A significant part of post-Friedberg recursion theory has been successfully generalized to recursion theory on an admissible ordinal \( \alpha \). Such a recursion theory has two properties seemingly important for priority arguments: it is an "infinite" theory and its domain is recursively wellordered. Kreisel ([5], pp. 172-173) has asked (with some persistence - see his reviews of [6] and [13] in Zentralblatt 1973 and 1976 respectively) whether these properties are significant for the existence of incomparable r.e. degrees. Recently Sy Friedman [4] has considered the first property by doing recursion theory over an arbitrary limit ordinal \( \beta \), thus dropping the admissibility criteria. His main result is the existence for many \( \beta \) of a pair of sets \( \Sigma_1 \) over \( L(\beta) \) such that neither is \( \beta \)-recursive in the other. We, on the other hand, are keeping admissibility while relaxing the requirement of a wellordered domain to that of a pre-wellordered domain, that is we are essentially studying recursion theory over resolvable admissible sets with urelements.

However, rather than restricting our attention to resolvable admissible sets, our approach in this paper is axiomatic. Starting with a precomputation theory in the sense of Moschovakis [6] with a computable selection operator, we add two axioms to obtain an infinite computation theory. The first asserts the existence of a prewell-order whose initial segments are uniformly "finite", while the second insures that all "computations" can be effectively generated and that this generation is matched up with the complexity of the domain as expressed by the prewellorder. The class of infinite computation theories coincides with the class of Friedberg theories as defined in [6].
It is doubtful (see Simpson [14]) whether the axioms for an infinite theory are quite adequate for giving a positive solution to Post's problem. A trivial but significant observation for $\alpha$-recursion theory (or for any recursively wellordered infinite theory) is that any $\alpha$-r.e. set bounded strictly below $\alpha^*$, the projectum of $\alpha$, is $\alpha$-finite. We call an infinite computation theory adequate whenever the analogous theorem holds. For adequate theories we prove a strong form of Sacks' splitting theorem [7,10], thereby supporting the conjecture that any of the usual finite injury priority arguments can be carried out for such theories.

In section 1 we give the axioms for an infinite computation theory and prove some elementary results. Section 2 introduces different notions of relative computability and gives sufficient conditions in terms of regularity and hyperregularity for the notions to coincide. Shore's blocking technique using $\Sigma_2$ functions is developed in section 3 while the proof of the splitting theorem, along with some of the usual corollaries, is given in section 4.

S.G. Simpson (see [14] and [13]) has independently studied recursion theory over resolvable admissible sets. In particular, he was the first to note that Shore's blocking technique could be used to obtain a version of the Friedberg-Muchnik theorem for what he calls thin admissible sets.
1. Infinite Computation Theories

We will be dealing with partial multivalued functions and functionals on some set $U$. An $n$-ary partial multivalued function is just an $(n+1)$-ary relation. Following the notation of Moschovakis [6] we mean by $f(x_1,\ldots,x_n) \rightarrow z$ that the partial multivalued function $f$ has $z$ as one of its values at $x_1,\ldots,x_n$, i.e. $(x_1,\ldots,x_n,z)$ is an element of the defining relation for $f$.

In case $f$ is singlevalued we may without confusion write $f(x_1,\ldots,x_n) = z$ for $f(x_1,\ldots,x_n) \rightarrow z$. In this paper partial multivalued functions on $U$ will simply be called functions, whereas a total singlevalued function will be called a mapping.

The notation used should easily be understood from the context keeping the following loosely defined conventions in mind: Functions on $U$ are denoted by $f,g,h,\ldots,p,q,r,\ldots$. $\alpha,\beta$ are reserved for ordinals and $i,j,m,n$ for elements in $N$. Remaining lower case latin and greek letters (except $\lambda,\mu$ and $\nu$ which will have their usual meanings) denote elements of $U$.

A computation domain is a structure $\mathcal{O} = \langle U,N,s,M,K,L \rangle$ where $U$ is a set, $N \subseteq U$, $\langle N,s|N \rangle$ is isomorphic to the natural numbers with the successor function, $M$ is a pairing function and $K$ and $L$ are inverses to $M$. The latter means that if $M(x,y) = z$ then $K(z) = x$ and $L(z) = y$. From $M,K$ and $L$ we define the tupling function $(\ldots)$ and its $i$:th inverse $(\ldots)_i$ in the usual fashion.

A set $\mathcal{O} \subseteq \bigcup\{U^n : n \geq 2\}$ is called a computation set on $\mathcal{O}$. For a computation set $\mathcal{O}$ we define the relation

$$\{e\}^n_\mathcal{O}(\overline{z}) \rightarrow z \text{ iff } lh(\overline{z}) = n \text{ & } (e,\overline{x},z) \in \mathcal{O}$$
where \( \text{lh}(\bar{x}) \) denotes the length of the sequence \( \bar{x} \). Thus \( \{\epsilon\}_\Theta^N \) defines an \( n \)-ary function for each \( \epsilon \in U \) and \( n \in N \). An \( n \)-ary function \( f \) on \( U \) is \( \Theta \)-computable if there is \( \epsilon \in U \) such that
\( f = \{\epsilon\}_\Theta^N \), in which case \( \epsilon \) is a \( \Theta \)-index for \( f \). An \( n \)-ary relation \( R \) on \( U \) is \( \Theta \)-semicomputable (\( \Theta \)-s.c.) with a \( \Theta \)-s.c. index \( \epsilon \) if \( R \) is the domain of a \( \Theta \)-computable function with \( \Theta \)-index \( \epsilon \). \( R \) is \( \Theta \)-computable with \( \Theta \)-index \( \epsilon \) in case its characteristic mapping \( c_R \) is \( \Theta \)-computable with \( \Theta \)-index \( \epsilon \). Finally we say that a consistent functional \( F(f_1, \ldots, f_k, \bar{x}) \), where \( f_i \) varies over \( n_i \)-ary functions, is \( \Theta \)-computable with \( \Theta \)-index \( \delta \) if
\[
\forall \epsilon_1, \ldots, \epsilon_k, \bar{x}, z(F(\{\epsilon_1\}_\Theta^n, \ldots, \{\epsilon_k\}_\Theta^n, \bar{x}) \rightarrow z \iff \{\delta\}_\Theta^{k+n}(\epsilon_1, \ldots, \epsilon_k, \bar{x}) \rightarrow z).
\]

The first step in putting some structure on a computation set \( \Theta \) is to require \( \Theta \) to be a precomputation theory in the sense of Moschovakis. For a precise definition we refer to [6]. Roughly speaking, \( \Theta \) is a precomputation theory if the constant mappings, the identity mapping, \( M, K, L \) and \( s \) are \( \Theta \)-computable. Furthermore the \( \Theta \)-computable functions must satisfy the usual closure and enumeration conditions in a uniform way. A basic fact of precomputation theories is the second recursion theorem.

The existence of a \( \Theta \)-computable selection operator is normally assumed in order for the \( \Theta \)-s.c. relations to behave nicely. A selection operator for \( \Theta \) is a function \( q \) such that (henceforth dropping \( n \) and \( \Theta \) from \( \{\epsilon\}_\Theta^n \) whenever possible)
\[
q(\epsilon) \downarrow \iff \exists x(\epsilon)(x) \downarrow \quad \text{and} \quad \forall z(q(\epsilon) \rightarrow z \Rightarrow \{\epsilon\}(z) \downarrow)
\]
where "\( \downarrow \)" means "is defined". Note that the existence of a
selection operator implies the existence of a uniform selection operator. That is there exists a \( \Theta \)-computable mapping \( p(n) \) such that for each \( n \in \mathbb{N} \), \( p(n) \) is a \( \Theta \)-index for an \( n \)-ary selection operator \( q^n \). The usual \( \vee \) notation for a uniform selection operator will be used, namely \( \forall z (\{e\}^n_\Theta (z, x) \downarrow) = q^n (e, x) \).

For a precomputation theory \( \Theta \) with a \( \Theta \)-computable selection operator, the \( \Theta \)-s.c. relations are closed under disjunctions and existential quantification and a relation is \( \Theta \)-computable iff it and its complement are \( \Theta \)-s.c. Furthermore \( \Theta \)-computable functions can be defined by cases in a general way.

In this setting one can define a well behaved notion of "finite". Following Moschovakis we say that a set \( K \) is \( \Theta \)-finite if the consistent functional

\[
E_K(f) = \begin{cases} 
0 & \text{if } \exists x \in K (f(x) = 0) \\
1 & \text{if } \forall x \in K (f(x) = 1)
\end{cases}
\]

is \( \Theta \)-computable. A \( \Theta \)-index for \( E_K \) is said to be a canonical \( \Theta \)-index for the \( \Theta \)-finite set \( K \). The usual properties (see [6]) of a generalized notion of finite hold uniformly.

Having asserted the existence of a selection operator it is too restrictive to require \( \Theta \) to be a single-valued theory as this would exclude some of the intended models. However when considering functions whose values are (canonical \( \Theta \)-indices for) \( \Theta \)-finite sets, then \( \Theta \) is essentially single-valued. In the lemma stated below let \( K_\eta \) denote the \( \Theta \)-finite set with canonical \( \Theta \)-index \( \eta \) in case \( \eta \) is such an index.

**Lemma 1.1.** Suppose \( r \) is a \( \Theta \)-computable function whose values are canonical \( \Theta \)-indices such that \( \forall x, \xi, \eta (r(x) \rightarrow \xi \& r(x) \rightarrow \eta \Rightarrow K_\xi = K_\eta) \).
Then there is a $\Theta$-computable mapping $q$ obtained uniformly from $r$ such that $\forall x, \eta(r(x)) \Rightarrow K_\eta = K_q(x)$.

We now list two additional axioms making $\Theta$ into an infinite computation theory.

**A1.** There is a $\Theta$-computable prewellorder $\preceq$ on $U$ such that initial segments of $\preceq$ are uniformly $\Theta$-finite.

Given a prewellorder $\preceq$ we let $x \prec y$ denote $\neg(y \preceq x)$ and $x \sim y$ denote $x \preceq y \& y \preceq x$.

**Definition 1.2.** A $(\preceq)$-enumeration of a set $W$ is a $\Theta$-computable mapping $\lambda \sigma W^\sigma$ (whose values are canonical $\Theta$-indices for the $\Theta$-finite sets $W^\sigma$) such that

(i) $\tau \preceq \sigma \Rightarrow W^\tau \subseteq W^\sigma$

(ii) $W = \bigcup\{W^\sigma : \sigma \in U\}$.

**A2.** There is a $\Theta$-computable mapping $p(n)$ such that for each $n \in N$, $p(n)$ is a $\Theta$-index for a $(\preceq)$-enumeration of the set

$$T_n = \{\langle e, \bar{x}, y \rangle : [e](\bar{x}) \rightarrow y \& lh(\bar{x}) = n\}.$$

**Definition 1.3.** Let $\Theta$ be a computation set over a computation domain $\mathcal{O}$. $\Theta$ is an infinite computation theory if

(i) $\Theta$ is a precomputation theory.

(ii) Equality on $U$ is a $\Theta$-computable relation.

(iii) $\Theta$ has a computable selection operator.

(iv) A1 and A2 hold for some prewellorder $\preceq$ on $U$.

A basic fact of infinite recursion theories, e.g. recursion on an admissible ordinal $\alpha$, is that computations can be coded effec-
tively into the domain in such a way that the complexity of the domain corresponds to the complexity of the coded computations. It therefore seems reasonable to assert the existence of a $\Theta$-computable partial order $\preceq$ whose initial segments are well-founded and uniformly $\Theta$-finite. Here we restrict ourselves to the case where $\preceq$ is a prewellorder. A2 then stipulates that all "computations" $\{e\}(\vec{x}) \rightarrow y$ can be effectively generated and that this generation is matched up with the complexity of the domain. Note that $U$ is not $\Theta$-finite for an infinite computation theory.

