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INTRODUCTION

In this paper we study formal deformations of graded algebras and corresponding problems in projective geometry. Given a graded algebra \( A \), we may forget the graded structure and deform (lift) \( A \) as an algebra. Clearly we also have a deformation theory respecting the given graded structure of \( A \). This deformation theory is closely related to the corresponding theory of \( X = \text{Proj}(A) \). One objective of this paper is to compare these three theories of deformation.

A basic tool is the cohomology groups of André and Quillen. Let \( S \rightarrow A \) be a graded ring homomorphism and let \( M \) be a graded \( A \)-module. We shall see that the groups

\[
H^i(S, A, M)
\]

are graded \( A \)-modules whenever \( S \) is noetherian and \( S \rightarrow A \) is finitely generated. In fact, if we let

\[
\mathcal{H}^i(S, A, M)
\]

correspond to \( S \)-derivations of degree \( \nu \), we shall prove that there are canonical isomorphisms

\[
\bigoplus_{\nu=-\infty}^{\infty} \mathcal{H}^i(S, A, M) \cong H^i(S, A, M)
\]

for every \( i \geq 0 \).

Deformations of \( A \) (forgetting the graded structure) are classified by the groups \( H^i(S, A, A) \) for \( i=1,2 \). Restricting to graded deformations, we shall see that they are classified by the subgroups

\[
oH^i(S, A, A)
\]

for \( i=1,2 \). These generalities are proved or at least stated in chapter 1.

Let \( \pi : R \rightarrow S \) be a graded surjection satisfying \((\ker \pi)^2 = 0\).
Since there is an injection

\[ \mathcal{H}^2(S, A, A \otimes \ker \Pi) \rightarrow \mathcal{H}^2(S, A, A \otimes \ker \Pi) \]

we deduce that \( A \) is liftable to \( R \) iff \( A \) is liftable as a graded algebra. We would like to generalize this result to arbitrary surjections of complete local rings. This seems difficult. However if we assume

\[ \gamma \mathcal{H}^1(S, A, A) = 0 \]

for \( \gamma > 0 \) or \( \gamma < 0 \) (called negative or positive grading respectively), then the statement above follows from 2.6 of chapter 2 when \( S \) is a field \( k \). In fact, let \( \mathfrak{A} \) be the category of artinian local \( V \)-algebras with residue fields \( k, V/m_V = k \), and let \( \text{Def}^0(A/k, -) \), resp \( \text{Def}(A/k, -) \), be the graded deformation functor, resp non-graded deformation functor on \( \mathfrak{A} \) with hulls \( R^0(A) \) and \( R(A) \) respectively. Consider the local \( V \)-morphism

\[ R(A) \rightarrow R^0(A) \]

Theorem 2.6 states that this morphism has a section whenever \( A \) has negative or positive grading. This follows from the existence of an isomorphisms.

\[ R(A) \sim R^0(A[T]) \]

Here \( \deg T = 1 \) if we have negative grading.

In chapter 3 we enter into projective geometry assuming the graded algebras to be positively graded and generated by elements of degree 1. We compare the groups

\[ \gamma \mathcal{H}^1(S, A, M) \]
with the corresponding groups $\Lambda_i^1(S, X, \tilde{M}(\nu))$ in projective geometry, $X = \text{Proj}(\mathbb{A})$. The groups $\Lambda_i^1(S, X, -)$ were introduced by Illusie in [I] and by Laudal [L1]. If $X$ is $S$-smooth, then

$$\Lambda_i^1(S, X, \tilde{M}) = \mathcal{H}^i(X, \theta_X \otimes S \tilde{M})$$

where $\theta_X$ is the sheaf of $S$-derivations on $X$. If the depth of $M$ with respect to the ideal

$$m = \bigoplus_{\nu=1}^{\infty} A_{\nu}$$

is sufficiently big, the groups

$$\mathcal{H}^i(S, A, M)$$

and

$$\Lambda_i^1(S, X, \tilde{M}(\nu))$$

coincide. For instance, if $\text{depth}_m A \geq 4$,

$$\mathcal{H}^1(S, A, A) \cong \Lambda^1(S, X, O_X(\nu))$$

and

$$\mathcal{H}^2(S, A, A) \cong \Lambda^2(S, X, O_X(\nu))$$

This implies that the deformations of $A$ and $X$ correspond uniquely to each other. When $\text{depth}_m A \geq 3$ a rigidity theorem of Schlessinger, see (2.2.6) in [K,L], is generalized by the injection

$$\mathcal{H}^1(S, A, A) \to \Lambda^1(S, X, O_X(\nu))$$

Now these depth conditions are usually rather crude, and the exact sequences in which these groups fit are in many cases a better tool.
In chapter 3 we also relate the groups corresponding to embeddings. Let \( \varphi: B \to A \) be a surjective morphism of graded \( S \)-algebras such that \( B_0 = A_0 = S \) and let

\[
f : X = \text{Proj}(A) \to Y = \text{Proj}(B)
\]

be the induced embedding. We would like to compare the groups \( \nu H^i(B,A,M) \) and \( A_i(S,f,\tilde{M}(\nu)) \). If \( f \) is locally a complete intersection, one knows that

\[
A_i(S,f,O_X(\nu)) \cong H^{i-1}(X,N_f(\nu))
\]

where \( N_f \) is the normal bundle of \( X \) in \( Y \). Again putting depth conditions on \( M \), we conclude that

\[
\nu H^i(B,A,M)
\]

and

\[
A_i(S,f,\tilde{M}(\nu))
\]

coincide. If \( \text{depth}_m A \geq 2 \), then

\[
\nu H^1(B,A,A) \cong A^1(S,f,\Theta_X(\nu))
\]

and

\[
\nu H^2(B,A,A) \cong A^2(S,f,O_X(\nu))
\]

From this follows that if \( B \) is \( S = k \)-free then

\[
\text{Def}^0(\varphi,-) \cong \text{Hilb}_X(-)
\]

on \( \mathcal{X} \) where \( \text{Def}^0(\varphi,-) \) is the graded deformation functor of \( \varphi \) and where \( \text{Hilb}_X(-) \) is the local Hilbert functor at \( X \). From this and the isomorphism

\[
R(A) \cong R^0(A[T])
\]

we generalize a theorem of Pinkham [P] as follows. If \( A \) has
negative grading and $\text{depth}_m A \geq 1$ and if $X = \text{Proj}(A[T])$ is the projective cone of $X$ in $\mathbb{P}^{n+1}_k = \text{Proj}(B[T])$, then there is a smooth morphism of functors

$$\text{Hilb}_X(-) \rightarrow \text{Def}(A/k,-)$$

In chapter 4 we investigate the conditions of negative and positive grading. We shall assume $A$ to be the minimal cone of a closed subscheme $X \subseteq \mathbb{P}^n_S$. By twisting the embedding we prove that the minimal cone $B$ of $X \subseteq \mathbb{P}^N_S$ for large $N$ very often has negative or positive grading. For instance, if $X$ is $S$-smooth $B$ will have negative grading. If $X$ is of pure dimension $\geq 2$ and locally Cohen-Macaulay, then $B$ will have positive grading. Combining these two results we deduce a theorem of Schlessinger [S3]. See also [M].

Using these results we find that the smooth unliftable projective variety of Serre [Se] gives rise to a graded $k$-algebra which is unliftable to characteristic zero. This is done in chapter 5. His example is of the form $X = Y/G$. $Y$ is a complete intersection of dimension 3 and the order of $G$ divides the characteristic.

The possibility of using this example to get an unliftable $k$-algebra may be looked upon as the beginning of this paper. The proof given here is due to O.A. Laudal and the author.

We end chapter 5 by proving that if $\text{ord}(G)$ did not divide the characteristic and if $Y$ was a complete intersection of dimension $\geq 3$ then $X = Y/G$ would have been everywhere liftable. This paper contains all the results of [K]. I would like to thank O.A. Laudal for reading the manuscript.
CHAPTER 1

Cohomology groups of graded algebras.

Rings will be commutative with unit. Let \( \text{S-alg} \) be the

category of \( S \)-algebras and

\[ \text{SF} \subseteq \text{S-alg} \]

the full subcategory of free \( S \)-algebras. Given an \( S \)-algebra

\( A \) and an \( A \)-module \( M \), we define

\[ H^i(S,A,M) = \lim_{i} \text{Der}_S(-,M) \]

where \( \text{Der}_S(-,M) \) is the functor on \( (\text{SF}/A)^0 \) with values in

\( \text{Ab} \) defined by

\[ \text{Der}_S(-,M)(F \to A) = \text{Der}_S(F,M) \]

\( M \) being an \( F \)-module via \( \varphi \).

