The generators of positive semigroups.

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Abstract: We characterise the infinitesimal generators of norm continuous one-parameter semigroups of positive maps on certain ordered spaces, with special reference to C^*-algebras.

The purpose of this note is to characterise the infinitesimal generators of norm continuous one-parameter semigroups of positive linear maps on certain ordered vector spaces. This research was inspired by known results for groups of positive maps in Hilbert spaces with self dual cones [1, Lemma 5.3], and for semigroups of completely positive, and indeed more generally locally completely positive maps on C^*-algebras [8,4]. The study of positive semigroups as representing dynamical systems in classical and quantum theory has received much attention on recent years. We refer the reader to [2, 5, 11] for accounts of this.

We begin with a result for C^*-algebras. But first we recall that if S is a set of states on a C^*-algebra A, then S is said to be <u>full</u> if $x \in A_h$ and $f(x) \geq 0$ for all f in S, then $x \geq 0$. Moreover S is said to be <u>invariant</u> if $f \in S$, and $x \in A$ satisfy $f(x^*x) \neq 0$, then $f[x^*(\cdot)x]/f(x^*x) \in S$. That (i) is equivalent to (vi) in the following Theorem was first shown in [12], for which we give an alternative proof.

<u>Theorem 1.</u>

Let L be a bounded self-adjoint linear map on a unital C^*-algebra A. Then the following conditions are equivalent:

(i) e^{tL} is positive, for all positive t.

(ii) $(\lambda - L)^{-1}$ is positive, for all large positive λ.

(iii) If $y \in A_+$, $a \in A$ satisfy $ya = 0$, then $a^*L(y)a \geq 0$.

(iv) For some full, invariant set of states S on A, that $y \in A_+$, $f \in S$ with $f(y) = 0$ imply $fL(y) \geq 0$.

(v) $L(x^2) + xL(1)x \geq L(x)\, x + xL(x)$, for all self adjoint x in A.

(vi) $L(1) + u^*L(1)u \geq L(u^*)u + u^*L(u)$, for all unitaries u in A.

Proof.

(iv) $\Rightarrow$ (iii). Let S be a full, invariant set of states satisfying (iv).

Let $y \in A_+$, $a \in A$ satisfy $ya = 0$. Then $f(a^*ya) = 0$ for all f in S. Hence by (iv), $f[a^*L(y)a] \geq 0$, for all f in S, since S is invariant, and thus $a^*L(y)a \geq 0$, since S is full.

(iii) $\Rightarrow$ (ii). Let $\lambda > \|L\|$. In order to show that $(\lambda - L)^{-1} \geq 0$, it is enough to show that if $x \in A_h$ satisfies $(\lambda - L)x \geq 0$, then $x \geq 0$. Let $x = x^+ - x^-$, where $x^+, x^- \in A_+$, and $x^+x^- = 0$. Thus $x^-L(x^+)x^- \geq 0$ by (iii). Then we have

$$\begin{aligned} 0 &\leq x^-[(1-L/\lambda)(x)]x^- \\ &= x^-x\, x^- - x^-\, [L/\lambda(x)]x^- \\ &= -(x^-)^3 - x^-\, [L/\lambda(x^+)]x^- + x^-[L/\lambda(x^-)]x^- \end{aligned}$$

Thus $0 \leq (x^-)^3 \leq x^-[L/\lambda(x^-)]x^-$,
and so $\|x^-\|^3 \leq \frac{\|L\|}{\lambda} \cdot \|x^-\|^3$.

Hence $x^- = 0$, as $\|L\|/\lambda < 1$.

(ii) $\Rightarrow$ (i) $e^{tL} = \lim_{n \to \infty} (1 - L/n)^{-n}$

(i) $\Rightarrow$(v) Let $L'(x) = L(x) - \frac{1}{2}[L(1)x + xL(1)]$. Then $e^{tL'} \geq 0$ for all $t \geq 0$ by the Lie-Trotter formula. (Note that the map $x \to -\frac{1}{2}[L(1)x + xL(1)]$ satisfies (i).) Since $e^{tL'}(1) = 1$, we have by differentiating Kadison's Schwarz inequality [6], namely

$$e^{tL'}(x^2) \geq e^{tL'}(x)^2$$

which is valid for all $t \geq 0$, and all self adjoint x , that

$$L'(x^2) \geq L'(x)\cdot x + x\cdot L'(x) \ , \ \forall \ x \in A_n \ .$$

Substituting for L' gives the desired result.

