1. INTRODUCTION.

Let $k$ be a field and $Y$ an algebraic scheme over $k$. By a deformation of $Y$ we mean a flat morphism $q: V \to W$ of algebraic schemes $V$ and $W$ such that $Y$ is isomorphic to the fiber of $q$ at some rational point of $W$. The deformation is called non-singular if the fiber of $q$ at each point in an open dense subset of $W$ is non-singular.

We shall in the following work deal with the problem of constructing deformations of an affine scheme defined by vanishing of Pfaffians of an alternating matrix (see section 2 for definitions).

In section 3 we show that the generic schemes $P_{2s}(X)$ (the scheme of all alternating $m \times m$-matrices whose Pfaffians of order $2s$ vanish) have properties like generic determinantal schemes: they are reduced and irreducible and $P_{2s-2}(X)$ is the singular locus of $P_{2s}(X)$. It is proved by Room (see 7, 10.4.4, p. 200) that the scheme $P_{2s}(X)$ has dimension \[\frac{m(m-1)-(m-2s+1)(m-2s+2)}{2}.\] We give another proof of this dimension formula.

In section 4 we use results proved by D. Laksov in 5 to construct deformations of schemes defined by vanishing of Pfaffians. Especially, we show that a Gorenstein point $Y$ in $\mathbb{A}^3 = \text{Spec}(k[x_1,x_2,x_3])$ has a non-singular deformation. To obtain this we use a structure theorem for Gorenstein ideals $I$ of height three in a regular local ring $R$. D.A. Buchsbaum and D. Eisenbud have in 1 showed that such ideals can be generated by Pfaffians of order $2n$ of an alternating $(2n+1) \times (2n+1)$-matrix with entries in $R$. 
From this we obtain an element \( d \) in \( k[x_1,x_2,x_3] \) such that 
\( Y \) is the closed subscheme of \( \text{Spec}(k[x_1,x_2,x_3]_d) \). \( k[x_1,x_2,x_3]_d \)
is the quotient ring of \( k[x_1,x_2,x_3] \) by the multiplicative 
set \( \{1,d,d^2,\ldots\} \) where the Pfaffians of an alternating 
matrix with entries in \( k[x_1,x_2,x_3]_d \) vanish. Using this 
matrix we construct a deformation \( q:V \to W \) of \( Y \) such that 
the fiber of \( q \) at all points in an open dense subset of \( W \) 
consists of distinct points each of multiplicity one.

A. Iarrobino and J. Emsalem have in 4 proved that certain 
types of local Gorenstein algebras of length \( n \) in three 
variables have no deformations to \( \text{Spec}(k[x]/(x^n)) \).
Together with our result on non-singular deformations of 
Gorenstein points in \( \mathbb{A}^3 \), this gives the existence of points 
in \( \mathbb{A}^3 \) which have non-singular deformations although 
they have no deformations to \( \text{Spec}(k[x]/(x^n)) \).

2. **Basic Properties of Pfaffians.**

Let \( R \) be a commutative ring. A square matrix with 
entries in \( R \) is called alternating if it is skew symmetric 
and if all its diagonal elements are zero.

Let \( M \) be an alternating \( n \times n \)-matrix with entries in 
\( R \). If \( n \) is an odd number, then \( \det(M) = 0 \). For \( n \) 
even, \( \det(M) = (\text{Pf}(M))^2 \), where \( \text{Pf}(M) \) is a polynomial 
function of the entries in \( M \) (see 1, Lemma 2.3 or 2, 
p. 82-84 or 6, Theorem 7, p. 373). The polynomial \( \text{Pf}(M) \) 
is called the Pfaffian of \( M \).

Denote by \( M_{i,j} \) the alternating \( (n-2) \times (n-2) \)-matrix 
obtained from \( M \) by deleting the \( i \)th and \( j \)th row and the 
\( i \)th and \( j \)th column. Then for any \( 1 \leq i \leq n \), the Pfaffian
of $M$ can be computed by the formula

$$\text{Pf}(M) = \sum_{j=1}^{n} (-1)^{j} m_{i,j} \text{Pf}(M_{i,j})$$  \hspace{1cm} (2.1)$$

where $m_{i,j}$ is the $(i,j)^{th}$ entry of $M$ (see 1, Lemma 2.4).