If \( S \to A \) is a graded \( S \)-algebra and if \( M \) is a graded

\( A \)-module, we may consider the category of graded \( S \)-algebras \n
\( \text{Sg-alg} \) and the corresponding category

\[ \text{SgF} \subseteq \text{Sg-alg} \]

of free graded \( S \)-algebras. Let

\[ k_{\text{Der}}_S(-,M) : \text{SgF}/A \to \text{Ab} \]

be the functor defined by

\[ k_{\text{Der}}_S(-,M)(F \to A) = k_{\text{Der}}_S(F,M) = \{ D \in \text{Der}_S(F,M) | D \text{ is graded of degree } k \} \]

Then we put

**Definition 1.1**

\[ kH^i(S,A,M) = \lim_{i} k_{\text{Der}}_S(-,M) \]

\[ \text{SgF}/A^0 \]
As mentioned in the introduction, the groups $H^i(S,A,-)$ and $\widehat{H}^i(S,A,-)$ classify formal deformations. Recall that if $R \rightarrow S$ is any surjection with nilpotent kernel, we say that an $R$-algebra $A'$ is a lifting or deformation of $A$ to $R$ if there is given a cocartesian diagram

$$
\begin{array}{ccc}
R & \longrightarrow & A' \\
\pi \downarrow & & \downarrow \\
S & \longrightarrow & A
\end{array}
$$

such that

$$\text{Tor}_1^R(A',S) = 0$$

Two liftings $A'$ and $A''$ are considered equivalent if there is an $R$-algebra isomorphism $A' \sim A''$ reducing to the identity on $A$. If $\phi : A \rightarrow B$ is a morphism of $S$-algebras and $A'$ and $B'$ are liftings of $A$ and $B$ respectively, we say that a morphism

$$\phi' : A' \rightarrow B'$$

is a lifting or deformation of $\phi$ with respect to $A'$ and $B'$ if $\phi \circ \text{id}_S = \phi$. We define graded liftings of graded algebras and graded liftings of graded morphisms in exactly the same way.

Assume that $R \rightarrow S$ satisfies $(\ker \pi)^2 = 0$

Then it is known that

**Theorem 1.2**

There is an element

$$\sigma(A) \in H^2(S,A,A \otimes \ker \pi)$$
which is zero if and only if $A$ can be lifted to $R$. If 
$\sigma(A) = 0$, then the set of non-equivalent liftings is a principal homogeneous space over $H^1(S, A, A \otimes \ker \pi)$

**Theorem 1.3**

There is an element 

$$\sigma(\varphi, A', B') \in H^1(S, A, B \otimes \ker \gamma)$$

which is zero if and only if $\varphi$ can be lifted to $R$ with respect to $A'$ and $B'$. If $\sigma(\varphi ; A', B') = 0$ then the set of liftings is a principal homogeneous space over 

$$H^0(S, A, B \otimes \ker \pi) = \text{Der}_S(A, B \otimes \ker \pi)$$

The elements $\sigma(A)$ and $\sigma(\varphi ; A', B')$ are called obstructions.

Then corresponding theorems in the graded case are

**Theorem 1.4**

There is an element 

$$\sigma_0(A) \in \oH^2(S, A, A \otimes \ker \pi)$$

which is zero if and only if $A$ can be lifted to a graded $R$-algebra. If $\sigma_0(A) = 0$, then the set of non-equivalent liftings is a principal homogeneous space over $\oH^1(S, A, A \otimes \ker \pi)$

**Theorem 1.5**

There is an element 

$$\sigma_0(\varphi ; A', B') \in \oH^1(S, A, B \otimes \ker \pi)$$

which is zero if and only if $\varphi$ can be lifted as a graded morphism to $R$ with respect to $A'$ and $B'$. Moreover, if $\sigma_0(\varphi ; A', B') = 0$, then the set of graded liftings is a principal homogeneous space over $\oH^0(S, A, B \otimes \ker \pi) = \o\text{Der}_S(A, B \otimes \ker \pi)$
In [L1] we find proofs of 1.2 and 1.3 and these can easily be carried over to the graded case.

If we want to compare the graded and non-graded theories of deformation, we need to know the relations between the groups $\text{H}^i(S, A, M)$ and $\text{H}^i(S, A, M)$ . This is given by the following theorem. A proof of this can also be found in [I].

**Theorem 1.6**

Let $S \to A$ be a graded ring homomorphism and let $M$ be a graded $A$-module. If $S$ is noetherian and $S \to A$ is finitely generated, then there is a canonical isomorphism

$$
\bigoplus_{k=-\infty}^{\infty} \mathbb{H}^i(S, A, M) \to \mathbb{H}^i(S, A, M)
$$

for every $i \geq 0$.

**Remark** In general, there is an injection

$$
\bigoplus_{k=-\infty}^{\infty} \mathbb{H}^i(S, A, M) \to \mathbb{H}^i(S, A, M)
$$

for every $i \geq 0$.

**Proof**

Let

$$(S \mathcal{F} / A)_{fg} \subseteq S \mathcal{F} / A$$

be the full subcategory defined by the objects $\phi : F \to A$ where $F$ is a finitely generated $S$-algebra.

Look at the diagram of categories

$$
\begin{array}{ccc}
(S \mathcal{F} / A)_{fg} & \to & S \mathcal{F} / A \\
\uparrow & & \uparrow \\
(S \mathcal{G} \mathcal{F} / A)_{fg} & \to & S \mathcal{G} \mathcal{F} / A
\end{array}
$$

where all functors are forgetful. These induce morphisms
I claim that these maps are all isomorphisms for \( i \geq 0 \).
This will prove 1.6 since there is a canonical isomorphism of functors
\[
\lim\left( t \right) \text{Der}_S(-,M) \cong \text{H}^i(S,A,M)
\]
\[
\left( (SF/A)_{fg} \right)
\]
\[
\lim\left( t \right) \text{Der}_S(-,M) \cong \text{H}^i(S,A,M)
\]
\[
\left( (SgF/A)_{fg} \right)
\]
\[
\lim\left( t \right) \text{Der}_S(-,M) \cong \text{H}^i(S,A,M)
\]
\[
\left( SgF/A \right)
\]

For \( i = 0 \), the contention of \( * \) is easily proved. For \( i > 0 \) let us prove that the right hand vertical morphisms are isomorphisms.

Let \( F \rightarrow A \) be a graded \( S \)-algebra surjection and let
\[
F_i = F \times F \times \cdots \times F \quad (i+1)-\text{times}
\]
Consider the complex
\[
\lim\left( q \right) \text{Der}_S(-,M) \rightarrow \lim\left( q \right) \text{Der}_S(-,M) \rightarrow \ldots \rightarrow \lim\left( q \right) \text{Der}_S(-,M) \rightarrow
\]
\[
\left( SF/F_i \right)
\]
where the differentials are the alternating sum of group-morphisms
\[
\lim\left( q \right) \text{Der}_S(-,M) \rightarrow \lim\left( q \right) \text{Der}_S(-,M)
\]
\[
\left( SF/F_{i-1} \right) \quad \left( SF/F_i \right)
\]
induced by the projections \( F_i \rightarrow F_{i-1} \). In this situation there
is a Leray spectral sequence given by the term

\[ E^p,q = H^p(\lim(q) \operatorname{Der}_S(-,M)) \leftarrow SF/F. \]

converging to

\[ \lim(\cdot) \operatorname{Der}_S(-,M) = H(\cdot)(S,A,M) \]

For a proof see (2.1.3) in [L1].

Similarly, there is a Leray spectral sequence with

\[ E^p,q = H^p(\lim(q) \operatorname{Der}_S(-,M)) \leftarrow SF/F. \]

converging to

\[ \lim(\cdot) \operatorname{Der}_S(-,M) \leftarrow SF/F. \]

To show that the morphisms

\[ \lim(1) \operatorname{Der}_S(-,M) \rightarrow \lim(1) \operatorname{Der}_S(-,M) \]

are isomorphisms, we use induction on \( i \). If it is an isomorphism for \( i \leq n \) and for every object \( A \) in \( \text{Sg}_{-\text{alg}} \), we conclude that the morphism

\[ E^p,q \rightarrow E^{p',q} \]

is an isomorphism for \( q \leq n \) and every \( p \).

Recall that

\[ E^0,q \subseteq \lim(q) \operatorname{Der}_S(-,M) = H^q(S,F,M) \leftarrow SF/F \]
Hence $E^0_{2,q} = 0$ for $q \geq 1$.

Since $F \in \text{obSGF}$, we get

$$\lim_{\text{SGF}/F} (q) \text{Der}_S(-,M) = 0 \quad \text{for } q \geq 1$$

as well. Since for $r \geq 2$ the differentials of the spectral sequence are of bidegree $(r,1-r)$, and since for $p$ and $q$ given, $E^p_r,q = E^p_r,q$ for some $r$, we easily deduce isomorphisms

$$E^p_\infty,q \to E^p_\infty,q$$

for every $p$ and $q$ with $p+q \leq n+1$. Hence there is an isomorphism

$$\lim_{\text{SGF}/A} (n+1) \text{Der}_S(-,M) \to \lim_{\text{SGF}/A} (n+1) \text{Der}_S(-,M)$$

Q.E.D.

