Torus actions on cohomology projective spaces

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1. In this note we outline some results on torus actions on cohomology projective spaces (CPS). X is a CPS with $X \sim P^r(q)$ if $H^*(X) = Q[a]/a^{r+1})$ as a Q-algebra, where $\deg a = q$ is an even number. (All cohomology is taken with rational coefficients.) If $r = 1$, X is a cohomology sphere (CS); if $r = 2$, X is a cohomology projective plane (CPP). Details and proofs of the results announced here will appear elsewhere.

Let X be the James reduced product of r copies of S^q, then $X \sim P^r(q)$, (James (6,7)). If the torus T acts linearly on S^q, this construction can be performed equivariantly, and gives examples of actions with:

(i) fixed point set connected, or

(ii) fixed point set consisting of $r+1$ isolated points. If H is a subtorus of T with fixed point set S^p, $p > 0$, then $F(H)$ is the James reduced product of r copies of S^p, and hence connected.

By Theorem 2 this orbit structure for X is in a sense the most generic, hence we call torus actions with the same cohomological orbit structure as this actions of standard type.

2. The torus T is assumed to act cohomology effectively on X, this means that $H^*(X,X^L) \neq (0)$ for any non-trivial subtorus L of T (see Chang and Skjelbred (3)). If X is a CPP, it is obvious that: (A) at most one component of the fixed point set F is a non-acyclic CPS (either a CS or a CPP). In (1) Chang and Comenetz show that if $q > 4$ and $rkT > \log_2 q$, then the action of T on a CPP X is of standard type. Such results can be proved also for a general CPS. However, in (8) Skjelbred applies a theorem of Grünwald and Sylvester to linear dependence relations among local geometric weights to prove the following theorem: If T acts cohomology effectively on a Poincaré duality space X such that $\dim H^*(X) = \dim H^*(F)$, F has two components F^1 and F^2 such that the restriction homomorphism $H^*(X) \to H^*(F^1)$ is onto, and $\dim F^1 \neq \dim F^2$ (as Poincaré duality spaces), then $rkT < 4$. When applied to torus actions on $X \sim P^2(q)$, this gives a condition on rkT which is endependent of q: (B) if $rkT > 4$, then the action of T is of standard type.

For a general CPS, $X \sim P^r(q)$, one expects a different situation. If $q = 2$ or $q = 4$, linear torus actions on complex projective spaces or quaternionic projective spaces demonstrate that tori of large ranks can have rich orbit structures, i.e. there can be many non-acyclic components of the fixed point set. From the work of Hsiang (4) and Hsiang and Su (5) it follows that the cohomological orbit structure of general torus actions on $X \sim P^r(2)$ or $X \sim P^r(4)$ is modelled after these linear examples. In particular there is the following theorem of Hsiang and Su (5): (C) If $X \sim P^r(4)$ and $rkT > 1$, then at most one component of the fixed point set is a CPS with a generator of degree four.

3. Let $H_T^*(X) = H^*(X_T)$ denote the equivariant cohomology of X; here $X_T \to B_T$ is the bundle associated to a universal bundle $E_T \to B_T$ by the given T-action on X. Our approach is to develop a relative version of some structure theory for equivariant cohomology, more precisely we need a linearity theorem for the primary decomposition of N im M where M and N are certain submodules of $H_T^*(X,F)$. If $N = (0)$, this is done in Chang and Skjelbred (2), for other cases see also Tomter (9).

By this theory and some elementary algebra, the following generalization of (C) can be proved:

Theorem 1.

Let T be a torus of rank at least 2 which acts cohomology effectively on $X \sim P^r(q)$, $q > 2$. Then there is at most one component of the fixed point set F of type $P^t(p)$ with $2p > q$. Furthermore, if $q > 4$, this can occur only if F is connected. Since there are in general more than two components of F here, Skjelbred's theorem does not apply directly. However, using the above structure theory and Theorem 1, the problem can be reduced to a similar application of the theorem of Grünwald and Sylvester. This gives the next theorem.

Theorem 2.

Let T be a torus of rank at least 6 which acts cohomology effectively on $X \sim P^r(q)$, $q > 4$. Then the action is of standard type.

Corollary. Let $G = SU(k)$, $k > 7$ act on $X \sim P^r(q)$ with $k(k-1) > q-2$. Then all orbits are finitely convered by Stiefel manifolds.

Remark. Theorem 2 shows that the dimensions $q = 2$ and $q = 4$ where there exist projective spaces of arbitrarily high dimensions, occupy a special position also from the point of view of symmetry groups on the space. Furthermore, Theorem 2 reduces the theory of cohomological orbit structure of actions of classical groups (of rank at least 6) on a space $X \sim P^r(q)$ with $q > 4$ to the theory of such actions on spaces $Y \sim S^q$. Actions of classical groups on cohomology spheres has been studied in detail by Hsiang (4).

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