A subset $B$ of the domain of a recursion theory is said to be regular if $B \cap K$ is "finite" (in the theory) whenever the set $K$ is "finite". Of course, in ordinary recursion theory every set is regular, so there the concept is not considered. However, when moving up to recursion theory on an admissible ordinal $\alpha$, non-regular $\alpha$-r.e. sets exist whenever $\alpha^* < \alpha$. In case $\alpha$ is inadmissible then there are non-regular $\alpha$-recursive sets.

When studying $\alpha$-r.e. degrees for an admissible ordinal $\alpha$ the obstacle of the existence of non-regular $\alpha$-r.e. sets is circumvented by the following theorem due to Sacks.

**Theorem 1 ([3]).** Suppose $\alpha$ is an admissible ordinal. Then every $\alpha$-r.e. set is of the same $\alpha$-degree as a regular $\alpha$-r.e. set.

Maass [1] has recently obtained a uniform version of theorem 1.

Let $\Theta$ be an infinite computation theory as defined in [6]. In this paper we prove the following analogue of theorem 1. (A weaker but for most degree theoretic purposes sufficient version
is proved in [7].

**Theorem 2.** Suppose $\mathcal{G}$ is an adequate infinite computation theory. Then every $\mathcal{G}$-s.c. set $B$ is of the same degree as a regular $\mathcal{G}$-s.c. set $D$. Furthermore $D$ may be chosen such that $\forall x(\forall y \sim x)\ (x \in D \Rightarrow y \in D)$.

**Remark:** Suppose $\mathcal{G}$ is the infinite computation theory over an adequate resolvable admissible set $\mathcal{A}$ with urelements. Then the theorem asserts that every $\mathcal{A}$-r.e. set is of the same $\mathcal{A}$-degree as a regular $\mathcal{A}$-r.e. subset of $\langle \mathcal{A} \rangle$, the ordinal of $\mathcal{A}$.

Thus we have that adequacy is a sufficient condition on $\mathcal{G}$ for the regular set theorem to hold. However, it is shown in [2] that the condition is not necessary. On the other hand, assuming AD, Simpson [4] has shown there is a $\mathcal{G}$ such that every regular $\mathcal{G}$-s.c. set is $\mathcal{G}$-computable.

The proof of theorem 1 was simplified by Simpson [5]. He utilized the wellordering of the domain in the form that every $\alpha$-r.e. non-$\alpha$-finite set has a 1-1 $\alpha$-recursive enumeration. The analogous property is false for arbitrary adequate computation theories. Thus our proof of theorem 2 is modelled after Sacks' original proof of theorem 1.

For definitions and notation the reader is referred to [6].

**Proof of theorem 2:** Let $B$ be a $\mathcal{G}$-s.c. non-$\mathcal{G}$-computable set. We are to find a regular $\mathcal{G}$-s.c. set $D$ such that $D \equiv B$. Let $B^* = \{ \xi : K_\xi \cap B \neq \emptyset \}$ where $\lambda K_\xi$ is a fixed enumeration of $\mathcal{G}$-finite sets. We have $K_\xi \cap B^* = \emptyset \iff \cup \{ K_\eta : \eta \in K_\xi \} \cap B = \emptyset$ and $K_\xi \cap B = \emptyset \iff \xi \notin B^*$. Thus $B^* \equiv B$. 
Let \( \pi: U \rightarrow L^{\sim} \) be a projection such that \( \pi(x) \rightarrow y_1 \) & \( \pi(x) \rightarrow y_2 \Rightarrow y_1 \sim y_2 \). Then

(1) \( K_y \cap B^* = \emptyset \Leftrightarrow U(\{K_\eta: \eta \in K_y\} \cap B = \emptyset \Leftrightarrow H_y \cap B^* = \emptyset \)

where \( \lambda yH_y \) is a \( \Theta \)-computable mapping whose values are (canonical \( \Theta \)-indices for) \( \Theta \)-finite sets such that \( \forall y (H_y \neq \emptyset) \),

\( \xi \in H_y \Rightarrow K_\xi = U(\{K_\eta: \eta \in K_y\} \) and

\( \xi_1, \xi_2 \in H_y \Rightarrow \xi_1 \sim \xi_2 \& \pi(\xi_1) \sim \pi(\xi_2) \). Because of (1) it is convenient to work with \( B^* \) instead of \( B \).

Let \( \lambda \sigma B^\sigma \) be a disjoint (\( \Lambda \))-enumeration of \( B^* \) such that \( \forall \sigma (B^\sigma \neq \emptyset) \) and \( \forall \sigma, x, y (x \in B^\sigma \& y \in B^\sigma \Rightarrow x \sim y \& \pi(x) \sim \pi(y)) \).