Let $R \to S$ be a graded surjection such that $(\ker \pi)^2 = 0$

It is easy to see that the injection

$$\sigma_0^S(A,A \otimes \ker \pi) \to \sigma^S(A,A \otimes \ker \pi)$$

maps the obstruction $\sigma_0(A)$ onto $\sigma(A)$. For definitions of the obstructions see [L1]. This proves

**Corollary 1.7**

Let $R \to S$ be a graded surjection such that $(\ker \pi)^2 = 0$.

If $A$ is a graded $S$-algebra, then $A$ can be lifted to $R$ iff $A$ can be lifted to $R$ as a graded algebra.

**Remark**

Let $F_A$ be the set of non-equivalent liftings of $A$ to
R and $F_A^0$ the corresponding set of graded liftings. If $A'$ is a graded lifting of $A$ to $R$, then there are isomorphisms and obvious vertical injections fitting into the diagram

\[
\begin{array}{ccc}
F_A & \sim & H^1(S,A,A) \\
\uparrow & & \uparrow \\
F_A^0 & \sim & H^1(S,A,A)
\end{array}
\]

Hence there is a projection

\[ p : F_A \to F_A^0 \]

Now 1.7 can be generalized as follows. Let

\[ \varphi : A \to B \]

be a graded $S$-algebra homomorphism. Assume there are liftings $A'$ and $B'$, not necessarily graded, of $A$ and $B$ such that $\varphi$ is liftable to $R$ with respect to $A'$ and $B'$. Then $\varphi$ admits a graded lifting to $R$ with respect $p(A')$ and $p(B')$. We omit the proof.

Similar results for graded $S$-modules and for graded module morphisms are valid.
CHAPTER 2

Defomation functors and formal moduli.

For the rest of this paper we shall deform only finitely generated algebras.

Let \( \Pi : R \rightarrow R' \) be a surjective ringhomomorphism. If \( (\ker \Pi)^2 = 0 \), then 1.7 say that \( A \) is liftable to \( R \) iff \( A \) is liftable to \( R \) as a graded algebra. We would like to drop the condition \( (\ker \Pi)^2 = 0 \) in 1.7. To do this we shall introduce defomation functors.

Let \( V \) be a noetherian local ring with maximal ideal \( m_V \) and residue field \( k = V/m_V \). Let \( \Gamma \) be the category whose objects are artinian local \( V \)-algebras with residue fields \( k \) and whose morphisms are local \( V \)-homomorphisms. Let \( S \) be a finitely generated \( k \)-algebra and assume that we can find graded liftings \( S_R \) of \( S \) to \( R \) for any \( R \in \text{obl} \) such that for any morphism \( \pi : R \rightarrow R' \) of \( \Gamma \) there is a morphism \( S_R \rightarrow S_{R'} \) with

\[
S_R \otimes_{R} R' \cong S_{R'}
\]

For each \( R \), fix one \( S_R \) with this property and let

\[
\varphi : S \rightarrow A
\]

be a finitely generated graded \( S \)-algebra. Relative to the choice of liftings \( S_R \) we define

\[
\text{Def}^0(A/S,R) = \left\{ \frac{S_R \rightarrow A'}{S_S \rightarrow A} \mid A' \text{ is a graded lifting of } A \text{ to } S_R \right\}/\sim
\]
It is easy to see that $\text{Def}^0(A/S,-)$ is a covariant functor on $\mathbb{L}$ with values in $\text{Set}_S$. This is the graded deformation functor or $A/S$. Correspondingly, we denote by $\text{Def}(A/S,-)$ the non-graded deformation functor of $A/S$.

Recall that a morphism of covariant functors

$$F \rightarrow G$$

on $\mathbb{L}$ is smooth iff the map

$$F(R) \rightarrow F(R') \times G(R)$$

is surjective whenever $R \rightarrow R'$ is surjective. The tangent space $t_F$ of $F$ is defined to be

$$t_F = F(k[\varepsilon])$$

when $k[\varepsilon] \in \text{ob} \mathbb{L}$ is the dual ring of numbers.

**Definition 2.1**

A pro-$\mathbb{L}$ object $R(A/S)$, or just $R(A)$ is called a hull for $\text{Def}(A/S,-)$ if there is a smooth morphism of functors

$$\text{Hom}_{\mathbb{L}}(R(A),-) \rightarrow \text{Def}(A/S,-)$$

on $\mathbb{L}$ which induces an isomorphism on their tangent spaces. $R^0(A)$ is similarly defined as the hull of $\text{Def}^0(A/S,-)$.

By 1.2 and 1.4 we see that

$$\text{Def}(A/S,k[\varepsilon]) = H^1(S,A,A)$$

$$\text{Def}^0(A/S,k[\varepsilon]) = _0H^1(S,A,A)$$

Look at the canonical morphism of functors

$$\text{Def}^0(A/S,-) \rightarrow \text{Def}(A/S,-)$$

and the corresponding $V$-morphism
If this morphism splits we have solved the problem mentioned at the beginning of this paragraph.

In [L1] we find a very general theorem describing these hulls. Following [L1] we notice that since $A$ is a finitely generated $S$-algebra, the group $H^i(S, A, A)$ for a given $i$ is finite as an $A$-module. We pick a countable basis $\{v_j\}$ for $H^i(S, A, A)$ as a $k$-vectorspace and define a topology on $H^i$ in which a basis for the neighbourhoods of zero are those subspaces containing all but a finite number of these $v_j$.

Let

$$H^i = \text{Hom}_k(H^i, k) \quad \text{for } i = 1, 2$$

and let

$$T_A^i, \text{ or just } T^i \quad i = 1, 2$$

be the completion of $\text{Sym}_v(H^i)$ in the topology induced by the topology on $H^i$, i.e. the topology in which a basis for the neighbourhoods of zero are those ideals containing some power of the maximal ideal and intersecting $H^i$ in an open subspace. If $H^i$ is a finite $k$-vectorspace then $T^i$ is a convergent power series algebra on $\mathfrak{t}$. The result we need is the following. See (4.2.4.) in [L1].

**Theorem 2.2**

There is a morphism of complete local rings

$$\sigma = \sigma(A) : T^2 \longrightarrow T^1$$

such that

$$R(A) \sim T^1 \overset{\wedge}{\otimes} T^2 \overset{\wedge}{v}$$
Short remark on the proof.

To simplify ideas, assume $V = k$ and $H^1(S, k, k)$ finite as a $k$-vector space. Let $\mathcal{L}_n \subset \mathcal{L}$ be the full subcategory of $\mathcal{L}$ consisting of objects $R$ satisfying $m^n_R = 0$. Put

$$T_n^1 = T_n^1 / m^n T_n$$

and $R_2 = T_2^1$. If $R \in \text{obj}_{\mathcal{L}_2}$, then by 1.2

$$\text{Def}(A/S, R) = H^1(S, k, A) \otimes_{m_R} \text{Hom}_k^c(T^1, R) = \text{Hom}_k^c(R_2, R)$$

Hence $R_2$ represents the functor $\text{Def}(A/S, -)$ on $\mathcal{L}_2$. Let $\mathcal{A}_2$ be the universal lifting of $A$ to $S_{R_2}$. If $\sigma_2 : T_2^2 \rightarrow k \rightarrow T_2^1$ is the composition,

then

$$R_2 = T_2^1 = T_2^1 \otimes_{T_2^2} k$$

By induction we shall assume that

$$\sigma_i : T_i^2 \rightarrow T_i^1 \quad ;2 \leq i \leq n-1$$

are constructed such that

$$R_i = T_i^1 \otimes_{T_i^2} k$$

and such that $A_2$ is liftable to $S_{R_1}$. Consider the following diagram

\[
\begin{array}{ccc}
T_2^2 & \xrightarrow{\sigma_n} & T_2^1 \\
\downarrow & & \downarrow \\
T_{n-1}^2 & \xrightarrow{\sigma_{n-1}} & T_{n-1}^1
\end{array}
\]

We shall try to construct $\sigma_n : T_n^2 \rightarrow T_n^1$ such that the diagram
above commutes. In fact it is enough to define $\sigma_n$ on $H^{2*}$ as a $k$-linear map. Let

$$R'_n = T'_n / \pi^{-1}(\alpha)$$

Then the diagram

$$
\begin{array}{ccc}
T'_n & \rightarrow & R'_n \\
\pi & \downarrow & \pi' \\
T_{n-1} & \rightarrow & R_{n-1}
\end{array}
$$

is commutative and $\ker \pi'$ is a $k$-module via $T'_n \rightarrow k$.

