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by

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Introduction.

Let R be a commutative noetherian ring and let M be a finitely generated R-module. In this paper we shall study the homology of the complex $X \otimes_R M$ where X is a differential, graded skew-commutative R-algebra of the form

$$X = R\langle V_1, \ldots, V_m; dV_i = v_i\rangle ,$$

that is an R-algebra obtained from R by successive adjunction of finitely many variables V_i killing cycles v_i in the sense of Tate [Ta]. If $H_o(X \otimes M)$ has finite length over R, then so has $H_i(X \otimes M)$ for all i and we can consider the Hilbertseries of $H(X \otimes M)$, which we denote by $|H(X \otimes M)|$. We have reasons to conjecture that this Hilbertseries is a rational function. The purpose of this paper is to prove the conjecture in two important cases. The fact that $|H(X \otimes M)|$ is rational in these cases has applications to the Poincaréseries of certain local rings. Recall that the Poincaré-series of a local ring R is the Hilbertseries of the graded vector-space $\mathrm{Tor}^R(k,k)$, where k is the residue firld of R. It is well known that it is possible to construct a minimal R-algebra resolution of k of the form

$$Y = R\langle V_1, V_2, \ldots, dV_i = v_i\rangle \qquad \text{cf. [Gu]}$$

For some classes of rings there exists a sub-R-algebra X of Y satsifying

(i) X is obtained from R by the adjunction of finitely many of the variables V_i,

(ii) X satisfies the assumptions of the cases 1 or 2 below,

(iii) the Poincaréseries of R can be written

$$P^R(t) = |X \otimes k|(t)[1 - t(|H(X)|(t) - 1)]^{-1}$$

Now $|X \otimes k|(t)$ is easily seen to be a rational function, hence the rationality of $P^R(t)$ follows from the rationality of $|H(X)|(t)$. - Besides Golodrings, examples of such rings can be found in [Gu"] and [Lö, corollary 2.7]. Recently Löfwall has found an example where the sub-R-algebra X is generated by variables of degree 1, 2 and 3.

Our method of proof is to define on $H(X \otimes M)$ a structure of a graded artinian module over a graded ring generated by certain derivations on X. The two cases considered are the following:

Case 1: R has prime characteristic. - In this case $H(X \otimes M)$ is an artinian module over a negatively graded commutative noetherian ring; the rationality of the Hilbertseries follows by a well known argument.

Case 2: No assumption is made on the characteristic, but the derivation associated with the adjunction of the variables are assumed to be extendable to derivations on X. - In this case $H(X \otimes M)$ is considered as a module over a ring which is not necessarily commutative. To obtain the rationality of the Hilbertseries in this case, we introduce the notion of π-rational modules.

Another application of the module structure introduced in the present paper has recently been obtained by Levin [Le] in his study of the Yoneda Ext-algebra of a local ring.

Notation and basic definitions.

R always denotes a commutative noetherian ring with unit. The term "graded" means $\mathbb{Z}$-graded. If H is a graded object, H_i denotes the homogeneous component of degree i. Graded modules will be left modules which are concentrated in non-negative degrees, that is the homogeneous components of negative degrees are all zero. The term "submodule" of a graded module means graded submodule. Maps are assumed to be homogeneous, but not necessarily of degree zero. deg j denotes the degree of the map j.
Graded rings are not assumed to be commutative, but they do have identityelement.

Let H be a graded module over the graded ring T. If each H_i is a T_o-module of finite length $l(H_i)$, we say that the Hilbert-series of H is defined. The Hilbertseries of H is the power-series

$$\Sigma_i l(H_i)t^i$$

It will be denoted by $|H|(t)$ or just $|H|$.

The term " R-algebra" will be used in the sense of Tate [Ta], i.e. a differential graded strictly skew-commutative algebra X over R, such that each X_i is a finitely generated R-module, $X_o = R \cdot 1$ and $X_i = 0$ for $i < 0$. d denotes the differential on X. The R-algebra $R\langle V_1,\dots,V_m; dV_i = v_i\rangle$ will briefly be denoted by $R\langle V_1,\dots,V_m\rangle$ if the values of dV_i are irrelevant. We will always assume that the variables have been adjoined according to increasing degrees, that is $\deg V_{i+1} \geq \deg V_i$ for all i. This assumption causes no loss of generality.

A <u>derivation</u> j on an R-algebra X is an R-linear map $j : X \to X$ commuting with the differential and satisfying

$$j(xy) = (-1)^{wq} j(x)y + xj(y)$$

where $w = \deg j$ and $y \in X_q$.

All tensorproducts are taken over R.