(v) $\Rightarrow$(iv) Let $y \in A_+$, $f \in A_+^*$, with $f(y) = 0$. Then $f(y^{\frac{1}{2}} z) = 0 = f(zy^{\frac{1}{2}})$ for all $z \in A$, by the Schwarz inequality. Thus $L(y) + y^{\frac{1}{2}}L(1)y^{\frac{1}{2}} \geq L(y^{\frac{1}{2}})y^{\frac{1}{2}} + y^{\frac{1}{2}}L(y^{\frac{1}{2}})$ implies that $fL(y) \geq 0$.

To show that (i) is equivalent to (vi), it is enough by a standard transformation to assume that $L(1) = 0$.

(i) $\Rightarrow$(vi) Since $e^{tL} \geq 0$, and $e^{tL}(1) = 1$, for all $t \geq 0$, we have $\|e^{tL}\| = 1$, $\forall t \geq 0$. Thus $\|e^{tL}(u)\| \leq 1$, for all unitaries u in A . Hence $e^{tL}(y^*)e^{tL}(u) \leq 1$, $\forall t \geq 0$, and so $L(u^*)u + u^*L(u) \leq 0$, for all unitaries u , by derivation.

(vi) $\Rightarrow$(i) This is contained in the proof of [8, Proposition 4], as observed in [12].

Not surprisingly, the above theorem is concerned mainly with the Jordan structure of the C^*-algebra. It can be used to simplify the proof of the following known result [8,4] for the C^*-structure. Note that if $A = M_n(\mathbb{C})$, the C^*-algebra of all $n \times n$ matries over $\mathbb{C}$, and $L(x) = x^t - x$, (where $x \to x^t$ is the transpose mapping) then L satisfies the conditions of Theorem 1, but not Theorem 2 if $n \geq 2$.

Theorem 2.

Let L be a bounded self-adjoint linear map on a C^*-algebra A. Then the following conditions are equivalent:

(i) $e^{tL}(x^*x) \geq e^{tL}(x^*)e^{tL}(x)$, $\forall x \in A$, $t \in \mathbb{R}^+$.

(ii) $L(x^*x) \geq L(x^*)x + x^*L(x)$, $\forall x \in A$.

Proof.

Suppose (ii) holds. Adjoin an identity 1 to A, and extend L to the enlarged algebra $\tilde{A}$ by putting $L(1) = 0$. Then by Theorem 2, e^{tL} is positive on $\tilde{A}$ for all positive t. The result follows from [4], or see [8] for a resolvent argument.

It is desirable to remove the initial hypothesis of boundedness of the generator in Theorem 1. We can modify the arguments of [7] to show:

Theorem 3.

Let L be a self adjoint linear map on a unital C^*-algebra A, with the following property: if $y \in A_+$, $f \in A_+^*$ satisfy $f(y) = 0$, then $f(Ly) \geq 0$. Then L is bounded, and e^{tL} positive for all positive t.

Proof.

Note that the map $x \to L(x) - \frac{1}{2}[L(1)x + xL(1)]$ satisfies the same conditional positivity property as L. Hence we can assume $L(1) = 0$. We will show that in this case, L is dissipative (in the sense of [9]) on A_h. i.e.

(1) $\lambda\|x\| \leq \|\lambda x - Lx\|, \quad \forall x \in A, \quad \lambda > 0$.

In order to show this for some x in A_h, we may assume, (by considering $-x$ if necessary) that there exists a positive f in A^*, such that $f(x) = \|x\|$, and $\|f\| = 1$. Then $f(\|x\| - x) = 0$, and so $f[L(\|x\|-x)] \geq 0$ by assumption, i.e. $f(Lx) \leq 0$.

