On the Classification of Nilpotent Lie Algebras

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Abstract. We establish a bijective correspondence between all central extensions of fixed Lie algebra $\mathcal{G}$ by $\mathbb{R}^k$ (with $k$-dimensional centers), and certain orbits in the set of all $k$-dimensional subspaces in the second cohomology group $H^2(\mathcal{G},\mathbb{R})$, under the canonical action of Aut($\mathcal{G}$). As an application we construct all six-dimensional nilpotent Lie algebras over $\mathbb{R}$ (finitely many). We also show that there are infinitely many mutually non-isomorphic two-step real Lie algebras of dimension nine.

1. Introduction. In this article we develop a method of constructing all nilpotent Lie algebras of dimension $n$ given those algebras of dimension $< n$, and their automorphism groups. Roughly speaking, we establish a bijective correspondence between all central extensions of a fixed Lie algebra $\mathcal{G}$ by $\mathbb{R}^k$ with $k$-dimensional centers, and certain orbits in the set of all $k$-dimensional subspaces in the second cohomology group $H^2(\mathcal{G},\mathbb{R})$, under the canonical action of Aut($\mathcal{G}$). For natural reasons we are working in the space $S$ of all bilinear forms on $\mathcal{G}$ with values in $\mathbb{R}^k$ satisfying the Jacobi identity, rather than in $H^2(\mathcal{G},\mathbb{R}^k)$. If $B \in S$, the action of Aut($\mathcal{G}$) on $B$ is simply given by

$$(\alpha, B) \rightarrow B^\alpha, \text{ where } B^\alpha(X,Y) = B(\alpha X, \alpha Y); X,Y \in \mathcal{G}, \alpha \in \text{Aut}(\mathcal{G}).$$
As an application of our procedure, we find all nilpotent Lie algebras over \( \mathbb{R} \) of dimension 6 (finitely many). Recall that Dixmier has classified all such algebras of dimension \( \leq 5 \), [1]. We will also show that there are infinitely many mutually non-isomorphic real two-step Lie algebras of dimension 9 with center of dimension 3. Even if our method is straightforward, the computations of \( \text{Aut}(\mathcal{G}) \)-orbits in \( H^2(\mathcal{G}, \mathbb{R}^k) \) are in general far from easy to carry out, and at the moment a complete classification of nilpotent Lie algebras seems to be out of reach.

2. Central extensions and automorphisms.

2.1. Let \( \mathcal{G} \) be a Lie algebra over \( \mathbb{R} \), \( B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k \) a skew symmetric bilinear form satisfying the Jacobi identity

\[
B([X,Y],Z) + B([Z,X],Y) + B([Y,Z],X) = 0, \quad \text{all } X,Y,Z \in \mathcal{G}.
\]

Such forms are said to be closed. If \( B \) is a closed form on \( \mathcal{G} \) we construct a Lie algebra on \( \mathcal{G} \oplus \mathbb{R}^k \), letting

\[
[[X',Y']] = \left( \begin{array}{c} [X,Y] \\ B(X,Y) \end{array} \right), \quad X,Y \in \mathcal{G}, \quad u,v \in \mathbb{R}^k.
\]

Denote this Lie algebra by \( \mathcal{G}(B) \).

Let now \( \mathcal{G} \) be a Lie algebra with center \( \mathcal{Z} \) and \( \dim \mathcal{Z} = k \), \( \gamma : \mathcal{G} \to \mathbb{R}^k \) be linear and such that \( \gamma(\mathcal{Z}) = \mathbb{R}^k \). We put \( \mathcal{G} = \mathcal{G}/\mathcal{Z} \) and get an isomorphism \( \mathcal{G} \cong \mathcal{G} \oplus \mathbb{R}^k \) where \( \tilde{X} \leftrightarrow (\frac{X}{u}) \), \( \gamma(\tilde{X}) = u \), and

\[
\gamma(X,Y) = \gamma(X',Y') \quad \text{where } X' + \tilde{Z} = X, \quad Y' + \tilde{Z} = Y.
\]

This shows \( \mathcal{G} \) and \( \mathcal{G}(B) \) are isomorphic. Hence each Lie algebra with center of dimension \( k \) is of the form \( \mathcal{G}(B) \) where \( B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k \).
2.2. Let \( B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k \) be a closed form. Then the center \( \tilde{\mathcal{G}} \) of \( \tilde{\mathcal{G}} = \mathcal{G}(B) \) is equal to
\[
\tilde{\mathcal{G}} = \mathbb{R}^k \oplus (\mathcal{G} \cap \mathcal{G}_B), \quad \text{where} \quad \mathcal{G}_B = \{X \in \mathcal{G} : B(X, \mathcal{G}) = (0)\}
\]

2.3. Given two such forms \( B_1, B_2 : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k \), and assume the extended algebras \( \mathcal{G}(B_1) \) and \( \mathcal{G}(B_2) \) are isomorphic and that their centers \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) both are equal to \( \mathbb{R}^k \). Let \( \alpha : \mathcal{G}(B_1) \to \mathcal{G}(B_2) \) be an isomorphism. Dividing with the common center \( \mathbb{R}^k \) we obtain an automorphism \( \alpha_0 : \mathcal{G} \to \mathcal{G} \). Let us fix a basis \( \{e_1, \ldots, e_n\} \) for \( \mathcal{G} \), and supplement it with a basis for \( \mathbb{R}^k \) to get a basis \( \mathcal{C} \) for \( \mathcal{G} \oplus \mathbb{R}^k \). We may realize \( \alpha \) as a matrix relative to \( \mathcal{C} \):

\[
(2.2) \quad \alpha = \begin{pmatrix} \alpha_0 & 0 \\ \phi & \psi \end{pmatrix}; \quad \alpha_0 \in \text{Aut}(\mathcal{G}), \quad \psi = \alpha|_{\mathcal{G}} \in \text{GL}(k), \quad \text{and} \quad \phi \in \text{Hom}(\mathcal{G}, \mathbb{R}^k).
\]

Now \( \alpha \) preserves the brackets, and writing \([\cdot, \cdot]_i \) for the products in \( \mathcal{G}(B_i) \), \( i = 1, 2 \), and \([\cdot, \cdot] \) for the product in \( \mathcal{G} \), we have
\[
\alpha\left[\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} Y \\ 0 \end{smallmatrix}\right)\right]_1 = \left[\alpha\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right), \alpha\left(\begin{smallmatrix} Y \\ 0 \end{smallmatrix}\right)\right]_2, \quad X, Y \in \mathcal{G},
\]
where \( \alpha\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha_0 X \\ \phi X \end{smallmatrix}\right) \), and hence
\[
\left[\alpha\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right), \alpha\left(\begin{smallmatrix} Y \\ 0 \end{smallmatrix}\right)\right]_2 = \left(\begin{smallmatrix} \alpha_0 X \\ \phi X \end{smallmatrix}\right) \left(\begin{smallmatrix} \alpha_0 Y \\ \phi Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha_0 X, \alpha_0 Y \\ \phi X, \phi Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} [\alpha_0 X, \alpha_0 Y] \\ \phi [X, Y] + \psi \end{smallmatrix}\right).
\]

and
\[
\alpha\left[\left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} Y \\ 0 \end{smallmatrix}\right)\right]_1 = \alpha_{B_1}(X, Y) = \alpha_0 [X, Y] \quad \text{for all} \quad X, Y \in \mathcal{G}.
\]

Hence
\[
(2.3) \quad B_2(\alpha_0 X, \alpha_0 Y) = \phi[X, Y] + \psi B_1(X, Y), \quad \text{all} \quad X, Y \in \mathcal{G}.
\]

In case \( B_1 = B_2 = B \), we get the following description of the automorphism group \( \text{Aut}(\mathcal{G}(B)) \).
2.4. Proposition. Let \( B \) be a closed form on the Lie algebra \( \mathcal{G} \) with values in \( \mathbb{R}^k \), and assume \( \mathcal{J}_B \cap \mathcal{J}(\mathcal{G}) = (0) \). Then the automorphism group \( \text{Aut} \mathcal{G}(B) \) of the extended algebra \( \mathcal{G}(B) \) consists of all linear operators of the matrix form \( \alpha = \begin{pmatrix} a & 0 \\ \phi & c \end{pmatrix} \) as in (2.2), where

\[
B(a_0 X, a_0 Y) = \psi B(X, Y) + \varphi [X, Y], \quad \text{all } X, Y \in \mathcal{G}.
\]