Let $A_{n-1}$ be any lifting of $A_2$ to $S_{R_{n-1}}$. The obstruction for lifting $A_{n-1}$ to $S_{R'_n}$ is given by

$$\sigma(A_{n-1}) \in H^2(S_{R_{n-1}}, A_{n-1}, A_{n-1} \otimes \ker \pi') \sim H^2(S, A, A) \otimes \ker \pi' \cong \text{Hom}(H^{2*}, \ker \pi')$$

Let $\sigma_n$ be any $k$-linear map fitting into the commutative diagram

$$
\begin{array}{ccc}
H^{2*} & \rightarrow & \sigma(A_{n-1}) \\
\sigma_n & \downarrow & \sigma_n \\
T'_n & \rightarrow & R'_n \supset \ker \pi'
\end{array}
$$

and put

$$R_n = R'_n / \text{im} \sigma(A_{n-1})$$

Thus killing the obstruction of lifting, we conclude that $A_{n-1}$ is liftable to $S_{R_n}$. Put

$$R(A) = \lim R_n \quad \text{and} \quad \sigma = \lim \sigma_n.$$
Lauald proves that this $R(A)$ is a hull for $\text{Def}(A/S, -)$.

If $V \rightarrow k$, just as in the general step, we let $V_2$ be the largest quotient of $V/m_{V^2}$ to which $S \rightarrow A$ is liftable. Any lifting $S_{V^2} \rightarrow A_2$ may serve as a zero point for the isomorphism

$$\text{Def}(A/S, R) \cong H^1(S, A, A) \otimes m_R$$

where $R \in \text{ob}_2$. For the rest we may proceed as before.

**Corollary 2.2.a**

Let $V$ be a regular local ring such that $S \rightarrow A$ is liftable to $V/m_{V^2}$. Then $R(A)$ is regular iff the composition

$$H^2^* \rightarrow T^2 \sigma \rightarrow T^1$$

is zero.

**Proof.**

It follows from the fact that the image of the composition is in $m_{T^1}^2$. Q.E.D.

Similar results are true for $R^0(A)$. If

$$o_{H^i}^* = \text{Hom}_k(o_{H^i}(S, A, A), k) \quad i=1,2$$

and

$$o_{T^i}^* \text{ A, or just } o_{T^i}^* \quad \text{ for } i=1,2$$

is the completion of $\text{Sym}_V(o_{H^i}^*)$ in the corresponding topology, then there is a morphism of complete local rings

$$\sigma_o = \sigma_o(A): o_{T^2} \rightarrow o_{T^1}$$

such that

$$R^0(A) \cong o_{T^1} \wedge V_{o_{T^2}}$$

The canonical injections
\[ oH^i = oH^i(S,A,A) \rightarrow H^i(S,A,A) = H^i \]

induces surjections

\[ T^i \rightarrow oT^i \quad \text{for } i = 1,2 \]

These surjections can be assumed to fit nicely into a commutative diagram

\[
\begin{array}{ccc}
T^2 & \rightarrow & oT^2 \\
\sigma & \downarrow & \sigma_o \\
T^1 & \rightarrow & oT^1
\end{array}
\]

in such a way that the induced morphism

\[ R(A) \rightarrow R^0(A) \]

makes the diagram

\[
\begin{array}{ccc}
\text{Def}^0(A/S,-) & \rightarrow & \text{Def}(A/S,-) \\
\uparrow & & \uparrow \\
\text{Hom}(R^0(A),-) & \rightarrow & \text{Hom}(R(A),-)
\end{array}
\]

commutative.

We shall only sketch a proof of this commutativity since we will not use it much. We need an easy lemma, see (4.2.3) in [L1].

**Lemma 2.3.**

Consider the commutative diagram

\[
\begin{array}{ccc}
R_1 & \rightarrow & R_2 \\
\pi_1 & \downarrow & \pi_2 \\
R_1 & \rightarrow & R_2 \\
& & \downarrow k
\end{array}
\]
whose objects and morphisms are in $\mathbb{L}$. Assume $\pi_1$ and $\pi_2$ surjective and $(\ker \pi_1)^2 = (\ker \pi_2)^2 = 0$. If $A_1$ is a lifting of $A$ to $S_{R_1}$, and $A_2 = A_2 \otimes_{R_2} R_2$, then

$$H^2(S_{R_1}, A_1, A_1 \otimes \ker \pi_1) \rightarrow H^2(S_{R_2}, A_2, A_2 \otimes \ker \pi_2)$$

maps the obstruction $\sigma(A_1)$ onto $\sigma(A_2)$.

**Proof of the commutativity.**

As in the "proof" of 2.2, let us assume $V = k$ and $H^1(S, A, A)$ finite as a $k$-vectorspace. We constructed $R_n$ and $\sigma_n$ in such a way that

$$R(A) = \lim R_n \quad \sigma = \lim \sigma_n$$

In the graded case we shall use the notations

$$R^0(A) = \lim^0 R_n \quad \sigma_0 = \lim (\sigma_0)_n$$

Now $R_2$ and $^0R_2$ represents this deformation functors on $\mathbb{L}_2$. If $A_2$ is the universal lifting of $A$ to $S_{R_2}$, we easily see that $A_2 \otimes _{R_2}^0R_2$ is the graded universal lifting to $S_{R_2}$.

Let $n \geq 3$ and let $A_{n-1}$ be a lifting of $A_2$ to $S_{R_{n-1}}$. By induction we may assume the commutativity of

$$\xymatrix{T^2_{n-1} \ar[r] \ar[d]^{\sigma_{n-1}} & ^0T^2_{n-1} \ar[d]^{(\sigma_0)_{n-1}} \\
^1T_{n-1} \ar[r] & ^0T^1_{n-1}}$$

and that $A_{n-1} \otimes _{R_{n-1}}^0R_{n-1}$ is a graded lifting of $A_2 \otimes _{R_2}^0R_2$. 
By 2.3 a commutative diagram

\[
\begin{align*}
H^2 \ ightarrow & \ H^2 \ + \\
\sigma_n & \downarrow \sigma_n \\
T_n & \rightarrow \ oT_n
\end{align*}
\]

is found, hence then is a commutative diagram

\[
\begin{align*}
R_n & \rightarrow \ oR_n \\
\pi_n & \downarrow \pi_n \\
R_{n-1} & \rightarrow \ oR_{n-1}
\end{align*}
\]

Since

\[
\ker \pi_n \rightarrow \ ker \ o\pi_n
\]

is a surjective map of $k$-vector spaces, we deduce from the surjectivity of

\[
\begin{align*}
H^1(S,A,A) \otimes \ker \pi_n & \rightarrow \ H^1(S,A,A) \otimes \ker o\pi_n \\
\end{align*}
\]

(using 1.2 and 1.7) that there is a lifting $A_n$ and $A_{n-1}$ to $S_{R_n}$ such that $A_n \otimes oR_n$ is a graded lifting of $A_{n-1} \otimes oR_{n-1}$.

The case $V \nmid k$ makes no trouble.

Q.E.D.

From this we get
Proposition 2.4

Let \( V \) be a regular local ring such that \( S \to A \) is liftable to \( V/m_V^2 \). Then

i) If \( R(A) \) is a regular local ring, so is \( R^0(A) \).

ii) If \( R^0(A) \) is regular, then the morphism

\[ R(A) \to R^0(A) \]

splits

Proof.

i) follows from the commutativity of the diagram

\[
\begin{array}{ccc}
H^2(-) & \to & oH^2(-) \\
\downarrow & & \downarrow \\
T^2 & \to & oT^2 \\
\sigma & \downarrow & \sigma_o \\
T^1 & \to & oT^1
\end{array}
\]

using 2.2.a

If \( R^0(A) \) is regular, then \( oT^1 = R^0(A) \).

The obvious surjection

\[ H^1 = H^1(S,A,A) = H^1(S,A,A) \to oH^1(S,A,A) = oH^1 \]

induces an injection

\[ oT^1 \to T^1 \]

which defines a one-sided inverse of \( R(A) \to R^0(A) \).

The surjections

\[ H^i \to oH^i \]

for \( i = 1,2 \)

induce morphisms
If the corresponding diagram

\[
\begin{array}{ccc}
0_{T^2} & \rightarrow & T^2 \\
\sigma & \downarrow & \sigma \\
0_{T^1} & \rightarrow & T^1
\end{array}
\]

commutes, then \( R(A) \rightarrow R^0(A) \) splits. In general there seem to be no reasons for this diagram to commute. However imposing some rather natural conditions on the graded algebra \( A \), the commutativity can be proved.

