

Spectral Properties of Positive Maps
on C^* -algebras

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Abstract: Perron-Frobenius type results are proved for discrete, Markovian, quantum stochastic processes.

§ 1. Introduction.

At the beginning of the century, Perron [19] and Frobenius [9,10] discovered many important spectral properties possessed by matrices positive entries. There now exists a vast literature extending some of their results to positive operators on a large class of ordered vector spaces. The most successful results being with compact operators and/or cones with a lattice ordering or a large interior. We refer the reader to [11,15,22,32] and the references quoted therein. Here we regard the original Perron-Frobenius theory as being concerned with the spectral properties of positive operators on finite dimensional, commutative C^* -algebras, and give a non-commutative version of this theory, at least for finite dimensional C^* -algebras.

The **first part** of the Perron-Frobenius story tells us that the spectral radius of a positive matrix with positive entries is an eigenvalue, possessing a positive eigenvector. Moreover, if the matrix is irreducible in a certain sense, then the spectral radius is a simple eigenvalue, and (apart from scalar multiplication) there are no other positive eigenvectors. It is this part of the theory which has received most attention by other authors, referred to above. In § 2 we give our generalization of this to finite dimensional C^* -algebras.

Perron and Frobenius also showed that the spectrum and eigenvectors of an irreducible positive matrix had certain multiplicative properties. This part of the theory has not received nearly as much attention, although Rota [20] and Schaefer [21] obtained some results in this direction for certain lattice ordered spaces, namely L^p -spaces and commutative C^* -algebras. Analogous results were obtained by Størmer for ergodic groups of automorphisms on von Neumann algebras [29]. In § 3 and § 4 we study multiplicative properties associated

with the spectrum of an irreducible positive operator on a finite dimensional C^* -algebra. In § 3, it is mainly the Jordan structure which is important, but the C^* -structure takes over in § 4 when we restrict attention to those maps which satisfy the Schwarz inequality which we call Schwarz maps.

In [4,5] Davies has proposed concepts of recurrence and transience for certain continuous time Markovian quantum stochastic processes. In § 3 we also propose definitions of recurrence and transience for discrete time Markovian quantum stochastic processes, which are different from those of Davies. (Davies' ideas easily carry over from continuous time to discrete time.) Discrete non-Markovian quantum stochastic processes have been studied recently by Accardi [1,2] and Lindblad [16].

§ 2. Positive eigenvectors.

Let A be a finite dimensional C^* -algebra, whose positive elements we denote by A_+ . If $x \in A$, we say that x is strictly positive, written $x > 0$, if there exists $\epsilon > 0$ such that $x \geq \epsilon$, (i.e. x is positive and invertible). Let τ be a faithful, normalized trace on A , which is uniquely determined modulo the strictly positive elements in the centre of A . We can identify A^* with A in a conjugate linear fashion, under the pairing

$$\langle x, y \rangle = \tau(y^*x) , \quad x, y \in A .$$

If φ is a linear map on A , we let φ' denote its adjoint on A under this identification. Thus

$$\langle \varphi(x), y \rangle = \langle x, \varphi'(y) \rangle , \quad x, y \in A .$$

Then φ is self-adjoint (respectively positive) if and only if φ'

is self adjoint (respectively positive). Note that when we say a linear map φ on A is self adjoint or positive, it is always meant in the C^* -sense, not the hilbert space sense. Thus φ self adjoint means $\varphi(x^*) = \varphi(x)^*$, $\forall x \in A$, not $\varphi = \varphi'$; and similarly φ positive means $\varphi(x^*x) \geq 0$, $\forall x \in A$, not $\langle \varphi(x), x \rangle \geq 0$, $\forall x \in A$. We say that φ is strictly positive, written $\varphi > 0$, if $\varphi(x) > 0$ for all non-zero x in A_+ . Note that if φ is strictly positive, then there exists $\epsilon > 0$, such that $\varphi \geq \epsilon$, but the converse is false. Moreover, φ is strictly positive if and only if $\langle \varphi(x), y \rangle > 0$, for all non-zero x, y in A_+ . Thus φ is strictly positive if and only if φ' is strictly positive.