2.5. Examples. The Heisenberg algebra \( \mathcal{G}_3 \) with non-zero brackets \([e_1, e_2] = e_3\) between the basis elements \( e_1, e_2, e_3 \) is a central extension of the abelian algebra \( \mathcal{G} = \mathbb{R} e_1 \times \mathbb{R} e_2 \) by \( \mathbb{R} e_3 \), given by the bilinear form \( B = B_{12} : (\Sigma x_1 e_i, \Sigma y_1 e_i) \rightarrow x_1 y_2 - x_2 y_1 \). Now \( \mathcal{J}_B = (0) \) and \( \text{Aut}(\mathcal{G}_3) \) consists of all operators

\[
\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u_1 & u_2 & c_0 \end{pmatrix}, \quad \text{where } ad - bc = c_0 \neq 0.
\]

The four dimensional nilpotent Lie algebra \( \mathcal{G}_4 \) given by \([e_1, e_2] = e_3, [e_1, e_3] = e_4\) is a central extension of \( \mathcal{G}_3 \) by \( \mathbb{R} e_4 \), given by the bilinear form \( B = B_{13} : (\Sigma x_1 e_i, \Sigma y_1 e_i) \rightarrow x_1 y_3 - x_3 y_1 \). Thus \( \mathcal{J}_B \cap \mathcal{J}(\mathcal{G}_3) = (0) \), and \( \text{Aut} \mathcal{G}_4 \) consists of all operators \( \alpha = \begin{pmatrix} a & 0 \\ \phi & c \end{pmatrix} \), \( a_0 \in \text{Aut} \mathcal{G}_3, \varphi \in \mathcal{G}_3^*, c_1 \in \mathbb{R} \), such that (2.4) is satisfied. This gives

\[
\alpha = \begin{pmatrix} a & 0 & 0 & 0 \\ c & d & 0 & 0 \\ u_1 & u_2 & ad & 0 \\ v_1 & v_2 & au_2 a^2 & c \end{pmatrix}, \quad \text{ad} \neq 0.
\]

On the basis of \( \text{Aut} \mathcal{G}_3 \) and \( \text{Aut} \mathcal{G}_4 \) one computes without difficulty the automorphism groups of all five-dimensional nilpotent Lie algebras. These can be found in Table 4.3. (except for \( \text{Aut}(\mathcal{G}_5, 1) \) which is not needed).
3. Central extensions of Lie algebras.

3.1. We continue our study of central extensions $\mathcal{G}(B)$ of a Lie algebra $\mathcal{G}$ given by closed forms $B : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^k$, and proceed to exclude forms $B$ such that $\mathcal{G}(B) \cong \mathbb{R} \times \mathcal{G}(B')$, where $B' : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^{k-1}$.

Let $J$ be the set of all linear maps $F : \mathbb{R}^k \to \mathbb{R}^k$ such that there exists a linear map $\varphi : \mathcal{G} \to \mathbb{R}^k$ with the property

$$F(B(X,Y)) = \varphi[X,Y], \text{ all } X,Y \in \mathcal{G}.$$  

Clearly $J$ is a left ideal of $\text{Hom}(\mathbb{R}^k, \mathbb{R}^k)$, i.e. $J = \text{Hom}(\mathbb{R}^k, \mathbb{R}^k) \cdot J$, and therefore $J$ is generated by a projection $\pi$; $J = \text{Hom}(\mathbb{R}^k, \mathbb{R}^k) \cdot \pi$. We have

$$\pi(B(X,Y)) = \varphi_\pi[X,Y], \text{ } \varphi_\pi \in \text{Hom}(\mathcal{G}, \mathbb{R}^k).$$

Put

$$B'(X,Y) = B(X,Y) - \varphi_\pi[X,Y] = (1-\pi)B(X,Y),$$

in particular $B'$ is cohomologous to $B$.

3.2. Lemma. Suppose we have an equation

$$F(B'(X,Y)) = \varphi[X,Y], \text{ all } X,Y \in \mathcal{G},$$

where $\varphi \in \text{Hom}(\mathcal{G}, \mathbb{R}^k)$ and $B'$ is given by (3.2).

Then $\varphi[X,Y] = 0$, all $X,Y \in \mathcal{G}$.

Proof. If the above equality holds, then

$$F^\circ (1-\pi)B(X,Y) = \varphi[X,Y], \text{ all } X,Y \in \mathcal{G},$$

so that $F^\circ (1-\pi) \in J$, and hence $F^\circ (1-\pi) = G \cdot \pi$, $G \in \text{Hom}(\mathbb{R}^k, \mathbb{R}^k)$, and $F = (G+F)^\circ \pi$; hence

$$F^\circ (1-\pi) = (G+F)^\circ \pi^\circ (1-\pi) = (G+F)^\circ (\pi-\pi) = 0.$$
2.3. It follows from (3.2) that \( \mathcal{G}(B) \) and \( \mathcal{G}(B') = \mathcal{G}(B - \varphi_\pi \alpha_\varphi) \) are isomorphic. Suppose \( \pi \neq 0 \) \((J \neq (0))\), then obviously

\[
\mathcal{G}(B') = \mathcal{G}(B'_0) \times \pi(\mathbb{R}^k)
\]

where \( \pi(\mathbb{R}^k) \) is abelian and

\[B'_0 = (1 - \pi) \cdot B : \mathcal{G} \times \mathcal{G} \rightarrow (1 - \pi) \mathbb{R}^k\].

**Corollary.** Let \( \mathcal{S}_B \cap \mathcal{J} = (0) \) and assume that \( \mathcal{G}(B) \) can not be written as \( \mathbb{R} \times \mathcal{G} \) for any Lie algebra \( \mathcal{G} \). For any pair of linear maps \( F : \mathbb{R}^k \rightarrow \mathbb{R}^k \), \( \varphi : \mathcal{G} \rightarrow \mathbb{R}^k \) such that \( F \circ B(X, Y) = \varphi[X, Y] \), we have \( F = 0 \).

We can always assume the corollary holds, since we are not interested in algebras of the form \( \mathbb{R} \times \mathcal{G} \).

2.4. In order to classify Lie algebras \( \mathcal{G} \) with no factor isomorphic to \( \mathbb{R} \), and with center \( \mathcal{J} \) of dimension \( k \) and \( \mathcal{G} / \mathcal{J} \cong \mathcal{G} \), we must consider closed forms \( B : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}^k \) as in Corollary 3.3. Let \( S \) be the vector space of all closed forms on \( \mathcal{G} \), and let \( S' \) be the subspace of all exact forms \( B(X, Y) = \varphi[X, Y] \) where \( \varphi : \mathcal{G} \rightarrow \mathbb{R} \) is linear, i.e. \( \varphi \in \mathcal{G}^* \). Then Corollary 3.3 is equivalent to the following: Let \( \pi_1, \ldots, \pi_k : \mathbb{R}^k \rightarrow \mathbb{R} \) be the coordinate functionals, then \( \pi_1 \circ B, \ldots, \pi_k \circ B \) are linearly independent in \( S / S' \). We know from §2.3 that \( \mathcal{G}(B_1) \cong \mathcal{G}(B_2) \iff B_2(\alpha_0 X, \alpha_0 Y) = \psi B_1(X, Y) + \varphi[X, Y] \)

where \( \alpha = \left( \begin{array}{c} \alpha_0 \\ \varphi \\ \psi \end{array} \right) \) is an isomorphism. Such an identity holds if and only if \( \pi_1 \circ B_1, \ldots, \pi_k \circ B_1 \) and \( \pi_1 \circ B_2 \circ \alpha_0, \ldots, \pi_k \circ B_2 \circ \alpha_0 \) generate the same subspace of \( S / S' \). We say that an \( \text{Aut}(\mathcal{G}) \)-orbit \( \Omega \) in the set of all \( k \)-dimensional subspaces of \( S / S' \) has no kernel in the center \( \mathcal{J} \) of \( \mathcal{G} \) if \( \mathcal{S}_B \cap \mathcal{J} = (0) \) for some (and hence for all)
B ∈ Ω. Up to this point we have made no use of the fact that \( \mathcal{G} \) was a real Lie algebra, thus we may state the following result for Lie algebras over arbitrary fields.