If we delete the same $n-2s$ rows and columns from $M$, we get an alternating $2s \times 2s$-matrix. The Pfaffian of this matrix is called "a Pfaffian of $M$ of order $2s".$ Denote by $\text{Pf}_{2s}(M)$ the ideal in $R$ generated by all Pfaffians of $M$ of order $2s$. By virtue of the formula (2.1) we get that

$$\text{Pf}_{2s+2}(M) \subset \text{Pf}_{2s}(M)$$  \hspace{1cm} (2.2)$$

where $1 \leq s \leq \frac{n}{2} - 1$.

If $I_t(M)$ is the ideal in $R$ generated by all minors of $M$ of order $t$, then for each $s$ with $1 \leq s \leq \frac{n}{2}$ we have that

$$I_{2s-1}(M) \subseteq \text{Pf}_{2s}(M)$$  \hspace{1cm} (2.3)$$

$$I_{2s}(M) \subseteq \text{Pf}_{2s}(M) \subseteq \text{Rad}(I_{2s}(M))$$  \hspace{1cm} (2.4)$$

where $\text{Rad}(I_{2s}(M))$ denotes the radical of the ideal $I_{2s}(M)$ (see 1, Corollary 2.6).

3. GENERIC PFAFFIANS.

Let $k$ be a field and let $x_{i,j}$, $1 \leq i < j \leq m$ be $m(m-1)/2$ algebraically independent elements over $k$ ( $m$ is a number $\geq 2$). If we put $x_{i,j} = 0$ and $x_{i,j} = -x_{j,i}$ for $i > j$, the $m \times m$-matrix $X = (x_{i,j})$ is alternating.

**Proposition 3.1.**

Let $Q$ be a minimal prime ideal in the polynomial ring
\[ P = k[x_1, x_2, \ldots, x_{m-1}, x_m] \] containing the ideal \( \text{Pf}_{2s}(X) \). Then the height of \( Q \) is \( (m-2s+2)(m-2s+1)/2 \).

**Proof:** We use induction on \( m \). For \( s=1 \) the statement in the proposition is obvious.

Suppose \( m \geq 4 \) and \( s \geq 2 \).

Since \( \text{Pf}_{2s}(X) \) is a homogeneous ideal we have that \( \text{Rad}(\text{Pf}_{2s}(X)) \subseteq (x_1, x_2, \ldots, x_{m-1}, x_m) \). But the closed subset \( \mathcal{V}(\text{Pf}_{2s}(X)) \) of the \( m(m-1)/2 \)-dimensional affine space contains points not in \( \mathcal{V}(x_1, x_2, \ldots, x_{m-1}, x_m) \), e.g. point \((1,0,\ldots,0)\). Hence \( \text{Rad}(\text{Pf}_{2s}(X)) \not\supseteq (x_1, x_2, \ldots, x_{m-1}, x_m) \) and we may suppose that \( x_1, x_2 \) is not in \( Q \).

Considered as a matrix with elements in the localized ring \( \mathbb{R}_{x_1, x_2} \) we can operate on the rows and columns of \( X \) until \( X \) has the form

\[
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & & & X'' \\
& & & & X'' \\
& & & & X'' \\
0 & 0 & & &
\end{pmatrix}
\]

where \( X'' \) is an alternating \((m-2) \times (m-2)\)-matrix with entries \( x_{j,i} + c_{i,j}, \ 3 \leq i < j \leq m \), and \( c_{i,j} \) consists of sums of elements from the first two rows of \( X \).