Following the notation of Barwise [1], let $A_m$ be a resolvable admissible set with urelements relative to a language $L^* = L(\epsilon, \ldots)$. By combining Moschovakis' characterization theorem for Friedberg theories with Gandy's theorem for $\Sigma^1_1$ inductive definitions over admissible sets (see Barwise [1] p. 208), it is easily verified that $A_m$ constitutes an infinite computation theory.

J. Stavi (unpublished) has shown the converse of Gandy's theorem to be false for some transitive sets. However, for resolvable transitive structures $A$ (closed under pairing and satisfying $\Delta^0_0$-separation) the converse is true, i.e. for such $A$, $A$ is admissible iff every $\Sigma^1_1$ inductive operator over $A$ has a $\Sigma^1_1$ fixed point. This result, due to A. Nyberg [16], gives some justification for axiom $A^1$.

In the sequel $\Theta$ will always denote an infinite computation theory over some computation domain $A$.

Lemma 1.4. Suppose $f$ is a $\Theta$-computable function. Then there is a $\Theta$-computable function $g$ obtained uniformly from $f$ such that $\text{dom } g = \text{dom } f$, $g \subseteq f$, and for each $\Theta$-finite set $K \subseteq \text{dom } f$ there
is a $\Theta$-finite set $N$ obtained uniformly from $K$ and $f$ such that $g(K) \subseteq N \subseteq f(K)$.

Before proceeding with the proof we need to introduce the $\mu$-operator. By $\mu z R(z,\vec{x})$ we mean a function whose values for $\vec{x}$ are some minimal $z$ such that $R(z,\vec{x})$. In particular, if $R$ is $\Theta$-computable we set

$$\mu z R(z,\vec{x}) = \nu z (R(z,\vec{x}) \land (\forall y < z) \neg R(y,\vec{x})).$$

Proof. Let $\lambda \sigma W^\omega$ be a $(\leq)$-enumeration of

$T_1 = \{ \langle \varepsilon, x, y \rangle : \{\varepsilon\}(x) = y \}$ and let $h(x) = \mu \sigma(\exists y < \sigma)(\langle \varepsilon, x, y \rangle \in W^\omega)$

where $\varepsilon$ is a $\Theta$-index for $f$. Whenever $f(x)$ is defined, let $N_x = \{ y < h(x) : \langle \varepsilon, x, y \rangle \in W^h(x) \}$. Note that $N_x$ is well-defined since $h(x) = \sigma$ $\land$ $h(x) = \tau \Rightarrow \sigma = \tau$. It follows from lemma 1.1 that a canonical $\Theta$-index for $N_x$ is obtained uniformly and single-valuedly from $\varepsilon$ and $x$. Let $g(x) = \nu y (y \in N_x)$. If $\Theta$-finite $K \subseteq \text{dom } f$, let $N = \bigcup \{ N_x : x \in K \}$.

An immediate corollary to lemma 1.4 is the existence of a "selection operator" which single-valuedly chooses a canonical $\Theta$-index for a non-empty subset of a non-empty $\Theta$-s.c. set. It is such a "selection operator", rather than the multi-valued one we assumed, which is needed for our arguments. In [3] Fenstad gives axioms for infinite computation theories which do not assert the existence of a selection operator, but where the existence of a "selection operator" as above nonetheless is a theorem. Thus Fenstad may and does restrict himself to single-valued theories.

Definition 1.5. A $(\leq)$-parametrization of $\Theta$-s.c. sets is a $\Theta$-computable mapping $\lambda \varepsilon \sigma W^\omega_\varepsilon$ such that
(i) \( \forall \varepsilon, \tau, \sigma (\tau \leq \sigma \Rightarrow W^\tau_\varepsilon \subseteq W^\sigma_\varepsilon) \)

(ii) for each \( \Theta\text{-s.c.} \) set \( W \) there is an \( \varepsilon \) such that \( W = \bigcup \{ W^\sigma_\varepsilon : \sigma \in U \} \).

Axiom A2 asserts the existence of a \((\preceq)\)-parametrization of \( \Theta\text{-s.c.} \) sets. Considering a fixed \((\preceq)\)-parametrization, we let \( W_\varepsilon \) denote \( \bigcup \{ W^\sigma_\varepsilon : \sigma \in U \} \). Note that a \( \Theta\text{-s.c.} \) index for \( W_\varepsilon \) is obtained uniformly from \( \varepsilon \), using the selection operator. Indices from a \((\preceq)\)-parametrization can therefore be used in explicit definitions of \( \Theta\text{-computable} \) functions.

Definition 1.6.

(i) A projection into \( W \) is a total \( \Theta\text{-computable} \) function \( p \) whose range is a subset of \( W \) such that if \( x \neq y \) then \( p(x) \cap p(y) = \emptyset \). (Here \( p(x) \) denotes the set \( \{ z : p(x) - z \} \).

(ii) \( \Theta \) is projectible into \( W \) if there is a projection into \( W \).

Lemma 1.7.

(i) Let \( W = \{ \varepsilon : W_\varepsilon \neq \emptyset \} \) for a given \((\preceq)\)-parametrization of \( \Theta\text{-s.c.} \) sets. Then \( \Theta \) is projectible into \( W \).

(ii) Suppose \( p \) is a projection. Then there is a \((\preceq)\)-parametrization of \( \Theta\text{-s.c.} \) sets such that \( \{ \varepsilon : W_\varepsilon \neq \emptyset \} \subseteq \text{ran } p \).

Proof.

(i) Define

\[
f(x) = \mu \sigma [ (\exists \varepsilon \leq \sigma) (x \in W^\sigma_\varepsilon \land (\forall y \in W^\sigma_\varepsilon) (y = x)]
\]

and let

\[
p(x) = \nu \varepsilon [ x \in W^f_\varepsilon (x) \land (\forall y \in W^f_\varepsilon (x)) (y = x)] .
\]
Then $p$ is clearly a projection into $W$.

(ii) Let $\lambda \in \sigma W^\sigma$ be any $(\leq)$-parametrization of $\Theta$-s.c. sets. Using lemma 1.4 we have a collection of $\Theta$-finite sets $K_x$, each obtained uniformly from $x$, such that $\emptyset \not\subseteq K_x \subseteq p(x)$. Let $W = \bigcup\{K_x : x \in U\}$ and let $\lambda \sigma W^\sigma$ be a $(\leq)$-enumeration of $W$. Define

$$r(\epsilon, \sigma) = \begin{cases} \forall x [\epsilon \in K_x] & \text{if } \epsilon \in W^\sigma \\ 0 & \text{if } \epsilon \not\in W^\sigma \end{cases}$$

Letting

$$W^\sigma_\epsilon = \begin{cases} r(\epsilon, \sigma) & \text{if } \epsilon \in W^\sigma \\ \emptyset & \text{if } \epsilon \not\in W^\sigma \end{cases}$$

we obtain a $(\leq)$-parametrization with the required property.

Due to the negative result in Simpson [14] it is reasonable to formulate yet another condition which isolates a subclass of infinite computation theories for which the priority argument can be carried out. The problem in the general case is that for any $(\leq)$-parametrization the set $\{\epsilon : W_\epsilon \neq \emptyset\}$ may be too "wide". Lemma 1.7 reduces the problem of finding a "narrow" $(\leq)$-parametrization to that of finding a "narrow" projection.

A $\Theta$-finite set $K$ is said to be strongly $\Theta$-finite if every $\Theta$-s.c. subset of $K$ is $\Theta$-finite.

Definition 1.8. An infinite computation theory $\Theta$ is said to be adequate if $\Theta$ is projectible into the field of a $\Theta$-s.c. prewellorder whose initial segments are uniformly strongly $\Theta$-finite.

In the sequel we assume that the prewellorder of definition 1.8 is $\leq$ or an initial segment of $\leq$, for some $\leq$ satisfying A1 and A2. The modifications necessary for the general case are left to the reader.
Let \( p_\xi \) be the unique order-preserving map from \( U \) onto the ordinal \(|\xi|\). Often we will be imprecise and write \( x \) when we mean \( p_\xi(x) \). Thus \( x < \alpha \) where \( \alpha \) is an ordinal stands for \( p_\xi(x) < \alpha \). Throughout the paper we use the following

**Convention:** \( L^\beta = \{ x \in U : x < \beta \} \).

**Definition 1.8.**

(i) The **projectum**(\( \xi \)), denoted \(|\xi|^*\), is the least ordinal \( \beta \) such that \( \Theta \) is projectible into \( L^\beta \).

(ii) The **r.e.-projectum**(\( \xi \)), denoted \(|\xi|^+\), is the least ordinal \( \beta \) for which there is a \( \Theta \)-s.c. non-\( \Theta \)-finite set \( W \subseteq L^\beta \).

Since the range of a projection is a \( \Theta \)-s.c. non-\( \Theta \)-finite set it follows that \(|\xi|^+ \leq |\xi|^*\). Thus, modulo our assumption after 1.8, \( \Theta \) is adequate if and only if \(|\xi|^+ = |\xi|^* = \text{limit ordinal}\).

Every computably wellordered \( \Theta \) is adequate since for such theories every \( \Theta \)-s.c. non-\( \Theta \)-finite set is the range of an injective \( \Theta \)-computable mapping. Any (choiceless) standard model of \( \mathcal{Z} \mathcal{F} \) constitutes a (non-wellorderable) adequate theory relative to the power set operator \( \mathcal{P} \). We may also use urelements to give some further examples of non-wellorderable adequate theories. Let \( \mathcal{M} = \langle M \rangle \) be an infinite structure without relations or let \( \mathcal{M} = \langle M, < \rangle \) be a dense linear ordering. Then \( \text{HYP}_\mathcal{M} \), the smallest admissible set above \( \mathcal{M} \) (defined in \([1]\)), as well as \( \text{HYP(HYP}_\mathcal{M}) \), \( \text{HYP(HYP(HYP}_\mathcal{M}) \)) and so on, can be shown to be adequate.
2. Relative Computability

Equivalent notions of Turing reducibility for ordinary recursion theory become distinct when considering recursion theory on an arbitrary admissible ordinal \( \alpha \). As Kreisel [5] emphasizes, the different notions fall into essentially two categories: those concerned with computability and those concerned with definability. Below, a notion from each will be defined (along with some auxiliary notions) corresponding to \( \leq_\alpha \) and \( \leq_{ca} \) for \( \alpha \)-recursion theory. We will then show that, as in the case of \( \alpha \)-recursion theory, the notions agree on regular hyperregular sets.

By an enumeration of \( \Theta \)-finite sets we mean a \( \Theta \)-computable mapping \( \lambda \xi K_\xi \) with the property that for each \( \Theta \)-finite set \( K \) there is \( \xi \) such that \( K = K_\xi \). Such an enumeration always exists since every \( \Theta \)-finite set is \( W_\epsilon^\sigma \) for some \( \epsilon \) and \( \sigma \). An enumeration can of course be chosen with somewhat care, e.g. we may require \( K_\xi \subseteq L_\xi \).

Definition 2.1. Let \( A \) and \( B \) be sets, \( f \) a function and \( \lambda \xi K_\xi \) a fixed enumeration of \( \Theta \)-finite sets.

(i) \( f \) is weakly \( \Theta \)-computable in \( B \) (denoted \( f \preceq_w B \)) if there is a \( \Theta \)-s.c. set \( W \) such that for all \( x, y \)

\[
f(x) \prec y \iff \exists \xi, \eta (\langle x, y, \xi, \eta \rangle \in W \land K_\xi \subseteq B \land K_\eta \cap B = \emptyset).
\]

\( A \) is weakly \( \Theta \)-computable in \( B \) (\( A \preceq_w B \)) in case \( c_A \preceq_w B \).