§ 1. π - rational modules

<u>Definition</u>. Let $\pi = \pi(t)$ be a polynomial in $\mathbb{Z}[t]$ such that $\pi(0) = \pm 1$. A powerseries $q(t)$ in $\mathbb{Z}[[t]]$ will be called π-rational if there exists a polynomial $r(t)$ such that $q(t) = r(t)\pi(t)^{-1}$.

A graded module H will be called π-rational if the Hilbertseries $|H|$ is defined, and for all graded submodules $N \subseteq H$, $|N|$ is π-rational. Observe that since $H_i = 0$ for $i < 0$, then for any integer n $t^n|H|(t)$ is π-rational if and only if $|H|(t)$ is π-rational.

<u>Lemma 1</u>. Let $0 \to H' \to H \to H'' \to 0$ be an exact sequence of graded modules. Let $\pi = \pi(t)$ be a polynomial in $\mathbb{Z}[t]$. Then H is π-rational if and only if H' and H'' are π-rational.

<u>Proof</u>. Assume that H' and H'' are π-rational. Let N be a submodule of H, then there exists an exact sequence

$$0 \to N' \xrightarrow{I} N \xrightarrow{J} N'' \to 0$$

where N' and N'' are submodules of H' and H'' respectively. According to the degrees of I and J there exists non-negative integers n', n and n'' such that

$$t^{n'}|N'| + t^{n''}|N''| = t^{n}|N|$$

It follows that $t^n|N|$ is π-rational, hence so is $|N|$. Conversely, assume that H is π-rational. It is easily seen that H' and H'' are π-rational.

Lemma 2. Let $H \xrightarrow{I} H' \xrightarrow{J} H'$ be an exact sequence of graded modules over the graded ring G. Assume that I has degree zero and that J denotes the multiplication map $J(x) = Jx$ where J is an element in G of negative degree $-w$. Let $\pi = \pi(t)$ be a polynomial and assume that H is artinian and π-rational. Then H' is artinian and π'-rational where $\pi'(t) = (1-t^w)\pi(t)$.

Proof. Clearly $H/\operatorname{Ker} I$ is artinian. By lemma 1 it is also π-rational. Hence we may assume without loss of generality that I is injective. It follows from lemma 1.2 in [Gu'] that H' is artinian. It remains to show that it is π'-rational.

Let N be a submodule of H'. We have exact sequences

(1) $$0 \to K_i \hookrightarrow J^{i-1}N \to J^iN \to 0 \qquad \text{for } i > 0$$

Since H' is artinian, there exists an integer s such that

$$J^sN = J^{s+1}N$$

So (1) yields an exact sequence

(2) $$0 \to K_{s+1} \hookrightarrow J^sN \xrightarrow{J} J^sN \to 0$$

As a submodule of H, K_{s+1} is π-rational. It follows from (2) that $|J^sN|$ is π'-rational. Using (1) it follows by descending induction on i that $|N|$ is π'-rational.

§ 2. The case of prime characteristic

In this section p denotes a fixed prime number and the ring R is assumed to have characteristic p, that is $pR = 0$.

Lemma 3. Let X be an R-algebra and let j be a derivation on X of even degree. Then so is j^p.

Proof. Let x and y be homogeneous elements in X. Then we have

$$j^p(xy) = \Sigma \binom{p}{k} j^k(x) j^{p-k}(y) = j^p(x)y + xj^p(y)$$

Theorem 1. Let X be an R-algebra obtained from R by the adjunction of a finite set of variables. Let M be a finitely generated R-module such that $H_0(X \otimes M)$ has finite length over R. Then $H(X \otimes M)$ is a graded artinian module over a commutative noetherian ring. Moreover, the Hilbertseries $|H(X \otimes M)|$ is a rational function.

Proof. Let $T_1, \ldots, T_n$ be the adjoined variables of degree 1, and $S_1, \ldots, S_r$ the variables of degree > 1.
Put

$$X^o = R\langle T_1, \ldots, T_n \rangle$$

and inductively

$$X^{q+1} = X^q \langle S^{q+1} \rangle \qquad \text{for } q \geq 0 .$$

We are going to construct commuting derivations $f_1, \ldots, f_m$ of even negative degree on X such that $H(X \otimes M)$ becomes a graded artinian module over the commutative, noetherian (negatively) graded ring $R[f_1, \ldots, f_m]$. This suffices since the rationality of $|H(X \otimes M)|$ then follows by a well known argument. Cf. lemma 1.3 in [Gu'].