We recall that a cone M in A_+ is called hereditary if $0 \leq x \leq y$, $y \in M$ implies $x \in M$, for each x in A . A C^* -subalgebra B of A is said to be hereditary if B_+ is hereditary in A_+ . If p is a projection in A , then pAp is hereditary in A . Conversely if B is a hereditary C^* -subalgebra of A , then there exists an unique projection p in A such that $B = pAp$. Also the map $M \rightarrow \text{lin}(M)$ is a bijection between hereditary cones in A_+ , and hereditary C^* -subalgebras of A . For details on these matters we refer the reader to [18].

Following Davies [4] (see also [6,8,13]) we say that the hereditary C^* -subalgebra pAp reduces the positive linear map φ (or simply that the projection p reduces φ) if φ leaves pAp globally invariant. Thus p reduces φ if and only if there exists $\lambda > 0$ such that $\varphi(p) \leq \lambda p$, and moreover p reduces φ if and only if it reduces φ' . We say that φ is irreducible if it is not reduced by any proper hereditary C^* -subalgebra. Thus φ is irreducible if and only if φ' is irreducible.

The following lemma is of a familiar type and allow us to deduce that an ergodic property is equivalent to irreducibility (c.f. [3,7,12,13,23,24]).

Lemma 2.1.

Let A be a finite dimensional C^* -algebra, realizable on a hilbert space of dimension n . A positive linear map φ on A is irreducible if and only if $(1+\varphi)^{n-1} > 0$.

Proof. Suppose φ is irreducible. Let $y \in A_+$ be non-zero. If $z = y + \varphi(y)$, we have $\ker(z) \subseteq \ker(y)$. Suppose $\ker(z) = \ker(y)$, i.e. $\ker \varphi(y) \supseteq \ker y$, or $\text{im } \varphi(y) \subseteq \text{im } y$. Let p be the projection on the range space of y . Then φ leaves the hereditary C^* -algebra pAp invariant. By irreducibility, $p = 1$, and y is invertible. Thus if y is not invertible $\dim \ker z < \dim \ker y$. Hence $\dim \ker(1+\varphi)^{n-1}y = 0$, and so $(1+\varphi)^{n-1} > 0$. The converse is clear.

Proposition 2.2.

Let A be a finite dimensional C^* -algebra, realizable on a hilbert space of dimension n . A positive linear map φ on A is irreducible if and only if for any x, y non-zero elements of A_+ , with $\langle x, y \rangle = 0$, there exists $k > 0$ such that $\langle \varphi^k(x), y \rangle > 0$. (In which case k may be chosen strictly smaller than n .)

Proof. If φ is reduced by a projection $p \in A$, then $\langle \varphi^k(p), 1-p \rangle = 0$, for all $k \geq 0$. If φ is irreducible, then by the previous lemma, $\langle (1+\varphi)^{n-1}x, y \rangle > 0$, for all non-zero x, y in A_+ . Hence by expansion, there exists k strictly smaller than n such that $\langle \varphi^k(x), y \rangle > 0$, if $\langle x, y \rangle = 0$. The Proposition is proved.

Let φ be an irreducible positive linear map on the C^* -algebra A . In order to produce positive eigenvectors, we follow Wielandt [33] and consider the real valued function

$$r_x = \sup\{\rho \in \mathbb{R} : \rho x \leq \varphi(x)\} \quad (2.1)$$

defined on A_+ . We will show that r attains its maximum value at a strictly positive element of A , which is uniquely determined up to scalar multiplication. Since $r_{\lambda x} = r_x$, for any $\lambda > 0$, and non-zero x , it is enough to restrict attention to the compact set $S = \{x \in A_+ : \tau(x) = 1\}$. However r is not necessarily continuous on S , so we restrict our attention even further. First, note that the range projection p of $\varphi(1)$ reduces φ , and hence by irreducibility, $p = 1$, and $\varphi(1)$ is invertible. Thus if $x \in A_+$, and $x \geq \epsilon$, for some $\epsilon > 0$, then $\varphi(x) \geq \epsilon \varphi(1)$. Hence $x > 0$ implies that $\varphi(x) > 0$.

But $r_x = \|\varphi(x)^{-\frac{1}{2}} x \varphi(x)^{-\frac{1}{2}}\|^{-1}$, for all non-zero x such that $\varphi(x)$ is invertible. In particular r is continuous on the strictly positive elements of A . Let N denote the compact set $(1+\varphi)^{n-1}S$, which is contained in the set of strictly positive elements in A .