(ii) \( A \) is \( \Theta \)-computable in \( B \) (denoted \( A \preceq B \)) if there is a \( \Theta \)-s.c. set \( W \) such that for all \( \gamma, \delta \)

\[
K_\gamma \hookrightarrow A \land K_\delta \cap A = \emptyset \iff \exists \xi, \eta (\langle \gamma, \delta, \xi, \eta \rangle \in W \land K_\xi \subseteq B \land K_\eta \cap B = \emptyset).
\]
The definitions are independent of the particular enumeration of $\Theta$-finite sets. We define the upper semi-lattice of degrees in the usual way using the transitive reducibility $\leq$. $A \equiv B$ denotes $A \leq B \& B \leq A$. The join of $\text{deg}(A)$ and $\text{deg}(B)$, $\text{deg}(A) \lor \text{deg}(B)$, is $\text{deg}(A \oplus B)$ where $A \oplus B = \{(x,0) : x \in A\} \cup \{(x,1) : x \in B\}$.

The notions of weakly $\Theta$-s.c. in and $\Theta$-s.c. in are easily abstracted from (i) and (ii) of definition 2.1. Thus $A$ is $\Theta$-s.c. in $B$ if there is a $\Theta$-s.c. set $W$ such that for each $\gamma$

$$K_\gamma \subseteq A \iff \exists \xi, \eta(\langle \gamma, \xi, \eta \rangle \in W \& K_\xi \subseteq B \& K_\eta \cap B = \emptyset).$$

The sets weakly $\Theta$-s.c. in $B$ are enumerated by putting

$$W_\emptyset^B = \{x : \exists \xi, \eta(\langle x, \xi, \eta \rangle \in W_\emptyset \& K_\xi \subseteq B \& K_\eta \cap B = \emptyset)\}.$$ 

It follows immediately from the definitions that a set is (weakly) $\Theta$-s.c. in $B$ iff both it and its complement are (weakly) $\Theta$-s.c. in $B$, and that a set is weakly $\Theta$-s.c. in $B$ iff it is the domain of a function weakly $\Theta$-computable in $B$.

To define a reducibility notion corresponding to definability is technically somewhat more complicated. From an infinite theory $\Theta$ and a set $B \subseteq U$ we construct a new theory $\Theta[B]$ and say that $f \leq_d B$ if $f$ is $\Theta[B]$-computable. In addition to the obvious requirement that $\Theta[B]$ should have the usual closure and enumeration properties, i.e. that $\Theta[B]$ should be a precomputation theory, we want $B$ to be $\Theta[B]$-computable, $\Theta \leq \Theta[B]$ ($\leq$ is the relation between precomputation theories given in [6]), and quantification over initial segments of $\leq$ to be $\Theta[B]$-computable. The latter means that the functional $E^z$ should be $\Theta[B]$-computable where

$$E^z(f, z) = \begin{cases} 0 & \text{if } \exists x < z (f(x) = 0) \\ 1 & \text{if } \forall x < z (f(x) = 1). \end{cases}$$
Furthermore Θ[B] should have a computable selection operator in order for the Θ[B]-s.c. relations and Θ[B]-finite sets to behave properly.

The theory Θ[B] will be the least fixed point of an inductive operator Γ defined by clauses 0-VIII. Clause 0 introduces the characteristic function of B and clause I makes Θ ≤ Θ[B] using axiom A2 for Θ. Clauses II-VI correspond to clauses IX'-XIII' in [6]. Finally, clauses VII and VIII introduce the functional E and a selection operator respectively. Having already opted for multi-valued theories we make the selection operator take all its possible values.

The β-th iteration of Γ is defined as Θ^β[B] = Γ(Θ^<β>[B]) where Θ^<β>[B] = \bigcup{Θ^γ[B] : γ < β}. Thus the least fixed point of Γ is Θ[B] = \bigcup_β Θ^β[B].

There is no need to give the detailed construction. We only note that all clauses have the following important property: A tuple (ε, x, z) is added to Θ^β[B] only if ε, x, z and (ε, x, z) are elements of L^β. For (ε, x, z) ∈ Θ[B], set |ε, x, z|_Θ[B] = least ordinal β such that (ε, x, z) ∈ Θ^β[B]. Using this notion of length of computations, Θ[B] is a computation theory in the sense of Moschovakis. One can show that Θ[B] is either an infinite theory or a Spector theory (defined in [3] and [6]) depending on whether U is Θ[B]-infinite or Θ[B]-finite.

In [9] Sacks defines α-recursion relative to a set B ⊆ a to be Σ₁-recursion on α relative to the structure <L(α,B), ε, B>, where L(α,B) is the result of relativizing L(α) to B by adding x ∈ B to the atomic formulas. We regard the theory Θ[B] as the relativization of an infinite theory Θ to a set B. Suppose Θ is a formulation of α-recursion theory. Then one can show that
\( \Theta[B] \) is an infinite theory if and only if \( \langle L(\alpha,B), \lambda, B \rangle \) is admissible, in which case the notions of \( \Theta[B] \)-finite and \( \Theta[B] \)-s.c. agree with \( a-B \)-finite and \( a-B \)-r.e.

**Definition 2.2.**

(i) \( f \leq_d B \) if \( f \) is \( \Theta[B] \)-computable.

(ii) \( A \leq_d B \) if \( c_A \) is \( \Theta[B] \)-computable.

**Lemma 2.3.** \( A \leq_d B \) if and only if \( \Theta[A] \leq \Theta[B] \).

**Corollary.** \( \leq_d \) is transitive.

The proof of the lemma is standard. The required mapping \( p \) is defined by cases using the second recursion theorem for \( \Theta[B] \). The if direction of \( (\varepsilon, x, z) \in \Theta[A] \iff (p(\varepsilon,n),x,z) \in \Theta[B] \) is shown by induction on \( |p(\varepsilon,n),x,z|_{\Theta[B]} \), while the only if direction is shown by induction on \( |\varepsilon,x,z|_{\Theta[A]} \).

Using the corollary we define \( d \)-degrees by \( d-deg(A) = \{ B : A \leq_d B \land B \leq_d A \} \). The \( d \)-degrees form an upper semi-lattice in the usual way.

**Lemma 2.4.** \( f \leq_w B \Rightarrow f \leq_d B \).

**Proof.** Let \( \lambda \notin \mathcal{K}_g \) be an enumeration (in \( \Theta \)) of \( \Theta \)-finite sets. It follows from the \( \Theta[B] \)-computability of \( E^\lambda \) and \( \Theta \leq \Theta[B] \) that \( K_g \subseteq B \) and \( K_\eta \cap B = \emptyset \) are \( \Theta[B] \)-computable relations. Suppose \( f \leq_w B \) using \( W \), i.e.

\[
f(x) \rightarrow y \iff \exists \xi, \eta(\langle x, y, \xi, \eta \rangle \in W \land K_g \subseteq B \land K_\eta \cap B = \emptyset).
\]

Recalling that \( \nu \) takes all its possible values in \( \Theta[B] \) we have

\[
f(x) = (\nu[\langle x, (y)_1, (y)_2, (y)_3 \rangle \in W \land K(y)_2 \subseteq B \land K(y)_3 \cap B = \emptyset] \).
\]
From the lemma we conclude that \( A \preceq B \Rightarrow A \preceq_w B \Rightarrow A \preceq_d B \).

None of the implications can be reversed since Driscoll [2] has shown that \( \preceq_w \) need not be transitive even on \( \Theta \)-s.c. sets.

We now introduce the analogues of two notions due to Sacks [8].

Recalling the definition of \( W^\mathcal{B}_\mathcal{C} \) let

\[
\sigma_{W^\mathcal{B}_\mathcal{C}} = \{ x : \exists \xi, \eta(\langle x, \xi, \eta \rangle \in W^\mathcal{G}_\mathcal{C} \land K^\xi \subseteq B \land K^\eta \cap B = \emptyset) \}.
\]

**Definition 2.5.**

(i) A set \( B \) is **regular** if \( B \cap K \) is \( \Theta \)-finite whenever \( K \) is \( \Theta \)-finite.

(ii) A set \( B \) is **hyperregular** if whenever \( K \subseteq W^\mathcal{B}_\mathcal{C} \) and \( K \) is \( \Theta \)-finite then there is \( \sigma \) such that \( K \subseteq \sigma_{W^\mathcal{B}_\mathcal{C}} \).

Hyperregularity has the following equivalent formulation in terms of functions: \( B \) is hyperregular if and only if whenever \( f \preceq \mathcal{W}_w \), \( K \subseteq \text{dom } f \) and \( K \) is \( \Theta \)-finite then \( \exists z(\forall x \in K)(3y < z)(f(x) \neq y) \).

Every \( \Theta \)-computable set is hyperregular (lemma 1.4) and every hyperregular \( \Theta \)-s.c. set is regular (proved in [15]). A useful characterization of the regular \( \Theta \)-s.c. sets is the following. Suppose \( \lambda \sigma W^\mathcal{G} \) is a \((\preceq)\)-enumeration of a set \( W \). Let

\[
V^\mathcal{G} = W^\mathcal{G} - \cup\{W^\mathcal{T} : \mathcal{T} < \sigma\}.
\]

For obvious reasons we say that \( \lambda \sigma V^\mathcal{G} \) is a disjoint \((\preceq)\)-enumeration of \( W \). Then \( W \) is regular if and only if

\[
(\forall \beta < |\mathcal{L}|)(\exists \tau)(\forall \mathcal{T} < \sigma)(V^\mathcal{T} \cap \beta^\mathcal{B} = \emptyset).
\]

The problem of non-regularity can be avoided in the usual way when studying \( \Theta \)-s.c. degrees for adequate theories.

**Theorem 2.6.** Suppose \( \Theta \) is an adequate theory. Then for every \( \Theta \)-s.c. set \( B \) there is a regular \( \Theta \)-s.c. set \( D \) such that \( B \equiv D \). \( D \) may be chosen such that \( \forall x(\forall y \sim x)(x \in D \Rightarrow y \in D) \).
The theorem is due to Sacks [8] for $\alpha$-recursion theory. Its proof (which we omit) in our more general setting is modelled on Sacks' original proof in [8]. A proof of a weaker, but for our purposes sufficient, version of theorem 2.6 can be found in [15].

Now we set out to show that for any sets $A$ and $B$, $A \subseteq B \iff A \subseteq_d B$ if $B$ is regular and hyperregular. Given disjoint sets $B_1$ and $B_2$ we obtain a theory $\Theta[B_1, B_2]$ by altering clause 0 in the definition of $\Theta[B]$ as follows:

0' If $(0,0), x, 0, \langle 0,0 \rangle, x, 0 \in L^\beta$ & $x \in B_1$
then $(0,0), x, 0 \in \Theta^\beta[B_1, B_2]$.

If $(0,0), x, 1, \langle 0,0 \rangle, x, 1 \in L^\beta$ & $x \in B_2$
then $(0,0), x, 1 \in \Theta^\beta[B_1, B_2]$.

Thus $\Theta[B] = \Theta[B, U-B]$. For each $\sigma, \xi, \eta$ and $m$ define

$$m^\sigma_{\xi, \eta} = \{ (e, \vec{x}, y) : (e, \vec{x}, y) \in \Theta^\sigma_{[K_\xi, K_\eta], \text{lh} (\vec{y}) = m} \}.$$ 

Lemma 2.7. $m^\sigma_{\xi, \eta}$ is $\Theta$-finite uniformly in $m, \sigma, \xi, \eta$.

Proof. $m^\sigma_{\xi, \eta}$ can be defined by induction on $\sigma$ with respect to $\subseteq$ considering all cases in the definition of $\Theta[K_\xi, K_\eta]$. 