Clearly the ideals \( \text{Pf}_{2s}(X') \) and \( \text{Pf}_{2s-2}(X'') \) in \( \mathbb{R}_{x_1, x_2} \) are equal. Using the formula (2.4) of section 2 we get that

\[
\text{Rad}(\text{Pf}_{2s}(X)) = \text{Rad}(I_{2s}(X)) \\
\text{Rad}(I_{2s}(X')) = \text{Rad}(\text{Pf}_{2s-2}(X'')) .
\]
But, considered as ideals in $P_{1,2}$, $\text{Rad}(I_{2s}(X))$ is equal to $\text{Rad}(I_{2s}(X'))$. Consequently the ideal $Q_{P_{1,2}}$ will be a minimal prime ideal containing $P_{2s-2}(X^n)$ in $P_{1,2}$. Thus, by induction, the height of $Q_{P_{1,2}}$ in $P_{1,2}$ (and hence also the height of $Q$ in $P$) is equal to $(m-2s+2)(m-2s+1)/2$, as required.

Q.E.D.

**Proposition 3.2.**
The affine scheme $P_{2s}(X) = \text{Spec}(k[x_1, \ldots, x_{m-1, m}]/P_{2s}(X)_{2 \leq 2s \leq m}$, has the following properties:

(A) $P_{2s}(X)$ is a reduced and irreducible subscheme of codimension $(m-2s+2)(m-2s+1)/2$ in the affine $m(m-1)/2$-dimensional space of all alternating $m \times m$-matrices.

(B) The scheme $P_{2s-2}(X)$ is the singular locus of the scheme $P_{2s}(X)$.

**Proof:** The codimension formula of (A) follows at once from Proposition 3.1. Moreover if $s=1$ both (A) and (B) is obvious. Suppose $s \geq 2$ and let $b_1, \ldots, b_1$ denote the Pfaffians of $X$ of order $2s-2$. Let $P_{2s}(X)_{b_i}, 1 \leq i \leq 1$, be the affine open subscheme of $P_{2s}(X)$ defined by

$$P_{2s}(X)_{b_i} = \text{Spec}(k[x_1, \ldots, x_{m-1, m}]/P_{2s}(X)_{k[x_1, \ldots, x_{m-1, m}]}_{b_i})$$

**Lemma 3.3.**

(i) $P_{2s}(X)_{b_i}$ is regular and irreducible.

(ii) $b_1, \ldots, b_1$ can be arranged in a sequence such that for each $2 \leq k \leq 1$, $P_{2}(X)_{b_k} \cap P_{2s}(X)_{b_t}$ is non-empty for at least one $t, 1 \leq t < k$. 

(iii) The union of the schemes $P_{2s}(X)_{b_1}$ is dense in $P_{2s}(X)$.

Proof of Lemma 3.3: Let $b$ be the Pfaffian of order $2s-2$ obtained by deleting the first $m-2s+2$ rows and the first $m-2s+2$ columns from $X$. Using the formula for expansion of Pfaffians along a row (see (2.1) in section 2) we get that $(m-2s+2)(m-2s+1)/2$ of the generators of $Pf_{2s}(X)$ can be written

$$bx_{i,j} + A_{i,j}, \quad 1 \leq i < j \leq m-2s+2$$

where $A_{i,j}$ is a polynomial in the variables $x_{u,v}$, $v \geq m-2s+3$. Indeed, $bx_{i,j} + A_{i,j}$ is the Pfaffian of $X$ of order $2s$ obtained by deleting the first $m-2s+2$ rows expect the $i$th and $j$th row and the first $m-2s+2$ columns except the $i$th and $j$th.

Let $I$ be the ideal in $k[x_1, 2, \ldots, x_{m-1}, m]$ generated by $bx_{i,j} + A_{i,j}, \quad 1 \leq i < j \leq m-2s+2$. The scheme $P_b' = \text{Spec}(k[x_1, 2, \ldots, x_{m-1}, m]_b / I_k[x_1, 2, \ldots, x_{m-1}, m]_b)$ is isomorphic to $\text{Spec}(k[x_1, 2, \ldots, x_{m-1}, m]_b / J)$ where $J$ is the ideal in $k[x_1, 2, \ldots, x_{m-1}, m]_b$ generated by $x_{i,j}$, $1 \leq i < j \leq m-2s+2$. This gives that $P_b'$ is a regular and irreducible affine scheme of dimension $[m(m-1)-(m-2s+2)(m-2s+1)]/2$.