Let $J_q : X^q \to X^q$ be the canonical derivation associated with the adjunction of the variable S_q, $q = 1, \ldots, r$. Cf. [Gu]. Let

$S_{q_1},\dots,S_{q_m}$ be the variables of even degree. Let t be an integer so large that

$$2p^t > \deg dS_{q_i} \qquad \text{for } i = 1,\dots,m.$$

Put

$$f_i = J_{q_i}^{p^t} \qquad \text{for } i = 1,\dots,m.$$

By lemma 3 each f_i is a derivation on X^{q_i}. Clearly f_i has degree less than or equal to $-2p^t$. Hence f_i vanish on all elements of degree less that $2p^t$. Since all the killed cycles dS_q $(q=1,\dots,r)$ have degree less that $2p^t$, it is possible to extend f_i successively to a derivation f_i on X in such a way that

$$f_i(S_l^{(k)}) = 0 \qquad \text{for all } l \neq q_i \text{ and all } k$$

(if $\deg S_l$ is odd $S_l^{(k)}$ shall mean S_l for $k=1$ and 0 for $k>1$). See the proof of Lemma in [Gu].

To see that f_i and f_j commute for all i and j, put

$$f_{ij} = f_i f_j - f_j f_i$$

f_{ij} is itself a derivation on X. It is easily seen that f_{ij} vanish on the R-algebra generators $S_l^{(k)}$, hence f_{ij} vanish on all of X.

The maps $f_1,\dots,f_r$ restrict to operators on X^q for all q. Since they are R-linear maps commuting with the differential on X^q, they operate on $H(X^q \otimes M)$ in the natural way. By induction on q we will show that $H(X^q \otimes M)$ is artinian as a module over the ring $G := R[f_1,\dots,f_m]$. For $q = 0$ this is trivial since X^o is a finite complex, in fact $X_t^o = 0$ for $t > n$. Observe also that

$$l(H_o(X^q \otimes M)) < \infty \text{ implies } l(H_i(X^q \otimes M)) < \infty \text{ for all } i.$$

Before we do the inductionstep, let us recall some basic facts about adjunction of variables. To simplify the notation, put

$$X^{q^*} := X^q \quad \text{if } \deg S_q \text{ is even}$$

$$X^{q^*} := X^{q-1} \quad \text{if } \deg S_q \text{ is odd}.$$

Put $J := J_q$ and observe that $\operatorname{Im} J = X^{q^*}$, thus we have an exact sequence of R-free complexes

$$(3) \qquad 0 \to X^{q-1} \hookrightarrow X^q \overset{J}{\to} X^{q^*} \to 0$$

For each i, the map f_i gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X^{q-1} & \hookrightarrow & X^q & \overset{J}{\to} & X^{q^*} & \to & 0 \\ & & \downarrow f_i & & \downarrow f_i & & \downarrow f_i & & \\ 0 & \to & X^{q-1} & \hookrightarrow & X^q & \overset{J}{\to} & X^{q^*} & \to & 0 \end{array}$$

which shows that the maps in the sequence

$$(4) \qquad H(X^{q-1} \otimes M) \to H(X^q \otimes M) \to H(X^{q^*} \otimes M)$$

induced from (3) are G-linear.

Now let $q > 0$, and assume that $H(X^{q-1} \otimes M)$ is artinian as a G-module. Let us first assume that $\deg S_q$ is odd. In this case we have $X^{q^*} = X^{q-1}$, so by (4), $H(X^q \otimes M)$ is artinian. Let us now assume that $\deg S_q$ is even. For all $s \geq 1$ we have an exact sequence of R-free complexes and G-linear maps

$$0 \to X^{q-1} \hookrightarrow J^{-s}(0) \to J^{-(s-1)}(0) \to 0$$

which gives rise to an exact sequence of G-modules

$$H(X^{q-1} \otimes M) \to H(J^{-s}(0) \otimes M) \to H(J^{-(s-1)}(0) \otimes M)$$

By induction on s it follows that $H(J^{-s}(0) \otimes M)$ is artinian for all s. From the split-exact sequence

$$0 \to \operatorname{Ker} f_i \hookrightarrow X^q \overset{f_i}{\to} X^q \to 0$$

we obtain an exact sequence of G-modules

$$H(\operatorname{Ker} f_i \otimes M) \to H(X^q \otimes M) \xrightarrow{f_i} H(X^q \otimes M)$$

Since $\operatorname{Ker} f_i = J^{-p^t}(0)$, $H(\operatorname{Ker} f_i \otimes M)$ is artinian.
Since $f_i \in G$ it follows from lemma 2 that $H(X^q \otimes M)$ is artinian.

§ 3. The case of extendable derivations

We now return to arbitrary characteristic. As before let X be an R-algebra obtained from R by the adjunction of a finite set of variables. Let the sub-R-algebras X^q and X^{q^*} and the canonical derivation $J^q : X^q \to X^q$ be as in the proof of theorem 1. We will say that a derivation j on X respects divided powers if for each adjoined variable S of even degree and each $k \geq 1$ we have

$$j(S^{(k)}) = S^{(k-1)} j(S)$$

Observe that if R contains the rationals, then any derivation respects devided powers.