**Definition 2.5**

We say that \( S \rightarrow A \) has negative grading (resp. positive grading) if

\[
\bigwedge^1(S, A, A) = 0 \quad \text{for} \quad \nu > 0
\]

(resp. \[
\bigwedge^1(S, A, A) = 0 \quad \text{for} \quad \nu < 0
\]

If \( A \) has positive or negative grading, then the diagram above commutes, proving

**Theorem 2.6**

If \( S \rightarrow A \) has negative or positive grading, then

\[
R(A) \rightarrow R^0(A)
\]

splits as a local \( V \)-homomorphism.

In the same direction we have the following more general result.

**Theorem 2.7**

Assume \( S \rightarrow A \) has negative (resp. positive) grading and put \( B = A[T] \) with \( \deg T = 1 \) (resp \( \deg T = -1 \)). Then there is
a $V$-isomorphism

$$R^0(B) \cong R(A)$$

We shall need some preparations.

Let $A$ and $B$ be graded $S$-algebras and

$$\Psi : B \to A$$

an $S$-algebra homomorphism, not necessarily graded. For every $i \geq 0$, $\Psi$ induces maps

$$\Psi^i : H^i(S, A, A) \to H^i(S, B, A)$$

$$\Psi_i : H^i(S, B, B) \to H^i(S, B, A)$$

Let $\Psi_i/0$ be the composed map

$$H^i(S, B, B) \to H^i(S, B, B) \to H^i(S, B, A)$$

Lemma 2.8

If $\Psi^i$ and $\Psi_i/0$ are isomorphisms for $i = 1$ and injections for $i = 2$, then there is a local $V$-isomorphism

$$R(A) \cong R^0(B)$$

Remark 2.8

Let $\pi : R \to R'$ be a surjection in $\mathbb{L}$ such that $\ker \pi$ is a $k$-module via $R \to k$. Look at

\[
\begin{array}{ccc}
R & \to & S_R \\
\downarrow \pi & & \downarrow \Psi \\
R' & \to & S_{R'} \\
\downarrow & & \downarrow \Psi' \\
k & \to & S \\
\end{array}
\]

where $A'$ and $\Psi'$ are liftings to $S_{R'}$ and $B'$ is a graded
lifting to $S_{R'}$. Consider the diagram

$$
\begin{array}{ccc}
H^2(S,B,B) \otimes \ker \pi & \longrightarrow & H^2(S,B,A) \otimes \ker \pi \\
\downarrow & & \downarrow \\
H^2(S,A,A) \otimes \ker \pi & \longrightarrow & H^2(S,B,A) \otimes \ker \pi \\
\end{array}
$$

By [L1], the obstructions for deforming $A'$ and $B'$ respectively map on the same element in $H^2(S,B,A) \otimes \ker \pi$.

**Proof of 2.8**

We shall use the notation

$$T^i_Y$$

for the completion of

$$\text{Sym}_Y(H^i(S,B,A)^*)$$

The morphisms

$$\psi^i : H^i(S,A,A) \longrightarrow H^i(S,B,A)$$

$$\psi^0 : H^i(S,B,B) \longrightarrow H^i(S,B,A)$$

induce morphisms

$$T^i_A \leftarrow T^i_Y$$

$$oT^i_B \leftarrow T^i_Y$$

which by the "proof" of 2.2 and by 2.8 a fit into a commutative diagram
The horizontal maps are surjections and isomorphisms by the assumptions of 2.8. Q.E.D

Remark 2.8 b

If $\psi^i$ is an isomorphism for $i = 1$ and an injection for $i = 2$ the morphism

$$(\psi^1)^{-1}\psi_1 : H^1(S, B, B) \to H^1(S, A, A)$$

induces a morphism

$R(A) \to R(B)$

Now we turn to the proofs of 2.6 and 2.7

Proof of 2.7

Let

$\psi : B \to A$

be the composition

$B = A[T] \to A[T]/(T-1) \cong A$

and let $j$ be the canonical injection

$j : A \to B$

Let $M$ be any $B$-module. $j$ induces maps

$\psi^i_M : H^i(S, B, M) \to H^i(S, A, M)$

Using the exact sequence

$$\to H^i(A, B, M) \to H^i(S, B, M) \to H^i(S, A, M) \to H^{i+1}(A, B, M) \to$$

and the fact that

$H^i(A, B, M) = 0$ for $i \geq 1$
we deduce that $j^i_M$ are isomorphisms for $i \geq 1$. However
$\psi^i$ are the inverse maps of $j^i_A$ for $i = 1, 2$.
Hence by 2.8 it is enough to prove that

$$\psi_{i/o} : oH^i(S, B, B) \longrightarrow H^i(S, B, A)$$

is an isomorphism for $i = 1$ and an injection for $i = 2$.
Look at the diagram

$$
\begin{array}{ccc}
 oH^i(S, B, B) & \longrightarrow & H^i(S, B, B) \\
 \downarrow j_B^i & & \downarrow j_B^i \\
 oH^i(S, A, B) & \longrightarrow & H^i(S, A, B)
\end{array}
$$

$\|$

$$
\begin{array}{ccc}
 H^i(S, A, A) \otimes k[T] & \longrightarrow & H^i(S, A, A)
\end{array}
$$

where the lower horizontal map is induced by sending $T$ to $1$. If $\deg T = 1$ and if $i \geq 1$, $(\psi^i)^{-1} \psi_{i/o}$ is given by the
composition

$$
\begin{array}{cc}
oH^i(S, B, B) \cong \bigoplus_{\nu = -\infty} \nu H^i(S, A, A)T^{-\nu} \cong \bigoplus_{\nu = -\infty} \nu H^i(S, A, A) \longrightarrow H^i(S, A, A)
\end{array}
$$

which is an injection for all $i \geq 1$. If $A$ has negative
grading, then by definition $\psi_{1/o}$ is an isomorphism. The
case $\deg T = -1$ is similar. Q.E.D.

Proof of 2.6

Let

$$\phi : B \longrightarrow B/(T) = A$$

be the canonical surjection. Then

$$\phi^i : oH^i(S, A, A) \longrightarrow oH^i(S, B, A)$$
are isomorphisms for $i \geq 1$. By 2.8 b there is a morphism $
abla^0(A) \rightarrow \nabla^0(B)$ deduced from the commutative diagram

$$
\begin{array}{ccc}
o_{T_A}^2 & \rightarrow & o_{T_B}^2 \\
\sigma_o(A) \downarrow & & \downarrow \sigma_o(B) \\
o_{T_A}^1 & \rightarrow & o_{T_B}^1
\end{array}
$$

The horizontal maps are induced by $(\phi^i)^{-1} \circ \phi i$. Moreover by 2.7 and its proof, the isomorphism

$$\nabla^0(B) \cong \nabla(A)$$

is deduced from the commutative diagram

$$
\begin{array}{ccc}
o_{T_B}^2 & \leftarrow & o_{T_A}^2 \\
\sigma_o(B) \downarrow & & \downarrow \sigma(A) \\
o_{T_B}^1 & \leftarrow & o_{T_A}^1
\end{array}
$$

The horizontal maps are induced by

$$(\psi^i)^{-1} o(\psi i/o) : oH^i(S,B,B) \rightarrow H^i(S,A,A)$$

However if $\deg T = 1$, this morphism is given by

$$oH^i(S,B,B) \sim \bigoplus_{\nu=-\infty} H^i(S,A,A) \rightarrow H^i(S,A,A)$$

which splits. Using a one-sided inverse, i.e. a projection for $i = 2$, a commutative diagram

$$
\begin{array}{ccc}
o_{T_B}^2 & \rightarrow & o_{T_A}^2 \\
\sigma_o(B) \downarrow & & \downarrow \sigma(A) \\
o_{T_B}^1 & \sim & o_{T_A}^1
\end{array}
$$
is found, inducing the isomorphism \( R^0(B) \cong R(A) \). We claim that the composed map

\[
R^0(A) \longrightarrow R^0(B) \cong R(A)
\]

is a one-sided inverse of \( R(A) \longrightarrow R^0(A) \). This is trivial if we look at the diagram

\[
\begin{array}{ccc}
\circ_{T_A}^2 & \longrightarrow & \circ_{T_B}^2 \\
\downarrow \sigma(A) & \circ & \downarrow \sigma(A) \\
\circ_{T_A}^1 & \longrightarrow & \circ_{T_B}^1 \\
\end{array}
\]

The composition of the horizontal maps are induced by the obvious projections

\[
\circ_i^H(S, A, A) \leftarrow H_i^i(S, A, A)
\]

since \((\phi^i)^{-1} \circ \phi_i \) are given by

\[
\circ_i^H(S, B, B) \cong \bigoplus_{v=-\infty}^{\infty} H_i^i(S, A, A)T^{-v} \longrightarrow \circ_i^H(S, A, A)
\]

sending \( T \) to 0. The case \( \deg T = -1 \) is similarly treated.

Q.E.D.