But $P_{2s}(X)_b$ is a closed subscheme of $P_b'$ of the same dimension as $P_b'$. Hence $P_b'$ and $P_{2s}(X)_b$ are equal, and $P_{2s}(X)_b$ is regular and irreducible.

To prove (ii) we must arrange $b_1, \ldots, b_{1}$ in a sequence such that for every $k$, $2 \leq k \leq 1$, $b_kb_t$ is not in $\text{Rad}(Pf_{2s}(X))$ for at least one $t$, $1 \leq t < k$. But, suppose $b_k$ and $b_t$ are Pfaffians of two submatrices of $X$ of size $2s-2$ which has $(2s-3)(2s-4)/2$ common entries.
Then any Pfaffian of $X$ of order $2s$ consists of sums of monomials such that each term in this sum contains a variable which is not in $b_k$ and $b_t$. Hence $b_kb_t$ is not in $\text{Rad} (\text{Pf}_{2s}(X))$.

On the other hand we can list $b_1, \ldots, b_1$ in a sequence such that for each $k \geq 2$, the matrix defining $b_k$ has $(2s-3)(2s-4)/2$ common entries with at least one of the matrices defining $b_1, \ldots, b_{k-1}$. This gives a proof of (ii).

Let $Q$ be a minimal prime ideal in $P_{2s}(X)$. From Proposition 3.1, we conclude that $Q$ is not in $P_{2s-2}(X)$. But the complement of $P_{2s-2}(X)$ in $P_{2s}(X)$ is equal to $\bigcup_{i=1}^{1} P_{2s}(X) b_i$, so this union is dense in $P_{2s}(X)$. Thus the last part of the lemma is shown.

We now complete the proof of Proposition 3.2.

Let $S$ be the singular locus of $P_{2s}(X)$ and denote by $f_1, \ldots, f_r$ the Pfaffians of $X$ of order $2s$. Using the expansion formula for Pfaffians (see (2.1) of section 2) we get that all entries in the Jacobian matrix $\left( \frac{\partial f_i}{\partial X_{u,v}} \right)$ are Pfaffians of $X$ of order $2s-2$ or zero. It follows at once that $P_{2s-2}(X) \subseteq S$. But the complement of $P_{2s-2}(X)$ in $P_{2s}(X)$ is equal to the union $\bigcup_{i=1}^{1} P_{2s}(X) b_i$, and each of the schemes $P_{2s}(X) b_i$ are regular (see (i) of Lemma 3.3). Therefore $S = P_{2s-2}(X)$ and (B) is proved.

Let $R$ be a noetherian ring and look at the following conditions about $R$ for $k = 0, 1, 2, \ldots$:

$(S_k)$ it holds that depth $(R_p) \geq \inf(k, \text{ht}(p))$ for all $p \in \text{Spec}(R)$.

$(R_k)$ if $p \in \text{Spec}(R)$ and $\text{ht}(p) \leq k$, then $R_p$ is
regular.

It is proved in EGA (see 3, Proposition 5.8.5, p. 108) that \( R \) is reduced if and only if \((R_0)\) and \((S_1)\) are satisfied.

Put \( R = k[\{x_1, 2, \ldots, x_{m-1,m}\}] / \text{Pf}_{2s}(X) \) and take a prime ideal \( Q \) in \( R \) with \( \text{ht}(Q) \leq 1 \). Then by Proposition 3.1, \( Q \) is not in \( P_{2s-2}(X) \) and it follows from statement (B) of the proposition that \( R_Q \) is regular. Hence both \((R_0)\) and \((S_1)\) holds for \( R \) and we have shown that \( P_{2s}(X) \) is reduced.

It remains to prove that \( P_{2s}(X) \) is irreducible.

Suppose \( P_{2s}(X) = Z_1 \cup Z_2 \) and suppose we have proved, that \( P_{2s}(X)_{b_i} = Z_1 \cap P_{2s}(X)_{b_i} \), \( 1 \leq i \leq k-1 \), \( 2 \leq k \leq l \).

We have that \( P_{2s}(X)_{b_k} = [P_{2s}(X)_{b_k} \cap Z_1] \cup [P_{2s}(X)_{b_k} \cap Z_2] \).