By an easy induction on $\sigma$ we have

Lemma 2.8. If $(e, \vec{x}, y) \in m^\sigma_{\xi, \eta}$ & $K_\xi \subseteq B$ & $K_\eta \cap B = \emptyset$ then $(e, \vec{x}, y) \in \Theta^\sigma_{\xi, \eta}[B]$. 

Theorem 2.9. Let $B$ be a regular set. Then (i)-(iii) below are equivalent.

(i) $B$ is hyperregular.

(ii) $\Theta[B]$ is an infinite theory.

(iii) $\forall f (f \leq_w B \iff f \leq_d B)$.

Proof.

(i) $\Rightarrow$ (ii). To show $\Theta[B]$ is an infinite theory it suffices to show $\Theta|\lessdot| [B] = \emptyset < \lessdot [B]$. Since $|\lessdot|$ is a limit ordinal we need only consider the case of universal quantification whose inductive clause is:

If $\langle 7,0, x, \langle \langle 7,0, e, x,1 \rangle \rangle \in \mathbb{L}^\emptyset$ and $(\forall y < x)[(\varepsilon, y, 1) \in \Theta^{\lessdot}[B]]$

then $\langle \langle 7,0, e, x,1 \rangle \rangle \in \Theta^\emptyset[B]$.

So suppose $(\varepsilon, y, 1) \in \Theta^{\lessdot}[B]$ for each $y < x$. It follows from the regularity of $B$ that for each $y < x$ there are $\sigma, \xi, \eta$ such that $\langle \varepsilon, y, 1 \rangle \in 7_{\xi, \eta}$ where $K_{\xi} \subseteq B$ and $K_{\eta} \cap B = \emptyset$. Letting $W = \{ \langle y, \xi, \eta \rangle \in \mathbb{L}^\emptyset : \langle \varepsilon, y, 1 \rangle \in 7_{\xi, \eta} \}$, this can be reformulated as $L^x \subseteq W^B$ where $\lambda \sigma W^\sigma$ is a $(\lessdot)$-enumeration of $W$. By the hyperregularity of $B$, $L^x \subseteq W^B$ for some $\tau$. But then $(\varepsilon, y, 1) \in \Theta^{\lessdot}[B]$ for each $y < x$ by lemma 2.8, and hence $\langle \langle 7,0, e, x,1 \rangle \rangle \in \Theta^{\lessdot}[B]$.

(ii) $\Rightarrow$ (iii). Suppose $f$ is $\Theta[B]$-computable with a $\Theta[B]$-index $\varepsilon$. Then by (ii), 2.8 and the regularity of $B$,

$f(\bar{x}) - y \iff \exists \beta < |\lessdot|((\varepsilon, \bar{x}, y) \in \Theta^\emptyset[B])$

$\iff \exists \sigma, \xi, \eta((\varepsilon, \bar{x}, y) \in m_{H^\sigma_{\xi, \eta}} \& K_{\xi} \subseteq B \& K_{\eta} \cap B = \emptyset)$. 

It follows that $f \leq_w B$. 

(iii) \(\Rightarrow\) (i). Assume (iii). Then every \(\Theta[B]-s.c.\) set has a \((\leq)\)-
enumeration in \(\Theta[B]\). For suppose \(V\) is \(\Theta[B]-s.c.\). Then \(V = \omega^B\)
for some \(\Theta-s.c.\) \(W\) by (iii). Put
\[V^\sigma = \{x < \sigma : \exists \xi, \eta < \sigma \ (\langle x, \xi, \eta \rangle \in W^\sigma \ \& \ K_\xi \subseteq B \ \& \ K_\eta \cap B = \emptyset)\}.\]
Then \(\lambda \sigma V^\sigma\) is a \((\leq)\)-enumeration in \(\Theta[B]\) of \(V\). It follows that
\(U\) is \(\Theta[B]-infinite\) (and in fact that \(\Theta[B]\) is an infinite theory).

Suppose a \(\Theta\)-finite set \(K \subseteq \omega^B\). Let \(f(x) = \mu \sigma \ [x \in \sigma W^B]\). Then
for each \(x \in K\), \(L^f(x)\) is \(\Theta[B]-finite\) uniformly in \(x\). Thus
\(M = \bigcup \{L^f(x) : x \in K\}\) is \(\Theta[B]-finite\) and hence bounded by some \(\sigma\).
Then \(K \subseteq \sigma W^B\), so \(B\) is hyperregular.

Note that regularity was not needed in going from (iii) via (ii)
to (i). The regular hyperregular sets can be characterized as those
sets \(B\) for which every \(\Theta[B]-finite\) set is \(\Theta-finite\). Of course,
whenever (iii) holds for \(B\) it follows that \(A \leq B \iff A \leq d B\).
Just let \(f(\gamma, \delta) = 0 \iff K_\gamma \subseteq A \ \& \ K_\delta \cap A = \emptyset\).

Before defining the jump of a set we introduce yet another
notion of reducibility.

Definition 2.10. A set \(A\) is many-one reducible to a set \(B\),
\(A \preceq_m B\), if there is a \(\Theta\)-computable mapping \(\lambda x H^B x\)
whose values are (canonical \(\Theta\)-indices for) non-empty \(\Theta\)-finite sets such that

(i) \(x \in A \iff H^B x \subseteq B\)
(ii) \(x \notin A \iff H^B x \cap B = \emptyset\).

Note that \(A \preceq_m B \Rightarrow A \preceq B\) and \(A \preceq_m B \ \& \ B \preceq_w C \Rightarrow A \preceq_w C\).

Following Shore [12] and Simpson [13] we want the jump of a set
\(B\) to be a \(\preceq_m\) complete set \(B'\) weakly \(\Theta-s.c.\) in \(B\), i.e. when-
ever $A$ is weakly $\Theta$-s.c. in $B$ then $A \leq_m B'$. Letting $\lambda \in W_\epsilon$ be a (not necessarily the) standard $(\leq_\lambda)$-parametrization of $\Theta$-s.c. sets we make the following definition.

**Definition 2.11.** The jump of a set $B$ is the set

$$B' = \{ \epsilon : \exists \xi, \eta \langle \xi, \eta \rangle \in W_\epsilon \land K_\xi \subseteq B \land K_\eta \cap B = \emptyset \}.$$ 

Our only requirement on the $(\leq_\lambda)$-parametrization used in the definition is that (iii) in proposition 2.12 below must hold. This is certainly the case for a $(\leq_\lambda)$-parametrization obtained from the standard one as in lemma 1.7.

**Proposition 2.12.**

(i) $B \leq_m B'$ but not $B' \leq_w B$ (so $B < B'$).

(ii) $B < D \iff B' \leq_m D'$.

(iii) $D$ is weakly $\Theta$-s.c. in $B \iff D \leq_m B'$.

(iv) $B'$ is weakly $\Theta$-s.c. in $B$.

Thus the jump is well defined and increasing on degrees. However, it may not be increasing on $d$-degrees as is readily seen by considering a non-hyperregular $d$-degree. This is not surprising since $\leq_d$ in general is a much stronger reducibility notion than $\leq$. The proper notion of "semi-computable in $B$" for $\leq_d$ is $\Theta[B]$-s.c. Thus we want the jump (in this connection called $d$-jump) of a set $B$ to be a complete $\Theta[B]$-s.c. set.

**Definition 2.13.** The $d$-jump of a set $B$ is the set

$$B^d = \{ \langle \epsilon, x \rangle : [\epsilon]_{\Theta[B]}(x) \downarrow \}.$$ 

It is easily verified that the analogue for the $d$-jump of proposition 2.12 holds. Of course, in case $B$ is regular and hyperregular then $B' =_m B^d$. 
It is clear that in case the domain of an infinite computation theory is not computably wellordered, one cannot consider a unique requirement at a given stage of a priority construction. There is thus a need to consider a \( \Theta \)-finite block of requirements at each stage. The obvious way to block requirements is in terms of the levels of the given prewellorder letting each level make up one block. This method suffices for \( \Theta \)-finite injury arguments where elements in at most one set of requirements can be injured more than a fixed finite number of times. In particular, a weak positive solution to Post's problem was obtained in [15] for every adequate theory using this method.

In proving the splitting theorem for an admissible ordinal \( \alpha \), Shore [11] developed a technique of blocking requirements into \( \sigma 2 \text{cf}(\alpha) \) many \( \alpha \)-finite sets. S.G. Simpson [14] was the first to note that this technique could also be used to prove a version of the Friedberg-Muchnik theorem for thin admissible sets. This led us to develop Shore's blocking technique for adequate theories \( \Theta \).

A set \( A \) is said to be \( \Sigma_0 \) and \( \Pi_0 \) if it is \( \Theta \)-computable. \( A \) is \( \Sigma_{n+1} \) if \( x \in A \iff \exists y ( (x,y) \in B ) \) where \( B \) is \( \Pi_n \), and \( A \) is \( \Pi_{n+1} \) if its complement is \( \Sigma_{n+1} \). A function \( f \) is \( \Sigma_n \) if its graph \( G_f = \{ (x,y) : f(x) = y \} \) is \( \Sigma_n \).

Let \( \mathcal{L} \) be the class of functions on \( U \) satisfying

\[ f(x_1, \ldots, x_i, \ldots, x_n) = z & f(x_1, \ldots, x'_i, \ldots, x_n) = z' \text{ and } x_i \sim x'_i \Rightarrow z \sim z'. \]

Functions in \( \mathcal{L} \) will be identified in the obvious way with partial single-valued functions on \( |\mathcal{L}| \). Thus by a function in \( \mathcal{L} \cap \Sigma_n \) we
will interchangeably mean a \( \Sigma_n \)-function in \( L \) or a function on \(|\xi|\)
induced by a \( \Sigma_n \)-function in \( L \). It is shown in [15] that \(|\xi|\) is
admissible and that every \(|\xi|\)-recursive function is in \( L_1 \cap L \).

Let \( f'(\alpha, \gamma) \) be a partial single-valued function on \(|\xi|\).
Then \( \lim_{\alpha} f'(\alpha, \gamma) = \delta \) iff \( \exists \beta (\forall \alpha \geq \beta) (f'(\alpha, \gamma) = \delta) \). For \( f, f' \in L \)
we say that \( \lim_{\sigma} f'(\sigma, x) \preceq f(x) \) if this is the case for the induced
functions on \(|\xi|\), where \( \preceq \) has its usual meaning.

**Lemma 3.1.** Let \( \Theta \) be an adequate theory. Suppose \( f \in L \cap L_2 \)
is total (on \(|\xi|\)). Then there is a total \( \Theta \)-computable function
\( f' \in L \) such that \( \lim_{\sigma} f'(\sigma, x) \preceq f(x) \).

**Proof.** Since \( G_f \) is \( L_2 \) it follows that \( f \leq^L A \), say using \( W \),
where \( A \) is \( \Theta \)-s.c. and (by theorem 2.6) regular. Let \( \lambda \sigma A^\Theta \) and
\( \lambda \sigma W^\Theta \) be \( (\leq) \)-enumerations of \( A \) and \( W \) respectively. Let \( N_x^\Theta \) be
the \( \Theta \)-finite set of minimal \( \eta < \sigma \) such that
\((\exists y < \sigma)(\exists x' \sim x) \langle x', y, \eta \rangle \in W^\sigma \land K^\eta \cap A^\sigma = \emptyset \).
Define
\[
f'(\sigma, x) = \begin{cases} \mu y [\exists \eta \in N_x^\Theta] (\exists x' \sim x) \langle x', y, \eta \rangle \in W^\sigma \quad \text{if} \quad N_x^\Theta \neq \emptyset \\ \sigma \quad \text{else .} \end{cases}
\]
Then \( f' \) is total and in \( L \cap L_1 \).