Define

\[ D^2 = \{ \sigma: (\exists \tau > \sigma)(B^\tau < B^\sigma \& \pi(B^\tau) < \pi(B^\sigma)) \} \] 

Note that expressions like \( \pi(B^\tau) < \pi(B^\sigma) \) make sense and are \( \Theta \)-computable. Clearly \( D^2 \) is \( \Theta \)-s.c. and \( U - D^2 \) is unbounded.

Claim 1: \( D^2 \) is regular.

Proof: Given \( \sigma_0 \) we show \( D^2 \cap L^{\sigma_0} \) is \( \Theta \)-finite. Having defined \( \sigma_0, \ldots, \sigma_n \) we choose, if possible, \( \sigma_{n+1} \) such that

\( \sigma_{n+1} > \sigma_n \) and \( \forall j \leq n \) (\( B^{\sigma_{n+1}} < B^{\sigma_j} \& \pi(B^{\sigma_{n+1}}) < \pi(B^{\sigma_j}) \)). By the well-foundedness of \( < \) the defined sequence is finite. Let \( \sigma_n \) be the last. Then

\[ D^2 \cap L^{\sigma_0} = \{ \sigma < \sigma_0: (\exists \tau \leq \sigma_n)(B^\tau < B^\sigma \& \pi(B^\tau) < \pi(B^\sigma) \& \tau > \sigma) \} \] 

One inclusion is obvious. So suppose \( \sigma \in D^2 \cap L^{\sigma_0} \). Choose \( \tau > \sigma \) such that \( B^\tau < B^\sigma \& \pi(B^\tau) < \pi(B^\sigma) \). If \( \tau \leq \sigma_n \) then all is well.

If \( \tau > \sigma_n \) then by the choice of \( \sigma_n \) there is \( j \leq n \) such that \( B^{\sigma_j} \not< B^\tau \& \pi(B^{\sigma_j}) \not< \pi(B^\tau) \). But then \( B^{\sigma_j} < B^\sigma \& \pi(B^{\sigma_j}) < \pi(B^\sigma) \) and \( \sigma < \sigma \leq \sigma_j \). Thus the inclusion from left to right holds.
Claim 2: \( D^2 \subseteq B^* \).

Proof: First we show

\[
\sigma \not\in D^2 \iff \pi^{-1}[\pi(B^{<\sigma}) \cap (L^{\pi(B^{<\sigma})} - \bigcup_{\tau < \sigma} \pi(B^{\tau}))] \cap B^* = \emptyset.
\]

Suppose the right hand side is false for a given \( \sigma \). Then there are \( x \) and \( \tau \) such that

\[
x \in \pi^{-1}[\pi(B^{<\sigma}) \cap (L^{\pi(B^{<\sigma})} - \bigcup_{\tau < \sigma} \pi(B^{\tau}))] \cap B^*.
\]

In particular \( \pi(x) \cap \pi(B^{<\sigma}) \neq \emptyset \) so \( x \in L^{\pi(B^{<\sigma})} \) (since \( \pi \) is a projection) and hence \( B^\tau < B^{<\sigma} \). Furthermore \( \pi(x) \cap (L^{\pi(B^{<\sigma})} - \bigcup_{\tau' < \sigma} \pi(B^{\tau'})) \neq \emptyset \) so \( \pi(B^{\tau'}) < \pi(B^{<\sigma}) \) and \( \tau > \sigma \). Thus \( \sigma \in D^2 \).

The converse of (2) follows by a similar argument. Using (2) we have

\[
K \cap D^2 = \emptyset \iff \bigcup_{\sigma \in K} \pi^{-1}[\pi(B^{<\sigma}) \cap (L^{\pi(B^{<\sigma})} - \bigcup_{\tau < \sigma} \pi(B^{\tau}))] \cap B^* = \emptyset,
\]

so \( D^2 \subseteq B^* \).

We now make an assumption and show that if the assumption holds then \( B^* \subseteq D^2 \). On the other hand if the assumption is false, we find \( \sigma \) such that \( B^* = B^* \cap L^\sigma \). It is then easy to find a regular \( \emptyset \)-s.c. set \( D \) such that \( B^* \cap L^\sigma = D \).

Define

\[
k_1(y) = \mu(\emptyset, \rho, \alpha, \sigma, \text{min } \pi([y : y \in \alpha])).
\]

\( k_1 \) is \( \emptyset \)-computable and total (by adequacy). Let

\[
k(y) = \mu(\emptyset, k_1(y), B^\sigma, \text{min } \pi([y : y \in k_1(y)])) \leq \pi(B^{<\sigma}) \cap B^* \subseteq D^2 \] .

Note that \( k \subseteq D^2 \).

Claim 2: If \( k \) is total then \( B^* \subseteq D^2 \).
Proof: Note that $H_y \cap B^* \neq \emptyset \iff H_y \subseteq B^*$. We show $H_y \subseteq B^* \iff H_y \subseteq \bigcup \{ B^\tau : \tau \prec k(y) \}$. It then follows from (1) that $B^* \subseteq D^2$. So let $\xi \in H_y \subseteq B^*$, say $\xi \in B^\tau$. We want to show $\tau \prec k(y)$. $B^\tau \prec k(y) \lessdot B^k(y)$ so $\tau \neq k(y)$. Suppose $\tau \succ k(y)$.

Then since $k(y) \notin D^2$ it must be that $\pi(B^k(y)) \leq \pi(B^\tau)$. But then $\pi(B^\tau) \sim \pi(H_y) < \min \pi(\{ y : y \sim k(y) \}) \leq \pi(B^k(y)) \leq \pi(B^\tau)$, a contradiction. Thus $\tau \prec k(y)$. 

Now we assume $k$ is not total. Choose $y$ such that $\forall \sigma([B^\sigma \prec k(y) \vee \pi(B^\sigma) < \min \pi(\{ y : y \sim k(y) \}) \vee \sigma \in D^2])$.

Let $B_Y^* = B^* \cap \bigcup \{ k_1(y) \}$. We will show $B_Y^* = B^*$. Clearly $B_Y^* \subseteq B^*$. By adequacy we can choose $\sigma_0$ such that $\tau \prec \sigma_0 \implies \pi(B^\tau) > \min \pi(\{ y : y \sim k_1(y) \})$. Thus

(3) $\forall \tau \succ \sigma_0 (B^\tau \prec k_1(y)) \vee \tau \in D^2$).

Let $B' = B^* - (\bigcup \{ B^\tau : \tau \lessdot \sigma_0 \} \cup \{ B^\tau : \tau \prec \sigma_0 \})$. Since clearly $B^* - B' \subseteq B_Y^*$, it suffices to show $B' \subseteq B_Y^*$ in order to show $B^* = B_Y^*$.

Claim 4: $B' \subseteq B_Y^*$.

Proof: We first show

(4) $\xi \in B' \iff \exists \sigma, \tau [\sigma_0 \lessdot \sigma \lessdot \tau \wedge \xi \in B^\sigma \wedge B^\tau \prec k_1(y) \lessdot B^\sigma$ & $\pi(B^\tau) < \pi(B^\sigma)]$.

The if direction is obvious. So suppose $\xi \in B'$. Then there is $\sigma \succ \sigma_0$ such that $\xi \in B^\sigma$ and, by (3) and the definition of $B'$, $\sigma \in D^2$. Thus there is $\tau_1 \succ \sigma$ such that $B^{\tau_1} \prec B^\sigma$ and $\pi(B^{\tau_1}) < \pi(B^\sigma)$. If $B^{\tau_1} \prec k_1(y)$ then we are done. If not, then $B^{\tau_1} \prec k_1(y)$ so $\tau_1 \in D^2$ by (3). Thus there is $\tau_2 \succ \tau_1$ such that
The sequence \( r_1, r_2, \ldots \) must be finite so eventually we obtain \( \tau_m \) such that \( B^{\tau_m} < k_1(\gamma) \). This proves (4).

Now suppose we have chosen the enumeration of \( \Theta \)-finite sets \( \lambda \in K_\xi \) to be repetitive in the following sense: Given any \( x \) then every \( \Theta \)-finite set has an index in \( U - L^x \). Then we can find a \( \Theta \)-computable mapping \( \lambda \eta G_\eta \) whose values are \( \Theta \)-finite sets such that

\[
(5) \quad K_\eta \cap B^* = \emptyset \iff (K_\eta - (L^\tau \cup \{B^\tau : \tau \leq \sigma_0 \})) \cap B^* = \emptyset \\
\iff G_\eta \cap B^* = \emptyset.
\]

Furthermore \( \lambda \eta G_\eta \) can be chosen to have the following properties:

\[
\forall \eta (G_\eta \neq \emptyset), \quad G_\eta \cap B^* \neq \emptyset \iff G_\eta \subseteq B^*, \quad G_\eta \subseteq B^* \iff G_\eta \subseteq B^*, \quad \text{and} \\
\xi_1, \xi_2 \in G_\eta \implies \xi_1 \sim \xi_2 \iff \pi(\xi_1) \sim \pi(\xi_2).
\]

Let \( F_\eta = \{ x \in L : n(x) < n(G_\eta) \} \), and let \( 1(\eta) = \mu \tau [(F_\eta - \cup \{B^\tau : \tau < \sigma \}) \cap B^*_Y = \emptyset] \). Then \( 1(\eta) \) is total by adequacy and \( 1 \leq_w B^*_Y \). Clearly \( 1(\eta) \) is a strict least upper bound for \( \{ \tau : B^\tau \subseteq F_\eta \} \). We show \( G_\eta \cap B^* = \emptyset \iff G_\eta \cap \cup \{B^\tau : \tau < 1(\eta) \} = \emptyset \). Combining this with (5) we then have \( B^* \leq B^*_Y \). So suppose \( \xi \in G_\eta \subseteq B^* \).

By (4) there is \( \sigma \) and \( \tau \) such that \( \sigma \prec \sigma \prec \tau, \xi \in B^\sigma, \quad B^\tau < k_1(\gamma) \) and \( \pi(B^\sigma) < \pi(B^\tau) \). If \( \sigma \geq 1(\eta) \) then \( \tau \geq 1(\eta) \) so \( \pi(G_\eta) \leq \pi(B^\tau) \).

But \( \pi(B^\tau) < \pi(B^\sigma) \sim \pi(G_\eta) \) so we have a contradiction. This shows \( \sigma < 1(\eta) \), which was all that remained to prove the claim.

Let \( C = \cup \{ n(x) : x \in B^*_Y \} \). It is easily seen that \( C = B^*_Y \) since \( B^*_Y \) is bounded. Let \( \lambda \sigma C^\sigma \) be a disjoint \( (\exists) \)-enumeration of \( C \) such that \( \forall \sigma (C^\sigma \neq \emptyset) \) and \( x, y \in C^\sigma \implies x \sim y \). Let \( D = \{ \sigma : (\exists \tau > \sigma)(C^\tau < C^\sigma) \} \), the deficiency set of \( C \). \( D \) is clearly
regular and $U - D$ is unbounded. We show $D \equiv B^*_Y$ thus completing the proof of the theorem.

We have $\sigma \notin D \iff (L^{C^\sigma}_Y - U[C^T : \tau < \sigma]) \cap C = \emptyset$ so $K \cap D = \emptyset \iff \bigcup_{\sigma \in K} (L^{C^\sigma}_Y - U[C^T : \tau < \sigma]) \cap C = \emptyset$. Thus $D \leq C \equiv B^*_Y$.

For the converse reducibility note that

\begin{equation}
K_\eta \cap B^*_Y = \emptyset \iff \bigcup \{K_\xi : \xi \in K_\eta \cap L, k_1(\gamma) \} \cap B = \emptyset \iff N_\eta \cap B' = \emptyset
\end{equation}

where $\lambda \eta N_\eta$ is a $\emptyset$-computable mapping having properties similar to those of $\lambda \eta G_\eta$. Let $f(\eta) = \mu \tau [C^T \geq \pi(N_\eta) \land \tau \notin D]$. $f$ is total by adequacy and $f \leq_w D$. Let

$g(\eta) = \mu \tau [\tau f(\eta)] - \bigcup \{B^\sigma : \sigma \prec \tau\} = \emptyset$. Then $g$ is total and $g \leq_w D$. We show $N_\eta \subseteq B' \iff N_\eta \subseteq \bigcup \{B^T : \tau \prec g(\eta)\}$. This together with (6) shows $B^*_Y \leq D$. So suppose $\xi \in N_\eta \subseteq B'$. By (4) there are $\sigma, \tau$ such that $\xi \in B^\sigma$, $\sigma \prec \tau$, $B^T \prec k_1(\gamma) \not\preceq B^\sigma$ and $\pi(B^T) \prec \pi(B^\sigma) \sim \pi(N_\eta)$. Thus $B^T \subseteq B^*_Y$ since $B^T \prec k_1(\gamma)$. Furthermore $B^T \subseteq \pi^{-1}(U[C^T' : \tau' f(\eta)])$ since $\pi(B^T) \prec \pi(N_\eta)$ and $D$ is a deficiency set for $C$. But then $\tau \prec g(\eta)$ so $\sigma \prec g(\eta)$.

As a final remark we note that the regular set produced is either $D^2$ or $D$. Both of these satisfy the last statement of the theorem.
Bibliography