Then r_x attains its maximum value r on N , at z say. Now suppose $x \in S$, then $\varphi(x) - r_x x \geq 0$.

Hence $(1+\varphi)^{n-1}[\varphi(x) - r_x x] \geq 0$,

i.e. $\varphi(y) - r_x y \geq 0$, where $y = (1+\varphi)^{n-1} x \in N$.

Thus $r_y \geq r_x$, and so

$$\begin{aligned} r &= \max\{r_x : x \in N\} = \max\{r_x : x \in S\} \\ &= \max\{r_x : x \in A_+\}. \end{aligned}$$

Note that if $\varphi(z) - rz \neq 0$, then as above we have $\varphi(u) - ru = (1+\varphi)^{n-1}[\varphi(z) - rz] > 0$, where $u = (1+\varphi)^{n-1} z$. Thus $r_u > r$, which contradicts the maximality of r .

In fact we have shown that if $\varphi(u) - ru \geq 0$, for some non-zero positive u , then $\varphi(u) = ru$.

The following Theorem in the commutative case was first shown in [10,11,19] by Perron and Frobenius.

Theorem 2.3.

Let φ be a positive irreducible linear map on a finite dimensional C^* -algebra A . Then the function defined on A_+ by

$$r_x = \sup\{\rho \in \mathbb{R} : \rho x \leq \varphi(x)\}$$

attains its maximum $r = r_z$ at a strictly positive element z of A , which is unique up to scalar multiplication. Moreover, r is a simple eigenvalue of φ with eigenvector z .

Proof. It only remains to show that r is a simple eigenvalue.

Suppose that z' is also an eigenvector, which can be taken to be self-adjoint. If $z^{-\frac{1}{2}} z' z^{-\frac{1}{2}} \notin \mathbb{C}$, we can find real λ such that $\lambda - z^{-\frac{1}{2}} z' z^{-\frac{1}{2}}$ is positive but not strictly positive. i.e.

$\lambda z - z' \geq 0$, but not $\lambda z - z' > 0$. Then

$(1+\varphi)^{n-1}(\lambda z - z') = (1+r)^{n-1}(\lambda z - z')$, and so $\lambda z - z' > 0$ by Lemma contrary to assumption. Thus z' is a scalar multiple of z .

We denote by $r = r(\varphi)$, and $z = z(\varphi)$ the characteristic number and characteristic vector respectively for φ . Since φ' is irreducible, we can consider $r' = r(\varphi')$, and $z' = z(\varphi')$, the characteristic number and characteristic vector respectively for φ' . Then

$$r\langle z, z' \rangle = \langle \varphi(z), z' \rangle = \langle z, \varphi'(z') \rangle = r'\langle z, z' \rangle.$$

Hence $r = r'$, since $z, z' > 0$. Moreover, if $\varphi(y) = \alpha y$, with y a non-zero element in A_+ , we have

$$\alpha \langle y, z \rangle = \langle \varphi(y), z' \rangle = \langle y, \varphi'(z') \rangle = r \langle y, z' \rangle \quad (2.2)$$

Hence $\alpha = r$, since $z' > 0$.

If we define the function r^x on A_+ by

$$r^x = \inf\{\sigma \in \mathbb{R} : \sigma x \geq \varphi(x)\}$$

we see that r^x attains its minimum value \hat{r} at an unique point $v \in N$, and if $x \in A_+$ satisfies $\hat{r}x \geq \varphi(x)$, then x is a scalar multiple of v . Moreover $\varphi(v) = \hat{r}v$, and hence by (2.2) $\hat{r} = r$, and thus v is a scalar multiple of z , since r is a simple eigenvalue of φ by Theorem 2.3. We summarise these results in the following Theorem:

Theorem 2.4.

Let φ be an irreducible, positive linear map on a finite dimensional C^* -algebra A . The following function defined on A_+ :

$$r^x = \inf\{\sigma \in \mathbb{R} : \sigma x \geq \varphi(x)\}$$

attains its maximum value r on exactly the direction given by z , where r, z are the characteristic number and vector of φ given in Theorem 2.3. The characteristic numbers of φ and φ' are equal. Furthermore, if $\varphi(y) = \alpha y$, for some non-zero positive y , and α in \mathbb{C} , then $\alpha = r$, and y is a scalar multiple of z .