Suppose \( f(\alpha) = \beta \) (on \(|\xi|\)). Choose \( x, y \) such that \( p_x(y) = \alpha \),
\( p_x(y) = \beta \) and \( f(x) \not\rightarrow y \), and choose \( \eta \) such that
\( \langle x, y, \eta \rangle \in W \land K^\eta \cap A = \emptyset \). By the regularity of \( A \) we can choose \( \sigma \)
sufficiently large so that \( y < \sigma \), \( \langle x, y, \eta \rangle \in W^\sigma \) and \( (U-A) \cap L^\eta = (U-A^\sigma) \cap L^\eta \). Suppose \( \tau \geq \sigma \). Then \( N_x^\tau \neq \emptyset \) since \( \eta \) is a candi-
date. Let \( \xi \in N_x^\tau \). There is \( x' \sim x \) and \( y' \) such that
\( \langle x', y', \xi \rangle \in W^\tau \land K^\xi \cap A^\tau = \emptyset \). Since \( \xi \leq \eta \) and (we may assume our enumeration of \( \Theta \)-finite sets to satisfy) \( K^\xi \subseteq L^\xi \), \( K^\xi \cap A = \emptyset \). But
then \( \langle x', y', \xi \rangle \) is a correct computation of \( f \), i.e. \( f(x') = y' \).
Since \( f \in L \) and \( x' \sim x \), we must have \( y' \sim y \). Thus
\[
\lim_{\sigma} f'(\sigma, a) = \beta.
\]

**Definition 3.2.** The \( \Sigma_2^\omega\)-cof(\( \alpha \)) is the least ordinal \( \beta \) for which there is a function \( f \in L \cap \Sigma_2 \) with domain \( \beta \) and range unbounded in \( \alpha \).

**Lemma 3.3.** Let \( \Theta \) be an adequate theory. Then \( \Sigma_2^\omega\)-cof(\( |\xi| \)) = \( \Sigma_2^\omega\)-cof(\( |\xi|^* \)).

**Proof.** Let \( k \in L \) be a total \( \Theta \)-computable function with range in \( |\xi|^* \) such that \( \{ \beta : k(\beta) < \alpha \} \) is bounded for each \( \alpha < |\xi|^* \).
Such a \( k \) can be defined from a \( (\xi) \)-enumeration of a \( \Theta \)-s.c. non-\( \Theta \)-computable set \( W \subseteq L |\xi|^* \). Suppose \( f \in L \cap \Sigma_2 \) with domain \( \beta \) is unbounded in \( |\xi| \). Then \( g(\alpha) = k(f(\alpha)) \) is an \( \Sigma_2 \) function unbounded in \( |\xi|^* \). Thus \( \Sigma_2^\omega\)-cof(\( |\xi|^* \)) \leq \Sigma_2^\omega\)-cof(\( |\xi| \)).

For the converse inequality suppose \( f \in L \cap \Sigma_2 \) with domain \( \beta \) is unbounded in \( |\xi|^* \). Let \( g(x) = \mu \sigma \left[ \forall \tau \exists \sigma (f(x) < k(\tau)) \right] \). Then \( g \in L \) and \( g \) is unbounded in \( |\xi| \). It follows from lemma 3.1 and some easily shown closure properties of \( \Sigma_n \) and \( \Pi_n \) sets that \( g \) is \( \Sigma_2^\omega \).

By a \( (\xi) \)-sequence of \( \Theta \)-s.c. sets we mean a \( \Theta \)-computable mapping \( r \) such that \( x \sim y \Rightarrow W_r(x) = W_r(y) \).

**Lemma 3.4.** Suppose \( \alpha < \Sigma_2^\omega\)-cof(\( |\xi| \)) and \( \langle I_x : x < \alpha \rangle \) is a \( (\xi) \)-sequence of \( \Theta \)-s.c. sets such that for each \( x < \alpha \), \( I_x \) is \( \Theta \)-finite. Then \( \cup \{ I_x : x < \alpha \} \) is \( \Theta \)-finite.
Proof. Let $\alpha$ be least for which such a sequence exists whose union is not $\Theta$-finite. Let $W = \bigcup\{I_x : x < \alpha\}$ and let $\lambda \sigma W^\sigma$ be a $(\leq)$-enumeration of $W$. Define $g(x) = \mu \sigma [I_x \subseteq W^\sigma]$. Then $g \in \mathcal{L} \cap \Sigma_2$ and $g$ is defined on $L^\alpha$. But $g(L^\alpha)$ is unbounded in $U$ since $W$ is not $\Theta$-finite, i.e. $\Sigma_2$-cof$(|\xi|) \leq \alpha$.

Assume for the remaining part of this section that $\Theta$ is an adequate theory. We are going to divide the projectum $L^{|\xi|^*}$ into $\Sigma_2$-cof$(|\xi|)$ many $\Theta$-finite blocks $M_\alpha$, each bounded strictly below $|\xi|^*$. Clearly $\Sigma_2$-cof$(|\xi|) \leq |\xi|^*$. Suppose first that $\Sigma_2$-cof$(|\xi|) = |\xi|^*$. In this case we let $M_\alpha = M_\alpha^\sigma = \{x : x \sim \alpha\}$ for each $\alpha < |\xi|^*$. Then each $M_\alpha$ is $\Theta$-finite uniformly in $\alpha$.

Now suppose $\Sigma_2$-cof$(|\xi|) < |\xi|^*$. We are going to define $\Theta$-finite approximations $M_\alpha^\sigma$ to our blocks $M_\alpha$ uniformly from $\sigma$ and $\alpha$. Furthermore $(\forall \alpha < \Sigma_2$-cof$(|\xi|)) (\exists \sigma) (\forall \tau \geq \sigma) (\forall \beta < \alpha) (M_\beta^T = M_\beta^\sigma)$, i.e. our approximation will be "tame".

Let $g : \Sigma_2$-cof$(|\xi|) \rightarrow |\xi|^*$ be a $\mathcal{L} \cap \Sigma_2$ function unbounded in $|\xi|^*$, and let $g' \in \mathcal{L}$ be $\Theta$-computable such that $\lim_\alpha g'(\sigma, \alpha) \sim g(\alpha)$ and $\text{rang } g' \in L^{|\xi|^*}$. These functions exist by 3.1 and 3.3. Define $h(\sigma, \alpha) = \mu \gamma [(\forall \beta < \alpha)(g'(\sigma, \beta) \leq \gamma)]$ and put $M_\alpha^\sigma = \{\epsilon : h(\sigma, \alpha) \leq \epsilon < h(\sigma, \alpha + 1)\}$. Note that a canonical $\Theta$-index for $M_\alpha^\sigma$ is obtained uniformly from $\alpha$ and $\sigma$ and that each $M_\alpha^\sigma$ is bounded strictly below $|\xi|^*$. To show $\lambda \sigma M_\alpha^\sigma$ is tame, let $I_\beta = \{\sigma : (\exists \tau > \sigma)(g'(\tau, \beta) \leq g'(\sigma, \beta))\}$. Fix $\alpha < \Sigma_2$-cof$(|\xi|)$. Then $\langle I_\beta : \beta < \alpha + 1 \rangle$ is a $(\leq)$-sequence of $\Theta$-s.c. sets such that each $I_\beta$ is $\Theta$-finite. Applying lemma 3.4 we obtain $\exists \sigma (\forall \beta < \alpha)(\forall \tau \geq \sigma)(g'(\tau, \beta) \geq g'(\sigma, \beta))$, i.e. $\exists \sigma (\forall \beta < \alpha)(\forall \tau \geq \sigma)(M_\beta^T = M_\beta^\sigma)$.

Let $M_\beta^\sigma = M_\beta^\sigma$ for sufficiently large $\sigma$. It remains to show
\[ \cup \{ \beta : \beta < \sum_{2}^\text{cof}(|\xi|) \} = |\xi|^*. \]

Fix \( \varepsilon < |\xi|^* \) and choose least \( \alpha \) for which \( \varepsilon < h(\sigma, \alpha) \) where \( \sigma \) is fixed and sufficiently large. Such \( \alpha \) exists since \( g \) is unbounded in \( |\xi|^* \). By the definition of \( h \) there is \( \beta < \alpha \) such that \( \varepsilon \leq g'(\sigma, \beta) \). But then \( \varepsilon < h(\sigma, \beta + 1) \), so by the choice of \( \alpha \), \( \alpha = \beta + 1 \) and \( h(\sigma, \beta) \leq \varepsilon \).

4. The Splitting Theorem

For parts (i) and (ii) of our main theorem we need assume \( \Theta \) has a reasonable pairing function. By this we mean that for each \( \alpha < |\xi|^* \) there is \( \beta < |\xi|^* \) such that \( L^\alpha \times L^\beta = \{(x,y):x,y \in L^\alpha \} \subseteq L^\beta \).

Surely any adequate \( \Theta \) that comes to mind has a reasonable pairing function.

**Theorem 4.1.** Suppose \( \Theta \) is an adequate theory with a reasonable pairing function. Let \( C \) be a regular \( \Theta \)-s.c. set and let \( D \) be a \( \Theta \)-s.c. non-\( \Theta \)-computable set. Then there are \( \Theta \)-s.c. sets \( A \) and \( B \) such that \( C = A \cup B \), \( A \cap B = \emptyset \), \( A \subseteq C \), \( B \subseteq C \) and

(i) \( \Theta[A] \) and \( \Theta[B] \) are adequate theories (so in particular \( A \) and \( B \) are hyperregular)

(ii) \( A' = B' = O' \)

(iii) \( D \nsubseteq A \) and \( D \nsubseteq B \).

Before proving theorem 4.1 we state some of its usual corollaries. First we need the following lemma.

**Lemma 4.2.** If \( A \) and \( B \) are disjoint regular \( \Theta \)-s.c. sets then \( \deg(A \cup B) = \deg(A) \lor \deg(B) \) and \( d\deg(A \cup B) = d\deg(A) \lor d\deg(B) \).
Proof. Clearly \( \text{A} \cup \text{B} \subseteq \text{A} \Theta \text{B} \). For the converse we note that 
\[ \text{U} - \text{A} = (\text{U} - \text{A} \cup \text{B}) \cup \text{B}. \]
Using the regularity of \( \text{B} \) we have 
\[ \text{K}_\gamma \cap \text{A} = \emptyset \iff \exists \eta (\text{K}_\eta \subseteq \text{K}_\gamma \land \text{K}_\eta \subseteq \text{B} \land \text{K}_\eta \cap (\text{A} \cup \text{B}) = \emptyset), \]
\( \text{A} \leq \text{A} \cup \text{B} \). The proof for \( \text{d} \)-degrees does not use regularity.

Let \( \alpha, \beta, \gamma, \delta \) vary over \( \Theta \)-s.c. degrees (\( \Theta \)-s.c. \( \text{d} \)-degrees) and let \( \alpha' \) denote the jump (the \( \text{d} \)-jump) of \( \alpha \).

Corollary 4.3.

(i) \( \forall \gamma > 0)(\exists \alpha, \beta)(\gamma = \alpha \vee \beta \land \alpha < \gamma \land \beta < \gamma \land \alpha \mid \beta) \).

(ii) \( \forall \delta)(\delta < \alpha' \Rightarrow \exists \alpha(\delta \mid \alpha \land \alpha' = \delta') \).

The proofs are entirely similar to the ones found in [10] and [11], using the main theorem and lemma 4.2.

Corollary 4.4.

(i) \( \exists \alpha, \beta(\delta < \alpha < \beta \land \alpha' = \beta') \).

(ii) \( \exists \alpha, \beta(\delta < \alpha < \beta \land \alpha' < \beta') \).

