g) \subseteq P_2(g) \subseteq \dots \subseteq P_{2n}(g) = q^{-1}(g)$$

such that each member in this stratification is the singular locus of the preceding. Moreover either $P_{2s}(g)$ has codimension $(2n-2s+3)(n-s+1)$ in Z or $P_{2s}(g)$ is empty. But since Z has dimension three $P_{2n-2}(g)$ is empty and hence $P_{2n}(g)$ is non-singular.

Q.E.D.

REMARK 4.4.

Iarrobino and Emsalem ask in 4 if a point Y in $\mathbb{A}^r$ which has non-singular deformations, has a deformation to $\mathrm{Spec}(k[x]/(x^n))$ too, i.e. a deformation $q : V \to W$ where the fiber of q at every point in an open dense subset of W is isomorphic to $\mathrm{Spec}(k[x]/(x^n))$.

But there exists a Gorenstein point in $\mathbb{A}^3$ which has no deformations to $\mathrm{Spec}(k[x]/(x^n))$ (see 4, Theorem 3.35). Thus, by virtue of Corollary 4.3 there is not, in general, a positive answer to the question.

R E F E R E N C E S :

1. D.A. BUCHSBAUM AND D. EISENBUD, Algebra structures for finite free resolutions, and some structure theorem for ideals of codimension 3, unpublished.

2. N. BOURBAKI, "Algèbre", Hermann, Paris, 1958.

3. A. GROTHENDIECK, Eleménts de géométrie algébrique, Publ. Math. de I.H.E.S., no. 20, 24, 28, 32 (1964,1965, 1966, 1967).

4. A. IARROBINO AND J. EMSALEM, Finite algebras having small tangent space; some zerodimensional generic singularities. unpublished.

5. D. LAKSOV, Deformation of determinantal varieties, Compositio Math. (1975), 273-292.

6. S. LANG, "Algebra", Addison-Wesley, Massachusetts, California, London, Sydney, Manila, 1971.

7. T.G. ROOM, "The geometry of determinantal loci," Cambridge University Press, Cambridge, 1938.