Note also that r is in fact the spectral radius of φ . For suppose $\varphi(u) = \alpha u$ for some non-zero $u \in A$, and some $\alpha \in \mathbb{C}$. Consider the positive map ψ given by

$$\psi(x) = \frac{1}{r} z^{-\frac{1}{2}} \varphi(z^{\frac{1}{2}} x z^{\frac{1}{2}}) z^{-\frac{1}{2}}, \quad x \in A.$$

Then $\psi(1) = 1$, and hence $\|\psi\| = 1$. If $v = z^{-\frac{1}{2}} u z^{-\frac{1}{2}}$, we see $\psi(v) = \frac{\alpha}{r} v$. Hence $|\alpha/r| \leq \|\psi\| = 1$.

Now consider an arbitrary positive linear map φ on A , and

let φ_n be a sequence of irreducible positive linear maps which converge to φ in norm (e.g. $\varphi_n = \varphi + 1/n \psi$, where ψ is a fixed irreducible positive linear map). Then since z_n are simple eigenvectors, it follows that z_n converge to a positive eigenvector of φ with eigenvalue r , such that r_n converges to r . Moreover, r is the spectral radius for φ , which we call the characteristic number for φ . We have thus recovered the following known theorem (see e.g. [22]).

Theorem 2.5.

Let φ be a positive linear map on a finite dimensional C^* -algebra A . If r is the spectral radius of φ , there is a non-zero positive element z in A such that $\varphi(z) = rz$.

Remark.

As alluded to in the introduction, Perron-Frobenius type results as in Theorem 2.3 and 2.4 are scattered throughout the literature. For example, the method of Wielandt [33] which we have adopted, has also been taken up by Mewborn [17].

§ 3. Stochastic maps.

Let A be a finite dimensional C^* -algebra, with faithful normalized trace τ as usual. Then with the identification described at the beginning of the previous section, the state space $S(A)$ of A may be identified with

$$S = \{x \in A_+ : \tau(x) = 1\}.$$

Thus the affine maps of $S(A)$ correspond to positive linear maps φ on A with the normalizing condition $\varphi'(1) = 1$. We say that a linear map φ on A is stochastic if it is positive and $\varphi'(1) = 1$.

(We will not distinguish between affine maps on $S(A)$, and stochastic maps on A .) The characteristic number of a stochastic map φ is 1, and by Theorem 2.5 (or the Markov-Kakutani theorem) there exist z in S such that $\varphi(z) = z$. If φ is irreducible, φ is unique and gives a faithful state.

If φ is a positive linear map on A , we let $F(\varphi)$ denote the fixed point set [8]:

$$F(\varphi) = \{x \in A : \varphi(x) = x\}.$$

Consider the case of a stochastic map φ on A , which is not necessarily irreducible. Let p_r be the projection onto the range of all elements in $F(\varphi)_+$. Since we are in finite dimensions, there is a \hat{z} in $F(\varphi)_+$, which has maximal range, and p_r is the projection onto this range. We call p_r the recurrent subspace, and we call the hereditary C^* -subalgebra $A_r \equiv p_r A p_r$ the recurrent subalgebra. Clearly the recurrent subalgebra reduces φ , and the restriction φ_r of φ to the recurrent subalgebra is stochastic i.e. $\tau \varphi_r(x) = \tau(x)$, for all x in A_r . We say that φ is recurrent if $p_r = 1$. φ_r is clearly recurrent.

Now consider a stochastic and recurrent map φ . Then there is a strictly positive \hat{z} in $F(\varphi)$. This implies that if $\varphi'(y) \geq y$ for some self-adjoint y , then $y \in F(\varphi')$. In fact,

$$\begin{aligned} \langle \hat{z}, \varphi'(y) - y \rangle &= \langle \hat{z}, \varphi'(y) \rangle - \langle \hat{z}, y \rangle \\ &= \langle \varphi(\hat{z}), y \rangle - \langle \hat{z}, y \rangle \\ &= 0, \text{ as } \hat{z} \in F(\varphi). \end{aligned}$$