3.5. Theorem. Let \( \mathcal{G} \) be a Lie algebra over a field \( K \). The isomorphism classes of Lie algebras \( \tilde{\mathcal{G}} \) with center \( \tilde{Z} \) of dimension \( k \), \( \tilde{\mathcal{G}} / \tilde{Z} \cong \mathcal{G} \), and without abelian direct factors, are in bijective correspondence with those \( \text{Aut}(\mathcal{G}) \)-orbits in the set of all \( k \)-dimensional subspaces of \( S/S' \) which have no kernel in \( J \).

Let \( \Lambda_2 \mathbb{R}^k \) be the set of all skew symmetric bilinear forms on \( \mathbb{R}^k \) with values in \( \mathbb{R} \), and let \( G_n(V) = \) the set of all \( n \)-dimensional subspaces of a vector space \( V \). As an application to the above theorem we get


Proof. We will consider only those algebras \( \mathcal{G} \) with \( \dim \mathcal{J} = 3 \) and with \( \mathcal{G} / \mathcal{J} \cong \mathbb{R}^6 \). By Theorem 3.5 the isomorphism classes of such algebras are in bijective correspondence with certain orbits of \( \text{GL}(6) \) in \( G_3(\Lambda_2 \mathbb{R}^6) \). Letting \( U \) be the Zariski-open subset of \( G_3(\Lambda_2 \mathbb{R}^6) \) of those 3-dimensional subspaces \( \mathcal{Q} \) with \( \mathcal{J}_\mathcal{Q} = (0) \), the orbits of \( \text{GL}(6) \) in \( U \) are in bijective correspondence with the isomorphism classes of Lie algebras which we consider. We have

\[
\dim U = \dim G_3(\Lambda_2 \mathbb{R}^6) = 3 \cdot (\binom{6}{2} - 3) = 36
\]

and \( \dim \text{GL}(6) = 36 \). However, since the center of \( \text{GL}(6) \) is acting trivially, the orbits are of dimension \( \leq 35 \). Since \( U \) is not a union of a finite number of analytic submanifolds of dimension less than \( \dim U \), the proof is complete.
4. Six-dimensional nilpotent Lie algebras with center of dimension one.

4.1. By virtue of Theorem 3.5 we can, at least in principle, classify all nilpotent Lie algebras of dimension $n$, given all such algebras of dimension $< n$, and their automorphism groups. As an application we shall work out this program for $n = 6$, and we start with the class of algebras having one-dimensional center. According to [1] the algebras of dimension 5 which we have to extend are the following

$$G_3 \times \mathbb{R}^2, G_4 \times \mathbb{R}, G_{5,1}, G_{5,2}, G_{5,3}, G_{5,4}, G_{5,5}, G_{5,6}.$$  

We shall illustrate the computations by means of two examples. The remaining six algebras are treated analogously, and the result, thirteen new algebras, is listed in the table 4.3.

If $\{e_1, \ldots, e_n\}$ is a fixed basis for the Lie algebra $G$, we let

$$B_{ij} : (\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_i) \rightarrow x_i y_j - x_j y_i, \quad 1 \leq i < j \leq n,$$

denote the elementary bilinear forms.

4.2. Extensions of $G_{5,4}$. $G_{5,4}$ is given by the following non-zero bracket-relations between the elements of a basis:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5.$$  

We first compute $\text{Aut}(G_{5,4})$. Now $G_{5,4}$ is a central extension of the Heisenberg algebra $G_3$ by $\mathbb{R}e_4 \times \mathbb{R}e_5$ determined by the bilinear form $B = (B_{13}, B_{23})$, and hence $G_B \cap G_3(\mathbb{R}^3) = \{0\}$. By (2.4) and (2.5) every $a \in \text{Aut} G_{5,4}$ can be written

$$a = \begin{pmatrix} a_0 & 0 \\ \varphi & b_0 e_0 \\ c_0 & y_0 \end{pmatrix}; \quad a_0 = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ u_1 & u_2 & e \end{pmatrix} \in \text{Aut}(G_3), e = ad-bc \neq 0, \quad \varphi \in \text{Hom}(G_3, \mathbb{R}^2).$$
Computing \( \alpha[X,Y] \) and \([aX,aY]\) (or using (2.4)) we find that

\[
\alpha = \begin{pmatrix}
a & b & 0 & 0 & 0 \\
c & d & 0 & 0 & 0 \\
u_1 & u_2 & e & 0 & 0 \\
b_1 & b_2 & au_2 - bu_1 & a e & b e \\
c_1 & c_2 & cu_2 - du_1 & c e & d e
\end{pmatrix}, \quad ad - bc = e \neq 0.
\]

Next we find a basis for the set \( S \) of all closed bilinear forms on \( \mathcal{G}_{5,4} \), and we can compute modulo the exact forms \( S' \). Thus \( B_{12}, B_{13}, B_{23} \) is a basis for the space \( S' \) checking the Jacobi-identity (2.1) on all the remaining 7 elementary forms \( B_{ij} \) we find the following basis for \( S/S' \): \( B_{25}, B_{14}, B_{24} + B_{15} \).

Before computing orbits under \( \text{Aut}(\mathcal{G}_{5,4}) \) we recall that only the forms \( B \) with \( \mathcal{J}_B \cap \mathcal{J}(\mathcal{G}_{5,4}) = (0) \) give new algebras with one-dimensional centers, in particular \( B_{25} \) and \( B_{14} \) do not satisfy this. For \( \alpha \in \text{Aut}(\mathcal{G}_{5,4}) \) we see (regarding the forms as matrices)

\[
\alpha^t(B_{24} + B_{15}) = 2aeacB_{14} + 2ebdB_{25} + e(ad+bc)(B_{24} + B_{15}), \quad \text{(modulo } S')\text{),}
\]

Hence \( B_{14} + B_{25} \) is not in the orbit of \( B_{24} + B_{15} \), however \( B_{14} + B_{25} \) satisfies \( \mathcal{J}_B \cap \mathcal{J}(\mathcal{G}_{5,4}) = (0) \). Now

\[
\alpha^t(B_{14} + B_{25}) = (a^2 + c^2) e B_{14} + (b^2 + d^2) e B_{25} + (ab+cd)e(B_{24} + B_{15}), \quad \text{(modulo } S')\text{),}
\]

and

\[
\alpha^t B_{25} = ec^2 B_{14} + ed^2 B_{25} + c d e(B_{24} + B_{15}), \quad \text{(modulo } S')\text{).}
\]

In particular the orbits \( \Omega(B_{25}) \) and \( \Omega(B_{14}) \) are identical and this orbit forms a 2-dimensional surface separating the two open orbits \( \Omega(B_{15} + B_{24}) \) and \( \Omega(B_{14} + B_{25}) \). The two last orbits correspond to two different isomorphism classes of six-dimensional algebras, and these are the only classes arising from \( \mathcal{G}_{5,4} \). Thus, within isomorphisms, the only extensions of \( \mathcal{G}_{5,4} \) are given by the bracket relations of...
\[ G_{5,4} \] together with the additional brackets,
\[
[e_1, e_5] = [e_2, e_4] = e_6 \quad (\Omega(B_{15} + B_{24}))
\]
and
\[
[e_1, e_4] = [e_2, e_5] = e_6 \quad (\Omega(B_{14} + B_{25})).
\]

We sketch the orbit space in \( S/S' \):

(4.1) Orbits in \( S/S' \cong H^2(G_{5,4}, \mathbb{R}) \) under \( \text{Aut}(G_{5,4}) \).

4.2. Extensions of \( G_{5,6} \). \( G_{5,6} \) is given by the non-zero brackets
\[
[e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_5, \ [e_2, e_3] = e_5.
\]

Hence \( G_{56} \) is a central extension of \( G_4 \) by \( \mathbb{R} e_5 \), and every \( a \in \text{Aut}(G_{5,6}) \) is of the form (2.2) where \( a_0 \in \text{Aut}(G_4) \) is determined by (2.6).