Theorem 2.7 can be generalized in the following way. Let

\[
C = A[T]_T = B_T
\]

be the localization of \( B \) in the multiplicative system \( \{1, T, T^2, \ldots\} \) and put \( \deg T = 1 \). Then for any finitely generated \( S \)-algebra \( A \), then is an isomorphism

\[
R^0(C) \cong R(A)
\]

We omit details of a proof.
The conditions of negative and positive grading on $S \rightarrow A$ are only reasonable if the graded ring $S$ sits in degree zero. However, if $S \rightarrow A$ is any graded morphism and $S$ is smooth, then

$$R(A) \rightarrow R^0(A)$$

splits if $S_0 \rightarrow A$ has negative or positive grading. In being more precise we shall assume that the "choice" of the liftings of $S_0$ and $S$ are compatible, i.e. for any $R \in \text{obl}$, there is a morphism $(S_0)_R \rightarrow S_R$ such that if $R \rightarrow R'$ is in $\Pi$, then there is a commutative diagram

$$
\begin{array}{ccc}
(S_0)_R & \rightarrow & S_R \\
\downarrow & & \downarrow \\
(S_0)_{R'} & \rightarrow & S_{R'}
\end{array}
$$

Then the maps

$$\text{Def}^0(A/S, -) \rightarrow \text{Def}^0(A/S_0, -)$$
$$\text{Def}(A/S, -) \rightarrow \text{Def}(A/S_0, -)$$

are well defined and they are easily seen to be smooth. Therefore the morphisms

$$R^0(A/S) \leftarrow R^0(A/S_0)$$
$$R(A/S) \leftarrow R(A/S_0)$$

are still smooth. These maps fit into a commutative diagram

$$
\begin{array}{ccc}
R^0(A/S) & \leftarrow & R^0(A/S_0) \\
\uparrow & & \uparrow \\
R(A/S) & \leftarrow & R(A/S_0)
\end{array}
$$
The right hand vertical morphism splits because \( S_0 \to A \) has negative or positive grading. By definition of smoothness the left hand vertical morphism also splits.
CHAPTER 3

Relations to projective geometry.

As we know the graded theory of algebras are closely related to projective geometry. In what follows we shall compare the groups $\nu H^i(S,A,M)$ with $A^i(S,X,\tilde{M}(\nu))$ when $X = \text{Proj}(A)$. Moreover if

$$\varphi : B \rightarrow A$$

is a surjective graded morphism and

$$f : \text{Proj}(A) \rightarrow \text{Proj}(B)$$

is the induced embedding, we shall relate the groups $\nu H^i(B,A,M)$ to $A^i(S,f,\tilde{M}(\nu))$.

Let $X$ be any $S$-scheme, $M$ any quasicoherent $O_X$-Module and let $f : X \rightarrow Y$ be a morphism of $S$-schemes. Then there are groups

$$A^i(S,X,M)$$

and

$$A^i(S,f,M)$$

for every $i \geq 0$. Using [L1] we shall summarize some properties needed in the sequel.

i) (3.1.12) in [L1] states that $A^i(S,X,M)$ is the abutment of a spectral sequence given by the term

$$E^2_{pq} = H^p(X,A^q(S,M))$$

If $U = \text{Spec}(A)$ is an open affine subscheme of $X$, the $O_X$-Module $A^q(S,M)$, or just $A^q(M)$, is given by

$$A^q(M)(U) = H^q(S,A,M(U))$$
If $X$ is affine, say $X = \text{Spec}(A)$, and $M = \tilde{M}$ for some $A$-module $M$, we deduce

$$A^i(S, X, M) = H^i(S, A, M)$$

If $X$ is $S$-smooth, we find

$$A^i(S, X, M) = H^i(X, \Theta_X \otimes_M S)$$

where $\Theta_X = A^0(0_X)$ is the sheaf of $S$-derivations.

ii) By (3.1.14) in [L1] $A^i(S, f, M)$ is the abutment of the spectral sequence given by

$$E^p_{ij} = H^p(Y, A^q(f, M))$$

If $V = \text{Spec}(B)$ is any open affine subscheme of $Y$, then by definition

$$A^q(f, M)(V) = A^q(B, f^{-1}(V), M)$$

Therefore if $f$ is affine, say $f^{-1}(V) = \text{Spec}(A)$,

$$A^q(B, f^{-1}(V), M) = H^q(B, A, H^0(f^{-1}(V), M))$$

iii) Let $Z \subseteq X$ be locally closed. By (3.1.16) there is an exact sequence

$$A^n(S, X, M) \longrightarrow A^n(S, X, M) \longrightarrow A^n(S, X - Z, M) \longrightarrow A^{n+1}(S, X, M)$$

where the groups $A^n(S, X, M)$ is the abutment of a spectral sequence given by the term

$$E^p_{ij} = A^p(S, X, H^n_M(M))$$

If $X = \text{Spec}(A)$ and $Z = V(I)$ for a suitable ideal $I \subseteq A$ we write

$$H^n_I(S, A, M) = A^n(S, X, \tilde{M})$$

iv) Let $f : X \rightarrow Y$ be an affine morphism of $S$-schemes.

By (3.2.3) there is a long exact sequence
Let $S$ be noetherian and let $A$ and $B$ be finitely generated, positively graded $S$-algebras generated by its elements of degree 1. Assume $A_0 = B_0 = S$. Let 
\[ \varphi : B \to A \]
be a surjective graded $S$-algebra morphism and let
\[ f : X = \text{Proj}(A) \to \text{Proj}(B) = Y \]
be the corresponding embedding. Put
\[ m = \bigoplus_{n=1}^{\infty} A_n \quad \text{and} \quad X' = \text{Spec}(A) - V(m) \]
Let
\[ \pi : X' \to X \]
be the obvious morphism. $\pi$ is an affine smooth surjection.

If $M$ is a graded $A$-module, we shall denote by $M_a$ the localization of $M$ in $\{1, a, a^2, \ldots\}$. Let $M(a)$ be the homogeneous piece of $M_a$ of degree zero.

Let $b \in B$ such that $a = \varphi(b)$. Since
\[ B(b) \to B_b \]
is flat, a theorem from [A] gives the isomorphism
\[ H^q(B(b), A(a), M_a) \cong H^q(B_b, A(a) \otimes_{B(b)} B_b, M_a) \]
However
\[ A(a) \otimes_{B(b)} B_b \cong A_a \]
Therefore
\[ H^q(B(b), A(a), M_a) \simeq H^q(B, A, M) \simeq H^q(B, A, M)_a \]
Hence
\[ H^q(B(b), A(a), M(a)) \simeq H^q(B, A, M)(a) \]
Put
\[ D_+(b) = \text{Spec}(B(b)) \subseteq Y \]
Then by (ii)
\[ A^q(f, \tilde{M}(\nu))(D_+(b)) = H^q(B(b), A(a), M(\nu)(a)) \]
proving
\[ A^q(f, \tilde{M}(\nu)) \simeq H^q(B, A, M)(\nu) \]
Using (i) we find
\[ A^q(B, \tilde{M})(D(a)) = H^q(B, A, M) = H^q(B, A, M)_a \]
Therefore
\[ A^q(B, \tilde{M}) \simeq H^q(B, A, M) \]
This proves
\[ \pi_*(A^q(B, \tilde{M})) \simeq \bigvee A^q(f, \tilde{M}(\nu)) \]

**Lemma 3.1**

With notations as above there is an isomorphism
\[ A^i(S, f, \tilde{M}(\nu)) \simeq \bigvee A^i(B, X', \tilde{M}) \]
where \( \bigvee A^i(B, X', \tilde{M}) \) is the homogeneous piece of \( A^i(B, X', \tilde{M}) \) of degree \( \nu \).

**Proof.**

Going back to the definitions of \( A^i(S, f, \tilde{M}(\nu)) \) and
\[ A^i(B,X',\overline{M}) \text{ in [L1] we deduce a morphism} \]
\[ A^i(S,f,\overline{M}(\nu)) \longrightarrow \sqrt{A^i(B,X',\overline{M})} \]

The corresponding morphism of spectral sequences
\[ H^p(X,A^q(f,\overline{M}(\nu))) \longrightarrow \sqrt{H^p(X',A^q(B,\overline{M}))} \cong \sqrt{H^p(X,A^q(B,\overline{M}))} \]
is an isomorphism for every \( p \) and \( q \)

**Q.E.D.**

**Theorem 3.2**

If \( \varphi : B \rightarrow A \) is surjective and if
\[
\text{depth}_mA \geq n
\]
then the morphisms
\[ \sqrt{H^i(B,A,M)} \longrightarrow A^i(S,f,\overline{M}(\nu)) \]
are isomorphisms for \( i < n \) and injections for \( i = n \)

**Proof**

By iii) there is a long exact sequence
\[ \rightarrow H^i_m(B,A,M) \rightarrow H^i(B,A,M) \rightarrow A^i(B,X',\overline{M}) \rightarrow H^{i+1}_m(B,A,M) \rightarrow \]
Since \( \text{depth}_mA \geq n \), we conclude that
\[ H^q_m(M) = 0 \quad \text{for} \quad q \leq n-1 \]
Moreover \( H^0(B,A,-) = 0 \) since \( \varphi \) is surjective. By the spectral sequence of iii) we deduce
\[ H^i_m(B,A,M) = 0 \quad \text{for} \quad i \leq n \]

**Q.E.D.**

**Corollary 3.3**

If \( \text{depth}_mA \geq 2 \)
\[ \sqrt{H^1(B,A,A)} \cong A^1(S,f,0_X(\nu)) \]
are isomorphisms and injections respectively.