But \( P_{2s}(X)_{b_k} \) is irreducible (see Lemma 3.3, (i)) and therefore equal to \( P_{2s}(X)_{b_k} \cap Z_1 \) or \( P_{2s}(X)_{b_k} \cap Z_2 \). Using that \( P_{2s}(X)_{b_k} \) intersects one of the schemes \( P_{2s}(X)_{b_i} \), \( 1 \leq i \leq k-1 \) (see Lemma 3.3, (ii)) and that \( P_{2s}(X)_{b_k} \) is non-singular (see Lemma 3.3, (i)) we conclude that \( P_{2s}(X)_{b_k} = P_{2s}(X)_{b_k} \cap Z_1 \).

Thus \( Z_1 \) contains the union of the schemes \( P_2(X)_{b_i} \), \( 1 \leq i \leq 1 \), and since this union is dense in \( P_{2s}(X) \) (see Lemma 3.3, (iii)) we have that \( P_{2s}(X) \) is irreducible.

Q.E.D.

**REMARK 3.4.**

If \( m = 2s+1 \) the scheme \( P_{2s}(X) \) is Cohen-Macaulay, i.e. the ring \( k[\{x_1, 2, \ldots, x_{2s+1}\}] / \text{Pf}_{2s}(X) \) is Cohen-Macaulay (see 1, Proposition 6.1).

For other values of \( s \) (except the trivial cases \( s = 1 \))...
or $2s=m$ it is not known if $P_{2s}(X)$ is Cohen-Macaulay or not.

4. CONSTRUCTION OF DEFORMATIONS OF SCHEMES DEFINED BY VANISHING OF PFAFFIANS.

Let $Z = \text{Spec}(A)$ be an affine open subset of the $p$-dimensional affine space $\mathbb{A}^p = \text{Spec}(k[Z_1, \ldots, Z_p])$. Put $\mathbb{A}^q = \text{Spec}(k[Y_1, \ldots, Y_q])$ and let $f: Z \to \mathbb{A}^q$ be a morphism of affine schemes. Denote by $f_j(Z)$ the image of $Y_j$ by the homomorphism $k[Y_1, \ldots, Y_q] \to A$ corresponding to the morphism $f$. Moreover, denote by $G = \text{Spec}(k[U_1, 1, U_1, 2, \ldots, U_p, q, V_1, \ldots, V_q])$ the affine space of $(p+1)\times q$-matrices and by $e$ the rational point of $G$ corresponding to the matrix with all entries equal to zero.

Define a homomorphism of rings

$$\psi: k[Y_1, \ldots, Y_q] \to A[U_1, 1, U_1, 2, \ldots, U_p, q, V_1, \ldots, V_q]$$

by $\psi(Y_j) = \sum_{i=1}^p U_{i,j} Z_i + V_j + f_j(Z)$. Let $F: G \times Z \to \mathbb{A}^q$ be the morphism of affine schemes corresponding to $\psi$.

Let $\Phi = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_c = D$ be a sequence of irreducible subschemes of $\mathbb{A}^q = M$ and suppose $D$ is Cohen-Macaulay. Moreover, assume that $D_i-1$ is the singular locus of $D_i$, $i=1, \ldots, c$.

Denote by $V$ the open subscheme of the scheme $F^{-1}(D) = (G \times Z)_{x,M,D}$ where the morphism

$$q_D: F^{-1}(D) \to G$$

induced by the projection of $G \times Z$ onto the first factor, is flat (see 3, IV.3, (11.1.1)).
For each rational point \( g \) of the scheme \( G \) we denote by \( f_g \) the restriction of the morphism \( F \) to the scheme \((g \times Z) \cong Z\). Note that by the associativity formula, the fiber \( q_D^{-1}(g) = g \times_G (g \times Z) x_M D \) is isomorphic to the inverse image \( f_g^{-1}(D) = (g \times Z) x_M D \) of \( D \) by \( f_g \).