From the relation (2.4) we derive
A basis for $S/S'$ is $\{B_{34} + B_{25}, B_{24} + B_{15}\}$, and a calculation of the action of $\text{Aut}(G_{5,6})$ on each of these forms yields two orbits

$\Omega = \{s(B_{15} + B_{24}) + t(B_{34} + B_{25}) : t \neq 0\}$

$\Omega^+ = \{s(B_{15} + B_{24}) : s > 0\}$.

In addition, we have the orbit $\Omega^- = \{s(B_{15} + B_{24}) : s < 0\}$. Of course, $\Omega^+$ and $\Omega^-$ define the same isomorphism class of algebras. Thus, within isomorphisms, there are two extensions of $G_{5,6}$ with one-dimensional center. They are given by the bracket relations of $G_{5,6}$ and the additional relations:

$[e_3, e_4] = [e_2, e_5] = e_6$ (\(\Omega\))

and

$[e_1, e_5] = [e_2, e_4] = e_6$ (\(\Omega^+\))

(4.2) Orbits in $S/S' \cong H^2(G_{5,6}, \mathbb{R})$ under $\text{Aut}(G_{5,6})$. 

\[
\begin{pmatrix}
a & 0 & 0 \\
c & a^2 & 0 \\
u_1 & u_2 & a^3 \\
b_1 & b_2 & au_2 \\
c_1 & c_2 & ab_2 + cu_2 + a^2 u_1 & a^2 u_2 + ca^3 & a^5
\end{pmatrix}, \ a \neq 0.
\]
### 4.3. Table. Six-dimensional real nilpotent Lie algebras $\mathfrak{g}$ with one-dimensional center $\mathfrak{g}$. $\mathfrak{g}/\mathfrak{z} = \mathfrak{g}$.

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>Basis for the set of closed non-exact forms on $\mathfrak{g}$ ($H^2(\mathfrak{g})$)</th>
<th>$\text{Aut}(\mathfrak{g})$</th>
<th>Representative for defining orbit in $H^2(\mathfrak{g})$</th>
<th>Product in $\mathfrak{g}$</th>
<th>$\mathfrak{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{g}_3 \times \mathbb{R}$</td>
<td>$B_{13}, B_{23}, B_{24}$, $B_{25}, B_{14}, B_{15}$, $B_{45}$</td>
<td>$\begin{pmatrix} a &amp; b &amp; 0 &amp; 0 &amp; 0 \ c &amp; d &amp; 0 &amp; 0 &amp; 0 \ b_1 &amp; b_2 &amp; c_0 &amp; b_3 &amp; b_4 \ c_1 &amp; c_2 &amp; 0 &amp; c_3 &amp; c_4 \ u_1 &amp; u_2 &amp; 0 &amp; u_3 &amp; u_4 \end{pmatrix}$ (c_0 = \text{ad} - bc \neq 0) (c_3 u_4 - u_3 c_4 \neq 0)</td>
<td>$B_{13} + B_{45}$</td>
<td>$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_4, e_5] = e_6$</td>
<td>$\mathfrak{g}_{6,1}$</td>
</tr>
<tr>
<td>$\mathfrak{g}_4 \times \mathbb{R}$</td>
<td>$B_{15}, B_{25}$, $B_{14}, B_{23}$</td>
<td>$\begin{pmatrix} a &amp; 0 &amp; 0 &amp; 0 \ c &amp; d &amp; 0 &amp; 0 \ u_1 &amp; u_2 &amp; \text{ad} &amp; 0 \ b_1 &amp; b_2 &amp; a u_2 &amp; a^2 d &amp; a \end{pmatrix}$ (v_1 v_2 0 0 c_0 = 0) (\text{ad} c_0 \neq 0)</td>
<td>$B_{14} + B_{25}$</td>
<td>$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_6$</td>
<td>$\mathfrak{g}_{6,2}$</td>
</tr>
<tr>
<td>$\mathfrak{g}_{5,1}$</td>
<td>$B_{13}, B_{24}$</td>
<td>$\begin{pmatrix} a &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ d &amp; e &amp; f &amp; 0 &amp; 0 \ g &amp; h &amp; k &amp; 0 &amp; 0 \ u_1 &amp; u_2 &amp; u_3 &amp; \text{ae} &amp; \text{af} \ v_1 v_2 \text{v_3} &amp; \text{ah} &amp; \text{ak} \end{pmatrix}$</td>
<td>$B_{25} + B_{34}$</td>
<td>$[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_2, e_5] = e_6$</td>
<td>$\mathfrak{g}_{6,4}$</td>
</tr>
<tr>
<td>$\mathfrak{g}_{5,2}$</td>
<td>$B_{14}, B_{15}, B_{24}$, $B_{35}, B_{25} + B_{34}$, $B_{23}$</td>
<td>$\begin{pmatrix} a &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ d &amp; e &amp; f &amp; 0 &amp; 0 \ g &amp; h &amp; k &amp; 0 &amp; 0 \ u_1 &amp; u_2 &amp; u_3 &amp; \text{ae} &amp; \text{af} \ v_1 v_2 \text{v_3} &amp; \text{ah} &amp; \text{ak} \end{pmatrix}$ (a \neq 0, \text{ek} \neq hf)</td>
<td>$B_{35} + B_{14}$</td>
<td>$[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_1, e_4] = e_6$, $[e_2, e_5] = e_6$</td>
<td>$\mathfrak{g}_{6,5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$B_{24} + B_{35}$</td>
<td>$[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$, $[e_2, e_4] = e_6$, $[e_3, e_5] = e_6$</td>
<td>$\mathfrak{g}_{6,6}$</td>
</tr>
<tr>
<td>$G_{5,3}$</td>
<td>$B_{15} - B_{24}$</td>
<td>$B_{15} - B_{34}$</td>
<td>$[e_1, e_2] = e_4$</td>
<td>$[e_1, e_4] = e_5$</td>
<td>$[e_1, e_5] = e_6$</td>
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<tr>
<td>$B_{24} - B_{13}$</td>
<td>(\begin{pmatrix} a &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; d &amp; 0 &amp; 0 &amp; 0 \ u_1 u_2 &amp; a_2 &amp; 0 &amp; 0 \ v_1 v_2 &amp; 0 &amp; ad &amp; 0 \ b_1 b_2 b_3 b_4 a_2 a_3 \end{pmatrix})</td>
<td>$b_4 = av_2 - du_1$, $ad \neq 0$</td>
<td>$B_{15} - B_{34}$</td>
<td>$[e_1, e_2] = e_4$</td>
<td>$[e_1, e_4] = e_5$</td>
</tr>
<tr>
<td>$G_{5,4}$</td>
<td>$B_{14} - B_{25}$</td>
<td>$B_{15} + B_{24}$</td>
<td>$[e_1, e_2] = e_3$</td>
<td>$[e_1, e_3] = e_4$</td>
<td>$[e_1, e_4] = e_6$</td>
</tr>
<tr>
<td>$B_{15} + B_{24}$</td>
<td>(\begin{pmatrix} a &amp; b &amp; 0 &amp; 0 &amp; 0 \ b &amp; d &amp; 0 &amp; 0 &amp; 0 \ c_1 c_2 &amp; e &amp; 0 &amp; 0 \ u_1 u_2 &amp; a_2 &amp; 0 &amp; 0 \ b_1 b_2 b_3 a_2 e_3 \end{pmatrix})</td>
<td>$e = ad - bc \neq 0$</td>
<td>$b_3 = au_2 - bu_1$, $c_2 = cu_2 - du_1$</td>
<td>$B_{25} + B_{14}$</td>
<td>$[e_1, e_2] = e_3$</td>
</tr>
<tr>
<td>$G_{5,5}$</td>
<td>$B_{15} - B_{23}$</td>
<td>$B_{15}$</td>
<td>$[e_1, e_2] = e_3$</td>
<td>$[e_1, e_3] = e_4$</td>
<td>$[e_1, e_4] = e_6$</td>
</tr>
<tr>
<td>$B_{15} + B_{23}$</td>
<td>(\begin{pmatrix} a &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ c &amp; d &amp; 0 &amp; 0 &amp; 0 \ u_1 u_2 &amp; ad &amp; 0 &amp; 0 \ b_1 b_2 a_2 &amp; 0 &amp; 0 &amp; 0 \ c_1 c_2 &amp; b_2 &amp; a_2 &amp; b_3 \end{pmatrix})</td>
<td>$ad \neq 0$</td>
<td>$B_{15} + B_{23}$</td>
<td>$[e_1, e_2] = e_3$</td>
<td>$[e_1, e_3] = e_4$</td>
</tr>
<tr>
<td>$G_{5,6}$</td>
<td>$B_{24} - B_{15}$</td>
<td>$B_{24} + B_{15}$</td>
<td>$[e_1, e_2] = e_3$</td>
<td>$[e_1, e_3] = e_4$</td>
<td>$[e_1, e_4] = e_6$</td>
</tr>
<tr>
<td>$B_{25} + B_{34}$</td>
<td>(\begin{pmatrix} a &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ c &amp; a^2 &amp; 0 &amp; 0 &amp; 0 \ u_1 u_2 &amp; a_3 &amp; 0 &amp; 0 \ b_1 b_2 a_2 &amp; 0 &amp; 0 &amp; 0 \ c_1 c_2 &amp; c_3 &amp; c_4 a_5 \end{pmatrix})</td>
<td>$B_{24} + B_{15}$</td>
<td>$[e_1, e_2] = e_3$</td>
<td>$[e_1, e_3] = e_4$</td>
<td>$[e_1, e_4] = e_6$</td>
</tr>
<tr>
<td>$G_{6,7}$</td>
<td></td>
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</tbody>
</table>
5. Six-dimensional nilpotent Lie algebras with center of dimension $\geq 2$.