Hence $\varphi'(y) = y$, since $\hat{z} > 0$. Now consider $u \in F(\varphi')_h$. Then since $\varphi'(1) = 1$, we have the Kadison-Schwarz inequality [14]:

$$\varphi'(x^2) \geq \varphi'(x)^2, \quad x \in A_h,$$

which shows that $\varphi'(u^2) \geq \varphi'(u)^2 = u^2$. Hence by the preceding $u^2 \in F(\varphi')$, (as in [8]). If ψ is any positive linear map on A , we let [6,8]:

$$M(\psi) = \{x \in A : x, x^*x, xx^* \in F(\psi)\}.$$

We have thus shown that $F(\varphi')_h \subseteq M(\varphi')$. In particular, $F(\varphi')_h$ is a Jordan algebra. Thus if $x \in F(\varphi')_h$, then $x^2 \in F(\varphi')_h$, and thus $(x^2 + \lambda x)^2 \in F(\varphi')_h$, for all real λ . Hence $x^3, x^4 \in F(\varphi')_h$, and in fact $x^n \in F(\varphi')_h$, for $n = 0, 1, 2, \dots$. In particular the spectral projections of x lie in $F(\varphi')$. Let p_1 be a minimal projection in $F(\varphi')$; then $1-p_1 \in F(\varphi')$, and hence we can find a resolution of the identity $\{p_1, \dots, p_k\}$ consisting of orthogonal minimal invariant projections (which of course is not necessarily unique). Each p_i reduces φ' and φ , and by minimality the restriction of φ' or φ to $p_i A p_i$ is irreducible. Hence for each i , there exists one unique z_i in $p_i A p_i \cap F(\varphi)$, such that z_i is invertible on the subspace given by p_i , and $\tau(z_i) = 1$. Then $z = \frac{1}{k} \sum_{i=1}^k z_i$ is a strictly positive element of $F(\varphi)$. We have thus proved the following theorems:

Theorem 3.1.

Let ψ be a positive identity preserving linear map on a finite dimensional C^* -algebra A . If there is a faithful state ω on A invariant under ψ (i.e. $\omega \circ \psi = \omega$) then there is a resolution of the identity $\{p_1, \dots, p_k\}$ of orthogonal projections in A such that $\psi(p_i) = p_i$, $i = 1, 2, \dots, k$, and the reduction of ψ by p_i is irreducible.

Remark.

Consider ψ ($= \varphi'$ in previous notation) a positive linear map as in the above theorem, which also satisfies the Schwarz inequality:

$$\psi(x^*x) \geq \psi(x)^* \psi(x) \quad , \quad \forall x \in A. \quad (3.1)$$

Then [6] $M(\psi)$ is actually a C^* -algebra, and the Jordan algebra $F(\psi)_h$ is the self-adjoint part of $M(\psi)$. It follows that k in the above theorem is unique, and the decomposition p_1, \dots, p_k is essentially unique (in an obvious sense). Maps satisfying (3.1) will be studied in further detail in §4.

Theorem 3.2.

Let A be a finite dimensional C^* -algebra, and φ an affine mapping of its state space $S(A)$, which is recurrent i.e. there exists a faithful invariant state. Then there is a maximal family of disjoint faces F_1, \dots, F_k in $S(A)$, such that each face is globally invariant under φ but the restriction of φ to each face is irreducible. There is a unique fixed point ω_i in each F_i , and F_i is the smallest face containing ω_i .

Consider an arbitrary stochastic map φ on a finite dimensional C^* -algebra. Let $p_t = 1 - p_r$ be the projection orthogonal to the recurrent subspace. We refer to p_t as the transient subspace. If $z \in F(\varphi)_+$, then

$$\langle \varphi(p_t), z \rangle = \langle p_t, \varphi(z) \rangle = \langle p_t, z \rangle = 0.$$

Thus $\langle \varphi'(p_t), p_r \rangle = 0$, since p_r is the projection on the ranges of $F(\varphi)_+$, and hence φ' and φ are reduced by p_t . We refer to the hereditary C^* -subalgebra $A_t \equiv p_t A p_t$ as the transient subalgebra. Let φ_t denote the restriction of φ to the transient

subalgebra. Then by Theorem 2.5, there exists a non-zero x_t in A_t^+ such that $\varphi_t(x_t) = r_t x_t$, where r_t is the spectral radius of φ_t . Since $\|\varphi\| = 1$, we have $0 \leq r_t \leq 1$. Note that in fact $r_t < 1$, since otherwise $x_t \in F(\varphi)_+$, and hence $p_r x_t = x_t$ by definition of the recurrent subspace, and $x_t = 0$. Thus since r_t is the spectral radius of φ_t , we have that $\varphi^n(x), (\varphi')^n(x) \rightarrow 0$, as $n \rightarrow \infty$, for all x in A_t . We have thus shown:

Theorem 3.3.