D. Laksov has in 5 proved that \( q_D \) and \( f_g \) have the following properties (see 5, Theorem 2 of section 3 and the proposition of section 4):

**PROPOSITION 4.1.** (D. Laksov)

If \( f_g^{-1}(D) \) is a subscheme of \( Z \) of pure codimension \( \text{codim} (D, M) \), then the following conditions hold:

(a) The fiber \( q_D^{-1}(c) \) is contained in \( V \).

(b) There exists an open dense subset \( U \) of \( G \) such that for each point \( g \) of \( U \) the following assertions hold:

(i) The fiber \( q_D^{-1}(g) \cong f_g^{-1}(D) \) is contained in \( V \).

(ii) Each scheme \( f_g^{-1}(D_i) \) in the sequence

\[
\emptyset = f_g^{-1}(D_0) \subseteq f_g^{-1}(D_1) \subseteq \ldots \subseteq f_g^{-1}(D_c) = f_g^{-1}(D)
\]

is of pure codimension \( \text{codim} (D_i, M) \) in \( Z \) (empty if \( \text{codim} (D_i, M) \) is greater than \( \dim M \)).

(iii) \( f_g^{-1}(D_{i-1}) \) is the singular locus of the scheme \( f_g^{-1}(D_i) \) for \( i = 1, \ldots, c \).

We are interested in the following special case: Let \( Y \) be a closed subscheme of pure codimension three in \( Z = \text{Spec}(A) \) defined by vanishing of Pfaffians of order \( 2n \) of an alter-
nating \((2n+1) \times (2n+1)\)-matrix \(F = (f_{i,j})\) with entries in \(A\).

Let \(M = \text{Spec}(k[x_1,2,\ldots,x_{2n},2n+1])\) be the affine \(n(2n+1)\)-dimensional space of alternating \((2n+1) \times (2n+1)\)-matrices. Denote by \(P_{2s}\) the scheme of all alternating \((2n+1) \times (2n+1)\)-matrices whose Pfaffians of order \(2s\) vanish \(0 \leq s \leq n\).

In section 3 we have proved the following:

\[ \mathcal{P} = P_0 \subseteq P_2 \subseteq \ldots \subseteq P_{2n} \]

is a sequence of irreducible subschemes of \(M\) and \(P_{2s-2}\) is the singular locus of \(P_{2s}\), \(s=1,\ldots,n\) (see Proposition 3.2). Moreover, \(P_{2n} = P\) is Cohen-Macanlay (see Remark 3.4).

Now, define a homomorphism of rings

\[ \phi : k[x_1,2,\ldots,x_{2n},2n+1] \rightarrow A \]

by sending \(x_{i,j}\) to \(f_{i,j}\), \(1 \leq i < j \leq 2n+1\). Then \(Y\) is the scheme theoretical inverse image of \(P\) by the morphism of affine schemes

\[ f : Z \rightarrow M \]

corresponding to \(\phi\).

Remember that \(\text{codim}(P,M)\) is three, and since \(V\) is supposed to have pure codimension three in \(Z\) we can use Proposition 4.1 to obtain the following result:

**Theorem 4.2.**

Let \(Z = \text{Spec}(A)\) be an affine open subset of the p-
Suppose $Y$ is the closed subscheme of $Z$ where the Pfaffians of order $2n$ of an alternating $(2n+1) \times (2n+1)$ matrix $F$ with entries in $A$ vanish. Moreover, suppose $Y$ has pure codimension three in $Z$.

Then there exists a flat morphism

$$q : V \to W$$

from an algebraic scheme $V$ to a regular, irreducible algebraic scheme $W$ and an open dense subset $U$ of $W$ such that:

(a) There exists a rational point $e$ in $W$ such that the scheme $Y$ is isomorphic to the fiber of $q$ at $e$.

(b) For each rational point $g$ of $U$ there exists an alternating $(2n+1) \times (2n+1)$ matrix $F(g)$ with entries in $A$ with the following properties:

(i) The fiber $q^{-1}(g)$ is isomorphic to $P_{2n}(F(g))$ (the closed subscheme of $Z$ where the Pfaffians of $F(g)$ of order $2n$ vanish).