5.1. Let $G_k(V)$ be the set of $k$-dimensional linear subspaces of a linear space $V$. Then by Theorem 3.5, the orbits of $\text{Aut}(\mathcal{G})$ in $G_k(H^2(\mathcal{G}))$ determine isomorphism classes of Lie algebras $\tilde{\mathcal{G}}$ with $\dim \tilde{\mathcal{G}} = \dim \mathcal{G} + k$. Here $G_k(V)$ is a real-analytic compact manifold, and the action of $\text{Aut}(\mathcal{G})$ is real-analytic. We will consider the case

$$\dim \tilde{\mathcal{G}} = 6, \quad \dim \mathcal{G} \leq 4,$$

assuming that $\mathcal{G}$ does not have a direct abelian factor algebra.

If $\dim \mathcal{G} \leq 3$, then $\mathcal{S}(\mathcal{G}) \subset \mathcal{A}_2(\mathcal{G})$ implies that $\dim H^2(\mathcal{G}) \leq \dim \mathcal{A}_2(\mathcal{G}) \leq 3$ with equality only for $\mathcal{G} = \mathbb{R}^3$. Thus we obtain the algebra $\tilde{\mathcal{G}} = \mathbb{R}^5(B_{12}, B_{23}, B_{13})$. When $\dim \mathcal{G} = 4$, we have $\mathcal{G} = \mathcal{G}_4, \mathbb{R} \times \mathcal{G}_3$, or $\mathbb{R}^4$.

5.2. $\mathcal{G} = \mathcal{G}_4$: The product in $\mathcal{G}_4$ is $[e_1, e_2] = e_3, [e_1, e_3] = e_4$, and hence $\mathcal{S}(\mathcal{G}_4) = (B_{12}, B_{13}) \subset \mathcal{S} = (B_{12}, B_{13}, B_{14}, B_{23})$, which gives the unique algebra $\tilde{\mathcal{G}} = \mathcal{G}_4(B_{14}, B_{23})$ of dimension 6.

5.3. $\mathcal{G} = \mathbb{R}^4$. In this case we have $H^2(\mathbb{R}^4) = \Lambda_2 \mathbb{R}^4$ of dimension 6 with basis $B_{ij}, 1 \leq i < j \leq 4$, and

$$\text{Aut}(\mathbb{R}^4) = \text{GL}(4).$$

$G_2(\Lambda_2 \mathbb{R}^4)$ is the union of 4 disjoint $\text{GL}(4)$-invariant sets $\Omega_i$, $1 \leq i \leq 4$. We first define the $\Omega_i$, and then show that they actually are orbits of $\text{GL}(4)$. Let $P$ be a two-dimensional subspace of $\Lambda_2 \mathbb{R}^4$.

$$\Omega_1 = \{ P : \mathcal{F}_P \neq (0) \}$$

$$\Omega_2 = \{ P : \mathcal{F}_P = (0) \} \quad \text{and} \quad P \text{ contains forms } B, B' \text{ with}$$

$$B \text{ of rank } 2 \quad \text{and} \quad B'| \mathcal{F}_B \neq 0 \}$$
\[ \Omega_3 = \{ P : \mathcal{J}_P = (0) \text{ and } P = (B, B') \text{ where } B \text{ is of rank 2 and } B' | \mathcal{J}_B = 0 \} \]

\[ \Omega_4 = \{ P : \text{any } B \in P \text{ is nondegenerate or trivial} \} \]

5.3.1. Clearly \( G_2(\Lambda_2 \mathbb{R}^4) \) is the union of the \( \Omega_i \). To show that \( \Omega_1 \) is an orbit, let \( P \in \Omega_1 \). Then \( \dim \mathcal{J}_P = 1 \), and hence, setting \( \mathbb{R}^3 = \mathbb{R}^4/\mathcal{J}_P \), we obtain a plane \( P' \in G_2(\Lambda_2 \mathbb{R}^3) \). The exterior product \( (\Lambda_2 \mathbb{R}^3) \otimes (\mathbb{R}^3)^* = \Lambda^2(\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^* \rightarrow \Lambda^2(\mathbb{R}^3)^* = \mathbb{R} \), defines a \( \text{GL}(3) \)-invariant isomorphism \( G_2(\Lambda_2 \mathbb{R}^3) = G_2(\mathbb{R}^3) \), and hence \( \text{GL}(3) \) is acting transitively on \( G_2(\Lambda_2 \mathbb{R}^3) \). Also \( \text{GL}(4) \) is acting transitively on the set of lines through the origin in \( \mathbb{R}^4 \), such as \( \mathcal{J}_P \), and hence it follows that \( \text{GL}(4) \) is acting transitively on \( \Omega_1 \). By the same reasoning, \( \text{SO}(4) \) is acting transitively on \( \Omega_1 \), and one can show that \( \Omega_1 = \text{SO}(4)/\mathbb{Z}_2 \times \text{O}(2) \).

5.3.2. For \( P \in \Omega_2 \), we will choose a basis for \( \mathbb{R}^4 \) such that there is a basis \( B, B' \) of \( P \) which has a standard matrix form in terms of the basis for \( \mathbb{R}^4 \). It will follow that \( \Omega_2 \) is an orbit of \( \text{GL}(4) \). To find such a basis, let \( B \) be of rank 2, and \( B' | \mathcal{J}_B \neq 0 \). Choose \( e_i \in \mathbb{R}^4 \) such that \( \mathcal{J}_B = (e_3, e_4) \) and \( \ker B'(e_3, -) \cap \ker B'(e_4, -) = (e_1, e_2) \). Since \( B' | \mathcal{J}_B \) is nonsingular, we have a basis for \( \mathbb{R}^4 \), in terms of which \( B = aB_{12} \) and \( B' = bB_{12} + cB_{34} \), \( a \neq 0 \). Hence \( P \) has standard form \( P = (B_{12}, B_{34}) \). It follows that the isotropy group of \( P \) has identity component \( (\text{GL}(2) \times \text{GL}(2))^0 \), and hence that \( \dim \Omega_2 = 16 - 8 = 8 = \dim G_2(\Lambda_2 \mathbb{R}^4) \) so that \( \Omega_2 \) is an open orbit. Because \( (B_{12}, B_{34}) \) is invariant under \( A = \text{diag}(-1, 1, 1, 1) \), it follows that \( \Omega_2 \) is connected.

5.3.3. When \( P \in \Omega_3 \), we have \( P = (B, B') \) where \( B \) is of rank 2 and \( B' | \mathcal{J}_B = 0 \). We will find a suitable basis for \( \mathbb{R}^4 \).
If $J_B' + J_B = \mathbb{R}^4$, we clearly have $B' = 0$. Hence, if $\dim J_B = 2$, we obtain $J_P = J_B \cap J_B \neq (0)$, contrary to the definition of $\Omega_3$.