Let $\lambda > 0$, then
$$\lambda f(x) \leq f(\lambda x - Lx) \leq \|f\|\|\lambda x - Lx\|$$
i.e. $\lambda\|x\| \leq \|f\|\|\lambda x - Lx\|$

This is enough to show that L is bounded. The original proof of this fact used semi-inner products [9]. A more elegant method has been given by Sullivan [11], which we repeat here for completeness. We show L is closed on A_h. Let $f_n \in A_h$, $f_n \to 0$, $Lf_n \to g$. Then for all h in A_h, $\lambda > 0$,

$$\lambda\|\lambda f_n + h\| \leq \|(\lambda - L)(\lambda f_n + h)\|$$

Letting $n \to \infty$, we have

$$\lambda\|h\| \leq \|\lambda(h-g) + L(h)\| ,$$

and then dividing by λ, and letting $\lambda \to \infty$, we see

$$\|h\| \leq \|h-g\|$$

for all h in A_h . Hence $g = 0$. Hence L is bounded on A_h , and thus on A . It follows from Theorem 1 that e^{tL} is positive for all $t \geq 0$. (This also follows from (1) , which shows that $(1 - L/\lambda)^{-1}$ is a contraction for $\lambda > \|L\|$, and thus positive since it preserves the identity).

One of the preceeding techniques is actually derived from a more general setting in certain ordered spaces. For this it is convenient to introduce a definition:

Definition: A cone E_+ in a Banach space E is said to have the nearest point property if for any x in E , there exists some y in E_+ such that $\|x-y\| = \text{dist}\,(x, E_+)$. In this case E_+ is closed.

A closed cone in a Hilbert space certainly has the nearest point property. The following Theorem thus improves Connes' characterisation of the generators of groups of positive maps for Hilbert spaces with self dual cones [1, Lemma 5.3]. The positive cone in any order unit space, in particular, the self-adjoint part of C^*-algebra, has the nearest point property. Thus the Theorem 4 also generalises the purely order theoretic part of Theorem 1.

Theorem 4.

Let E be a real Banach space, and E_+ a cone in E with the nearest point property. Then if L is a bounded linear map on E , the following conditions are equivalent:

(i) e^{tL} is positive for all positive t .

(ii) $(\lambda-L)^{-1}$ is positive for all large positive λ .

(iii) If $x \in E_+$, $f \in E_+^*$ satisfy $f(x) = 0$, then $fL(x) \geq 0$.

Proof.

We show (iii) $\Rightarrow$ (ii). The rest is standard. Let $\lambda > \|L\|$. To show $(\lambda - L)^{-1} \geq 0$, assume $x \in E$ and $(\lambda - L)x \in E_+$. Assuming (iii) holds, we have to show $x \in E_+$. If not, then there exists $y \in E_+$, with $\|y-x\| = \text{dist}(x, E_+) = d > 0$. A Hahn-Banach theorem applied to the closed convex set E_+ , and the interior of the ball $S(x,d)$, with centre x , radius d, now provide $f \in E_+^*$ such that $f \leq 0$ on $S(x,d)$, and $f(x) < 0$. This implies $d\|f\| \leq -f(x)$. Since $y \in E_+ \cap S(x,d)$, $f(y) = 0$, and so by (iii) $fL(y) \geq 0$. Then

$$
\begin{aligned}
0 \leq f[(1 - L/\lambda)x] &= f(x) - 1/\lambda fL(x) \\
&= f(x) - 1/\lambda f(L(x-y)) - 1/\lambda fL(y) \\
&\leq - d\|f\| + 1/\lambda \|f\| \|L\| \|x-y\| \\
&= d\|f\| [\|L\|/\lambda - 1] \\
&< 0 \text{ , a contradiction.}
\end{aligned}
$$

The finite dimensional case of the above Theorem was proved in [10]. Following Schneider and Vidyasagar, the maps satisfying (iii) of Theorem 4 may be called crosspositive. Intuitively, this means that the orbit $\{e^{tL}x : t \in \mathbb{R}\}$ crosses the boundary of E_+ in a positive sense, that is, into E_+ (or at least not out of it).

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