Let φ be an affine mapping of the state space of a finite dimensional C^* -algebra A . Then there exist two maximal disjoint faces F_t and F_r , called the transient and recurrent faces respectively, invariant under φ . The restriction of φ to F_r is recurrent, and if $\omega \in F_t$, then $\varphi^n(\omega) \rightarrow 0$, as $n \rightarrow \infty$.

Remark.

The problem of showing relaxation to the unique equilibrium state has been studied in [4,5,8,25,26] for continuous time irreducible processes. These techniques go over to the discrete case. In fact, suppose ψ is an irreducible affine map on the state space of a finite dimensional C^* -algebra A , with unique equilibrium state z . Then Lemma 2.1 and the proof of [4, Theorem 14] show that $[(1+\psi)/2]^n(\omega) \rightarrow z$ as $n \rightarrow \infty$, for all ω in $S(A)$. Similarly, the relaxation results of [8] go over to the discrete case. In fact $\psi^n(\omega) \rightarrow z$ as $n \rightarrow \infty$, for all $\omega \in S(A)$. This is seen as in [6, Lemma 6] say, by considering the bilinear form:

$$\begin{aligned} d(x,y) &= \langle z, y^*x - \psi'(y^*)\psi'(x) \rangle \\ &= \langle z, \psi'(y^*x) - \psi'(y^*)\psi'(x) \rangle, \quad x,y \in A, \end{aligned}$$

(c.f. [30] for behaviour at infinity of discrete and continuous contraction semigroups on hilbert spaces).

Suppose φ is an irreducible stochastic map on a finite dimensional C^* -algebra A , and that -1 is in the spectrum of φ' . Then there exists a self adjoint x of norm one such that $\varphi'(x) = -x$. Hence by the Schwarz inequality $\varphi'(x^2) \geq x^2$, and so $x^2 = 1$, by Theorem 2.3. Thus there is a projection p in A such that $x = 1 - 2p$, and $\varphi'(1-p) = p$. Thus $(\varphi')^2$ and φ^2 are reducible, with $(\varphi')^2(p) = p$, $(\varphi')^2(1-p) = 1-p$. φ and φ' take pAp into $(1-p)A(1-p)$, and also take $(1-p)A(1-p)$ into pAp . Since p reduces φ^2 , there exists a non-zero positive z_1 in pAp , such that $\varphi^2(z_1) = z_1$. Then $z_2 = \varphi(z_1) \in (1-p)A(1-p)$, and $\varphi^2(z_2) = z_2$, so that we have shown:

Theorem 3.4.

Let φ be an irreducible stochastic map on a finite dimensional C^* -algebra A . If -1 is in the spectrum of φ' , there is a proper projection p in A such that p and $1-p$ reduce φ^2 . Moreover φ takes the hereditary C^* -subalgebra pAp into $(1-p)A(1-p)$, and there are non-zero z_1 in $(pAp)_+$, and z_2 in $[(1-p)A(1-p)]_+$, such that $\varphi(z_1) = z_2$, and $\varphi(z_2) = z_1$.

Let φ be a positive irreducible map on a finite dimensional C^* -algebra with $\varphi(1) = 1$. Then the symmetrised Schwarz inequality [27, Lemma 7.3] says that

$$D(x,y) \equiv \varphi(xy^* + y^*x) - \varphi(x)\varphi(y^*) - \varphi(y^*)\varphi(x)$$

defines a positive sesquilinear form on A . Hence as in the proof of [6, Theorem 3.1] we have that $D(x,x) = 0$ for some x in A implies that $D(x,y) = 0$ for all y in A . (Notice that the results of Broise recorded in [28] follow easily from this observation.) Let now $\varphi(u) = \alpha u$, for some α in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, u in A .