It follows that $\dim J_B = 0$, that is, $B'$ is nondegenerate, and we can choose a basis for $\mathbb{R}^4$ such that $J_B = (e_3, e_4)$ and $\text{Ker} B'(e_3, -) = (e_3, e_4, e_2)$, $B'(e_1, e_3) = 1$, $\text{Ker} B'(e_4, -) = (e_3, e_4, e_1)$, and $B'(e_2, e_4) = 1$.

Then $B = aB_{12}$, $B' = B_{13} + B_{24}$ and $P = (B_{12}, B_{13} + B_{24})$ and hence $\Omega_3$ is a single orbit. One can show that the Lie algebra of the isotropy group of $P$ (the isotropy algebra, see 5.4.3) is

$$\mathfrak{g}_P = \left\{ \begin{pmatrix} A & 0 \\ B & cI - At \end{pmatrix} : A, B \in \text{gl}(2), c \in \mathbb{R} \right\},$$

and hence that $\dim \Omega_3 = 16 - 9 = 7$. $\Omega_3$ has two components because the orientation of $\mathbb{R}^4$ defined by the chosen basis $(e_1, e_2, e_3, e_4)$ depends only on $P$.

5.4.3. In order to show that $\Omega_4$ is a single orbit, we will do as follows: First we will find a connected subset $C$ of $\Omega_4$ such that every $\text{GL}(4)$-orbit in $\Omega_4$ meets $C$. Next we will show that each point in $C$ lies in an open $\text{GL}(4)$-orbit. It then follows that $C$ lies in a single orbit, which now must equal $\Omega_4$. To find the set $C$, let $P = (B, B') \in \Omega_4$, and choose a plane $\pi_1 \subset \mathbb{R}^4$ such that $B|\pi_1 \neq 0$ and $B'|\pi_1 \neq 0$. Then there are unique planes $\pi_2, \pi_3 \subset \mathbb{R}^4$ such that $B(\pi_1, \pi_2) = 0$ and $B'(\pi_1, \pi_3) = (0)$. There is a number $c$ such that $B'|\pi_1 = cB|\pi_1$. It follows that $(B' - cB)|((\pi_1 + (\pi_2 \cap \pi_3)) = 0$. Since $B' - cB$ is nondegenerate, $\dim(\pi_1 + (\pi_2 \cap \pi_3)) \leq 2$, and hence $\pi_2 \cap \pi_3 = (0)$. It follows that $\pi_i \cap \pi_j = (0)$ for $1 \leq i < j \leq 3$.

Lemma. $\text{GL}(4)$ is acting transitively on the set of triples $(\pi_1, \pi_2, \pi_3)$ where $\pi_i$ are planes in $\mathbb{R}^4$ such that $\pi_i \cap \pi_j = (0)$ for $i \neq j$. 
Proof. Clearly, $GL(4)$ is acting transitively on the set of pairs $(\pi_1, \pi_2)$ where $\pi_1 \cap \pi_2 = (0)$. Hence it suffices to show that $GL(2)^2 = GL(\pi_1) \times GL(\pi_2)$ is acting transitively on the set of all planes $\pi_3$ with $\pi_3 \cap \pi_1 = \pi_3 \cap \pi_2 = (0)$. Because $\pi_3 \cap \pi_2 = (0)$, there is a linear map $\phi: \pi_1 \to \pi_2$ such that

$$\pi_3 = \{v + \phi(v) : v \in \pi_1\},$$

and $\phi$ is an isomorphism because $\text{Ker} \phi = \pi_1 \cap \pi_3 = (0)$.

For $(A_1, A_2) \in GL(2)^2 \subset GL(4)$, we have

$$(A_1, A_2)(\pi_3) = \{A_1 v + A_2 \phi(v) : v \in \pi_1\} = \{v + A_2 \phi A_1^{-1}(v) : v \in \pi_1\},$$

and since the action $(A_1, A_2)\phi = A_2 \phi A_1^{-1}$ on the set of linear isomorphisms $\phi: \pi_1 \to \pi_2$ is transitive, the proof is complete.

It follows from the lemma that there is some $P' = (B_1, B_2)$ in the $GL(4)$-orbit of $P$ such that $B_1(\pi_1, \pi_2) = (0)$ and $B_2(\pi_1, \pi_3) = (0)$ where

$$\pi_1 = (e_3, e_4), \quad \pi_2 = (e_1, e_2), \quad \text{and} \quad \pi_3 = (e_1 + e_3, e_2 + e_4).$$

The only $B_1, B_2$ satisfying this condition are

$$B_1 = aB_{12} + bB_{34}, \quad ab \neq 0, \quad \text{and}$$

$$B_2 = cB_{12} + d(B_{12} + B_{22} + B_{14}), \quad d \neq 0,$$

where $B_2$ is nondegenerate for $d \neq 0$. It follows that

$$P' = (B_{12} + b'B_{34}, c'B_{34} + B_{23} + B_{14}), \quad b' \neq 0.$$

Choose $x > 0$ and $\epsilon = \frac{1}{x}$ such that $x^2 b' = -\epsilon$ and set $a = -xc'$. Then the matrix $\text{diag}(1, 1, x, -x)$ will transform $P'$ to

$$P'' = (B_{12} + \epsilon B_{34}, \alpha B_{34} + B_{23} + B_{14}) = P(a, \epsilon).$$

$$\det(s(B_{12} + \epsilon B_{34}) + t(\alpha B_{34} + B_{23} + B_{14})) = (s(\epsilon s + \alpha t) + t^2)^2 =$$

$$= ((t + as/2)^2 - (\alpha^2 - 4\epsilon) s^2/4)^2.$$
This polynomial in \( s \) and \( t \) contains a linear factor if and only if \( \alpha^2 \geq 4 \varepsilon \), and \( P(\alpha, \varepsilon) \) will then contain a degenerate form. Hence there is a connected subset \( C \) of \( \Omega_4 \),

\[
C = \{ P(\alpha, 1) : |\alpha| < 2 \} = \{ P(\alpha, \varepsilon) : P(\alpha, \varepsilon) \in \Omega_4 \}
\]

To complete the proof that \( \Omega_4 \) is a single orbit, it suffices to show that each of the planes \( P(\alpha, 1) \), \( |\alpha| < 2 \), lies on an open orbit in \( G_2(\Lambda_2 \mathbb{R}^4) \).

For \( B \in \Lambda_2 \mathbb{R}^4 \) and \( A \in \mathrm{gl}(4) \), we set \( B^A(v, w) = B(e^tA_v, e^tA_w)^t = B(Av, w) + B(v, Aw) \). Then the isotropy algebra of a plane \( P \subset \Lambda_2 \mathbb{R}^4 \) is

\[
\mathcal{J}_P = \{ A \in \mathrm{gl}(4) : B^A \in P \text{ for all } B \in P \}.
\]

Setting \( P = (B_1, B_2) \), the conditions \( B_1^A, B_2^A \in P \) are a system of linear equations in the entries of \( A \), and the rank of this system equals \( \dim \mathrm{gl}(4) - \dim \mathcal{J}_P \), which is equal to \( \dim \Omega(P) \). To write down those equations for the plane \( P(\alpha, \varepsilon) \), we note that, with \( A = (a_{ij}) \),

\[
(B_{ij})^A = \sum_{k=1}^4 (a_{ik} B_{kj} - a_{jk} B_{ki}).
\]