Theorem 2. Assume that the canonical derivations $J_{q_1}, \ldots, J_{q_m}$ associated with the adjunction of variables of even degree can all be extended to derivations on X respecting devided powers. Let M be a finitely generated R-module such that $H_o(X \otimes M)$ has finite length over R. Then $H(X \otimes M)$ is an artinian module over the tensoralgebra $T = T_R(J_{q_1}, \ldots, J_{q_m})$. Moreover, there exists a polynomial $f(t)$ with integral coefficients such that

$$|H(X \otimes M)|(t) = f(t)\left[\prod_{i=1}^{m} (1-t^{w_i})\right]^{-1}$$

where $w_i = -\deg J_{q_i}$.

Proof. For each $i = 1,\dots,m$ we fix an extension of J_{q_i} to a derivation on X which respects divided powers. By abuse of notation the extended derivation will also be denoted by J_{q_i}. Since it respects divided powers and since we have assumed that the variables have been adjoined according to increasing degrees, it is easily seen that J_{q_i} operates on every subalgebra X^q. Hence $H(X^q \otimes M)$ becomes a module over the tensoralgebra T in the obvious way.

Fix i. Put $J = J_{q_i}$. It is straight forward to check that we have a commutative diagram for all q

$$\begin{array}{ccccccccc} 0 & \to & X^{q-1} & \hookrightarrow & X^q & \overset{J_q}{\to} & X^{q^*} & \to & 0 \\ & & \downarrow J & & \downarrow J & & \downarrow J & & \\ 0 & \to & X^{q-1} & \hookrightarrow & X^q & \overset{J_q}{\to} & X^{q^*} & \to & 0 \end{array}$$

Since the rows are split-exact we obtain an exact sequence of graded T-modules

$$(5) \qquad H(X^{q-1} \otimes M) \to H(X^q \otimes M) \to H(X^{q^*} \otimes M)$$

By induction on q we will prove that $H(X^q \otimes M)$ is an artinian T-module which is π_q-rational, where

$$\pi_q(t) = \prod_J (1-t^{-\deg J})$$

J running through the set $\{J_{q_i} \mid q_i \leq q \text{ and } 1 \leq i \leq m\}$.
If this set is empty, we take the product to be equal to 1.
We first consider the case $q = 0$. We have $\pi_o = 1$. X^o is a finite complex with $l(H_i(X^o \otimes M)) < \infty$ for all i. Hence $|H(X^o \otimes M)|$ is a polynomial. Clearly $H(X^o \otimes M)$ is an artinian T-module which is 1-rational.

Now let $q \geq 1$ and assume that $H(X^{q-1} \otimes M)$ is artinian and π_{q-1}-rational. Let us first assume that $\deg S_q$ is odd. In this case (5) reads

$$H(X^{q-1} \otimes M) \to H(X^q \otimes M) \to H(X^{q-1} \otimes M)$$

Then clearly $H(X^q \otimes M)$ is artinian as a T-module. From lemma 1 it follows that it is also π_{q-1}-rational. Since no new variable of even degree has been adjoined, we have $\pi_q = \pi_{q-1}$.

We will now assume that $\deg S_q$ is even. In this case we have $q = q_i$ for suitable i. Now (5) reads

$$H(X^{q-1} \otimes M) \to H(X^q \otimes M) \to H(X^q \otimes M)$$

where the right hand map is left multiplication by the element J_{q_i} in T. Put $w = -\deg J_{q_i}$. It follows from lemma 2 that $H(X^q \otimes M)$ is artinian over T and $(1-t^w)\pi_{q-1}(t)$-rational. Since

$$\pi_q(t) = (1-t^w)\pi_{q-1}(t)$$

it follows that $H(X^q \otimes M)$ is π_q-rational.

<u>Corollary</u>. Let $R, \mathfrak{m}$ be a local noetherian ring, and let $Y = R\langle V_1, V_2, \ldots, V_m, \ldots \rangle$ be a minimal R-algebra resolution of $R/\mathfrak{m}$ as constructed in [Gu]. Put

$$Y^m = R\langle V_1, \ldots, V_m; dV_i = v_i \rangle$$

and let M be a finitely generated R-module. Then if the Hilbert-series $|H(Y^m \otimes M)|$ is defined, it is a rational function.

<u>Proof</u>. In [Gu] it is shown how the canonical derivations associated with the adjunction of the variables can be extended to derivations on Y^m. It is immediate from the construction of the extended derivations that they respect divided powers. Hence the corollary follows from theorem 2.

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