(iii) \( \exists \alpha, \beta(\alpha \mid \beta \land \alpha' = \beta' = (\alpha \vee \beta)'), \)

(iv) \( \exists \alpha, \beta(\alpha \mid \beta \land \alpha' = \beta' = \alpha \vee \beta') \).

We now proceed with the proof of theorem 4.1. Our description of the construction will be in terms of \( \text{A} \) only whenever the description in terms of \( \text{B} \) is analogous. In case \( \lambda \sigma \mathcal{H}^\sigma \) is a \( (\leq) \)-enumeration of \( \Theta \)-finite sets we use the notation \( \mathcal{H}^{\leq \sigma} = \bigcup \{ \mathcal{H}^\tau : \tau < \sigma \} \). By theorem 2.6 we may assume \( \text{D} \) to be regular and satisfy 
\[ \forall x (\forall y \sim x)(x \in \text{D} \Rightarrow y \in \text{D}). \]
Let \( \lambda \sigma \mathcal{D}^\sigma \) be a \( (\leq) \)-enumeration of \( \text{D} \) and let \( \lambda \sigma \mathcal{C}^\sigma \) be a disjoint \( (\leq) \)-enumeration of \( \text{C} \). We are going to define \( (\leq) \)-enumerations \( \lambda \sigma \mathcal{A}^\sigma \) and \( \lambda \sigma \mathcal{B}^\sigma \) of \( \text{A} \) and \( \text{B} \) induc-
tively on the prewellorder $\preceq$. If $\sigma \sim \tau$ then the set constructions at stage $\sigma$ and stage $\tau$ will be identical though the indices used may differ. At stage $\sigma$, $C^\sigma$ will be added to precisely one of $A^\sigma$ and $B^\sigma$. Thus $A$ and $B$ will be $\Theta$-s.c., $C = A \cup B$ and $A \cap B = \emptyset$. Furthermore $A \subseteq C$ and $B \subseteq C$. For let $q(\xi) = \mu \sigma [(K_\xi - C^\sigma) \cap C = \emptyset]$. Then $q \leq_\omega C$ and $q$ is total by the regularity of $C$. Clearly $K_\xi \cap A = \emptyset \iff K_\xi \cap A^q(\xi) = \emptyset$, so $A \subseteq C$.

In order to satisfy (i) and (ii) of the theorem, some care is needed in choosing a $(\preceq)$-parametrization $\lambda \in \omega \sigma^\xi_\varepsilon$ of $\Theta$-s.c. sets, besides requiring $\{ \varepsilon : W_\varepsilon \neq \emptyset \} \subseteq L^{|\xi|^*}$. First of all we want $\lambda \varepsilon \sigma W^\xi_\varepsilon$ to be repetitive in the following sense: For each $\alpha, \varepsilon < |\xi|^*$ there is $\delta < |\xi|^*$ and $\sigma$ such that $\alpha < \delta$ and $\forall \tau \geq \sigma(W^\tau_\varepsilon = W^\delta_\varepsilon)$. Then we want definition 2.11 of the jump to make sense for our choice of $(\preceq)$-parametrization. Let $\lambda \varepsilon \sigma V^\xi_\varepsilon$ be a $(\preceq)$-parametrization obtained as in lemma 1.7 from the standard one, such that $\{ \varepsilon : V_\varepsilon \neq \emptyset \} \subseteq L^{|\xi|^*}$. Let $V^\xi_\varepsilon = V^\sigma_{(\varepsilon)}_1$. Then $\lambda \varepsilon \sigma W^\xi_\varepsilon$ has the required properties.

To make $\Theta[A]$ and $\Theta[B]$ into adequate theories, the construction is split into two cases.

**Definition 4.5.** Suppose $\beta < |\xi|$. Then

$$\text{cof}(\beta) = \mu \alpha [\exists \Theta \text{-computable } q : L^\beta \rightarrow L^\alpha \text{ such that } \forall \varepsilon \in L^\beta \exists \gamma < \beta (q^{-1}(\varepsilon) \subseteq L^\gamma)].$$

**Remark.** Since $\beta < |\xi|$, $q^{-1}(\varepsilon)$ may be considered a $\Theta$-finite set with an index obtained uniformly from $\varepsilon$. Note that $\text{dom } q = L^\beta$.

If $|\xi|^* = |\xi|$ or $|\xi|^* < |\xi|$ and $\text{cof}(|\xi|^*) < |\xi|^*$ then attempts are made to preserve computations $x \in W^A_\varepsilon$ for $x < \varepsilon$. In case $|\xi|^* < |\xi|$ and $\text{cof}(|\xi|^*) = |\xi|^*$, additional attempts are
made to preserve computations on initial segments of $L^{|\xi|}$.

Assume we have shown $A \leq^m 0'$. $A'$ is weakly $\Theta$-s.c. in $A$ by 2.12. By the hyperregularity of $A$, $A'$ is in fact $\Theta$-s.c. in $A$ and hence $\Theta$-s.c. in $0'$. Let $A^S$ denote the jump of $A$ using the standard $(\leq)$-parametrization $\lambda \in V_\xi$. Then

$$K_\delta \cap A^S = \emptyset \iff \neg(\exists \eta \in \bigcup \{V_\varepsilon : \varepsilon \in K_\delta\})(K_\eta \cap A = \emptyset) \iff f(\delta) \in A^S$$

where $f$ is a $\Theta$-computable mapping giving a standard index for $\bigcup \{V_\varepsilon : \varepsilon \in K_\delta\}$. Thus $(U - A^S)$ is weakly $\Theta$-s.c. in $0'$ iff $(U - A^S)$ is $\Theta$-s.c. in $0'$. Both $A'$ and $A^S$ satisfy 2.12 (iii) and (iv), so $A' \equiv^m A^S$. Thus $A^S \leq_w 0'$ since $A' \leq_w 0'$, and hence $(U - A^S)$ is $\Theta$-s.c. in $0'$. But then (again using $A' \equiv^m A^S$) $(U - A')$ is $\Theta$-s.c. in $0'$. Since both $A'$ and its complement are $\Theta$-s.c. in $0'$, $A' \leq 0'$. Thus it suffices to make $A' \leq_w 0'$ in order to satisfy (ii).

To make $A' \leq_w 0'$, attempts are made to preserve computations showing $\varepsilon \in A'$ by creating a requirement for such a computation. Then one can effectively from $0'$ look through the list of requirements to determine whether or not $\varepsilon \in A'$.

Finally, to insure that for no $\varepsilon$, $(U - D) = W_\varepsilon^A$, we use the usual approach of trying to preserve computations $x \in W_\varepsilon^A$ for minimal $x$ not in $D$. In case $(U - D) = W_\varepsilon^A$ for some $\varepsilon$ we would eventually preserve a correct computation for each $x \in W_\varepsilon^A$, i.e. $W_\varepsilon^A$ would be $\Theta$-s.c. Thus computations $x \in W_\varepsilon^A$ will eventually stop being preserved. However we need have $\Theta$-finite blocks of requirements to settle down by some stage of the construction. Towards this end we use Shore's technique of letting each block play the role of a single requirement in trying to preserve a computation $x \in W_\varepsilon^A$ for $x \notin D$.
and some \( \epsilon \) in the block considered. Furthermore, to avoid the problem of never finishing creating requirements with arguments from a fixed level of \( \leq \), we utilize the fact that \( D \) was chosen to have the property \( \forall x(\forall y \sim x)(x \in D \Rightarrow y \in D) \). Thus there is a need to create a requirement preserving a computation \( x \in W^A_\epsilon \) only if no other computation \( y \in W^A_\epsilon \) for \( y \sim x \) is being preserved.

Let \( M^\sigma_\alpha \) and \( M_\alpha \) for \( \alpha < \Sigma_2 - \text{cof}(|\Xi|) \) be the \( \Theta \)-finite blocks described in section 3. We will create sets \( R^_,i_\alpha \) \((R^B,i_\alpha)\) of requirements for \( i < 3 \). \( R^_,0_\alpha \) will insure that \( \Theta[A] \) is adequate, \( R^_,1_\alpha \) that \( A' \leq_w 0' \), and \( R^_,2_\alpha \) that \( D \not\preceq_w A \). \( S_\alpha \) denotes the set of \( A \)-requirements (i.e. requirements in \( \bigcup\{R^_,i_\alpha : i < 3\} \)) injured during the construction. \( R^_,i_\alpha \) and \( S^\sigma_\alpha \) denote the \( \Theta \)-finite parts of \( R^_,i_\alpha \) and \( S_\alpha \) obtained by stage \( \sigma \). Each requirement will be of the form \( \langle \epsilon, x, F \rangle \) where \( F \) is (a canonical \( \Theta \)-index for) a \( \Theta \)-finite set. Such a requirement in \( R^_,i_\alpha \) is called an \( \epsilon - A \) requirement or an \( \alpha - A \) requirement (at \( \sigma \)) in case \( \epsilon \in M^\sigma_\alpha \) \((\epsilon \in M_\alpha^\sigma)\). It is said to have argument \( x \). In case \( F \cap A^\sigma = \emptyset \) it is said to be active at \( \sigma \), else it is inactive. \( \epsilon \in M^\sigma_\alpha \) is an inactive \( \alpha - A \) reduction procedure at \( \sigma \) in case there is an active \( \epsilon - A \) requirement in \( R^\sigma_\alpha \) preserving a computation \( x \in W^A_\epsilon \) for some \( x \in D^\sigma \), i.e. there is \( \langle \epsilon, x, F \rangle \in R^\sigma_\alpha \) \( S^\sigma_\alpha \) such that \( \exists \eta < \sigma(\langle x, \eta \rangle \in W^\sigma_\epsilon \land K_\eta \subseteq F \land x \in D^\sigma) \). If no such requirement exists, then \( \epsilon \) is an active \( \alpha - A \) reduction procedure at \( \sigma \).

Let \( r : |\Xi| \rightarrow \Sigma_2 - \text{cof}(|\Xi|) \) be a \( \Theta \)-computable function such that \( (\forall \alpha < \Sigma_2 - \text{cof}(|\Xi|))(\forall \beta)(\exists \gamma > \beta)(r(\gamma) = \alpha) \), where \( \alpha, \beta \) and \( \gamma \) vary over \( |\Xi| \). \( r \) indicates which part of the construction to concern ourselves with at a given stage.
The construction at stage $\sigma$: Suppose $r(\sigma) = \alpha$. We describe only the construction of $A$-requirements, the construction of $B$-requirements being analogous.

First we construct requirements making $\Theta[A]$ adequate. The construction is split into two cases.

**Case 1**: $|\xi| * = |\xi| \text{ or } \text{cof}(|\xi| *) < |\xi| * < |\xi|$

Let

$$K^\sigma = \{ \langle \epsilon, x \rangle \in M^\sigma \times \bigcup \{ M^\beta : \beta \leq \alpha \} : (\exists \eta < \sigma)(\langle x, \eta \rangle \in \check{w}_\epsilon \wedge K_{\eta} \cap A^\omega = \emptyset)$$

and $(\forall w \in \check{R}_{A,0} - \check{S}_A)((w)_1 \neq \epsilon \vee (w)_2 \neq x)$. Thus $\langle \epsilon, x \rangle \in K^\sigma$ only if there is a computation $x \in \check{w}_\epsilon$ which is not already being preserved by an active requirement. A requirement for each $\langle \epsilon, x \rangle \in K^\sigma$ preserving such a computation will be created.