Setting \( B_1 = B_{12} + \varepsilon B_{34} \) and \( B_2 = \alpha B_{34} + B_{23} + B_{14} \), we have

\[
B_1^A = (a_{23} - \varepsilon a_{41}) B_{13} + (-a_{14} + \varepsilon a_{32}) B_{24}
\]

\[
+ (a_{11} + a_{22}) B_{12} + \varepsilon (a_{33} + a_{44}) B_{34}
\]

\[
- (a_{23} + \varepsilon a_{42}) B_{23} + (a_{24} + \varepsilon a_{31}) B_{14},
\]

and

\[
B_2^A = (-a a_{41} + a_{21} + a_{43}) B_{13} + (a a_{22} + a_{34} + a_{12}) B_{24}
\]

\[
+ (-a_{31} + a_{42}) B_{12} + (a a_{33} + a a_{44} - a_{24} + a_{13}) B_{34}
\]

\[
+ (-a a_{42} + a_{22} + a_{33}) B_{23} + (a a_{31} + a_{11} + a_{44}) B_{14}.
\]
The condition \( B_1^A \in (B_1, B_2) \) is expressed by the 4 equations

\[
\begin{align*}
(1) \quad & a_{23} - \varepsilon a_{41} = 0 \\
(2) \quad & a_{32} - \varepsilon a_{14} = 0 \\
(3) \quad & a_{24} + \varepsilon a_{31} + a_{13} + \varepsilon a_{42} = 0 \\
(4) \quad & a_{33} + a_{44} - a_{11} - a_{22} + \alpha \varepsilon a_{13} + \varepsilon a_{42} = 0,
\end{align*}
\]

and the condition \( B_2^A \in (B_1, B_2) \) by

\[
\begin{align*}
(5) \quad & a_{21} - \varepsilon a_{41} + a_{43} = 0 \\
(6) \quad & a_{12} + \alpha a_{32} + a_{34} = 0 \\
(7) \quad & a_{11} - a_{22} + a_{44} - a_{33} + \alpha \varepsilon a_{42} + a_{31} = 0 \\
(8) \quad & a_{24} - \varepsilon a_{31} + \alpha a_{22} - \alpha a_{44} - a_{13} + (\varepsilon - \alpha^2) a_{42} = 0.
\end{align*}
\]

The system \( S_1 \) formed by the equations numbered (1), (2), (5), and (6) involve only the 8 variables \( a_{12}, a_{21}, a_{23}, a_{32}, a_{14}, a_{41}, a_{34}, \) and \( a_{43} \) and this system is of rank 4.

The system \( S_2 \) formed by the equations numbered (3), (4), (7), and (8) involve only the 8 remaining variables \( a_{13}, a_{31}, a_{24}, a_{42}, \) and \( a_{ii} \). The linear space generated by the coefficient columns of \( S_2 \) is generated by the coefficient columns of \( a_{24}, a_{11}, a_{22}, \) and \( a_{13} \), and the coefficient matrix of those variables is

\[
M = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & -1 & -1 & \varepsilon a \\
0 & 1 & -1 & 0 \\
1 & 0 & \alpha & -1
\end{pmatrix}
\]

with \( \det M = \varepsilon \alpha^2 - 4 \).

Hence \( S_2 \) has rank 4 if \( \alpha^2 \neq 4\varepsilon \), and has rank 3 if \( \alpha^2 = 4\varepsilon \), and the whole system (1) through (8) has rank 8 if \( \alpha^2 \neq 4\varepsilon \) and has rank 7 if \( \alpha^2 = 4\varepsilon \).

It follows that \( \dim \Omega(P(\alpha, \varepsilon)) = 8 \) when \( \alpha^2 \neq 4\varepsilon \), and hence that each \( P(\alpha, \varepsilon) \) contained in \( C \) lies on an open orbit, completing the proof.
5.4. Theorem. \( G_2(\Lambda_2 \mathbb{R}^4) \) is the union of four \( \text{GL}(4) \)-orbits \( \Omega_i \), \( 1 \leq i \leq 4 \), such that

\[
(B_{12}, B_{23}) \in \Omega_1, \quad \text{dim} \Omega_1 = 5,
\]
\[
(B_{12}, B_{34}) \in \Omega_2, \quad \text{dim} \Omega_2 = 8,
\]
\[
(B_{12}, B_{13} + B_{24}) \in \Omega_3, \quad \text{dim} \Omega_3 = 7, \quad \text{and}
\]
\[
(B_{12} + B_{34}, B_{23} + B_{14}) \in \Omega_4, \quad \text{dim} \Omega_4 = 8.
\]

Remark. To determine the orbit \( \Omega_i \) to which a given plane \( P \subseteq \Lambda_2 \mathbb{R}^4 \) belongs, one can proceed as follows. Let \( e_1, \ldots, e_4 \) be the standard basis of \( \mathbb{R}^4 \) and define a symmetric bilinear form on \( \Lambda_2 \mathbb{R}^4 \) by

\[
\langle B_1, B_2 \rangle = \frac{1}{4} \sum_{\sigma} \text{sgn}(\sigma)B_1(e_{\sigma(1)}, e_{\sigma(2)})B_2(e_{\sigma(3)}, e_{\sigma(4)}),
\]

the summation extending over all permutations of \( \{1, 2, 3, 4\} \). In terms of the basis \( B_{ij} \) for \( \Lambda_2 \mathbb{R}^4 \), we have \( \langle B_\sigma(1)\sigma(2), B_\sigma(3)\sigma(4) \rangle = \text{sgn}(\sigma) \) and \( \langle B_{ij}, B_{jk} \rangle = 0 \), and hence \( \langle \cdot, \cdot \rangle \) is a nondegenerate symmetric form on \( \Lambda_2 \mathbb{R}^4 \) of signature \((3,3)\). Restricting this form to \( P \), we obtain \( \langle \cdot, \cdot \rangle_P \), and the orbit containing \( P \) is characterized as follows.

\[
\langle \cdot, \cdot \rangle_P = 0 \iff P \in \Omega_1,
\]
\[
\langle \cdot, \cdot \rangle_P \text{ is of rank one} \iff P \in \Omega_3,
\]
\[
\langle \cdot, \cdot \rangle_P \text{ is nonsingular, indefinite} \iff P \in \Omega_2,
\]
\[
\langle \cdot, \cdot \rangle_P \text{ is nonsingular and definite} \iff P \in \Omega_4.
\]

The orbits \( \Omega_3 \) and \( \Omega_4 \) both have two components corresponding to those \( P \) with \( \langle \cdot, \cdot \rangle_P \) positive or negative semidefinite.

5.5. \( G = \mathbb{R} \times G_3; \) \( [e_1, e_2] = e_3, \ e_4 \) is central,

\[
S(G) = (B_{13}, B_{23}, B_{14}, B_{24}, B_{12}), \quad S'(G) = (B_{12}).
\]

The automorphisms of \( G \) have matrix form
The automorphism induced by $A$ in $H^2(\mathcal{G})$ does not depend on $r, s, t,$ and $u$. Hence, it suffices to take $A = \varphi \psi = \psi \varphi$ where

$$
\varphi = \begin{pmatrix}
  a & b & 0 & 0 \\
  c & d & 0 & 0 \\
 r & s & \delta & \beta \\
 t & u & 0 & \alpha
\end{pmatrix}, \quad \psi = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
 0 & 0 & 1 & \beta \\
 0 & 0 & 0 & \alpha
\end{pmatrix},
$$

and we have

(5.1) \hspace{1cm}
B_{1j} \circ \varphi = \delta(ab_{1j} + bB_{2j}) , \hspace{1cm} B_{13} \circ \psi = B_{13} + \beta B_{14} , \hspace{1cm}
B_{2j} \circ \varphi = \delta(cB_{1j} + dB_{2j}) , \hspace{1cm} B_{14} \circ \psi = \alpha B_{14} , \hspace{1cm} i = 1, 2, \ j = 3, 4.