Then

$$D(u,u) = \varphi(uu^*+u^*u) - uu^* - u^*u \geq 0.$$

Hence $D(u,u) = 0$ by Theorem 2.3, so that $u^*u+uu^* \in \mathbb{C}$, and

$$\varphi(xu+ux) = \varphi(x)\alpha u + \alpha u\varphi(x), \text{ for all } x \text{ in } A.$$

In particular if $\varphi(v) = \beta v$, for some $\beta \in \mathbb{C}$, $v \in A$, then

$$\varphi(uv+vu) = \alpha\beta(vu+uv).$$

Thus if $uv+vu \neq 0$, we have $\alpha\beta \in \text{sp}(\varphi)$. In particular, $\varphi(u^n) = \alpha^n u^n$, for all $n \geq 0$, so that either $\alpha^n \in \text{sp}(\varphi)$, or $u^n = 0$. If u is invertible, then $\alpha^m = u^m = 1$ for some $m \geq 1$, since there can only be a finite number of eigenvalues.

Theorem 3.5.

Let φ be an irreducible positive map of a finite dimensional C^* -algebra such that $\varphi(1) = 1$. If $\alpha \in \text{sp}(\varphi) \cap \mathbb{T}$, and u is an eigenvector for α , we can normalize u such that $u^*u+uu^* = 1$; and moreover

$$\varphi(xu+ux) = \varphi(x)\alpha u + \alpha u\varphi(x), \quad \forall x \in A.$$

If $\beta \in \text{sp}(\varphi)$, with corresponding eigenvector v , then

$$\varphi(vu+uv) = \alpha\beta(vu+uv).$$

Thus either $uv+vu = 0$, or $\alpha\beta \in \text{sp}(\varphi)$. In particular, either $u^k = 0$, or $\alpha^k \in \text{sp}(\varphi)$, with eigenvector u^k , for all $k \geq 0$.

§4. Schwarz maps.

In this section we show that the multiplicative properties of irreducible stochastic maps which were obtained in the previous section, can be improved if we impose a stronger positivity property.

These results are then a non-commutative analogue of those obtained by Perron and Frobenius [10,11,19]. (They have also been generalized in [20,21] for certain lattice algebras.) These results were also obtained by Størmer [29] for ergodic groups of automorphisms.

Let φ be a linear map on a C^* -algebra A . We say that φ is a Schwarz map if $\varphi(1) = 1$, and

$$\varphi(x^*x) \geq \varphi(x)^* \varphi(x), \quad \text{for all } x \text{ in } A. \quad (4.1)$$

If $\alpha \in \mathbb{C}$, we let $M^\varphi(\alpha)$ denote the spectral subspace $\text{Ker}(\varphi - \alpha)$.

Lemma 4.1.

Let φ be an irreducible Schwarz map on a finite dimensional C^* -algebra A , then

(i) $M^\varphi(1) = \mathbb{C}$

(ii) For any α in $\text{sp}(\varphi) \cap \mathbb{T}$, $M^\varphi(\alpha)$ consists of scalar multiples of a unitary element u . Moreover

$$\varphi(ux) = \alpha u \varphi(x), \quad \text{for all } x \text{ in } A.$$

(iii) $M^\varphi(\alpha)M^\varphi(\beta) \subseteq M^\varphi(\alpha\beta)$, for all α in \mathbb{T} , β in \mathbb{C} .

Proof. (i) is a consequence of Theorem 2.3.

(ii) Let $u \in M^\varphi(\alpha)$, $u \neq 0$, where $\alpha \in \text{sp}(\varphi) \cap \mathbb{T}$. Then by (4.2), $\varphi(u^*u) \geq u^*u$, and hence $u^*u \in \mathbb{C}$, by Theorem 2.3. Similarly $uu^* \in \mathbb{C}$, and so u is a scalar multiple of a unitary operator. If D is the positive sesquilinear form $(x,y) \rightarrow \varphi(xy^*) - \varphi(x)\varphi(y)^*$, we know that $D(u,u) = 0$. Hence $D(u,x) = 0$, for all x in A , and so (ii) follows.

(iii) This follows from (ii).

Let φ be an irreducible Schwarz map. It follows from the above lemma that the eigenvalues on the unit circle form a discrete group Γ , with generator $\gamma = \exp(2\pi i/m)$ say, for some integer m . Moreover Γ acts on $\text{sp}(\varphi)$ by multiplication, since $M^\varphi(\alpha\beta) \neq 0$, for all $\alpha \in \Gamma$, $\beta \in \text{sp}(\varphi)$, by (ii) and (iii) of Lemma 4.1. Moreover if u is unitary in $M^\varphi(\gamma)$, then $\varphi(u^k) = \gamma^k u^k$. Hence $u^m = 1$, and $\text{sp}(u) \subseteq \Gamma$. Thus the spectral decomposition of u is

$$u = \sum_{k=0}^{m-1} \gamma^k p_k,$$

where the spectral projections p_k lie in A . From $\varphi(u) = \gamma u$, it follows that $\varphi(p_k) = p_{k+1}$, $k = 0, 1, \dots, m-2$, and $\varphi(p_{m-1}) = p_0$. We have thus shown:

Theorem 4.2.