(ii) Each scheme $P_{2s}(F(g))$ in the sequence

$$\phi = P_0(F(g)) \subseteq P_2(F(g)) \subseteq \ldots \subseteq P_{2n}(F(g))$$

is empty or of pure codimension $(2n-2s+3)(n-s+1)$ in $Z$.

(iii) $P_{2s-2}(F(g))$ is the singular locus of the scheme $P_{2s}(F(g))$, $1 \leq s \leq n$. 
Let $Y$ be a Gorenstein point in $\mathbb{A}^3$, i.e. $Y = \text{Spec}(k[x_1, x_2, x_3]/I)$ where $k[x_1, x_2, x_3]/I$ is a local Gorenstein ring of dimension zero. Then $Y$ has non-singular deformations.

Proof of the corollary: We will show that there exists an element $d$ in $k[x_1, x_2, x_3]$ such that $k[x_1, x_2, x_3]/I$ is isomorphic to $k[x_1, x_2, x_3]/d/Ik[x_1, x_2, x_3]d$ and such that the ideal $Ik[x_1, x_2, x_3]d$ is generated by Pfaffians of an alternating matrix with entries in $k[x_1, x_2, x_3]d$.

First, localizing in the maximal ideal $Q$ containing $I$, we can write $k[x_1, x_2, x_3]/I$ as a quotient of the local ring $k[x_1, x_2, x_3]Q$ by the ideal $Ik[x_1, x_2, x_3]Q$. We then use the Pfaffian structure of Gorenstein ideals of height three in regular local rings (see [1], Theorem 2.1): If $R$ is a regular local ring and $J$ is a Gorenstein ideal in $R$ of height three (i.e., $R/J$ is a Gorenstein ring of dimension $\dim R - 3$) then there exists an alternating $(2n+1) \times (2n+1)$-matrix $N$ with entries in $R$ such that $J$ is equal to $\text{Pf}_{2n}(N)$. Thus we get that the ideal $Ik[x_1, x_2, x_3]Q$ is generated by the Pfaffians of order $2n$ of an alternating matrix $F'$ with entries in $k[x_1, x_2, x_3]Q$. If we multiply each entry in $F'$ by the product of the denominators of the entries in $F'$ we get an alternating matrix $F$ with entries in $k[x_1, x_2, x_3]$ such that $\text{Pf}_{2n}(F') = \text{Pf}_{2n}(F)k[x_1, x_2, x_3]Q$.

Since $Ik[x_1, x_2, x_3]Q = \text{Pf}_{2n}(F)k[x_1, x_2, x_3]Q$ we can find an element $d$ in $k[x_1, x_2, x_3]$, $d$ not in $Q$, such that $Ik[x_1, x_2, x_3]d = \text{Pf}_{2n}(F)k[x_1, x_2, x_3]d$. 

By Theorem 4.2 with $Z = \text{Spec}(k[x_1, x_2, x_3])$ we can construct a deformation

$$q : V \to W$$

where the fiber of $q$ at all points $g$ in an open dense subset of $W$ has a stratification

$$\psi = P_0(g) \subseteq P_2(g) \subseteq \ldots \subseteq P_{2n}(g) = q^{-1}(g)$$

such that each member in this stratification is the singular locus of the preceding. Moreover either $P_{2n-2}(g)$ has codimension $(2n-2s+3)(n-s+1)$ in $Z$ or $P_{2n}(g)$ is empty. But since $Z$ has dimension three $P_{2n-2}(g)$ is empty and hence $P_{2n}(g)$ is non-singular.

Q.E.D.

REMARK 4.4.

Iarrobino and Emsalem ask in 4 if a point $Y$ in $\mathbb{A}^r$ which has non-singular deformations, has a deformation to $\text{Spec}(k[x]/(x^n))$ too, i.e. a deformation $q : V \to W$ where the fiber of $q$ at every point in an open dense subset of $W$ is isomorphic to $\text{Spec}(k[x]/(x^n))$.

But there exists a Gorenstein point in $\mathbb{A}^3$ which has no deformations to $\text{Spec}(k[x]/(x^n))$ (see 4, Theorem 3.35). Thus, by virtue of Corollary 4.3 there is not, in general, a positive answer to the question.
REFERENCES:


