Abstract.

$E_k(x_2, \ldots, x_n)$ is defined by $E_k(a_2, \ldots, a_n) = 1$ if $\sum_{i=2}^{n} a_i = k$, else $E_k(a_2, \ldots, a_n) = 0$. We determine the periods of the sequences generated by the shift register with the feedback function $x_1 + E_k(x_2, \ldots, x_n) + E_{k+1}(x_2, \ldots, x_n) + E_{k+2}(x_2, \ldots, x_n)$ over the field $GF(2)$. We indicate also how to find the periods when the feedback function is $x_1 + E_k(x_2, \ldots, x_n) + \ldots + E_{k+p}(x_2, \ldots, x_n)$ where $p > 2$. 
1. Introduction.

In this paper we study only shift registers over the field \( \mathbb{GF}(2) = \{0, 1\} \) characterized by \( 1 + 1 = 0 + 0 = 0 \) and \( 1 + 0 = 1 \). Let \( S(x_2, \ldots, x_n) \) be a symmetric polynomial. A symmetric shift register of \( n \) stages with feedback function \( x_1 + S(x_2, \ldots, x_n) \) is the function \( \theta : \{0, 1\}^n \to \{0, 1\}^n \) defined by

\[
\theta(x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_1 + S(x_2, \ldots, x_n)) .
\]

If \( \theta^S(a_1, \ldots, a_n) = (a_1, \ldots, a_n) \), \( s \) is a period of \( (a_1, \ldots, a_n) \) with respect to \( \theta \). These periods are equal to the periods of the sequences \( \{a_t\}_{t=1}^{\infty} \) satisfying the nonlinear difference equation

\[
a_{n+t} = a_t + S(a_{t+1}, \ldots, a_{t+n-1}) \text{ for } t > 0 .
\]

For a general treatment of nonlinear shift registers see \cite{1}.

We shall in this paper extend the results of Kjeldsen \cite{2} and Søren\cite{3}. I am grateful to K. Kjeldsen who inspired me to study symmetric shift registers.

The weight \( w(\vec{a}) \) of a vector \( \vec{a} = (a_1, \ldots, a_n) \) is defined by \( w(\vec{a}) = \sum_{i=1}^{n} a_i \). We define \( E_k(x_2, \ldots, x_n) \) for \( k \in \{0, 1, \ldots, n-1\} \) by

\[
E_k(a_2, \ldots, a_n) = 1 \text{ if } w(a_2, \ldots, a_n) = k, \text{ else } E_k(a_2, \ldots, a_n) = 0 .
\]

The polynomials \( E_k \) are very important. In \cite{3} we showed that all symmetric polynomials are of the form \( \sum_{k \in \Delta} E_k \) for some \( \Delta \subset \{2, \ldots, n\} \). Besides,
if the periods of $E_k + \ldots + E_{k+p}$ for $p \geq 0$ are known, the periods of all symmetric shift registers can be determined.

In this paper we determine the periods when $S = E_k + E_{k+1} + E_{k+2}$. In [3] we determined the periods when $S = E_k$ and $S = E_k + E_{k+1}$. By using Thm. 2.2 in [3] we therefore know the periods of all $S$ of the form $S = \Sigma_{k \in \Delta} E_k$, where $\Delta \subset \{2, \ldots, n\}$ has the property

$$k, k+1, k+2 \in \Delta \Rightarrow k-1, k+3 \notin \Delta.$$ 

Besides this paper gives probably all ideas needed to solve the general case $S = E_k + \ldots + E_{k+p}$ for $p > 2$. In Section 4 we will indicate how to treat the general case.

In Section 2 we state the results. In Section 3 and 5 we prove them. Section 3 contains the main lines of the proofs and Section 5 contains the technical lemmas which are needed. In section 4 we indicate the general situation by an example.

We denote $\hat{a} = (a_1, \ldots, a_n) \in \{0, 1\}^n$ also by $\hat{a} = a_1 \ldots a_n$. We denote finite sequences of numbers by capitol letters (also the empty sequence). For $s \in \{0, 1, \ldots\}$ we define $s(A) = A \ldots A$ where $A$ appears $s$ times. We let $1_t = 1 \ldots 1$ (resp. $0_t = 0 \ldots 0$) denotes a string of $t$ consecutive 1's (resp. 0's). We refer to the index of notation in the end of this paper.

2. Main results.

In this section we introduce the concept of blocks and
the main results. In the proofs we show how the blocks of a vector $A = a_1 \ldots a_n$ moves by using $\theta$.

Definition 2.1.
Let $A = a_1 \ldots a_n \in \{0, 1\}^n$. We put $a_{n+1} = a_{n+2} = a_{n+3} = 0$, hence $a_1 \ldots a_{n+3} = A000$. We define the 3-blocks in $A$ by the following inductive procedure:
Suppose $i=0$ or that the 3-blocks in $a_1 \ldots a_i$ are defined.
Let $j$ be the least number $>i$ such that $a_j \ldots a_{n+3}$ starts with $11s(01)1$ for some $s \geq 0$. If such a $j$ does not exist, we stop the procedure.
Let $p$ be the least number $>j$ such that $a_p \ldots a_{n+3}$ starts with $00s(10)0$ for some $s \geq 0$.
By definition $a_j \ldots a_{p-1}$ is a 3-block in $A$. We have now defined the 3-blocks in $a_1 \ldots a_{p-1}$, and we continue the procedure.

Definition 2.2.
Let $A = a_1 \ldots a_n \in \{0, 1\}^n$. Isolated 1's outside 3-blocks and isolated 0's inside 3-blocks are called 1-blocks.
11 outside 3-blocks and 00 inside 3-blocks are called 2-blocks.

We illustrate the definitions by two examples. We put one * above the 1-blocks, one line above the 2-blocks and one line below the 3-blocks.

$$(2.1) \quad 0110100111011011000100110001110 .$$

$$$(2.2) \quad 1101010001100111001001110011010011 .$$
The next theorem is the main result of this paper.

Theorem 2.3.
Suppose \( n \) and \( k \) are positive integers such that \( 0 \leq k \leq n-3 \). Suppose \( \theta : \{0,1\}^n \to \{0,1\}^n \) is defined by

\[
\theta(x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_{n+1}) \text{ where } x_{n+1} = x_1 + E_k(x_2, \ldots, x_n) + E_{k+1}(x_2, \ldots, x_n) + E_{k+2}(x_2, \ldots, x_n).
\]

We suppose \( A = a_1 \ldots a_n \) is such that \( w(A) = k+3 \) and \( A \) contains both 1-, 2- and 3-blocks.

We let \( \gamma_i \) be equal to the number of \( i \)-blocks in \( A \) for \( i = 1, 2, 3 \). We let \( a \) and \( b \) be the minimal positive integers such that

\[
(2.3) \quad a(2n+4-4\gamma_1-6\gamma_2-8\gamma_3) = b(n+1-2\gamma_1-2\gamma_2-2\gamma_3).
\]

Then \( p \) defined by

\[
p = a(n+2-2\gamma_2-4\gamma_2-4\gamma_3)(n+3)+4a\gamma_2+2b\gamma_1
\]

is a period for \( A \). That means \( \theta^p(A) = A \).

The next theorem treats the situation that \( A = a_1 \ldots a_n \) does not contain 3 different types of blocks.

Theorem 2.4.
\( \theta \) is defined as in Thm. 2.3. We suppose \( A = a_1 \ldots a_n \) satisfies \( w(A) = k+3 \). We let \( \gamma_i \) be equal to the number of \( i \)-blocks of \( A \) for \( i = 1, 2, 3 \).

a) \( A \) contains only 1- and 2-blocks. Then the following is a period

\[
(n+1-2\gamma_1-2\gamma_2)(n+2)+2\gamma_1.
\]
b) A contains only 1- and 3-blocks. Then the following is a period

\[(n+1-2\gamma_1-2\gamma_3)(n+3)+4\gamma_1\].

c) A contains only 2- and 3-blocks. Then the following is a period

\[(n+2-4\gamma_2-4\gamma_3)(n+3)+4\gamma_2\].

d) If A contains only i-blocks, n+i is a period for i = 1,2,3.

We do not prove Thm. 2.4. It can be proved by using the distance functions defined in Def. 3.13 and the same ideas as in the proof of Lemma 3.15. Besides, the proof is similar to the proof of Thm. 4.4 in [3].

If \(w(A) \in \{k,k+1,k+2,k+3\}\), there exist in almost all cases an integer \(q\) such that \(w(\theta^q(A)) = k+3\). Then we use Thm. 2.3 or Thm. 2.4 to find a period of \(\theta^q(A)\). If \(w(A) < k\) or \(w(A) > k+3\), we prove easily that \(\theta^n(A) = A\).

Now we illustrate by three examples how Thm. 2.3 is used.

Let \(n = 12, k = 3\) and \(A = \underbrace{000000101100}_\text{12}\). We use Thm. 2.3 on \(\theta^2(A) = 000101100111\). Since \(\gamma_1 = \gamma_2 = \gamma_3 = 1\), (2.3) implies \(10a = 7b\). We get \(a=7, b=10\) and the period equal to

\[7 \cdot (12+2-2-4-4) \cdot 15+4 \cdot 7+2 \cdot 10 = 468\].

The example (2.1) satisfies the hypothesis of the theorem with \(k = 13\). In this example \(n = 32, \gamma_1 = 5\),
\( \gamma_2 = 2 \) and \( \gamma_3 = 2 \). (2.3) implies \( 20a = 15b \). We get \( a = 3, \ b = 4 \) and the period equal to 904.

The example (2.2) satisfies the hypothesis of the theorem with \( k = 13 \). In this example \( n = 32, \gamma_1 = 4, \gamma_2 = 3 \) and \( \gamma_3 = 2 \). (2.3) implies \( 18a = 15b \). We get \( a = 5, \ b = 6 \) and the period equal to 1158.

**Corollary 2.5.**

\( \theta \) is as in Thm. 2.3. We suppose \( A = a_1 \ldots a_n \) satisfies

\[ w(A) = k+3. \]

Then the minimal period of \( A \) with respect to \( \theta \) is less than \( n^3 \).

We prove Cor. 2.5 in the end of Section 5.

Quite often the periods we find in Thm. 2.3 and Thm. 2.4 are the minimal periods. However, we have not found any good hypothesis which implies minimality. By studying the proofs we think it is possible to find such a hypothesis. The next corollary is a simple example.

**Corollary 2.6.**

\( \theta \) is as in Thm. 2.3. We suppose \( A = a_1 \ldots a_n \) satisfies

\[ w(A) = k+3, \] and \( A \) contains 1 i-block for \( i = 1, 2, 3 \).

Then the period we find in Thm. 2.3 is the minimal period of \( A \).

We prove Cor. 2.6 in the end of Section 5.

3. **Main lines of the proofs.**

In this section we prove Thm. 2.3. The proofs of the lemmas in this section are done in Section 5. We suppose
n and $k$ are positive integers such that $k \leq n - 3$. The proof of Thm. 2.3 is easier if we suppose $A = a_1 \ldots a_n$ satisfies the next condition.

**Condition 3.1.**

Let $A = a_1 \ldots a_n \in \{0, 1\}^n$. $A$ satisfies **Condition 3.1.** if

1) $w(A) = k + 3$

2) $A$ contains 1-, 2- and 3-blocks.

3) $A$ does not start with a 1-block or a 2-block.

4) $A$ ends with a 3-block.

**Lemma 3.2.**

If $A = a_1 \ldots a_n$ satisfies 1) and 2) in Cond. 3.1, there exists an integer $q$ such that $\theta^q(A)$ satisfies Cond. 3.1.

Later in this section we define an integer $k(A)$ which is dependent of $A$. If $A$ satisfies Cond. 3.1, we prove that $\theta^{n+3+k(A)}(A)$ satisfies Cond. 3.1. In the proof of Thm. 2.3 we regard $A_0 = A$, $A_1 = \theta^{n+3+k(A)}(A)$, $A_2 = \theta^{n+3+k(A_1)}(A_1)$, etc. At last we find an integer $s$ such that $A = A_{s+1}$.

Then the following is a period for $A$:

$$s \sum_{i=0}^{n+3+k(A_i)} = (s+1)(n+3) + \sum_{i=0}^{s} k(A_i).$$

We calculate $s$ and $\sum_{i=0}^{s} k(A_i)$ and get the wanted period.

The idea of the proof is to examine the blocks of $\theta^{n+3+k(A)}(A)$ when we know the blocks of $A$. Usually an
i-block moves $k(A)+3-i$ places to the left by applying $\varphi^{n+3+k(A)}$ on $A$. Because the blocks move with different velocities, they will meet sometimes. Therefore we must examine what happens when the blocks meet. In addition we must examine what happens when 1-blocks and 2-blocks inside a 3-block reach the left endpoint of the 3-block. We must also examine what happens when a block reaches the first place in $A$. In that case the block cannot move to the left. Besides, we will prove that a 3-block does not change size by applying $\varphi^{n+3+k(A)}$ on $A$. As a measure of the size of a 3-block $B$ we will define the mass $m(B)$ of $B$.

First we study how the blocks move by applying $\varphi^{n+2}$. Before we formulate the next lemma we need some definitions.

**Definition 3.3.**
Let $A = a_1 \ldots a_n$ and $B = a_{s} \ldots a_{t}$ be a piece of $A$. We define the left endpoint of $B$ by $l(B) = l(A,B) = s$ and the right endpoint of $B$ by $r(B) = r(A,B) = t$.

**Definition 3.4.**
Let $B$ be a 3-block. We define the mass of $B$ by

$$m(B) = (\text{the number of 1's in } B) - (\text{the number of 0's in } B).$$

**Definition 3.5.**
a) Let $B_3$ be a 3-block in $A$. Suppose $A$ is of the form

$$A = CB_300s(10)0C_1 \ldots C_tD$$

where $s \geq 0$, and $C_1 = 10$ or $C_1 = 11t(01)00$ for some $t \geq 0$. 
By definition the 1- and 2-blocks in 00s(10)0C1 ... Cp meet B3 by applying e^{n+2}.

b) Let B3 be a 3-block. We suppose B3 is of the form

\[ B_3 = 11s(01)1C_1 ... C_p \]

where s \geq 0, and C_i = 01 or C_i = 00t(10)11 for some t \geq 0.

By definition the 1- and 2-blocks in 11s(01)1C_1 ... C_p jump out of B3 by applying e^{n+2}.

c) Suppose B2 is a 2-block in A which does not meet or jump out of a 3-block by applying e^{n+2}. Suppose r(A, B_2) = s. If there are 1-blocks on the places s+2, s+4, ..., s+2t, we say that these 1-blocks meet B2.

**Lemma 3.6.**
Suppose A satisfies Cond. 3.1, and let A* = e^{n+2}(A) \in \{0,1\}^{n+1}.

a) Suppose B1 is a 1-block in A. Then there exists a 1-block B_1* in A* such that

\[ r(A*, B_1*) = r(A, B_1) - 1 - y - 2z \]

where y = 1 if B_1 meets or jumps out of a 3-block by applying e^{n+2}, y = 0 otherwise, and z = 1 if B_1 meets a 2-block by applying e^{n+2}, z = 0 otherwise.

b) Suppose B2 is a 2-block in A. Then there exists a 2-block B_2* in A* such that

\[ r(A*, B_2*) = r(A, B_2) - 2y + 2z \]

where y = 1 if B_2 meets or jumps out of a 3-block by applying
\( \theta^{n+2} \), \( y = 0 \) otherwise, and \( z \) is equal to the number of 1-blocks which meet \( B_2 \) by applying \( \theta^{n+2} \).

c) Suppose \( B_3 \) is a 3-block in \( A \). Then there exists a 3-block \( B_3^* \) in \( A^* \) such that

\[
r(A^*, B_3^*) = r(A, B_3) + 1 + 2\beta_1 + 4\beta_2
\]

where \( \beta_i \) is the number of \( i \)-blocks which meet \( B_3 \) by applying \( \theta^{n+2} \).

\[
l(A^*, B_3^*) = l(A, B_3) + 1 + 2\beta_1 + 4\beta_2
\]

where \( \beta_i \) is the number of \( i \)-blocks which jump out of \( B_3 \) by applying \( \theta^{n+2} \). Besides \( m(B_2^*) = m(B_3) \).

\[d)\ w(A^*) = k + 3. \] All the blocks in \( A^* \) arise from one of the blocks in \( A \) as in a), b) and c).

e) \( A^* \) is of the form

\[
A^* = s(10)0C_1 \ldots C_pD
\]

where \( s \geq 0 \), \( C_i = 10 \) or \( C_i = 11t(01)00 \) for some \( t \geq 0 \), and \( D \) starts with \( 0 \) or a 3-block.

**Definition 3.7**

Let \( A \) and \( A^* = \theta^{n+2}(A)1 \) be as in the previous lemma.
Suppose \( A^* = s(10)0C_1 \ldots C_pD \) is as in Lemma 3.6.e.

\[a)\ \text{We define} \ k(A) = r(A^*, C_p) - 1. \]

\[b)\ \text{We define} \ \varphi(A) = \theta^{n+3+k(A)}(A). \]

c) By definition the 1-blocks and 2-blocks in \( A \), which correspond to blocks in \( s(10)0C_1 \ldots C_p \), circle around by applying \( \varphi \), and meet \( \hat{B}_3 \) by applying \( \varphi \), where \( \hat{B}_3 \) is the last 3-block in \( A \).
We observe that $k(A) = 2y_1 + 4y_2$ where $y_i$ is the number of $i$-blocks which circle around by applying $\phi$. Besides, $k(A)$ is the least integer $s$ such that $\theta^{n+3+s}(A)$ satisfies Cond. 3.1.

The next definitions and lemma describe what happens to the blocks in $A$ when we apply $\phi = \theta^{n+3+k(A)}$ in case $A$ satisfies Cond. 3.1.

**Definition 3.8.**
Suppose $A = a_1 \ldots a_n$ satisfies Cond. 3.1, and let $\phi = \theta^{n+3+k(A)}$.

If two blocks in $A$ meet by applying $\theta^{n+2}$, we also say that the two blocks meet by applying $\phi$.

If a 1-block or a 2-block $B$ jumps out of a 3-block by applying $\theta^{n+2}$, we say that $B$ jumps out by applying $\phi$.

Before the lemma we must define precisely the concept that a block moves (to the left). We calculate modulo $n$, therefore place $0 = \text{place } n$, place $(-1) = \text{place } (n-1)$, etc.

**Definition 3.9.**
Suppose $A = a_1 \ldots a_n$ satisfies Cond. 3.1, and $B$ is an $i$-block in $A(i=1,2,3)$.

Then $B$ moves $q$ places (to the left) by applying $\phi$ means: There exists an $i$-block $B^{**}$ in $\phi(A)$ such that

$r(\phi(A), B^{**}) = r(A, B) - q \pmod{n}$.

**Lemma 3.10.**
Suppose $A = a_1 \ldots a_n$ satisfies Cond. 3.1.
a) Let \( B_1 \) be a 1-block in \( A \). As the main rule \( B_1 \) moves \( k(A)+2 \) places by applying \( \varphi \). In addition we have:
- If \( B_1 \) meets a 3-block, it moves 1 place extra.
- If \( B_1 \) jumps out of a 3-block, it moves 1 place extra.
- If \( B_1 \) meets a 2-block, it moves 2 places extra.
- If \( B_1 \) circles around, it moves -1 place extra.

b) Let \( B_2 \) be a 2-block in \( A \). As a main rule \( B_2 \) moves \( k(A)+1 \) places by applying \( \varphi \). In addition we have:
- If \( B_2 \) meets a 3-block, it moves 2 places extra.
- If \( B_2 \) jumps out of a 3-block, it moves 2 places extra.
- \( B_2 \) moves -2 places for each 1-block which meets \( B_2 \) by applying \( \varphi \).
- If \( B_2 \) circle around, it moves -2 places extra.

c) Let \( B_3 \) be a 3-block in \( A \). As a main rule \( B_3 \) moves \( k(A) \) places by applying \( \varphi \). In addition we have:
- \( B_3 \) moves -4 places for each 2-block which meets \( B_3 \) by applying \( \varphi \).
- \( B_3 \) moves -2 places for each 1-block which meets \( B_3 \) by applying \( \varphi \).

d) Again let \( B_3 \) be a 3-block in \( A \). \( B_3 \) corresponds to a 3-block \( B_3^{**} \) in \( \varphi(A) \) as in c). Then
\[
1(\varphi(A), B^{**}) = 1(A,B) - k(A) + 2y_1 + 4y_2
\]
where \( y_i \) = the number of \( i \)-blocks which jump out of \( B_3 \) by applying \( \varphi \).

**Definition 3.11.**

Suppose \( A \) satisfies Cond. 3.1. By lemma 3.10 a block
B in A corresponds to a block $B^{**}$ in $\varphi(A)$. We denote $B^{**}$ by $\varphi(B)$.

**Lemma 3.12.**

Suppose $A$ satisfies Cond. 3.1. Then $\varphi(A)$ satisfies Cond. 3.1, and all blocks in $\varphi(A)$ are equal to $\varphi(B)$ for some block $B$ in $A$.

If $B_3$ is a 3-block in $A$, then $m(B_3) = m(\varphi(B_3))$.

We illustrate lemma 3.10 by seven examples. We put an asterisk above the 1-blocks, a line above the 2-blocks, and a line below the 3-blocks.

**Example 1.** $(k=10, k(A)=0)$

\[
\begin{align*}
A &= \mbox{00010100111001110111} \\
\varphi(A) &= \mbox{00010100111001110111}.
\end{align*}
\]

**Example 2.** $(k=7, k(A)=0)$

\[
\begin{align*}
A &= \mbox{0010011110011110111} \\
\varphi(A) &= \mbox{0010011110011110111}.
\end{align*}
\]

**Example 3.** $(k=8, k(A)=0)$

\[
\begin{align*}
A &= \mbox{1110001011100111} \\
\varphi(A) &= \mbox{1110001011100111}.
\end{align*}
\]

**Example 4.** $(k=5, k(A)=0)$

\[
\begin{align*}
A &= \mbox{001110001000111} \\
\varphi(A) &= \mbox{001110001000111}.
\end{align*}
\]

**Example 5.** $(k=7, k(A)=0)$

\[
\begin{align*}
A &= \mbox{001110110001110111} \\
\varphi(A) &= \mbox{001110110001110111}.
\end{align*}
\]

**Example 6.** $(k=1, k(A)=2)$

\[
\begin{align*}
A &= \mbox{0100000111} \\
\varphi(A) &= \mbox{0100000111}.
\end{align*}
\]
Example 7. \( k=2, k(A)=4 \) \( A = 0 \begin{array}{cccccccc} T & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \end{array} \)
\( \theta^{n+2}(A) \) \( 1 = 0 \begin{array}{cccccccc} T & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \end{array} \)
\( \varphi(A) = \theta^{n+3+k(A)}(A) = 0 \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ \end{array} \).

We also illustrate the proof of Thm. 2.3 by an example with \( k=3 \).
\[
A = 00T10*00000111
\]
\[
\varphi(A) = 0*00000000000111 = \theta^{n+3}(A)
\]
\[
\theta^{n+2}(\varphi(A)) \) \( 1 = 100110000000111
\]
\[
\varphi^2(A) = 0000110110111 = \theta^2(n+3)+6(A)
\]
\[
\varphi^3(A) = 000010*T0000111 = \theta^3(n+3)+6(A)
\]
\[
\varphi^4(A) = 0*000T0000111 = \theta^4(n+3)+6(A)
\]
\[
\theta^{n+2}(\varphi^4(A)) \) \( 1 = 0000110000111
\]
\[
\varphi^5(A) = 000T00000110111 = \theta^5(n+3)+8(A)
\]
\[
\varphi^6(A) = 0*T00000100111 = \theta^6(n+3)+8(A)
\]
\[
\theta^{n+2}(\varphi^6(A)) \) \( 1 = 01100010000111
\]
\[
\varphi^7(A) = 0*000011011111 = \theta^7(n+3)+12(A)
\]
\[
\theta^{n+2}(\varphi^7(A)) \) \( 1 = 10000001100111
\]
\[
\varphi^8(A) = 0000T000110111 = \theta^8(n+3)+14(A)
\]
\[
\varphi^9(A) = 000T000100111 = \theta^9(n+3)+14(A)
\]
\[
\varphi^{10}(A) = 00T01*00001111 = \theta^{10}(n+3)+14(A)
\].

Putting \( n=13 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = 1 \) in (2.3) we get
\( 12a = 8b \), and hence \( a=2 \) and \( b=3 \). By Thm. 2.3 the period is
\[
2(13+2-4-4)(n+3)+4\cdot2+2\cdot3 = 10(n+3)+14
\]
This is in accordance with the calculations in the example.
Part 1 of the proof of Thm. 2.3:

We prove in this first part the existence of two integers \( a \) and \( b \) satisfying (2.3) such that

\[
a(n+2-2\gamma_1-4\gamma_2-4\gamma_3)(n+3) + 4a\gamma_2 + 2b\gamma_1
\]

is a period.

In the second part we prove that \( a \) and \( b \) can be chosen minimal.

Because of Lemma 3.2 we can suppose that \( A \) satisfies Cond. 3.1.

We consider \( A, \varphi(A), \varphi^2(A), \ldots \). There clearly exist integers \( s_1 < s_2 \) such that \( \varphi^{s_1}(A) = \varphi^{s_2}(A) \). Putting \( s = s_2 - s_1 \), we get \( \varphi^s(A) = A \).

We suppose \( A \) contains the blocks \( E_1, \ldots, E_x \), numbered from left to right, that is \( r(A, E_i) < r(A, E_{i+1}) \) for \( i = 1, \ldots, x-1 \).

Consider \( A = \varphi^s(A) = \varphi^{2s}(A) = \ldots \). Because of the finiteness there exist \( p < q \) such that

\[
r(\varphi^{ps}(A), \varphi^{ps}(E_1)) = r(\varphi^{qs}(A), \varphi^{qs}(E_1)) \text{ for } i=1, \ldots, x.
\]

Putting \( t = qs - ps \), we get

\[
r(\varphi^t(A), \varphi^t(E_1)) = r(A, E_i) \text{ for } i=1, \ldots, x.
\]

This means that every 1-block (2-block) circles exactly the same number of times around by applying \( \varphi^t \). Let \( b \) be the number of times every 1-block (2-block) circles around by applying \( \varphi^t \). By Lemma 3.10 the 3-block do not circle around at all. Therefore we get that every 1-block,
2-block and 3-block moves respectively \( nb \), \( na \) and 0 places by applying \( \varphi^t \).

Using Lemma 3.10 we get by applying \( \varphi^t \):

Each 1-blocks \( B_1 \) moves (the number of places)

\[
\sum_{i=0}^{t-1} (2 + k(A)) \quad \text{(the main rule)}
\]

\(+ by_3 \quad \text{(} B_1 \text{ meets every 3-block } b \text{ times})
\]

\(+ by_3 \quad \text{(} B_1 \text{ jumps out of every 3-block } b \text{ times})
\]

\(+ 2(b-a)y_2 \quad \text{(} B_1 \text{ meets every 2-block } (b-a) \text{ times})
\]

\(- b \quad \text{(} B_1 \text{ moves } -1 \text{ place every time } B_1 \text{ circles around}).
\]

Hence,

\[
(3.1) \quad nb = 2t + \sum_{i=0}^{t-1} k(A) + 2by_3 + 2(b-a)y_2 - b.
\]

Each 2-block \( B_2 \) moves (the number of places)

\[
\sum_{i=0}^{t-1} (1 + k(A)) \quad \text{(the main rule)}
\]

\(+ 2ay_3 \quad \text{(} B_2 \text{ meets every 3-block } a \text{ times})
\]

\(+ 2ay_3 \quad \text{(} B_2 \text{ jumps out of every 3-block } a \text{ times})
\]

\(- 2(b-a)y_2 \quad \text{(} B_2 \text{ meets every 1-block } (b-a) \text{ times})
\]

\(- 2a \quad \text{(} B_2 \text{ moves } -2 \text{ places every time } B_2 \text{ circles around}).
\]

Hence,
(3.2) \[ na = t + \sum_{i=0}^{t-1} k(\phi^i(A)) + 4\gamma_3 - 2(b-a)\gamma_1 - 2a. \]

Each 3-block \( B_3 \) moves (the number of places)
\[
\sum_{i=0}^{t-1} k(\phi^i(A)) \quad \text{(the main rule)}
\]
- \( 2b\gamma_1 \) (\( B_3 \) meets every 1-block \( b \) times)
- \( -4a\gamma_2 \) (\( B_3 \) meets every 2-block \( a \) times).

Hence,
\[
0 = \sum_{i=0}^{t-1} k(\phi^i(A)) - 2b\gamma_1 - 4a\gamma_2.
\]

Hence,
\[
(3.3) \quad \sum_{i=0}^{t-1} k(\phi^i(A)) = 2b\gamma_1 + 4a\gamma_2.
\]

(This follows also from the definition of \( k(\phi^i(A)) \), which implies that \( k(\phi^i(A)) = 2\gamma_1 + 4\gamma_2 \) where \( \gamma_j \) = the number of \( j \)-blocks in \( \phi^i(A) \) circling around by applying \( \phi \).)

(3.1) and (3.3) imply
\[
nb = 2t + 2b\gamma_1 + 4a\gamma_2 + 2b\gamma_3 + 2b\gamma_2 - 2a\gamma_2 - b.
\]

Hence
\[
(3.4) \quad 2t = b(n+1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3) - 2a\gamma_2.
\]

(3.2) and 3.3) imply
\[
na = t + 2b\gamma_1 + 4a\gamma_2 + 4a\gamma_3 - 2b\gamma_1 + 2a\gamma_1 - 2a.
\]

Hence
(3.5) \[ t = a(n+2 - 2\gamma_1 - 4\gamma_2 - 4\gamma_3) . \]

(3.4) and (3.5) imply (2.3):

\[ b(n+1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3) = a(2n+4 - 4\gamma_1 - 6\gamma_2 - 8\gamma_3) . \]

\( \phi(\varphi^i(A)) = 6^{n+3+k(\varphi^i(A))} (\varphi^i(A)) . \) Hence \( \phi^t \) is equal to \( \theta \) applied

\[ \sum_{i=0}^{t-1} (n+3+k(\varphi^i(A))) = t(n+3) + \sum_{i=0}^{t-1} k(\varphi^i(A)) \] times.

(3.3) and (3.5) imply that \( \phi^t \) is equal to \( \theta \) applied

\[ t(n+3) + 2b\gamma_1 + 4a\gamma_2 = a(n+2-2\gamma_1-4\gamma_2-4\gamma_3)(n+3) + 2b\gamma_1 + 2a\gamma_2 \] times

which is a period for \( A \). The proof of the first part is complete.

The main concept of the second part of the proof is the definitions of distances between blocks. We calculate modulo \( n \). We write \( \text{card} \ m \) to denote the number of elements in \( m \) where \( m \) is a set.

**Definition 3.13.**

Suppose \( B \) and \( C \) are two blocks in \( A = a_1 \ldots a_n \). If \( B \) is to the left of \( C \), we define

\[ m(B,C) = m = \{ a_{r(C)+1}, \ldots, a_n \} \cup \{ a_1, \ldots, a_{r(B)-1} \} \] and

\[ z(B,C) = z = 1 \], else

\[ m(B,C) = m = \{ a_{r(C)+1}, \ldots, a_{r(B)-1} \} \] and \( z(B,C) = z = 0 \).

If \( a_i \in m \), we say that \( a_i \) is between \( B \) and \( C \).
If B is a 1-block we define
\[ \chi(B, C) = \chi = 2 \cdot \text{(the number of 1-blocks between B and C)} \]
\[ + 2 \cdot \text{(the number of 2-blocks between B and C)} \]
\[ + (\text{the number of endpoints } a_i \text{ between B and C, of 3-blocks}) \cdot z. \]

If B is a 2-block or 3-block we define
\[ \chi(B, C) = \chi = 2 \cdot \text{(the number of 1-blocks between B and C)} \]
\[ + 4 \cdot \text{(the number of 2-blocks between B and C)} \]
\[ + 2 \cdot (\text{the number of endpoints } a_i \text{ between B and C, of 3-blocks}) \cdot 2z. \]

We define \( d(B, C) = \text{card } M - \chi \).

Before proving the second part of Thm. 2.3 we need 5 lemmas concerning distances between blocks.

**Lemma 3.14.**
Suppose \( A \) satisfies Cond. 3.1, and contains \( \gamma_i \) i-blocks for \( i=1,2,3 \). Suppose further that \( B_i \) and \( C_i \) are i-blocks in \( A \), and \( \hat{B}_3 \) is the last 3-block in \( A \).

a) If \( B_1 \) and \( B_2 \) meet by applying \( \varphi \), we have
\[ d(B_1, B_2) = 1 \text{ and } d(\varphi(B_1), \varphi(B_2)) = n+1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3, \]
otherwise
\[ d(\varphi(B_1), \varphi(B_2)) = d(B_1, B_2) - 1 + z \]
where \( z = 1 \) if \( B_2 \) jumps out of a 3-block or meet a 3-block \( \hat{B}_3 \) by applying \( \varphi \).

b) \( B_2 \) and \( B_3 \) meet by applying \( \varphi \) if and only if
\[ d(B_2, B_3) = 4. \]
In this case
\[ d(\varphi(B_2), \varphi(B_3)) = n+5 - 2\gamma_1 - 4\gamma_2 - 4\gamma_3, \]
otherwise

\[ d(\varphi(B_2), \varphi(B_3)) = d(B_2, B_3) - 1. \]

c) \[ d(\varphi(B_1), \varphi(C_1)) = d(B_1, C_1). \]

d) \[ d(\varphi(B_2), \varphi(C_2)) = d(B_2, C_2). \]

e) \[ d(\varphi(B_3), \varphi(C_3)) = d(B_3, C_3). \]

**Lemma 3.15.**

We suppose \( A \) satisfies Cond. 3.1, and \( B_i \) is an \( i \)-block for \( i=1,2 \). If \( t \) is a multiple of \( n+2 - 2\gamma_1 - 4\gamma_2 - 4\gamma_3 \) and \( d(\varphi^t(B_1), \varphi^t(B_2)) = d(B_1, B_2) \), then \( \varphi^t(A) = A \).

**Lemma 3.16.**

We suppose \( A \) satisfies Cond. 3.1, and \( B_i \) is an \( i \)-block in \( A \) for \( i=1,2 \). Moreover, we suppose that \( r \) and \( s \) are multiples of \( n+2 - 2\gamma_1 - 4\gamma_2 - 4\gamma_3 \).

If \( B_1 \) and \( B_2 \) meet \( < c \) times by applying \( \varphi^t \) on \( \varphi^r(A) \), then \( B_1 \) and \( B_2 \) meet \( \leq c \) times by applying \( \varphi^t \) on \( \varphi^s(A) \).

**Lemma 3.17.**

Suppose \( A \) satisfies the hypothesis of Thm. 2.3, and let \( s = n+2 - 4\gamma_1 - 4\gamma_2 - 2\gamma_3 \). Moreover, \( B_i \) is an \( i \)-block for \( i=2,3 \).

Then \( B_2 \) meets \( B_3 \) once, and jumps out of \( B_3 \) once, by applying \( \varphi^s \) on \( A \).

**Lemma 3.18.**

We suppose \( A \) satisfies Cond. 3.1, and that each \( 1 \)-block \( B_1 \) meets each \( 2 \)-block \( c \) times, and each \( 2 \)-block \( B_2 \) meets each \( 3 \)-block \( B_3 \) \( a \) times by applying \( \varphi^s \). We also suppose
$\varphi^s(A) = A$.

Then each 1-block $B_1$ circles around $c+a$ times by applying $\varphi^s$.

**Part 2 of the proof of Thm. 2.4:**

We suppose $A$ satisfies Cond. 3.1, and that $a, b$ are the minimal numbers which satisfies (2.3).

From the first part of the proof where exist integers $a', b'$ which satisfies (2.3), and if $t = a'(n+2-2\gamma_1-4\gamma_2-4\gamma_3)$ (See (3.5)), then $\varphi^t(A) = A$. Moreover, each 1-block meets each 2-block in $A$ $c' = b' - a'$ times by applying $\varphi^t$.

There exists a $q > 0$ such that $a' = aq$ and $b' = bq$.

We define

$$t_i = ai(n+2-2\gamma_1-4\gamma_2-4\gamma_3) \text{ for } i=1,\ldots,q.$$  

Hence,

(3.6) \hspace{1cm} $\varphi^{tq}(A) = A$.

(3.7) \hspace{1cm} Each 1-block meets each 2-block $qc = qb - qa$ times by applying $\varphi^{tq}$ on $A$.

We prove

(3.8) \hspace{1cm} Each 1-block meets each 2-block $c=b-a$ times by applying $\varphi^{t1}$ on $A$.

Suppose (3.8) is not true. By (3.7) there exist a 1-block $B_1$, a 2-block $B_2$ and $i, j \in \{0, \ldots, q-1\}$ such that $\varphi^{ti}(B_1)$ meets $\varphi^{ti}(B_2) < c$ times by applying $\varphi^{t1}$, and $\varphi^{tj}(B_1)$ meets $\varphi^{tj}(B_2) > c$ times by applying $\varphi^{t1}$. Lemma
3.16 with \( t = t_1 \) gives a contradiction.

Next we show that \( d(B_1, B_2) = d(\phi^{t_1}(B_1), \phi^{t_1}(B_2)) \) where \( B_1 \) is an \( i \)-block. Lemma 3.17 implies

\[(3.9) \quad \text{Each 2-block meets each 3-block a times and jumps out of each 3-block a times by applying } \Phi^{t_1} \text{ on } A.\]

Let \( \Omega = \{0, \ldots, t_1 - 1\} \). Then (3.9) and Lemma 3.14.a) imply

\[(3.10) \quad \text{There exist } (\gamma_3 - 1)a + \gamma_3 a \text{ numbers } i \in \Omega \text{ such that } d(\phi^{i+1}(B_1), \phi^{i+1}(B_2)) = d(\phi^i(B_1), \phi^i(B_2)).\]

\[(3.8) \text{and Lemma 3.14a)} \text{ imply}\]

\[(3.11) \quad \text{There exist } c = b - a \text{ numbers } i \in \Omega \text{ such that } d(\phi^i(B_1), \phi^i(B_2)) = 1 \text{ and } d(\phi^{i+1}(B_1), \phi^{i+1}(B_2)) = n + 1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3. \text{ In this case } \phi^i(B_1) \text{ meets } \phi^i(B_2) \text{ by applying } \phi.\]

\[(3.10), (3.11) \text{ and Lemma 3.14a) imply}\]

\[(3.12) \quad \text{There exist } t_1 - c - 2\gamma_3 a + a \text{ numbers } i \in \Omega \text{ such that } d(\phi^{i+1}(B_1), \phi^{i+1}(B_2)) = d(\phi^i(B_1), \phi^i(B_2)) - 1.\]

By (3.11) \( d(\phi^i(B_1), \phi^i(B_2)) \) changes first from \( d(B_1, B_2) \) to 1, then \( (c - 1) \) times from \( n + 1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3 \) to 1, and finally from \( n + 1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3 \) to \( d(\Phi^{t_1}(B_1), \Phi^{t_1}(B_2)) \).

Hence by (3.12)

\[
t_1 - c - 2\gamma_3 a + a = (d(B_1, B_2) - 1) + (c - 1)(n + 1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3 - 1) + (n + 1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3 - d(\Phi^{t_1}(B_2), \Phi^{t_1}(B_2))).
\]

Since \( t_1 = a(n + 2\gamma_1 - 4\gamma_2 - 4\gamma_3) \) and \( c = b - a \), we get
\[ d(B_1, B_2) - d(\varphi^{t_1}(B_1), \varphi^{t_1}(B_2)) = a(n+3-2\gamma_1-4\gamma_2-6\gamma_3) - (b-a) \]
\[ -(b-a)(n-2\gamma_1-2\gamma_2-2\gamma_3) \]
\[ = a(2n+4-4\gamma_1-6\gamma_2-8\gamma_3) - b(n+1-2\gamma_1-2\gamma_2-2\gamma_3) = 0 \]

by (2.3). Hence,

(3.13) \[ d(B_1, B_2) = d(\varphi^{t_1}(B_1), \varphi^{t_2}(B_2)). \]

(3.13) and Lemma 3.15 imply that \( A = \varphi^{t_1}(A) \). By Lemma 3.18 each 1-block circles around \( b=a+c \) times by applying \( \varphi^{t_1} \). Besides, each 2-block circles around \( a \) times by applying \( \varphi^{t_1} \). Hence,

\[ \sum_{i=0}^{t_1-1} k(\varphi^i(A)) = 2b\gamma_1+4a\gamma_2 \]

As in the end of the first part of the proof we get that \( a(n+2-2\gamma_1-4\gamma_2-4\gamma_3)(n+3)+2b\gamma_1+4a\gamma_2 \) is a period. The proof is complete.

**Proof of Cor. 2.5:**

In the case that \( A \) contains only two different types of blocks, the proof is easy by using Thm. 2.4.

Suppose \( A \) contains 3 different types of blocks, therefore \( n \geq 9 \). We suppose that \( a, b \) are the minimal positive integers which satisfy (2.3). We have

\[ a \leq n+1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3 \leq n-5 \]

and

\[ b \leq 2n+4 - 4\gamma_1 - 6\gamma_2 - 8\gamma_3 \leq 2n-14 \]

The period \( p \) in Thm. 2.3 satisfies
\[ p = a(n+2-2\gamma_1-4\gamma_2-4\gamma_3)(n+3) + 4a\gamma_2 + 2b\gamma_1 \]
\[ \leq (n-5)(n-8)(n+3) + 4(n-5)\frac{n}{4} + 2(2n-14)\frac{n}{2} \]
\[ = n^3 - 7n^2 - 18n + 120 < n^3 \quad \text{since} \quad n \geq 9 \]

We have used the fact that \( \gamma_1 \leq \frac{n}{2} \) and \( \gamma_2 \leq \frac{n}{4} \).

**Proof of Cor. 2.6:**

We suppose \( A \) satisfies Cond. 3.1. Then \( \varphi_i(A) \) satisfies Cond. 3.1. for all \( i \).

It is easy to see that \( \theta(\varphi_i(A)), \ldots, \theta^{n+3+k(\varphi_i(A)) - 1}(A) \) do not satisfy Cond. 3.1. Therefore the minimal period \( p \) satisfies \( \theta^p = \varphi^q \) for some \( q \); that is, \( \theta^p(A) = \varphi^q(A) = A \) for some \( q \).

We suppose the 1-block and the 2-block circles respectively \( b \) and a times around by applying \( \varphi^q \) on \( A \). Then it is easy to see that the 1-block meets the 2-block \( c = b-a \) times by applying \( \varphi^q \) on \( A \). As in the first part of the proof of Thm. 2.3 we see that \( p \) is as in the theorem.

4. The general situation.

In this section we will indicate by an example how to treat the general situation \( E_k + \ldots + E_{k+p} \) for \( p > 2 \).

We suppose \( p=3 \). As in the case \( p=2 \) we must define the concepts: i-block (for \( i=1,2,3,4 \)), \( \theta \), \( \varphi \), \( \kappa(A) \), meet, jump out, circle around, and "Cond. 3.1." Specially, \( \varphi(A) = \theta^{n+4+k(A)}(A) \)

We suppose \( A \in \{0,1\}^n \) satisfies "Cond. 3.1", and contains 1 i-block \( B_i \) for \( i=1,2,3,4 \). Then we can show the following:

As a main rule \( B_1 \) moves \( 3+k(A) \) places by applying \( \varphi \). \( B_1 \) moves in addition:
2 places if $B_1$ meets $B_2$,
1 place if $B_1$ meets $B_3$ or $B_4$,
1 place if $B_1$ jumps out of $B_3$ or $B_4$,
-1 place if $B_1$ circles around.

As a main rule $B_2$ moves $2+k(A)$ places by applying $\phi$.

$B_2$ moves in addition:
-2 places if $B_1$ meets $B_2$,
2 places if $B_2$ meets or jumps out of $B_3$,
2 places if $B_2$ meets or jumps out of $B_4$,
-2 places if $B_2$ circles around.

As a main rule $B_3$ moves $1+k(A)$ places by applying $\phi$.

$B_3$ moves in addition:
-2 places if $B_1$ meets $B_3$,
-4 places if $B_2$ meets $B_3$,
3 places if $B_3$ meets or jumps out of $B_4$,
-3 places if $B_3$ circles around.

As a main rule $B_4$ moves $k(A)$ places by applying $\phi$.

$B_4$ moves in addition:
-2 places if $B_1$ meets $B_4$,
-4 places if $B_2$ meets $B_4$,
-6 places if $B_3$ meets $B_4$.

We suppose next that $A = \omega^s(A)$, and that the 1-block, 2-block and 3-block respectively circles around $a, b$ and $c$ \(s-1\) times. Let $K = \sum_k^s k(\omega^i(A))$. By applying $\phi^s$ to $A$, $B_1$ moves the following number of places:
3s+K (the main rule)
-a (B₁ circles around a times)
+2(a-b) (B₁ meets B₂ (a-b) times)
+2(a-c) (B₁ meets and jumps out of B₃ 2(a-c) times)
+2a (B₁ meets and jumps out of B₄ 2a times).

Hence

(4.1) \( na = 3s+K-a + 2(a-b) + 2(a-c) + 2a \).

In the same way, by studying \( B₂, B₃ \) and \( B₄ \) we get the equations:

(4.2) \( nb = 2s+K - 2b - 2(a-b) + 4(b-c) + 4b \).

(4.3) \( nc = s+K - 3c - 2(a-c) - 4(b-c) + 6c \).

(4.4) \( 0 = K - 2a - 4b - 6c \).

From (4.4) we see that \( K = 2a + 4b + 6c \). Putting this into (4.1), (4.2) and (4.3), we get

(4.5) \( 3s = a(n-7) - 2b - 4c \).

(4.6) \( 2s = b(n-12) - 2c \).

(4.7) \( s = c(n-15) \).

Hence,

(4.8) \( a(n-7) - 2b - 4c = 3c(n-15) \).

(4.9) \( b(n-12) - 2c = 2c(n-15) \).

As in the end of the first part of the proof of Thm. 2.3
we can show that \( \varphi^g \) is equal to \( \theta \) applied

\[ (4.10) \quad p = s(n+4) + K = c(n-15)(n+4) + 2a + 4b + 6c \]

times (We use (4.4) and (4.7)). \( p \) is therefore a period for \( A \).

Let us check the above result on the following example: \( n=19 \), \( k=7 \) and \( A = 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \). Calculations show that the period of \( A \) is \( p = 748 \).

Putting \( n=19 \) into (4.8) and (4.9) we then get

\[ (4.11) \quad 12a - 2b - 4c = 12c, \]
\[ (4.12) \quad 7b - 2c = 8c. \]

The smallest integers satisfying (4.11) and (4.12) are \( a=11, b=10, c=7 \). We put these into (4.10), and again obtain \( p = 7 \cdot 4 \cdot 23 + 2 \cdot 11 + 4 \cdot 10 + 6 \cdot 7 = 748 \) as a period.

5. Proofs of Lemmas from Section 3.

Throughout this section, \( k,n \) and \( \theta \) are as in Thm. 2.3.

Definition 5.1.

If \( a=1 \), then \( a'=0 \). If \( a=0 \), then \( a'=1 \). Moreover, for every \( C = c_1 \ldots c_t \in \{0,1\}^t \), we define \( C' = c_1' \ldots c_t' \).

Lemma 5.2.

If \( A = a_1 \ldots a_n \), then \( \theta(A) = a_2 \ldots a_n a_1' \) whenever \( w(a_2 \ldots a_n) \in \{k,k+1,k+2\} \), \( \theta(A) = a_2 \ldots a_n a_1 \) otherwise.

The proof is obvious.
Definition 5.3.
Suppose $A = a_1 \ldots a_n$ and $C = c_s \ldots c_r$.
If $C$ is outside all the 3-blocks in $A$ and $C=10$ or $11t(01)00$ for some $t \geq 0$, $C$ is an $H$-block in $A$.
If $C$ is inside a 3-block in $A$ and $C=01$ or $00t(10)11$ for some $t \geq 0$, $C$ is a $K$-block in $A$.

Lemma 5.4.
Suppose $A \in \{0,1\}^n$ and $w(A) = k+3$.

a) If $A=10C$, then $\theta^2(A) = 001$.

b) If $A = 11t(01)00C$, then $\theta^{4+2t}(A) = C00t(10)11$.

c) Suppose
\[ B_3 = 11s_0(01)1C_1 \ldots C_p^1s_1C_{p+1}^1s_2 \ldots C_{p+q}^1s_{q+1}, \]
\[ G = 00f(10)0D_1 \ldots D_r \text{ and } A = B_3G, \]
where $s_i > 0$, each $C_i$ is a $K$-block, each $D_i$ is an $H$-block and $s_i,f \geq 0$. Furthermore, let
\[ \tilde{B}_3 = 00s_0(10)0C_1' \ldots C_p'1s_1C_{p+1}'s_2 \ldots C_{p+q}'s_{q+1}. \]
\[ \tilde{G} = 11f(01)1D_1' \ldots D_r' = G'. \]
\[ y = r(A,B_3) \text{ and } z = r(A,G). \]

Then we have
\[ \theta^y(A) = GEB_3, \theta^z(A) = \tilde{E}B_3\tilde{G}, w(\theta^y(A)) = k \text{ and } w(\theta^z(A)) = k+3. \]
Proof.

a) and b) follows from Lemma 5.2.

c) Let \( n_i = r(A, C_i) \) and \( m_i = r(A, D_i) \). We use Lemma 5.2 many times. The vectors in the following equations have weight \( k \).

\[
\begin{align*}
\theta^{3+2s_0}(A) &= C_1 \ldots \text{EOOs}_0(10)0 \\
\theta^{n_1}(A) &= C_2 \ldots \text{EOOs}_0(10)OC_1' \\
\theta^{n_p}(A) &= 1s_1 \ldots \text{EOOs}_0(10)OC_1' \ldots C_p' \\
\theta^{n_p+s_1}(A) &= C_{p+1} \ldots \text{EOOs}_0(10)OC_1' \ldots C_p'1s_1 \\
\theta^{n_p+1}(A) &= 1s_2 \ldots \text{EOOs}_0(10)OC_1' \ldots C_p'1s_1C_{p+1} \\
\theta^{n_p+q}(A) &= 1s_{q+1} \ldots \text{EOOs}_0(10)OC_1'1s_1C_{p+1} \ldots C_{p+q} \\
\theta^y(A) &= 00f(10)0D_1 \ldots D_x\tilde{E}B_3 = G \cdot \tilde{B}_3 \\
\end{align*}
\]

The vectors in the following equations have weight \( k+3 \).

\[
\begin{align*}
\theta^{y+3+2f}(A) &= D_1 \ldots D_x\tilde{E}B_3\tilde{f}(01)1 \\
\theta^{m_1}(A) &= D_2 \ldots D_x\tilde{E}B_3\tilde{f}(01)1D_1' \\
\theta^z(A) = \theta^m r(A) &= \tilde{E}B_3\tilde{f}(01)1D_1' \ldots D_r' = \tilde{E}B_3\tilde{G} \\
\end{align*}
\]

Proof of Lemma 3.2:

(5.1) If \( A = D01 \) and \( w(A) = k+3 \), then \( \theta^{-2}(A) = 10D \).

(5.2) If \( A = \text{DOos}(10)11 \) and \( w(A) = k+3 \) where \( s \geq 0 \), then \( \theta^{-(4+2s)}(A) = 11s(01)00D \).
Suppose $A$ satisfies 1) and 2) in Def. 3.1, and $A = CD$ where $C$ ends with a 3-block and $D$ does not contain any 3-block. We define $p_1 = n - r(A, C)$. Then $A_1 = \theta^{-p_1}(A)$ ends with a 3-block. (5.1) and (5.2) implies that $w(A_1) = k+3$. Therefore $A_1$ satisfies 1), 2) and 4) in Cond. 3.1.

Suppose $A_1 = C_1 \ldots C_p E B_3$ where $C_i = 10$ or $C_i = 11s(01)00$, $B_3$ is a 3-block and $E$ starts with 0 or a 3-block. Let $p_2 = r(A, C_p)$. $\theta^{p_2}(A_1) = E B_3 C_1' \ldots C_p'$. Then $B_3 C_1' \ldots C_p'$ becomes a 3-block in $\theta^{p_2}(A_1)$. Therefore $\theta^{p_2}(A_1)$ satisfies Cond. 3.1.

**Proof of Lemma 3.6:**

We observe that $A$ has the form

$$A = 0s_1 Q_1 0s_2 Q_2 0s_3 \ldots 0s_p Q_p$$

where $s_i \geq 0$, and $Q_i$ has one of the following forms for $i < p$

(5.3) $Q_i = 10$ where $Q_i$ is outside all the 3-blocks in $A$, and the 1-block in $Q_i$ does not meet any block by applying $\theta^{n+2}$ on $A$.

(5.4) $Q_i = 11t(01)00$ where $t \geq 0$, $Q_i$ is outside all the 3-blocks in $A$, and the blocks in $Q_i$ do not meet any 3-block in $A$ by applying $\theta^{n+2}$ on $A$.

(5.5) $Q_i = B_3 G$ where $B_3$ and $G$ are as in Lemma 5.4. c).

Furthermore,
If \( Q_i \) is of the form (5.3) or (5.4), then \( s_1 > 0 \).

If \( Q_i \) is of the form (5.5) and \( 0 < i < p \), then

\[ Q_{i+1} \text{ is of the form (5.5) or } s_{i+1} > 0. \quad Q_p = B_3 \]

where \( B_3 \) is as in Lemma 5.4, c).

By Lemma 5.4

\[ \theta^n(A) = 0 s_1 \tilde{Q}_1 0 s_2 \tilde{Q}_2 0 s_3 \ldots 0 s_p \tilde{Q}_p \]

where \( \tilde{Q}_i \) is defined as follows:

Case 1: If \( Q_i \) is as in (5.3), then \( \tilde{Q}_i = 01 \).

Case 2: If \( Q_i \) is as in (5.4), then \( \tilde{Q}_i = 00t(10)11 \).

Case 3: If \( Q_i = B_3 G \) is as in (5.5), then \( \tilde{Q}_i = \tilde{B}_3 \tilde{G} \)

as in Lemma 5.4, c).

Case 4: If \( i=p \), \( \tilde{Q}_p = \tilde{B}_3 \) is as in Lemma 5.4, c)

(see (5.6)).

Furthermore, Lemma 5.4, c) implies \( w(\theta^n(A)) = k \). Since \( A \) starts with 01,11 or 00, \( \theta^n(A) \) starts with 00. Hence \( \theta^{n+2}(A) \) is of the form

\[ \theta^{n+2}(A) = C \tilde{Q}_p 11 \]

and \( w(\theta^{n+2}(A)) = k+2 \).

Next, we prove

\[ A \text{-block in } A^* = \theta^{n+2}(A) 1 \text{ is contained in } \tilde{Q}_p 1 \]

or \( \tilde{Q}_i \) where \( Q_i \) is as in (5.5).

Let \( Q_i \) be as in (5.5). By (5.6), \( A = HQ_i0t(10)M \) or \( A = Q_iK \) where \( K \) starts with a 3-block. In both cases \( \tilde{Q}_i \)

is followed by \( 00t(10)0 \) for some \( t \geq 0 \). If \( Q_i \) is as in

(5.3) or (5.4), no 3-block in \( A^* \) can start at any position.
We conclude that (5.8) is true.

Case 1: We denote the 1-block in $Q_i = 10$ by $B_1$. The number 1 in $Q_i = 01$ is in position $r(A, B_1) + 1$ in $\theta^n(A)$, and is preceded and followed by 0. Therefore, there is a 1-block $B^*$ in position $r(A, B_1) - 1$ in $A^* = \theta^{n+2}(A)$. This is in accordance with a) since $B_1$ do not meet any block by applying $\theta^{n+2}$ on $A$.

Case 2: We denote the 2-block in $Q_i$ by $B_2$ and the 1-blocks by $B_1^1, \ldots, B_1^t$, such that $Q_i = B_2 B_1^1 0 \ldots B_1^t 0 0$. Since $\tilde{Q}_i = 0 0 t(10) 1 1$ is followed by 00, there are 1-block in the positions $r(A, B_1^1) - 1, \ldots, r(A, B_1^t) - 1$ and a 2-block in the position $r(A, B_2) + 2t + 2$ in $\theta^n(A)$. Therefore, there are 1-blocks in the positions $r(A, B_1^1) - 3, \ldots, r(A, B_1^t) - 3$ in $A^* = \theta^{n+2}(A)$. This is in accordance with a) since the 1-blocks meet $B_2$ by applying $\theta^{n+2}$. Furthermore, there is a 2-block in the position $r(A, B_2) + 2t$. This is in accordance with b), since $B_2$ meet t 1-blocks by applying $\theta^{n+2}$ on $A$.

Case 3: $Q_i = B_2 G$ and $\tilde{Q}_i = \tilde{B}_2 \tilde{G}$ where

\[
B_2 = 11 s_0 (01) 1 C_1 \ldots C_p 1 s_1 C_{p+1} 1 s_2 \ldots C_{p+q} 1 s_{q+1}
\]
\[
G = 0 0 f(10) 0 D_1 \ldots D_r
\]
\[
\tilde{B}_2 = 0 0 s_0(10) 0 C_1' \ldots C_p' 1 s_1 C_{p+1}' 1 s_2 \ldots C_{p+q}' 1 s_{q+1}
\]
\[
\tilde{G} = 11 f(01) 1 D_1' \ldots D_r'
\]

where $s_i > 0$, $C_i$ is a K-block, $D_i$ is an H-block and $s_i, f \geq 0$.

We divide Case 3 into 9 subcases.
Case 3a: Suppose $1 \leq i \leq p$. Suppose $C_i = 01 = B_1'.1$ where $B_1$ is a 1-block which jumps out of $B_3$. Then $C_i' = 10, C_i''$ is preceded by a 0 and is outside all the 3-blocks in $\theta^2(A)$. Therefore there is a 1-block in $\theta^3(A)$ in position $r(A, B_1')$, hence a 1-block in $A^*$ in position $r(A, B_1') - 2$.

Case 3b: Suppose $1 \leq i \leq p$ and $C_i = 00t(10)11$. $C_i' = 11t(01)00$ is outside all the 3-blocks in $\theta^2(A)$. As in Case 3a, the blocks in $C_i$ do not move by applying $\theta^R$. Therefore if $B$ is a block in $C_i$, there is a block $B^*$ of the same type in $A^*$ such that $r(A^*, B^*) = r(A, B) - 2$. Since the block $B$ jumps out of the 3-block $B_3$ by applying $\theta^{n+2}$, this is in accordance with Lemma 3.6 a) and b).

Case 3c: The 1-blocks in $s_0(01)$ move as the 1-block in Case 3a.

Case 3d: We define $B^*_3$ and $F$ by

$$B^*_3 = 1s_1 C'_{p+1} 1s_2 \ldots 1s_{q+1} 11t(01)1D'_1 \ldots D'_r = 11F,$$

hence $C'_i = 00s_0(10)0C'_1 \ldots C'_p B^*_3$. First we prove that $B^*_3$ starts with $11t(01)1$ for some $t \geq 0$. If $s_1 \geq 2$, $C'_{p+1} = 11t(01)00$ or $C'_{p+1} = 1s_{q+1}$ is the empty set, the claim is trivially true. Therefore, we suppose $s_1 = 1$ and $C'_{p+1} = 10$. If we move from the left to the right in $F$, we reach two consecutive 1's before we reach two consecutive 0's. Hence, $B^*_3$ starts with $11t(01)1$ for some $t \geq 0$.

Next we observe that $B^*_3$ does not contain any piece of the form $00s(10)0$. By (5.8) $B^*_3$ is a 3-block in $A^*$. We now observe that:
\[ m(B_3) = 3 + s_1 + \ldots + s_{q+1} = m(B^*_3), \]
\[ r(A^*,B^*_3) = r(A,B) + 3 + 2\beta_1 + 4\beta_2 - 2 \]

where \( \beta_i \) is the number of \( i \)-blocks in \( 11f(01)1D', \ldots, D' \)

\[ \gamma = \text{the number of } i \text{-blocks which meet } B_3 \text{ by applying } \theta^{n+2}, \]

\[ l(A^*,B^*_3) = l(A,B) + 3 + 2\beta_1 + 4\beta_2 - 2 \]

where \( \beta_i \) is the number of \( i \)-blocks in \( 00s_0(10)00', \ldots, 0' \)

\[ \gamma = \text{the number of } i \text{-blocks which jump out of } B_3 \text{ by applying } \theta^{n+2}. \]

**Case 3e:** Suppose \( p < i \leq p+q \) and \( C_1 = 01 = B_10 \) where

\( B_1 \) is a 1-block in \( A \) contained in \( B_3 \). Then \( C'_1 = 10 \),

\( C'_i \) is followed by a 1 and \( C'_i \) is contained in \( B^*_3 \). The

0 in \( C'_1 \) is a 1-block in \( A^* \). Hence, there is a 1-block

in \( A^* \) in the position \( r(A,B_1) - 1 \). This is in accordance

with the lemma since \( B_1 \) does not meet or jump out of any

block by applying \( \theta^{n+2} \).

**Case 3f:** Suppose \( p < i \leq p+q \) and \( C_i = 00t(10)11 = B_2'B_1'B_2' \ldots, 1B_2'B_1'B_2' \),

where \( B_2 \) is a 2-block and \( B_1 \) are

1-blocks. \( C'_i = 11t(01)00 = 11B_1'B_2' \ldots, 1B_2'B_1'B_2' \),

where \( B_1 \) are 1-blocks and \( B_2 \) is a 2-block in \( A^* \). \( r(A^*,B_1) = r(A,B_1) - 3 \) and \( r(A^*,B_2) = r(A,B_2) + 2t \).

This is in accordance with the lemma, since \( B_1 \) meets a 2-block and

\( B_2 \) meets \( t \) 1-blocks by applying \( \theta^{n+2} \).

**Case 3g:** Suppose \( D_i = 10 = B_10 \) where \( B_1 \) is a 1-block

which meets \( B_3 \) by applying \( \theta^{n+2} \). \( D'_i = 01 = B^*_11 \) is

contained in \( B^*_3 \), and \( B^*_1 \) is a 1-block in \( A \). \( r(A^*,B^*_1) = r(A,B_1) - 2 \).
Case 3h: The 1-blocks in $f(10)$ move as the 1-block in case 3g.

Case 3i: Suppose $D_1 = 11t(01)00 = B_2 O B_0 B_1^i B_2^i \ldots O B_0^t$ where $B_2$ is a 2-block and $B_1^i$ are 1-blocks in $A$. Then $D_1^i = 00t(10)11 = B_*^2 1 B_1^i 1 B_2^i \ldots 1 B_1^i 11$ where $B_*^i$ are 1-blocks and $B_*^2$ is a 2-block in $A_*$. $r(A_*, B_1^i) = r(A, B_1^i) - 2$ and $r(A_*, B_*^2) = r(A, B_2) - 2$. This is in accordance with the lemma, since $B_1^i$ and $B_2$ meet $B_*^2$ by applying $e^{n+2}$.

Case 4: This case is treated like Case 3a, ..., Case 3f.

Specially, there is a 3-block $B_*^3$ in $A_*$ such that $r(A_*, B_*^3) = n+1$.

The proof of Lemma 3.6 a), b), c) and d) is now complete.

Suppose $Q_1$ is of the form (5.5). Then $\tilde{Q}_1$ starts with $00s_0(10)0$ and e) is satisfied.

Next, suppose $Q_1$ is of the form (5.3) or (5.4). By (5.6) $s_1 > 0$. $A$ is of the form $0 s_1 C_1 \ldots C_e D_3$ where $D_3$ starts with 0 or a 3-block, and $C_i = 10$ or $C_i = 11t(01)00$ for some $t \geq 0$. $\theta^n(A) = 0 s_1 C_1 ^i \ldots C_e D_3$ where $D_3$ starts with $00s(10)0$ for some $s \geq 0$, and e) is satisfied.

The proof of Lemma 3.6 is complete.

Proof of Lemma 3.10. We denote the last 3-block in $A$ by $B_3$. We let $A^* = \theta^{n+2}(A)1 = s(10)OC_1 \ldots C_e D$ be as in Lemma 3.6.e). Besides, we denote $A^*$ by $A^* = a^*_{1} \ldots a^*_{n+1}$ and put $r = r(A^*, C_p)$. Then

$$\phi(A) = A^{n+3+\theta}(A) = a^*_{r+1} \ldots a^*_{n} s(01)1 C_1 ^r \ldots C_e ^r =$$
We suppose \( \hat{B}_3^* = a^*_n \ldots a^*_{n+1} \). From (5.7) in the proof of Lemma 3.6 we get that \( a^*_{n-1} = a^*_n = 1 \). Therefore,

\[ \hat{B}_3^* = a^*_s \ldots a^*_n s(01)_1 \ldots C_p = a^*_s \ldots a^*_{n+1} \]

is a 3-block in \( \Phi(A) \).

Since (the number of 1's in \( s(01)_1 \ldots C_p \)) - (the number of 0's in \( s(01)_1 \ldots C_p \)) = 1 , \( m(\hat{B}_3^*) = m(\hat{B}_3^{**}) \). We observe that \( k(A) = r-1 = 2\beta_1 + 4\beta_2 \) where \( \beta_i \) is the number of i-blocks which meet \( \hat{B}_3^* \) by applying \( \Phi \). Hence,

\[ r(\hat{B}_3^{**}) = n = r(\hat{B}_3^*) - (k(A) - 2\beta_1 - 4\beta_2) \]

Next let \( B_i \) be an i-block in \( A \) which corresponds to a block \( \hat{B}_i^* \) in \( a^*_1 \ldots a^*_r \). We prove that \( B_i \) corresponds to an i-block in \( \Phi(A) \) such that \( r(B_i^{**}) = n+r(B_i^*) - (k(A)+1) \).

If \( B_1^* = a_j^* = 1 \), then \( B_1^{**} = a_j^* = 0 \) is a 1-block in \( \Phi(A) \) and

\[ (5.9) \quad r(B_1^{**}) = n-r+j = n+j-(k(A)+1) = n+r(B_1^*)-(k(A)+1) \]

Analogously, there exists a 2-block \( B_2^{**} \) in \( \Phi(A) \) such that

\[ (5.10) \quad r(B_2^{**}) = n+r(B_2^*) - (k(A)+1) \]

By Lemma 3.6.a) and (5.9) \((y,z \text{ are defined in Lemma 3.6.a})\)

\[ (5.11) \quad r(B_1^{**}) = n+r(B_1) - ([k(A)+2]+y+2z+1-1) \]

We add and subtract 1 to indicate that \( B_1 \) both circles around and meets \( \hat{B}_3^* \) by applying \( \Phi \). (5.11) is in accordance with
Lemma 3.10.a). By Lemma 3.6.b) and (5.10) \((y,z)\) are defined in Lemma 3.6.b))

\[(5.12)\quad r(B_2^{**}) = n+r(B_2) - ([k(A)+1]+2y-2z+2-2)\].

We add and subtract 2 to indicate that \(B_2\) both circles around and meets \(\hat{B}_3\) by applying \(\phi\). (5.12) is in accordance with Lemma 3.10.b).

Suppose \(B_i\) is an \(i\)-block in \(A\) different from \(\hat{B}_3\), which does not circle around by applying \(\phi\), and corresponds to \(B_i^*\) in \(A^*\). Since \(\phi(A) = _g(n+2)+(1+k(A))(A)\), there exists an \(i\)-block \(B_i^{**}\) in \(\phi(A)\) such that

\[(5.13)\quad r(B_i^{**}) = r(B_i^*) - k(A)-1, \quad l(B_i^{**}) = l(B_i^*)-k(A)-1\] and
\[m(B_i^{**}) = m(B_i^*)\,.

By (5.13) and Lemma 3.6 the Lemma is true for \(B_i\).

Finally, \(l(B_3^{**}) = l(\hat{B}_3^*) - k(A)-1\). Therefore, by Lemma 3.6 we get that \(d)\) in the Lemma is true for \(B_3 = \hat{B}_3\).

The proof of Lemma 3.12 follows easily from the proof of Lemma 3.10.

**Lemma 5.5.**

Suppose \(B\) and \(C\) are blocks in \(A = a_1 \ldots a_n\) and specially that \(B\) is a 2-block. Furthermore, suppose \(\hat{B}_3\) is the last 3-block in \(A\). Let \(\mathcal{M} = \mathcal{M}(B,C)\) be as in Def. 3.13. We then define

\[\mathcal{N} = \mathcal{N}(B,C) = \bigcup \{D \subseteq \mathcal{M} : D \text{ is an } H\text{-block or a } K\text{-block in } A\}\]
u\{a_i, a_{i+1}\} \subseteq M : a_i \in D \text{ is a left endpoint of a 3-block in } A^1_i$

u\{a_i, a_{i+1}\} \subseteq M : a_{i-1} \in D \text{ is a right endpoint of a 3-block } \uparrow \hat{B}_3 \text{ in } A^1_i.

If $C \not\subseteq \hat{B}_3$, then $d(B, C) = \text{card } M - \text{card } \mathcal{N}$, while $C = \hat{B}_3$ implies $d(B, \hat{B}_3) = \text{card } M - \text{card } \mathcal{N} + 2$. Besides, all the sets in the union in this lemma are disjoint.

**Proof:** By studying the definitions of blocks we observe that all the sets in the union in the lemma are disjoint. Hence,

$$\text{card } \mathcal{N} = 2(\text{the number of 1-blocks between } B \text{ and } C) + 4(\text{the number of 2-blocks between } B \text{ and } C) + 2(\text{the number of endpoint } a_i \uparrow a_n \text{, between } B \text{ and } C \text{, of 3-blocks}).$$

If $C \not\subseteq \hat{B}_3$, then $T = (\text{the number of endpoints, between } B \text{ and } C \text{, of 3-blocks}) - 2z$ is equal to $(\text{the number of endpoints } a_i \uparrow a_n \text{, between } B \text{ and } C \text{, of 3-blocks})$, else $T = (\text{the number of endpoints } a_i \uparrow a_n \text{, between } B \text{ and } C \text{, of 3-blocks}) - 2$, where $z$ is as in Def. 3.13. Therefore, $\chi = \text{card } \mathcal{N}$ if $C \not\subseteq \hat{B}_3$, and $\chi = \text{card } \mathcal{N} - 2$ otherwise.

**Proof of Lemma 3.14:** In this proof, $B_i$ and $C_i$ denote $i$-blocks. Furthermore, "meet", "jump out" and "move" mean meet by applying $\circ$ etc.

a) Suppose $B_1^1, \ldots, B_1^t$ meet $B_2^t$. By Def. 3.5 and 3.8 we can suppose
and that $B_1^i$ and $B_2$ cannot meet any 3-block $\hat{B}_3$. From Lemma 3.10, if $B_1^i$ meets $\hat{B}_3$, then $B_1^i$ moves 1 position in addition. Moreover, $B_1^i$ also circles around, hence moves -1 position in addition. Analogously with $B_2$. Lemma 3.10 implies
\begin{align*}
r(\varphi(B_1^i)) &= r(B_1^i) - (k(A) + 2 + 2) = r(B_2) + 2i - k(A) - 4 \\
r(\varphi(B_2)) &= r(B_2) - (k(A) + 1 - 2t).
\end{align*}
Hence,
\begin{equation}
r(\varphi(B_1^i)) - r(\varphi(B_2)) = 2i - 3 - 2t.
\end{equation}
By (5.14) and (5.15) we get
\begin{align*}
d(B_1^i, B_2) &= (r(B_1^i) - r(B_2) - 1) - 2(i - 1) = r(B_2) + 2i - r(B_2) - 1 - 2i + 2 = 1 \\
\text{card } \mathcal{M}(\varphi(B_1^i), \varphi(B_2)) &= r(\varphi(B_1^i)) - 1 + n - r(\varphi(B_2)) = n - 4 + 2i - 2t \\
d(\varphi(B_1^i), \varphi(B_2)) &= n - 4 + 2i - 2t - 2(\gamma_1 - (t - i + 1)) - 2\gamma_2 - \gamma_3 + 1 = n + 1 - 2\gamma_1 - 2\gamma_2 - 2\gamma_3.
\end{align*}
This is in accordance with the first part of a).

Suppose $B_1$ and $B_2$ do not meet, and let
\begin{align*}
\mathcal{M} &= \mathcal{M}(B_1, B_2), x = x(B_1, B_2), z = z(B_1, B_2), \mathcal{M}_{\varphi} = \mathcal{M}(\varphi(B_1), \varphi(B_2)), \\
\chi_{\varphi} &= x(\varphi(B_1), \varphi(B_2)), z_{\varphi} = z(\varphi(B_1), \varphi(B_2)).
\end{align*}
We calculate \text{card } \mathcal{M}_{\varphi} and \text{card } \mathcal{M}_{\varphi} according to the following table: 

By Lemma 3.10 we must decrease \text{card } \mathcal{M}_{\varphi} and \text{card } \mathcal{M}_{\varphi} according to the following table:
Decrease \( \text{card}(\mathcal{M}_\varphi) \) by Decrease \( \chi_\varphi \) by

<table>
<thead>
<tr>
<th>Rule Description</th>
<th>( \text{card}(\mathcal{M}_\varphi) )</th>
<th>( \chi_\varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>The main rule</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( B_1 ) meets a 2-block</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(5.16) ( B_1 ) meets ( \hat{B}_3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( B_1 ) meets a 3-block ( \nmid \hat{B}_3 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( B_1 ) jumps out of a 3-block</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>A 1-block meets ( B_2 )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(5.17) ( B_2 ) meets ( \hat{B}_3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( B_2 ) meets a 3-block ( \nmid \hat{B}_3 )</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>( B_2 ) jumps out of a 3-block</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

(5.16) follows in this way: If \( B_1 \) meets \( \hat{B}_3 \), both \( \mathcal{M}_\varphi \) and \( \chi_\varphi \) decrease by 1. However, \( B_1 \) also circles around, hence \( \mathcal{M}_\varphi \) increases by 1. Besides, \( \chi_\varphi \) increases by 1 since \( z = 1 \) and \( z_\varphi = 0 \). (5.17) follows in the same way. Conclusion: \( \text{card}\mathcal{M}_\varphi - \chi_\varphi = \text{card}\mathcal{M} - \chi \) if \( B_2 \) meet a 3-block \( \nmid \hat{B}_3 \) or jumps out of a 3-block, else

\[ \text{card}\mathcal{M}_\varphi - \chi_\varphi = (\text{card} \mathcal{M} - \chi) - 1. \]

Hence, a) is proved.

b) Suppose \( A = a_1 \ldots a_n, \mathcal{M} = \mathcal{M}(B_2, B_3) \) and \( \mathcal{N} = \mathcal{N}(B_2, B_3) \) (see Lemma 5.5). In the following and asterisk below \( a_i \) means: \( a_i \in \mathcal{M} \) and \( a_i \notin \mathcal{N} \). We observe

(5.18) If \( B_2 = a_1 a_{i+1} \), then \( a_1 \in \mathcal{M} \) and \( a_1 \notin \mathcal{N} \).

First we suppose \( B_3 \nmid \hat{B}_3 \), hence

(5.19) \( A = DB_200s(10)0C_1 \ldots C_pE \) where \( E \) starts with 0 or a 3-block and \( C_i \) are H-blocks.
If $B_2$ meets $B_3$, then $B_2$ is contained in $C_1 \ldots C_p$.

(5.18) and (5.19) imply by Lemma 5.5 that $d(B_2, B_3) = 4$.

If $B_2$ does not meet $B_3$, we have two cases

(5.20) $A = DB_3 OS(10)OC_1 \ldots C_p OF$ or $A = DB_3 OS(10)OC_1 \ldots C_p 1t(01)1F$

Besides, $B_2$ is contained in $F$ or $D$. (5.18) and (5.20) imply by Lemma 5.5 that $d(B_2, B_3) \geq 5$.

Finally we suppose $B_3 = B_3$. Moreover, we suppose $B_2^*$ and $B_3^*$ in $A^* = D^n+2(A)1$ correspond to $B_2$ and $B_3$. We now prove that

(5.21) $d(B_2^*, B_3^*) = d(B_2, B_3)$.

Suppose $m = m(B_2, B_3), \chi = \chi(B_2, B_3), m^* = m(B_2^*, B_3^*)$ and $\chi^* = \chi(B_2^*, B_3^*)$. We calculate card $m^*$ and $\chi^*$ by the following procedure: First put card $m^* = \text{card } m$ and $\chi^* = \chi$. By Lemma 3.6 we must decrease card $m^*$ and $\chi^*$ according to the following table:

<table>
<thead>
<tr>
<th>Decrease card $m^*$ by</th>
<th>Decrease $\chi^*$ by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 1-block meet $B_2$ by</td>
<td>$-2$</td>
</tr>
<tr>
<td>applying $\delta^{n+2}$</td>
<td></td>
</tr>
<tr>
<td>$B_2$ meet a 3-block by applying $\delta^{n+2}$</td>
<td>$2$</td>
</tr>
<tr>
<td>$B_2$ jumps out of a 3-block by applying $\delta^{n+2}$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

Hence, $d(B_2^*, B_3^*) = \text{card } m^* - \chi^* = \text{card } m - \chi = d(B_2, B_3)$.

Next we prove

(5.22) $B_2^* \in A^*$ circles around (this is equivalent to "$B_2$ meets $B_3$") if and only if $d(B_2^*, B_3^*) = 4$. 
A* has the following form as in Lemma 3.12.e.

(5.23) \[ s(10)OC_1 \ldots C_p \] where D starts with a 0 or a 3-block
and \( C_i \) are H-blocks. If \( B_2 \) meets \( \hat{B}_3 \), \( B_2^* \) is contained in
\( C_1 \ldots C_p \). Putting \( \mathcal{N}^* = \mathcal{N}(B_2^*,\hat{B}_3^*) \) we get by (5.18), (5.23)
and Lemma 5.5 that \( \text{card } \mathcal{M}^* - \text{card } \mathcal{N}^* = 2 \). If \( B_2 \) does not
meet \( \hat{B}_3 \), we show as in the case \( B_3 = B_3 \) that
\( \text{card } \mathcal{M}^* - \text{card } \mathcal{N}^* \geq 3 \). By Lemma 5.5 \( d(B_2^*,\hat{B}_3^*) = \text{card } \mathcal{M}^* - \text{card } \mathcal{N}^* + 2 \)
and the proof of (5.22) is complete.

Combining (5.21) and (5.22) we get: \( B_2 \) meets \( \hat{B}_3 \) if
and only if \( d(B_2^*,\hat{B}_3^*) = 4 \).

Suppose \( B_2 \) meets \( B_3 \neq \hat{B}_3 \) (the case \( B_3 = \hat{B}_3 \) is treated
in the same way), and that there are \( T_i \) i-blocks between
\( B_2 \) and \( B_3 \). Moreover, we suppose \( A = EB_200s(10)OC_1 \ldots C_i C_{i+1} \)
where \( C_j \) are H-blocks and \( C_{i+1} = B_2t(01)00 \). Observing
that \( \text{card}(00s(10)OC_1 \ldots C_i) = 3+2T_1+4T_2 \), we get
\[
\text{r}(B_2) - \text{r}(B_3) = 5 + 2T_1 + 4T_2 .
\]

Supposing there are \( s_1 \) i-blocks which meet \( B_3 \) we get:\[
\text{r}(\varphi(B_2)) = \text{r}(B_2) - 1 - 2 + k(A) ,
\]
\[
\text{r}(\varphi(B_3)) = \text{r}(B_3) + 2s_1 + 4s_2 - k(A) ,
\]
\[
\text{card } \mathcal{M}(\varphi(B_2),\varphi(B_3)) = [\text{r}(\varphi(B_2)) - 1] + \text{n-r}(\varphi(B_3))
= n-4-2s_1-4s_2+(r(B_2)-r(B_3))
= n+1+2(T_1-s_1)+4(T_2-s_2) .
\]
\[
\chi(\varphi(B_2),\varphi(B_3)) = 2(\gamma_1-(s_1-T_1)) + 4(\gamma_2-(s_2-T_2)) + 2(2\gamma_3-1)-2
= 2\gamma_1 + 4\gamma_2 + 4\gamma_3 - 4 - 2(s_1-T_1) - 4(s_2-T_2) .
\]
\[
\text{d}(\varphi(B_2),\varphi(B_3)) = \text{card } \mathcal{M}(\varphi(B_2),\varphi(B_3)) - \chi(\varphi(B_2),\varphi(B_3)) = n+5-2\gamma_1 - 4\gamma_2 - 4\gamma_3 .
\]
The last part of b), and the parts c), d) and e) are proved by using a procedure and a table as in the proof of a).

**Definition 5.6**

Suppose B and C are two blocks in \( A = a_1 \ldots a_n \). If B is to the left of C, we define

\[
\overline{m}(B, C) = \overline{m} = \{a_{l(C)+1}, \ldots, a_{n}\} \cup \{a_1, \ldots, a_{l(B)-1}\} \quad \text{and} \quad z(B, C) = z = 1,
\]

else

\[
\overline{m}(B, C) = \overline{m} = \{a_{l(C)+1}, \ldots, a_{l(B)-1}\} \quad \text{and} \quad z(B, C) = z = 0.
\]

We define "between", \( \overline{e}(B, C) = \overline{e} \) and \( \overline{d}(B, C) \) as in Def. 3.13 by using \( \overline{m} \) instead of \( m \).

**Lemma 5.7.**

Suppose \( B_i \) is an i-block for \( i = 2, 3 \). Then \( B_2 \) jumps out of \( B_3 \) if and only if \( \overline{d}(B_2, B_3) = 2 \). In this case

\[
\overline{d}(\varphi(B_2), \varphi(B_3)) = n + 3 - 2Y_1 - 4Y_2 - 4Y_3,
\]

otherwise

\[
\overline{d}(\varphi(B_2), \varphi(B_3)) = \overline{d}(B_2, B_3) - 1.
\]

The proof of Lemma 5.7 is similar to the proof of Lemma 3.14.b). We only indicate the proof on an example: \( n = 14 \), \( k = 3 \) and

\[
\begin{align*}
A &= 0 0 0 1 0 0 0 1 1 1 0 0 1 1, \\
\varphi(A) &= 0 1 0 0 0 0 0 1 1 0 0 1 1.
\end{align*}
\]
Denoting the $i$-blocks in $A$ by $B_i$, we observe that
\[ \text{card}(B_2, B_3) = 2, \ z(B_2, B_3) = 0, \ \bar{x}(B_2, B_3) = 0, \]
\[ \text{card}(\varphi(B_2), \varphi(B_3)) = 9 = n-5, \ z(\varphi(B_2), \varphi(B_3)) = 1, \]
\[ \bar{x}(\varphi(B_2), \varphi(B_3)) = 2y_1 + 4(y_2-1) + 2(2y_3-1) - 2 = 2+0+2-2 = 2. \]
Hence, $d(B_2, B_3) = 2$ and

\[ d(\varphi(B_2), \varphi(B_3)) = (n-5)-(2y_1+4(y_2-1)+2(2y_3-1)-2) = n+3-2y_1-4y_2-4y_3. \]

**Lemma 5.18**

Suppose $B_i$ is an $i$-block in $A$ for $i = 2, 3$, $A$ satisfies Cond. 3.1 and let $s = n+2 - 2y_1 - 4y_2 - 4y_3$. Then

\[ d(B_2, B_3) = d(\varphi^s(B_2), \varphi^s(B_3)) \quad \text{and} \quad d(B_2, B_3) = d(\varphi^s(B_2), \varphi^s(B_3)). \]

**Proof:** We show first that

\[ (5.25) \quad 4 \leq d(B_2, B_3) \leq n+5 - 2y_1 - 4y_2 - 4y_3. \]

We choose $p$ as the least integer such that $\varphi^{-p}(B_2)$ meets $\varphi^{-p}(B_3)$ by applying $\varphi$. By Lemma 3.14.b)

\[ d(\varphi^{-(p-1)}(B_2), \varphi^{-(p-1)}(B_3)) = n+5-2y_1-4y_2-4y_3. \]

Hence,

\[ d(B_2, B_3) = (p-1) + (n+5-2y_1-4y_2-4y_3) = n+5-2y_1-4y_2-4y_3. \]

\[ 4 \leq d(B_2, B_3) \text{ is obvious. Putting } T = d(B_2, B_3) \text{ we get} \]

\[ d(\varphi^T(B_2), \varphi^T(B_3)) = 4. \]

\[ d(\varphi^{T-3}(B_2), \varphi^{T-3}(B_3)) = n+5-2y_1-4y_2-4y_3 = s+3. \]

\[ d(\varphi^s(B_2), \varphi^s(B_3)) = (s+3) - (s-T+3) = T = d(B_2, B_3). \]

Since $\varphi^s = \varphi^{(s-T+3)} \circ \varphi^{T-3}$. $\bar{d}(B_2, B_3) = \bar{d}(\varphi^s(B_2), \varphi^s(B_3))$ follows in the same way.
Definition 5.9.

"Between" is used in the same way as in Def. 3.13. Suppose \( B \) and \( C \) are blocks in \( A \). Then

\[
y_i(B, C) = \text{the number of } i\text{-blocks between } B \text{ and } C \quad (i=1,2),
\]
\[
y_3(B, C) = \text{the number of endpoints between } B \text{ and } C , \text{ of } 3\text{-blocks}.
\]

Moreover, we order the positions in \( A \) relatively to \( B \) in this way: \( r(B) < r(B) + 1 < \ldots < n < 1 < \ldots < r(B) - 1 \).

Lemma 5.10

Suppose \( A \) satisfies Cond. 3.1. Moreover, let \( B_i^* \) be an \( i\)-block for \( i=1,2 \) and \( d(B_1^*, B_2^*) = d(\varphi^P(B_2^*), \varphi^P(B_2^*)) \).

Then \( d(B_1^*, B_2^*) = d(\varphi^P(B_1^*), \varphi^P(B_2^*)) \) for every \( 1\)-block \( B_1^* \).

Proof: Suppose \( r(B_2^*) < r(B_1^*) < r(B_1) \) relatively to \( B_2^* \). Then \( z(B_1^*, B_2^*) = z(B_1^* , B_1^*) + z(B_1^*, B_2^*) \), \( m(B_1^*, B_2^*) = m(B_1^*, B_2^*) + m(B_1^*, B_1^*) + 1 \) and

\[
x(B_1^*, B_2^*) = 2(y_1(B_1^*, B_2^*)+y_1(B_1^*, B_1^*)+1)+2(y_2(B_1^*, B_2^*)+y_2(B_1^*, B_1^*))
\]
\[
+ (y_3(B_1^*, B_2^*)+y_3(B_1^*, B_1^*)) + z(B_1^*, B_2^*) = x(B_1^*, B_2^*) + x(B_1^*, B_1^*) + 2.
\]

Hence,

\[
(5.26) \quad d(B_1^*, B_2^*) = d(B_1^*, B_1^*) + d(B_1^*, B_2^*) - 1.
\]

By Lemma 3.14 c) \( d(\varphi^P(B_1^*), \varphi^P(B_2^*)) = d(B_1^*, B_1^*) \). Since \( r(B_2^*) < r(B_1^*) < r(B_1) \), \( d(B_1^*, B_2^*) < d(B_1^*, B_1) \).

Hence, \( d(\varphi^P(B_1^*), \varphi^P(B_2^*)) < d(\varphi^P(B_1^*), \varphi^P(B_1)) \), which implies \( \varphi^P(B_2^*) < \varphi^P(B_1^*) < \varphi^P(B_1) \) relatively to \( \varphi^P(B_2^*) \). Similar to (5.26), we get \( d(\varphi^P(B_1^*), \varphi^P(B_2^*)) = d(\varphi^P(B_1^*), \varphi^P(B_1^*)) + d(\varphi^P(B_1^*), \varphi^P(B_2^*)) - 1 \).
Hence, \( d(B_1, B_2^*) = d(\phi(B_1), \phi(B_2^*)) \).

If \( r(B_2^*) < r(B_1) < r(B_1^*) \) relatively \( B_2^* \), we show similar to (5.26) that

\[
\begin{align*}
\hat{d}(B_1, B_2^*) &= \hat{d}(B_1, B_1^*) + \hat{d}(B_1, B_2) - 1, \\
\hat{d}(\phi^p(B_1^*), \phi^p(B_2^*)) &= \hat{d}(\phi^p(B_1^*), \phi^p(B_1)) + \hat{d}(\phi^p(B_1), \phi^p(B_2^*)) - 1.
\end{align*}
\]

This implies by Lemma 3.14. c) that \( \hat{d}(B_1, B_2^*) = \hat{d}(\phi^p(B_1), \phi^p(B_2^*)) \).

**Lemma 5.11.**

Suppose \( A \) satisfies Cond. 3.1, and \( B_i \) is an \( i \)-block for \( i = 1, 2, 3 \). Then

\[
\begin{align*}
\hat{d}(B_2, B_3) + \hat{d}(B_3, B_2) &= (n-2) - [2\gamma_1 + 4(\gamma_2 - 1) + 2(2\gamma_3 - 1) + 2], \\
\hat{d}(B_2, B_3) + \hat{d}(B_3, B_2) &= (n-2) - [2\gamma_1 + 4(\gamma_2 - 1) + 2(2\gamma_3 - 1) + 2], \\
\hat{d}(B_1, B_2) + \hat{d}(B_2, B_1) &= (n-2) - [2(\gamma_1 - 1) + 2(\gamma_2 - 1) + 2\gamma_3 + 1].
\end{align*}
\]

**Proof:** We observe that \( \mathcal{M}(B_2, B_3) + \mathcal{M}(B_3, B_2) = n-2 \) and \( \chi(B_2, B_3) + \chi(B_3, B_2) = [2\gamma_1 + 4(\gamma_2 - 1) + 2(2\gamma_3 - 1) + 2] \). Hence, the first equality is true. The other equalities are proved in the same way.

**Proof of Lemma 3.15.** \( C_i \) denotes an arbitrary \( i \)-block.

Lemmas 5.8, 5.11 and 3.13.d) imply

\[
\begin{align*}
\hat{d}(\phi^t(C_2), \phi^t(B_2)) &= \hat{d}(C_2, B_2), \hat{d}(\phi^t(C_3), \phi^t(B_2)) = \hat{d}(C_3, B_2), \\
\hat{d}(\phi^t(C_2), \phi^t(B_2)) &= \hat{d}(C_2, B_2), \hat{d}(\phi^t(C_1), \phi^t(B_2)) = \hat{d}(C_1, B_2).
\end{align*}
\]

(5.27)

Let \( A = DB_2E = a_1 \ldots a_n, \phi^t(A) = \mathcal{F}_\phi^t(B_2)G = b_1 \ldots b_n, i = r(B_2) \)

and \( j = r(\phi^t(B_2)) \). We then get
$B_2$ is contained in $C_3$ if and only if $d(B_2, C_3) > \bar{d}(B_2, C_3)$

(5.28) \[ d(\phi^t(B_2), \phi^t(C_3)) > \bar{d}(\phi^t(B_2), \phi^t(C_3)) \iff \phi^t(B_2) \text{ is contained in } \phi^t(C_3). \]

We suppose there exist a minimal integer $q$ such that $a_{i+q} \neq b_{j+q}$. Without loss of generality we can suppose $a_{i+q} = 1$. Hence,

(5.29) \[ a_i = b_j, \ldots, a_{i+q-1} = b_{j+q-1}. \]

(5.27), (5.28) and (5.29) imply for $0 < q' < q$

\[ l(C_3) = i+q' \implies l(\phi^t(C_3)) = j+q', r(C_3) = i+q' \]

(5.30) \[ r(\phi^t(C_3)) = j+q', r(C_2) = i+q' \implies r(\phi^t(C_2)) = j+q', r(C_1) = i+q' \implies r(\phi^t(C_1)) = j+q'. \]

In particular, we have $a_{i+q}$ is contained in a 3-block if and only if $b_{j+q}$ contained in a 3-block. Thus (5.29) and (5.30) give a contradiction. For example, if $a_{i+q} = 1 = C_1$ is a 1-block, then $b_{j+q} = \phi^t(C_1) = 0$ is a 1-block. This gives a contradiction since $b_{j+q}$ is not contained in any 3-block. Without loss of generality we can suppose $i \geq j$.

We have therefore proved that $a_i = b_j, \ldots, a_n = b_{j+(n-i)}$.

By (5.30) $n = r(\hat{B}_3) = r(\phi^t(\hat{B}_3)) = j+n-i$. Hence, $j=i$ and $E = G$.

$D = F$ is proved in the same way by using $\bar{d}(B_2, C_3) = \bar{d}(\phi^t(B_2), \phi^t(C_3))$, $\bar{d}(B_2, C_3) = d(\phi^t(B_2), \phi^t(C_3))$, $d(B_2, C_2) = d(\phi^t(B_2), \phi^t(C_2))$ and $d(B_2, C_1) = d(\phi^t(B_2), \phi^t(C_1))$.

**Proof of Lemma 3.16:** If $\phi^r(A) = \phi^s(A)$, the Lemma is tri-
vial. We suppose \( \psi^r(A) \neq \psi^s(A) \). If there exists an \( i \) such that \( d(\psi^{r+i}(B_1),\psi^{r+i}(B_2)) = d(\psi^{s+i}(B_1),\psi^{s+i}(B_2)) \), we get by Lemma 3.15 that \( \psi^{r+i}(A) = \psi^{s+i}(A) \). Hence, \( \psi^r(A) = \psi^s(A) \) which is a contradiction. Therefore
\[
d(\psi^{r+i}(B_1),\psi^{r+i}(B_2)) \neq d(\psi^{s+i}(B_1),\psi^{s+i}(B_2)) \text{ for all } i.
\]
We observe by Lemma 3.14 a): If \( \phi^i(B_1) \) and \( \phi^j(B_2) \) do not meet by applying \( \phi \), \( d(\phi^i(B_1),\phi^j(B_2)) \) "decreases" by 0 or 1. Hence:
\[
\text{If } d(\psi^{s+i}(B_1),\psi^{s+i}(B_2)) > d(\psi^{r+i}(B_1),\psi^{r+i}(B_2)),
\]
\[
\phi^{r+i}(B_1) \text{ meets } \phi^{r+i}(B_2) \text{ "before" } \phi^{s+i}(B_1) \text{ meets } \phi^{s+i}(B_2), \text{ else } \phi^{s+i}(B_1) \text{ meets } \phi^{s+i}(B_2) \text{ "before" } \phi^{r+i}(B_1) \text{ meets } \phi^{r+i}(B_2).
\]

We suppose \( t_1, \ldots, t_q (q < c) \) are the integers such that \( \psi^{r+t_i}(B_1) \) meets \( \phi^{r+t_i}(B_2) \) by applying \( \phi \). We prove the following 3 claims by using (5.31):

\[
(5.32) \quad \psi^s(B_1) \text{ meets } \psi^s(B_2) \text{ at most once by applying } \phi^{t_1+1}.
\]
\[
(5.33) \quad \psi^{s+t_i+1}(B_1) \text{ meets } \psi^{s+t_i+1}(B_2) \text{ once by applying } \psi^{t_i+t_i-1}.
\]
\[
(5.34) \quad \psi^{s+t_q+1}(B_1) \text{ meets } \psi^{s+t_q+1}(B_2) \text{ at most once by applying } \psi^{t-t_q+1}.
\]

The Lemma now follows easily from (5.32), (5.33) and (5.34).

Proof of (5.32): If \( d(\psi^s(B_1),\psi^s(B_2)) > d(\phi^r(B_1),\phi^r(B_2)) \), then \( \psi^s(B_1) \) does not meet \( \psi^s(B_2) \) by applying \( \phi^{t_1+1} \). Otherwise, let \( y \) be the least integer such that \( \psi^s(B_1) \) meets \( \psi^s(B_2) \) by applying \( \phi^y \). Then
$d(\psi^{s+y}(B_1), \psi^{s+y}(B_2)) > d(\psi^{r+y}(B_1), \psi^{r+y}(B_2))$, and $\psi^{r+y}(B_1)$ meets $\psi^{r+y}(B_2)$ "before" $\psi^{s+y}(B_1)$ meets $\psi^{s+y}(B_2)$.

Proof of (5.33): Let $y$ be the least integer such that $\psi^{s+t_i+1}(B_1)$ meets $\psi^{s+t_i+1}(B_2)$ by applying $\psi^y$. Then $d(\psi^{s+t_i+1+y}(B_1), \psi^{s+t_i+1+y}(B_2)) > d(\psi^{r+t_i+1+y}(B_1), \psi^{r+t_i+1+y}(B_2))$, and $\psi^{r+t_i+1+y}(B_1)$ meets $\psi^{r+t_i+1+y}(B_2)$ "before" $\psi^{s+t_i+1+y}(B_1)$ meets $\psi^{s+t_i+1+y}(B_2)$.

The proof of (5.34) is analogous.

The proof of Lemma 3.17 follows from the proof of Lemma 5.8.

The proof of Lemma 3.18 is obvious since each 2-block meets each 3-block $a$ times, each 1-block meets each 2-block $c$ times and $A = \psi^s(A)$.
INDEX OF NOTATION

- \( E_k \) - The introduction
- \( w(A) \) - The introduction
- \( i \)-block - Def. 2.1, 2.2.
- \( 0 \) - Thm. 2.3.
- \( \gamma_i \) - Thm. 2.3.
- \( l(B) = l(A,B) \) - Def. 3.3.
- \( r(B) = r(A,B) \) - Def. 3.3.
- \( m(B) \) - Def. 3.4.
- \( \text{meet} \) - Def. 3.5, 3.7 and 3.8.
- \( \text{jump out} \) - Def. 3.5, 3.8.
- \( B_3 \) - Def. 3.7.
- \( k(A) \) - Def. 3.7.
- \( \varphi(A) \) - Def. 3.7.
- \( \text{circle around} \) - Def. 3.7.
- \( \text{card} = \text{"the number of elements in"} \)

REFERENCES