Let us apply this result to the case \( S = k, k \) a field. We denote by

\[
\text{Hilb}_f(-)
\]

the local Hilbert functor relative to \( Y \) at \( f \), defined on the category \( \mathfrak{M} \). (See the beginning of chapter 2 and use \( V = k \)). Let

\[
\text{Def}^0(\varphi, -)
\]

by the functor \( \text{Def}^0(A/B, -) \) defined in chapter 2 using trivial liftings of \( B \).

**Corollary 3.4**

If \( \text{depth}_m A \geq 2 \), then there is an isomorphism of functor

\[
\text{Def}^0(\varphi, -) \sim \text{Hilb}_f(-)
\]

on \( \mathfrak{M} \).

**Proof**

Both functors are prorepresentable. By (2.2) \( \text{Def}^0(\varphi, -) \) is prorepresented by

\[
\text{Sym}(\varpi^1(B, A, A)^* \wedge \otimes k
\]

\[
\text{Sym}(\varpi^2(B, A, A)^*)
\]

Using (5.1.1) in [L1], \( \text{Hilb}_f(-) \) is prorepresented by the object

\[
\text{Sym}(A^1(f, O_X)^* \wedge \otimes k
\]

\[
\text{Sym}(A^2(f, O_X)^*)
\]
The natural morphism of functors
\[ \text{Def}^0(\varphi, -) \to \text{Hilb}_t(-) \]
corresponds to a morphism between their prorepresenting objects. This is nothing but the morphism induced by the natural maps in (3.2).

Q.E.D.

Assume B to be k-free and
\[ f : X \subset \mathbb{P}^n_k \]
to be the induced embedding. In this case Hilb_r(\cdot) is also denoted by Hilb_X(\cdot). Both Hilb_X(\cdot) and Def^0(\varphi, -) are easily defined on \( \mathfrak{A} \) for \( V \) arbitrary, and by the same arguments as before there is an isomorphism of functors
\[ \text{Def}^0(\varphi, -) \cong \text{Hilb}_X(\cdot) \]
on \( \mathfrak{A} \) whenever \( \text{depth} \ A \geq 2 \). Even if \( \text{depth} \ A \geq 1 \) we deduce this isomorphism in some cases. In fact, the sequence
\[ 0 \to oH^1(B, A, A) \to A^1(k, f, O_X) \to oH^1(B, A, H^1_m(A)) \to oH^2(B, A, A) \to A^2(k, f, O_X) \]
is exact. The isomorphism therefore follows from
\[ oH^1(B, A, H^1_m(A)) = 0 \]
Recall that if \( I = \ker \varphi \subset B \)
\[ oH^1(B, A, H^1_m(A)) = o\text{Hom}_A(I/I^2, H^1_m(A)) \]
Furthermore \( X \subset \mathbb{P}^n_k = P \) and if \( n \geq 2 \)
\[ H^1_m(A) \cong \mu H^1(P, \mathbb{F}(v)) \]
If we define \( c \) by
\[ c = \max \{ v | H^1(P, \mathbb{F}(v)) \neq 0 \} \]
and

\[ s = \min \{ \deg f_i \mid \{ f_1, \ldots, f_r \} \text{ is a minimal set of generators of } I \} \]

then

\[ \partial H^i(B, A, H^1_m(A)) = 0 \quad \text{for } c < s \]

In [E] we find more or less a direct proof of (3.4).

So far we have concentrated on deformations of embeddings. One may ask for the relationship between the groups

\[ \bigwedge^H_i(S, A, M) \]

and

\[ A^i(S, X, \tilde{M}(\nu)) \]

This is given by our next theorem

**Theorem 3.5**

There are canonical morphisms

\[ \bigwedge^H_i(S, A, M) \longrightarrow A^i(S, X, \tilde{M}(\nu)) \]

for any \( i \geq 0 \) and any \( \nu \). If \( n \geq 1 \) and if \( \text{depth}_M \geq n+2 \), then the morphisms above are bijective for \( 1 \leq i < n \) and injective for \( i = n \).

**Proof.**

Consider the following two exact sequences

\[ \rightarrow H^i_m(S, A, M) \longrightarrow H^i(S, A, M) \longrightarrow A^i(S, X', \tilde{M}) \longrightarrow H^{i+1}_m(S, A, M) \longrightarrow \]

\[ \rightarrow A^i(S, \Pi, \tilde{M}) \longrightarrow A^i(S, X', \tilde{M}) \longrightarrow A^i(S, X, \Pi \ast \tilde{M}) \longrightarrow A^{i+1}(S, \Pi, \tilde{M}) \]

with

\[ \Pi : X' = \text{Spec}(A) - V(m) \longrightarrow X = \text{Proj}(A) \]
as before. The spectral sequence given by
\[ E^{p,q}_2 = H^p(X, \mathbb{A}^q(n, \tilde{M})) \]
converges to
\[ A^{p+q}(S, \pi, M) \]
\(A^q(n, \tilde{M})\) is defined by
\[ A^q(n, \tilde{M})(D(a)) = A^q(A(a), A_a, M_a) \]
and it is easy to see that
\[ A^q(n, \tilde{M}) = \begin{cases} 0 & q \neq 0 \\ \pi_\ast \tilde{M} & q = 0 \end{cases} \]
Since depth \( M \geq n+2 \) then
\[ A^i(S, \pi, M) = H^i(X, \pi_\ast \tilde{M}) = \oplus H^i(X, M(\nu)) = 0 \quad \text{for} \ 1 \leq i \leq n \]
Furthermore
\[ H^i_M(M) = 0 \quad \text{for} \ i \leq n+1 \]
implying that
\[ H^i_M(S, A, M) = 0 \quad \text{for} \ i \leq n+1 \]
The theorem now follows from the two exact sequences stated at the beginning of this proof. \( Q.E.D. \)

**Corollary 3.6.**

If depth \( A \geq 3 \) and
\[ A^1(S, X, 0_X(\nu)) = 0 \]
for every \( \nu \), then
\[ H^1(S, A, A) = 0 \]
In the smooth case
\[ A^1(S, X, O_X(\nu)) \cong H^1(X, \theta_X(\nu)) \]
and (3.6) reduces to a rigidity theorem of Schlessinger; see (2.2.6) in [K,L]. See also [SV].

**Corollary 3.7**

If \( \text{depth}_m A \geq 4 \) and
\[ A^2(S, X, O_X(\nu)) = 0 \]
for every \( \nu \), then
\[ H^2(S, A, A) = 0 \]
If \( X \) has only a finite number of nonsmooth points, then
\[ H^1(X, A^1(O_X(\nu))) = 0 \]
Moreover if the non-smooth points are complete intersections
\[ H^0(X, A^2(O_X(\nu))) = 0 \]

In this case we conclude
\[ A^2(S, X, O_X(\nu)) \cong H^2(X, \theta_X(\nu)) \]

We will end this chapter by proving a geometric variant of (2.7) due to Pinkham [P]. We also need (3.4).

Let \( R \) be \( k \)-free and \( \varphi : R \to A \) be surjective, corresponding to \( X = \text{Proj}(A) \subseteq \mathbb{P}^n_k \). Look at the diagram

\[
\begin{array}{ccc}
R[T] & \rightarrow & R \\
\downarrow \varphi & & \downarrow \varphi \\
B = A[T] & \rightarrow & A
\end{array}
\]

where \( \bar{\varphi} = \varphi \otimes \text{id}_k[T] \) and where the horizontal maps are induced by sending \( T \) to 1. Put \( \deg T = 1 \).
Clearly
\[ \text{Def}^0(\overline{\sigma},-) \rightarrow \text{Def}^0(B/k,-) \]
is smooth. Hence
\[ R^0(B) \rightarrow R^0(\overline{\sigma}) \]
is smooth. If \( A \) has negative grading
\[ R(A) \sim R^0(B) \]
The composition
\[ R(A) \sim R^0(B) \rightarrow R^0(\overline{\sigma}) \]
is therefore smooth.
Moreover if depth \( A \geq 1 \) then depth \( B \geq 2 \). Using (3.4) we find
\[ \text{Def}^0(\overline{\sigma},-) \sim \text{Hilb}_X(-) \]
whenever \( X = \text{Proj}(B) \). This proves
Theorem 3.8
Let \( X \) be a closed subscheme of \( \mathbb{P}_k^n \) and let \( A \) be its minimal cone. If
\[ X = \text{Proj}(A[T]) \]
is its projective cone in \( \mathbb{P}_k^{n+1} \) and if \( A \) has negative grading, then there is a smooth morphism of functors
\[ \text{Hilb}_X(-) \rightarrow \text{Def}(A/k,-) \]
on \( \mathcal{I} \) (\( V \) arbitrary)
CHAPTER 4

Positive and negative grading

In this paragraph we shall see that if

\[ X \subseteq \mathbb{P}_S^N \]

is closed and satisfies some weak conditions, then after a suitable twisting the minimal cone of the corresponding embedding will have positive or negative grading.