Letting

$$\check{R}_{\epsilon,0}^\sigma = \check{R}_{A,0}^\sigma \cup \{ \langle \epsilon, x, \check{R}_{\epsilon,0}^\sigma \rangle : \langle \epsilon, x \rangle \in K^\sigma \}.$$ 

**Case 2**: Let

$$K^\sigma = \{ \langle \epsilon, x \rangle \in M^\sigma \times \check{L}_\xi^*: (\exists \eta < \sigma)(\langle x, \eta \rangle \in \check{w}_\epsilon \wedge K_{\eta} \cap A^\omega = \emptyset)$$

and $(\forall w \in \check{R}_{A,0}^\sigma - \check{S}_A)((w)_1 \neq \epsilon \vee (w)_2 \neq x)$

and $[(\forall y < x)(\exists w \in \check{R}_{A,0}^\sigma - \check{S}_A)((w)_1 = \epsilon \wedge (w)_2 = y) \vee x \in \bigcup \{ M^\beta : \beta \leq \alpha \}].$

To show that $A$ is hyperregular in this case, we need preserve computations on initial segments of $\check{L}_\xi^*$. In addition, in order to show $\Theta[A]$ is adequate, we need preserve computations $x \in \check{w}_\epsilon$ for
Next we construct requirements making $A' \subseteq \emptyset '$. Let

$$
I^\sigma = \{ \varepsilon \in M^\sigma_\emptyset : (\exists \eta \in W^\sigma_\varepsilon)(K_{\eta} \cap A^{< \sigma} = \emptyset) \& (\forall \nu \in R^\sigma_{A,1}, S^\sigma_A)(\nu, \varepsilon) \}.
$$

Letting $G^\sigma_\varepsilon = \cup\{ K_{\eta} : \eta \in W^\sigma_\varepsilon \& K_{\eta} \cap A^{< \sigma} = \emptyset \}$ we put

$$
R^\sigma_{A,1} = R^\sigma_{A,1} \cup \{ \langle \varepsilon, 0, G^\sigma_\varepsilon \rangle : \varepsilon \in I^\sigma \}.
$$

Finally we construct requirements making $D \subseteq_\emptyset A$. Let $R^\sigma$ be the $\emptyset$-finite set of minimal $x$ such that for each $x' - x$, $x' \not\in D^\sigma$ and $\neg(\exists \langle \varepsilon, x', F \rangle \in R^\sigma_{A,2} - S^\sigma_A)$ ("$\varepsilon$ is an active $\alpha - A$ reduction procedure at $\sigma$"). Next let

$$
N^\sigma = \{ \langle \varepsilon, x \rangle \in M^\sigma_A \times H^\sigma : " \varepsilon \text{ is an active } \alpha - A \text{ reduction procedure at } \sigma " \& (\exists \eta < \sigma)(\langle x, \eta \rangle \in W^\sigma_\varepsilon \& K_{\eta} \cap A^{< \sigma} = \emptyset) \}.
$$

Letting $F^\sigma_\varepsilon = \cup\{ K_{\eta} : (\exists x \in H^\sigma)(\langle x, \eta \rangle \in W^\sigma_\varepsilon \& K_{\eta} \cap A^{< \sigma} = \emptyset) \}$ we put

$$
R^\sigma_{A,2} = R^\sigma_{A,2} \cup \{ \langle \varepsilon, x, F^\sigma_\varepsilon \rangle : \langle \varepsilon, x \rangle \in N^\sigma \}.
$$

To establish our priorities let

$$
J^\sigma_A = \{ \langle \varepsilon, x, F \rangle \in R^\sigma_{A,2} - S^\sigma_A : F \cap C^\sigma \not= \emptyset \} \text{ where } R^\sigma_A = \cup\{ R^\sigma_{A,i} : i < 3 \}.
$$

$J^\sigma_A$ is the set of active $A$-requirements which would be injured in case $C^\sigma$ were added to $A$. Using the notation $(H)_1 = \{(w)_1 : w \in H\}$, define $f^A_\emptyset(\sigma) = \mu \beta [\cup J^\sigma_A \cap M^\sigma_\emptyset \not= \emptyset]$ in case such $\beta$ exists and let $f^A_\emptyset(\sigma) = |2|$ otherwise. It is clear from the definition of the blocks $M^\sigma_\emptyset$ (considering the split in that definition) that $f^A_\emptyset$ and $f^B_\emptyset$ may be viewed as $\emptyset$-computable functions. If $f^A_\emptyset(\sigma) \leq f^B_\emptyset(\sigma)$, let $B^\sigma = B^{< \sigma} \cup C^\sigma$ and $A^\sigma = A^{< \sigma}$. If $f^B_\emptyset(\sigma) < f^A_\emptyset(\sigma)$, let $A^\sigma = A^{< \sigma} \cup C^\sigma$ and $B^\sigma = B^{< \sigma}$.

To complete the construction let $S^\sigma_A = \{ \langle \varepsilon, x, F \rangle \in R^\sigma_A : F \cap A^\sigma \not= \emptyset \}$. 

$x \in \cup\{ M^\sigma_{\beta} : \beta \leq \alpha \}$. $F^\sigma_{\varepsilon,x}$ and $R^\sigma_{A,0}$ are defined as in the previous case.
Lemma 4.6. For each \( \alpha < \Sigma_2\)-cof(\(|\xi|\)) the set of \( \alpha-A \) and \( \alpha-B \) requirements is \( \Theta \)-finite.

Proof. The proof is by induction on \( \alpha \). Fix \( \alpha < \Sigma_2\)-cof(\(|\xi|\)) and assume the set of \( \beta-A \) and \( \beta-B \) requirements is \( \Theta \)-finite for each \( \beta < \alpha \). By the tameness of our blocking there is a stage \( \sigma \) by which all blocks \( M^\beta_\sigma \) for \( \beta \leq \alpha \) have settled down. Let

\[
I_\beta = \{ \sigma > \sigma_\circ : (\exists \omega \in R^\sigma_A \cup R^\sigma_B - R^\sigma_A \cup R^\sigma_B)((w)_1 \in M^\beta_\sigma) \}.
\]

Then \( I_\beta \) is \( \Theta \)-finite for each \( \beta < \alpha \) by our induction hypothesis so \( \bigcup \{ I_\beta : \beta < \alpha \} \) is \( \Theta \)-finite by Lemma 3.4. Thus, using the regularity of \( C \), we can assert the existence of \( \sigma_1 \geq \sigma_\circ \) such that all \( \beta \)-requirements for \( \beta < \alpha \) have been created by \( \sigma_1 \) and no such \( \beta \)-requirement will meet \( C^\tau \) for \( \tau \geq \sigma_1 \). It follows that \( f_A(\tau) \geq \alpha \) and \( f_B(\tau) \geq \alpha \) for \( \tau \geq \sigma_1 \) and hence, by our priorities, no \( \alpha-A \) requirement will be injured beyond \( \sigma_1 \).

Now we show the existence of \( \sigma_2 \geq \sigma_1 \) beyond which no \( \alpha-A \) requirement in \( R^\alpha_A \) is created. Let

\[
T_1 = \{ \epsilon \in M_\alpha : (\exists \sigma \geq \sigma_1)(\exists \omega \in R^\sigma_A, -R^\sigma_Ax)((w)_1 = \epsilon) \}.
\]

\( T_1 \) is \( \Theta \)-s.c. and hence, by the adequacy of \( \Theta \), \( \Theta \)-finite. After \( \sigma_1 \) only permanent \( \alpha-A \) requirements are created. As is readily seen from the definition of \( I^\sigma \), at most one permanent \( \epsilon \)-requirement is created for each \( \epsilon \in M_\alpha \). Thus the existence of \( \sigma_2 \) follows from \( T_1 \) being \( \Theta \)-finite.

Next we show the existence of \( \sigma_3 \geq \sigma_2 \) beyond which no \( \alpha-A \) requirement in \( R^\alpha_A, 0 \) is created. We need consider two cases.

\( |\xi| \ast = |\xi| \) or \( \text{cof}(|\xi| \ast) < |\xi| \ast < |\xi| \): The set

\[
\{(\epsilon, x) \in M_\alpha \times \bigcup \{ M^\beta_\sigma : \beta \leq \alpha \} : (\exists \sigma \geq \sigma_2)(\exists \omega \in R^\sigma_A, -R^\sigma_A, 0)((w)_1 = \epsilon, (w)_2 = x) \}
\]
is $\Theta$-finite by adequacy and the assumption on the pairing function. The existence of $\sigma_3$ then follows as above.

$$\text{Cof}(|\xi|) = |\xi| < |\xi| :$$ Let

$$T_0 = \{ \epsilon \in M_\alpha : (\forall x < |\xi|^*)(\exists \omega \in R^\sigma_{A_0} - S^\sigma_A)((w)_1 = \epsilon \& (w)_2 = x) \}.$$  

$T_0$ is the set of $\epsilon \in M_\alpha$ for which there is a permanent $\epsilon$-requirement with argument $x$ for each $x \in L|\xi|^*$. $T_0$ is $\Theta$-finite by adequacy and hence there is $\sigma_2 \succeq \sigma_2$ by which stage all such requirements are created.

Suppose there is $\gamma < |\xi|^*$ such that if an $\alpha-A$ requirement in $R_{A,0}$ is created beyond $\sigma_1$ then its argument is less than $\gamma$. Then the existence of $\sigma_3 \succeq \sigma_2$ follows just as in the former case.

Suppose no such $\gamma$ exists. For each $x \in L|\xi|^*$ let

$$q(x) = \nu \epsilon [(\exists \omega \geq \sigma_2)(\exists \omega \in R^\sigma_{A_0} - R^\sigma_{A_0})(w)_1 = \epsilon \& (w)_2 \geq x \& \epsilon \in M_\alpha)].$$

Then $q : L|\xi|^* \to M_\alpha$ is total. Fix $\epsilon \in M_\alpha$. If there is a permanent $\epsilon$-requirement with argument $x$ for each $x \in L|\xi|^*$ then $q^{-1}(\epsilon) = \emptyset$ by our choice of $\sigma_2$. Else there is $x < |\xi|^*$ such that there is no permanent $\epsilon$-requirement with argument $x$. If $x \in \cup\{M_\beta : \beta \leq \alpha\}$ then $q^{-1}(\epsilon) \subseteq \cup\{M_\beta : \beta \leq \alpha\}$, else $q^{-1}(\epsilon) \subseteq L^{\leq (x)+1}$. In either case $q^{-1}(\epsilon)$ is bounded strictly below $|\xi|^*$. But then $\text{cof}(|\xi|^*) < |\xi|^*$, contradicting our case hypothesis.

Finally we show the existence of $\sigma \succeq \sigma_3$ beyond which no $\alpha-A$ requirement in $R_{A,2}$ is created. First note that an $\alpha-A$ reduction procedure inactive at some $\tau \succeq \sigma_1$ will remain inactive forever, since no $\alpha-A$ requirement is injured beyond $\sigma_1$. The set of $\alpha-A$ reduction procedures which become inactive beyond $\sigma_1$ is $\Theta$-s.c. and hence $\Theta$-finite. Thus there is $\sigma_4 \succeq \sigma_3$ beyond which no $\alpha-A$ reduction procedure is made inactive.
Suppose $\sigma_4 < \sigma < \tau$ and $r(\sigma) = r(\tau) = \alpha$. From the choice of $\sigma_4$ it is easily seen that $H^\sigma < H^\tau$ (i.e. $x \in H^\sigma$ & $y \in H^\tau \Rightarrow x \not\approx y$). Moreover, if an $\alpha$-$A$ requirement is created at $\sigma$ then $H^\sigma < H^\tau$. It follows that either the set of $\alpha$-$A$ requirements is $\Theta$-finite or for each $x \notin D$ there is a permanent $\alpha$-$A$ requirement $\langle \epsilon, x', F \rangle$ where $x' \sim x$ and $\epsilon$ is a reduction procedure active beyond $\sigma_4$. If the latter were the case $D$ would be $\Theta$-computable contrary to our hypothesis. For then

$$x \notin D \iff (\exists \tau > \sigma_4)(\exists x' \sim x)(\exists \langle \epsilon, x', F \rangle \in R^\tau_A, g \cdot S^\tau_A)$$

("$\epsilon$ is an active $\alpha$-$A$ reduction procedure at $\sigma$").