**Theorem.** There are 7 orbits of $\text{Aut}(\mathbb{R} \times \mathcal{G}_3)$ in $G_2(H^2(\mathbb{R} \times \mathcal{G}_3))$. We write them as $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_3, \mathcal{Q}_1, \mathcal{Q}_3, \mathcal{Q}_4$, and $\mathcal{Q}_4^c$ where the subscript is the dimension of the orbit, and

\[
\{(B_{14}, B_{24})\} = \mathcal{O}_0
\]

\[
(B_{14}, B_{13}) \in \mathcal{O}_1
\]

\[
(B_{14}, B_{23}) \in \mathcal{O}_3
\]

\[
(B_{13}, B_{23}) \in \mathcal{Q}_1
\]

\[
(B_{13}, B_{23} + B_{14}) \in \mathcal{Q}_3
\]

\[
(B_{13}, B_{23} + B_{24}) \in \mathcal{Q}_4
\]

\[
(B_{13} + B_{24}, B_{23} - B_{14}) \in \mathcal{Q}_4^c
\]

**Proof.** It follows from (5.1) that $P_4 = (B_{14}, B_{24})$ is $\text{Aut}(\mathcal{G})$-invariant. Let $X$ be the closed subspace of $G_2(H^2(\mathcal{G}))$ consisting of the planes $P$ with $P \cap P_4 \neq \emptyset$. It is not difficult to see that $X$ is the disjoint union of the following three sets,
\[ \mathcal{O}_0 = \{P_4\} \]
\[ \mathcal{O}_1 = \{(sB_{14} + tB_{24}, sB_{13} + tB_{23}) : (s, t) \neq (0, 0)\} \]
\[ \mathcal{O}_3 = \{(sB_{14} + tB_{24}, u(tB_{14} - sB_{24}) + sB_{13} + tB_{23}) : (s, t) \neq (0, 0), (u, st_1 - ts_1) \neq (0, 0)\} \]

By directly computing the orbit through \((B_{14}, B_{13})\) and \((B_{14}, B_{23})\), using formulas (5.1), we find that they are \(\mathcal{O}_1\) and \(\mathcal{O}_3\). Clearly \(\dim \mathcal{O}_1 = 1\), and \(\dim \mathcal{O}_3 = \dim X = 3\).

Next we have to consider the planes \(P\) with \(P \cap P_4 = \langle 0 \rangle\). Let \(\pi : H^2(\mathcal{G}) \rightarrow (B_{13}, B_{23})\) be the projection with kernel \(P_4\). Then \(\pi(P) = (B_{13}, B_{23})\) and hence \(P = (B_{13} + v_1, B_{23} + v_2)\) where \(v_1, v_2 \in P_4\) are uniquely determined by \(P\). Setting
\[ v_1 = a_{11}B_{14} + a_{21}B_{24}, \quad A(P) = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \]
\[ v_2 = a_{12}B_{14} + a_{22}B_{24}, \]
the matrix \(A(P)\) is uniquely determined by \(P\), and every 2 by 2 matrix \(A\) is of the form \(A(P)\) for a unique \(P\). Setting \(A(P)\varphi = A(P^o\varphi)\) and \(A(P)\psi = A(P^o\psi)\), a direct computation, using (5.1) shows that, for any \(A\),
\[ A\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} A \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A\psi = aA + \beta I. \]

This defines an action of \(\text{Aut}(\mathcal{G})\) in the affine space of 2 by 2 matrices, and it is not difficult to see that the orbits are

- \(Q_1\): scalar matrices,
- \(Q_3\): matrices with only one eigenvalue, but which are not scalar,
- \(Q_4\): matrices with two real eigenvalues, and
- \(Q_4^c\): matrices with complex eigenvalues.

Here \(Q_4\) and \(Q_4^c\) are open orbits. Also, \(\dim Q_3 = 3\) because \(Q_3 \cup Q_1\) is the variety in \(\mathbb{R}^4\) defined by \((a_{11} - a_{22})^2 = 4a_{12}a_{21}\).
Choosing representatives for the orbits such as \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), we obtain representatives for the corresponding orbits in \( \mathbb{G}_2(\mathbb{H}^2(\mathfrak{g})) \),

\[
\begin{align*}
(B_{13}, B_{23}) &\in Q_1, (B_{13}, B_{23} + B_{14}) \in Q_3 \\
(B_{13}, B_{23} + B_{24}) &\in Q_4, (B_{13} + B_{24}, B_{23} - B_{14}) \in Q_4^C.
\end{align*}
\]

This completes the proof of the theorem. We note that \( \sigma_0 \) and \( Q_1 \) are degenerate orbits in the sense that \( \mathcal{H}_p \cap \mathfrak{g} = (e_3) \neq (0) \) and \( \mathcal{H}_{B_{13}} \cap \mathcal{H}_{B_{23}} \cap \mathfrak{g} = (e_4) \neq (0) \). Hence, there are 5 Lie algebras with center of dimension 2 obtained from the remaining 5 orbits.

5.6. Table. Six-dimensional real nilpotent Lie algebras \( \tilde{\mathcal{G}} \) with center \( \tilde{\mathfrak{g}} \) of dimension \( \geq 2 \). \( \mathcal{G} = \mathfrak{g}/\tilde{\mathfrak{g}} \).

<table>
<thead>
<tr>
<th>( \mathcal{G} )</th>
<th>Defining orbit in ( \mathbb{G}_k(\mathbb{H}^2(\mathfrak{g})) ); ( k = 2, 3 )</th>
<th>Product in ( \mathfrak{g} ) (only non-zero brackets are given).</th>
<th>( \mathfrak{g} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R}^3 )</td>
<td>( k = 3 )</td>
<td>( \mathcal{H}^2(\mathfrak{g}) = {e_1, e_2} = e_4 ) ( [e_2, e_3] = e_5 ) ( [e_1, e_3] = e_6 )</td>
<td>( \mathcal{G}_{6, 14} )</td>
</tr>
<tr>
<td>( \mathbb{R}^4 )</td>
<td>( k = 2 )</td>
<td>( \Omega_2 ) ( [e_1, e_2] = e_5 ) ( [e_3, e_4] = e_6 )</td>
<td>( \mathcal{G}<em>{3} \times \mathcal{G}</em>{3} )</td>
</tr>
<tr>
<td>( \Omega_3 )</td>
<td>( [e_1, e_2] = e_5 ) ( [e_1, e_3] = [e_2, e_4] = e_6 )</td>
<td>( \mathfrak{g}_{6, 15} )</td>
<td></td>
</tr>
<tr>
<td>( \Omega_4 )</td>
<td>( [e_1, e_2] = [e_3, e_4] = e_5 ) ( [e_2, e_3] = [e_1, e_4] = e_6 )</td>
<td>( \mathfrak{g}_{6, 16} )</td>
<td></td>
</tr>
<tr>
<td>( \mathbb{R} \times \mathcal{G}_3 )</td>
<td>( k = 2 )</td>
<td>( \sigma_1 ) ( [e_1, e_2] = e_3 ) ( [e_1, e_3] = e_5 ) ( [e_1, e_4] = e_6 )</td>
<td>( \mathfrak{g}_{6, 17} )</td>
</tr>
<tr>
<td>( \sigma_3 )</td>
<td>( [e_1, e_2] = e_3 ) ( [e_2, e_3] = e_5 ) ( [e_1, e_4] = e_6 )</td>
<td>( \mathfrak{g}_{6, 18} )</td>
<td></td>
</tr>
<tr>
<td>( Q_3 )</td>
<td>([e_1, e_2] = e_3 \quad [e_1, e_3] = e_5 )</td>
<td>([e_1, e_4] = [e_2, e_3] = e_6 )</td>
<td>( G_{6,19} )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>( Q_4 )</td>
<td>([e_1, e_2] = e_3 \quad [e_1, e_3] = e_5 )</td>
<td>([e_2, e_3] = [e_2, e_4] = e_6 )</td>
<td>( G_{6,20} )</td>
</tr>
<tr>
<td>( Q_4^c )</td>
<td>([e_1, e_2] = e_3 \quad [e_1, e_3] = [e_2, e_4] = e_5 )</td>
<td>([e_2, e_3] = [e_1, e_4] = e_6 )</td>
<td>( G_{6,21} )</td>
</tr>
<tr>
<td>( G_4 )</td>
<td>( k = 2 )</td>
<td>([e_1, e_2] = e_3 \quad [e_1, e_3] = e_4 )</td>
<td>( G_{6,22} )</td>
</tr>
</tbody>
</table>

Clearly the real nilpotent Lie algebras of dimension six containing direct factors are, within isomorphisms,

\[
G_1^6; G_3 \times G_1^3; G_4 \times G_1^2; G_5, i \times G_1, 1 \leq i \leq 6; G_3 \times G_3.
\]

Theorem. Every six-dimensional real nilpotent Lie algebra with no nontrivial direct factor is isomorphic to one of the following algebras

\[
G_{6,1}, G_{6,2}, \ldots, G_{6,22}.
\]

These algebras are pairwise nonisomorphic.

**Reference**