Let φ be an irreducible Schwarz map on a finite dimensional C^* -algebra A . Then $\text{sp}(\varphi) \cap \mathbb{T}$ forms a discrete subgroup Γ of the unit circle \mathbb{T} . Each eigenvalue in Γ is simple, with corresponding eigenvectors which are scalar multiples of a unitary element in A . These eigenvectors form an abelian group isomorphic with Γ . If $|\Gamma| = m$, $\gamma = \exp(2\pi i/m)$, and u is unitary in $M^\varphi(\gamma)$, then $\text{sp}(u) = \Gamma$, and u has spectral resolution:

$$u = \sum_{k=0}^{m-1} \gamma^k p_k$$

where $\varphi(p_k) = p_{k+1}$, $k = 0, \dots, m-2$, $\varphi(p_{m-1}) = p_0$. Thus $\text{sp}(\varphi) \cap \mathbb{T} = \{1\}$ if and only if φ^n is irreducible for all n .

Let ψ be an irreducible stochastic map on a finite dimensional C^* -algebra A such that ψ' is a Schwarz map. Then $\Gamma = \text{sp}(\psi') \cap \mathbb{T} \subseteq \text{sp}(\psi') = \text{sp}(\psi)$, and so Γ acts on $\text{sp}(\psi)$. In

fact, if $v \in M^{\psi'}(\alpha)$, where $\alpha \in \Gamma$, we have

$$\begin{aligned} \langle v\psi(x), z \rangle &= \tau(z^*v\psi(x)) = \langle \psi(x), v^*z \rangle \\ &= \langle x, \psi'(v^*z) \rangle = \langle x, \bar{\alpha}v^*\psi'(z) \rangle \\ &= \alpha \langle vx, \psi'(z) \rangle = \alpha \langle \psi(vx), z \rangle \end{aligned}$$

for all z in A . Thus $\bar{\alpha}v\psi(x) = \psi(vx)$, $\forall v \in M^{\psi'}(\alpha)$, $x \in A$. In particular if $x \in M^{\psi}(\beta)$ for some $\beta \in \mathbb{C}$, then $\psi(v^*x) = \alpha\beta(v^*x)$, and similarly $\psi(xv^*) = \alpha\beta xv^*$. If $u, \gamma, \{p_0, \dots, p_{m-1}\}$ are as in Theorem 4. for the Schwarz map ψ' , then each p_k reduces ψ^m , and the ψ^m reduced by p_k is irreducible. Thus there exist unique z_0, \dots, z_{m-1} in $p_0 A p_0, \dots, p_{m-1} A p_{m-1}$ respectively such that $\varphi(z_k) = z_{k+1}$, $\psi(z_{m-1}) = z_0$, $\tau(z_k) = 1$, and $z_k > 0$ in $p_k A p_k$, for $k = 0, 1, 2, \dots, m-2$. Then $z = \sum z_k$ is the unique invariant state for ψ . We summarise this as follows:

Theorem 4.4.

Let ψ be an irreducible affine map on the state space of a finite dimensional C^* -algebra, such that its associated ψ' is a Schwarz map. Then 1 is the only eigenvalue on the unit circle if and only if ψ^m is irreducible for any $m \geq 1$. In any case, the eigenvalues on the unit circle form a discrete subgroup Γ which operates on $sp(\psi)$. If $|\Gamma| = m$, then there is a maximal family of disjoint faces of $S(A)$, F_0, \dots, F_{m-1} such that $\psi(F_k) = F_{k+1}$, $k = 0, \dots, m-2$, and $\psi(F_{m-1}) = F_0$. If ω is the unique invariant state for ψ , then $\omega = \frac{1}{m} \sum \omega_k$, where $\omega_k \in F_k$, $\psi(\omega_k) = \omega_{k+1}$, $k = 0, \dots, m-2$, $\psi(\omega_{m-1}) = \omega_0$, and F_k is the minimal face containing ω_0 .

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