This completes the proof that the set of $\alpha$-$A$ requirements is $\Theta$-finite. Using the regularity of $C$ choose $\sigma_5 > \sigma_4$ sufficiently large for all $\alpha$-$A$ requirements to have been created and such that no $C^\tau$ will meet an $\alpha$-$A$ requirement for $\tau \geq \sigma_5$. No $\alpha$-$B$ requirement is injured beyond $\sigma_5$ since $f_A(\tau) > \alpha$ whenever $\tau \geq \sigma_5$. To show that the set of $\alpha$-$B$ requirements is $\Theta$-finite we can thus repeat the above argument with $B$ in place of $A$ starting with $\sigma_5$ in place of $\sigma_1$.

Lemma 4.7. $A$ and $B$ are hyperregular.

Proof. The proof splits into three cases.

$|<| = |<|$ : Suppose $H \subseteq W^A_\epsilon$ where $H$ is $\Theta$-finite. We need to show the existence of $\tau$ such that $H \subseteq \tau W^A_\epsilon$. Recall that our $(\leq)$-parametrization of $\Theta$-s.c. sets was chosen to be repetitive. Choose $\beta_0$ such that $H \subseteq \cup \{M_\gamma : \gamma < \beta_0\}$ and choose $\alpha \geq \beta_0$ for which there is $\delta \in M_\alpha$ such that $W_\epsilon = W_\delta$. Let $\sigma$ be sufficiently large for all $\alpha$-$A$ requirements to have settled down. Then for
each \( x \in H \) there is a permanent \( \delta \)-requirement with argument \( x \) in \( R^\sigma_{A,0} \). For if this was not the case for some \( x \in H \), choose \( \eta \) such that \( \langle x, \eta \rangle \in W_\delta \) and \( K_\eta \cap A = \emptyset \). Let \( \tau > \sigma \) be such that \( r(\tau) = \alpha \) and \( \langle x, \eta \rangle \in W_\delta^\tau \). Then \( \langle \delta, x \rangle \in K^\tau \) so a \( \delta \)-requirement with argument \( x \) would be put into \( R^\tau_{A,0} \) contradicting the choice of \( \sigma \). Let \( x \in H \) and choose \( \langle \delta, x, F \rangle \in R^\sigma_{A,0} - S^\sigma_A \). Then there is \( \eta \) such that \( \langle x, \eta \rangle \in W_\delta^\sigma \) and \( K_\eta \subseteq F \). But \( \langle \delta, x, F \rangle \) is a permanent requirement so \( F \cap A = \emptyset \), i.e. \( x \in \sigma_{W_\delta}^A \). Thus \( H \subseteq \sigma_{W_\delta}^A \). Choose \( \tau \) such that \( W_\delta^\sigma \subseteq W_\epsilon^\tau \). Then \( H \subseteq \tau_{W_\epsilon}^A \).

Before proceeding to the remaining cases we note that by easy manipulations using a projection function one can show the following:

If \( |\bar{\xi}|^* < |\xi| \) then a set \( A \) is hyperregular iff for every \( \epsilon \),
\[
L|\bar{\xi}|^* \leq W_\epsilon^A \Rightarrow \exists \epsilon (L|\bar{\xi}|^* \leq \sigma_{W_\epsilon}^A).
\]

**Cof(|\bar{\xi}|^*) = |\bar{\xi}|^* < |\xi|:** Suppose \( L|\bar{\xi}|^* \leq W_\epsilon^A \) and let \( \epsilon \in M_\alpha \). Choose \( \sigma \) sufficiently large for all \( \alpha - A \) requirements to have settled down. Recall from the construction that in this case we attempted to preserve computations on initial segments of \( L|\bar{\xi}|^* \).

Thus using an argument similar to the one above there is for each \( x \in L|\bar{\xi}|^* \) a permanent \( \epsilon \)-requirement with argument \( x \) in \( R^\sigma_{A,0} \) preserving a correct computation \( x \in W_\epsilon^A \). Thus \( L|\bar{\xi}|^* \leq \sigma_{W_\epsilon}^A \).

**Cof(|\bar{\xi}|^*) < |\bar{\xi}|^* < |\xi|:** Let \( \text{cof}(|\bar{\xi}|^*) = \gamma \) and let \( q : L|\bar{\xi}|^* \rightarrow L^\gamma \) be as in definition 4.5. Recalling the remark following that definition we view \( q^{-1}(x) \) as a set \( \Theta \)-finite uniformly in \( x \). Define the \( \Theta \)-computable mapping \( \lambda \in \sigma_{V_\epsilon}^\sigma \) by
\[
V_\epsilon^\sigma = V_\omega \cup \{ \langle x, \eta \rangle \in L^\gamma \times L^\sigma : (\forall y \in q^{-1}(x)) (\exists \xi < \sigma) (\langle y, \xi \rangle \in W_\epsilon^\sigma \land K_\xi \subseteq K_\eta) \}
\]
where
\[
V_\omega = \{ \langle x, \eta_0 \rangle : x \in L^\gamma \land q^{-1}(x) = \emptyset \} \quad \text{and} \quad K_{\eta_0} = \emptyset.
\]
Claim: \( L^{\leq \zeta} \subseteq W_e^A \iff L^\gamma \subseteq V_e^A \).

To prove the claim assume \( L^{\leq \zeta} \subseteq W_e^A \) and let \( x \in L^\gamma \). If \( q^{-1}(x) = \emptyset \) then \( x \in V_e^A \). Suppose \( q^{-1}(x) \neq \emptyset \). Then \( q^{-1}(x) \) is bounded strictly below \( \| \zeta \|^* \). Let \( \alpha, \delta \) and \( \sigma_0 \) be such that \( q^{-1}(x) \subseteq \bigcup \{ M_\beta : \beta \leq \alpha \} \), \( \delta \in M_\alpha \) and \( \forall \tau \geq \sigma_0 (W_e^\tau = W_\delta^\tau) \). Choose \( \sigma \geq \sigma_0 \) sufficiently large for all \( \alpha - A \) requirements to have settled down. Then as in the first case there is a permanent \( \delta \)-requirement in \( R_A^\sigma,0 \) with argument \( y \) for each \( y \in q^{-1}(x) \). Let \( K_\eta = \cup \{ F : \langle \delta, y, F \rangle \in R_A^\sigma,0 - S_A^\sigma \& y \in q^{-1}(x) \} \). Then \( \langle x, \eta \rangle \in V_e^\tau \) for \( \tau \geq \sigma \) and \( \tau > \eta \). Furthermore \( K_\eta \cap A = \emptyset \) since only permanent requirements were used to obtain \( K_\eta \). It follows that \( x \in V_e^A \).

Conversely assume \( L^\gamma \subseteq V_e^A \) and let \( y \in L^{\leq \zeta} \). Choose \( x, \eta \) and \( \sigma \) such that \( y \in q^{-1}(x) \), \( \langle x, \eta \rangle \in V_e^\sigma \) and \( K_\eta \cap A = \emptyset \). Then there is \( \xi \) such that \( \langle y, \xi \rangle \in W_e^\sigma \) and \( K_\xi \subseteq K_\eta \). Thus \( y \in W_e^\sigma \).

Suppose \( L^{\leq \zeta} \subseteq W_e^A \). By the claim, \( L^\gamma \subseteq V_e^A \). Choose \( \alpha \) and \( \delta \) such that \( L^\gamma \subseteq \bigcup \{ M_\beta : \beta \leq \alpha \} \), \( \delta \in M_\alpha \) and \( V_e = W_\delta^\tau \). By the usual argument there is \( \sigma \) such that \( L^\gamma \subseteq W_\delta^\sigma \). Let \( \tau \) be such that \( W_\delta^\sigma \subseteq V_e^\tau \). Then \( L^\gamma \subseteq \tau V_e^A \) so by the last half of the proof of the claim, \( L^{\leq \zeta} \subseteq V_e^A \).

Lemma 4.8. \( \Theta[A] \) and \( \Theta[B] \) are adequate theories.

Proof. \( \Theta[A] \) is an infinite theory by theorem 2.9 since \( A \) is hyperregular and regular. Clearly \( \| \zeta \|^* \geq \| \zeta \|^*_{\Theta[A]} \). We show \( \| \zeta \|^*_{\Theta[A]} \geq \| \zeta \|^*_{\Theta[B]} \). Let \( \nu \subseteq L^\beta \) be a \( \Theta[A] \)-s.c. set where \( \beta < \| \zeta \|^*_{\Theta[B]} \). Then, again using theorem 2.9, \( \nu \) is weakly \( \Theta \)-s.c. in \( A \). Let \( \alpha \) and \( \delta \) be such that \( \nu \subseteq \bigcup \{ M_\beta : \beta \leq \alpha \} \), \( \delta \in M_\alpha \) and \( \nu = W_\delta^\tau \). A permanent \( \delta \)-requirement with argument \( x \) is put into \( R_A^\sigma,0 \) for each
Let \( \sigma \) be sufficiently large for all \( \alpha \)-requirements to have settled down. Then
\[
x \in V \iff (\exists w \in R_A^\sigma, 0 - S_A^\sigma)((w)_1 = \delta \& (w)_2 = x),
\]
so \( V \) is \( \Theta \)-finite and hence \( \Theta[B]-finite \).

**Lemma 4.9.** \( A' \equiv B' \equiv 0' \).

**Proof.** As already remarked, it suffices to show \( A' \leq_w 0' \).

Let \( q(\epsilon) = \mu \sigma[(\forall \tau \geq \sigma)(\forall w \in (R_A^\tau - R_A^\sigma) \cup (S_A^\tau - S_A^\sigma))((w)_1 > \epsilon)] \). \( q \) is defined on all of \( L|_2 \) by lemma 4.6. Furthermore \( q \leq_w 0' \) since \( q \) is a \( \Sigma_2 \)-function. Clearly \( \epsilon \in A' \iff \epsilon \in (R_A^q(\epsilon) - S_A^q(\epsilon))_1 \), and hence \( A' \leq_w 0' \).

**Lemma 4.10.** \( D \models_A A \) and \( D \models_B B \).

**Proof.** Suppose \( (U - D) = \mathsf{W}_A^\epsilon \). Choose \( \alpha \) and \( \sigma_o \) such that \( \epsilon \in M_\alpha \), all \( \alpha - A \) requirements have settled down by stage \( \sigma_o \) and no \( \delta \in M_\alpha \) becomes an inactive \( \alpha - A \) reduction procedure beyond \( \sigma_o \). Note that \( \epsilon \) is an active \( \alpha - A \) reduction procedure at \( \sigma_o \), for else an erroneous computation would be preserved. Choose a minimal \( x \notin D \) such that there is no \( x' \sim x \) for which \( \langle \delta, x', \theta \rangle \in R_A^\sigma \cap (S_A^\sigma)_1 \), where \( \delta \) is an active \( \alpha - A \) reduction procedure at \( \sigma_o \). By the regularity of \( D \) there is \( \sigma_1 \geq \sigma_o \) such that \( L^x \cap D = L^x \cap D^{\sigma_1} \).

Let \( \tau > \sigma_1 \) be such that \( x \in \mathsf{W}_\epsilon^A \) and \( r(\tau) = \alpha \). Then \( H^\tau = \{ x' : x' \sim x \} \) and \( \langle \epsilon, x \rangle \in N^\tau \). It follows that an \( \epsilon \)-requirement with argument \( x \) will be created at \( \tau \), contradicting the fact that \( \tau \geq \sigma_o \).
Bibliography