Suppose \( S \) noetherian and let

\[ A = \bigoplus_{\nu=0}^{\infty} A_\nu \]

be a graded \( A_0 = S \) algebra of finite type, generated by \( A_1 \). Denote by \( m \) the augmentation ideal of \( A \); i.e.

\[ m = \bigoplus_{\nu=1}^{\infty} A_\nu \]

Assume moreover

\[ \text{depth}_m A \geq 1 \]

Let \( M \) be any graded \( A \)-module and put

\[ M(d) = \bigoplus_{\nu=0}^{\infty} M_{d\nu} \]

In what follows we shall relate the groups

\[ H^i(S, A, M) \]

to the groups

\[ H^i(S, A(d), M(d)) \]

**Lemma 4.1**

If \( X' = \text{Spec}(A) - V(m) \) and \( X'(d) = \text{Spec}(A(d)) - V(m(d)) \)

then the groups
are isomorphic for every \( i \)

**Proof**

The canonical morphism \( A(d) \hookrightarrow A \) induces a morphism of schemes

\[
X' \longrightarrow X'(d)
\]

thus a homomorphism

\[
A^i(S, X', \tilde{M}) \longrightarrow A^i(S, X'(d), \tilde{M})(d)
\]

It suffices to prove that the corresponding morphism of spectral sequences

\[
H^p(X', \Lambda^q(\tilde{M}))(d) \longrightarrow H^p(X'(d), \Lambda^q(\tilde{M})(d))
\]

is an isomorphism for every \( p \) and \( q \).

Consider the commutative diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X'(d) \\
\downarrow \pi & & \downarrow \pi \\
\text{Proj}(A) = X & \sim & X(d) = \text{Proj}(A(d))
\end{array}
\]

Then

\[
H^p(X', \Lambda^q(\tilde{M}))(d) \sim H^p(X, \pi_* \Lambda^q(\tilde{M}))(d) \sim \underset{\nu}{\frac{\text{HP}(X, \Lambda^q(\tilde{M}))(d\nu)}{\nu}} \sim
\]

\[
\frac{\text{HP}(X(d), \Lambda^q(\tilde{M})(d)) \nu}{H^p(X(d), \pi_* \Lambda^q(\tilde{M})(d))} \sim H^p(X'(d), \Lambda^q(\tilde{M})(d))
\]

This will prove

**Theorem 4.2**

Let \( n \) be an integer and assume \( \text{depth}_m M \geq n + 2 \)

Then
are isomorphic for \( i \leq n \) and for every \( d \geq 1 \).

**Proof**

Consider the exact sequences

\[
\begin{align*}
&\rightarrow H^i_m(S, A, M)(d) \rightarrow H^i(S, A, M)(d) \rightarrow A^i(S, \tilde{X}, \tilde{M})(d) \rightarrow H^{i+1}_m(S, A, M)(d) \\
&\rightarrow H^i_{m(d)}(S, A(d), M(d)) \rightarrow H^i(S, A(d), M(d)) \rightarrow A^i(S, \tilde{X}(d), \tilde{M}(d)) \rightarrow H^{i+1}_m(S, A(d), M(d)) \\
&\rightarrow \cdots \rightarrow \rightarrow H^i(S, A(d), M(d)) \rightarrow A^i(S, \tilde{X}(d), \tilde{M}(d)) \rightarrow H^{i+1}_m(S, A(d), M(d)) \rightarrow \cdots
\end{align*}
\]

Since \( \text{depth}_M \geq n+2 \) is equivalent to the conditions

\[
\forall i \leq n \\
H^0(X, \tilde{M}(\nu)) = 0
\]

we easily deduce

\[
\text{depth}_M(d) \geq n+2
\]

Hence

\[
H^i_m(S, A, M) = H^i_{m(d)}(S, A(d), M(d)) = 0 \quad \text{for } i \leq n+1
\]

Q.E.D.

We are specially interested in (4.2) for the case \( n = 1 \) and \( M = A \). Let \( R \) be a graded \( S \)-free algebra, generated by \( R_1 \), such that

\[
A = R/I
\]

Put

\[
P = P_S^N = \text{Proj}(R)
\]

If \( N \geq 2 \),
Furthermore by assumption $\text{depth } A \geq 1$, thus

$$H^0_m(A) = 0$$

Recall also

$$H^{i+1}_m(A) = H^i_t(X, O_X(t)) = H^i_t(O_X(t)) \quad \text{for } i \geq 1$$

**Proposition 4.3**

Let $d \geq 1$ and assume

$$H^1_t(X, O_X(t+1)) = 0 \quad H^1_t(P, \mathcal{F}(t+1)) = 0$$

for all $t \geq d$. Then there is a natural isomorphism

$$d^\nu H^1_t(S, A, A) \cong H^1_t(S, A(d), A(d))$$

for $\nu \geq 1$

**Proof**

Consider the long exact sequences

$$\cdots \rightarrow H^1_t(S, A, A)(d) \rightarrow H^1_t(S, A, A)(d) \rightarrow A^1(S, X'(d), O_{X'(d)}) \rightarrow H^2_t(S, A, A)(d) \rightarrow \cdots$$

$$\cdots \rightarrow H^1_t(S, A(d), A(d)) \rightarrow H^1_t(S, A(d), A(d)) \rightarrow A^1(S, X'(d), O_{X'(d)}) \rightarrow H^2_t(S, A(d), A(d)) \rightarrow \cdots$$

By assumption we have

$$d^\nu H^1_t(S, A, A) = d^\nu H^0_t(S, A, H^1_t(A)) = d^\nu \text{Der}_S(A, H^1_t(A)) = 0$$

$$H^1_t(d)(S, A(d), A(d)) = \nu \text{Der}_S(A(d), H^1_t(\mathcal{F}(dt))) = 0$$

since $\nu \geq 1$. 

Futhermore
\[ d_v H^0(S, A, H^2_m(A)) = d_v \text{Der}_S(A, \Omega^1_X(0_X(t))) = 0 \]
\[ v H^0(S, A(d), H^2_{m(d)}(A)) = v \text{Der}_S(A(d), \Omega^1_X(0_X(dt))) = 0 \]

Since
\[ H^1(R, A, H^1_m(A)) \rightarrow H^1(S, A, H^1_m(A)) \]
is surjective and since
\[ d_v H^1(R, A, H^1_m(A)) = d_v \text{Hom}_A(I/I^2, \Omega^1_X(0_X(t))) = 0 \]
we find
\[ d_v H^1(S, A, H^1_m(A)) = 0 \quad \text{for } v \geq 1 \]

Similarly we prove that
\[ v H^1(S, A(d), H^1_{m(d)}(A(d))) = 0 \quad \text{for } v \geq 1. \]

Hence
\[ d_v H^2_m(S, A, A) = 0 \quad \text{for } v \geq 1 \]
\[ v H^2_m(d)(S, A(d), A(d)) = 0 \quad \text{for } v \geq 1 \]

The exact sequences above together with (4.1) prove the proposition

Q.E.D.

**Corollary 4.4**

If \( \text{depth}_mA \geq 2 \) and if \( v \) is an integer such that
\[ H^1(X, 0_X(dv+1)) = 0 \]
then
\[ d_v H^1(S, A, A) = 0 \quad \text{implies} \quad v H^1(S, A(d), A(d)) = 0 \]
Proof.

By assumption

\[ H^1_m(A) = H^1_{m(d)}(A(d)) \]

Moreover

\[ (d \nu + 1)H^2_m(A) = H^1(X, O_X(d \nu + 1)) = 0 \]

Thus

\[ d \nu H^2_m(S, A, A) = d \nu \text{Der}_S(A, H^2_m(A)) = 0 \]

Using the long exact sequences of the proof of (4.3) we find a diagram

\[
\begin{array}{c}
0 \rightarrow d \nu H^1(S, A, A) \rightarrow d \nu A^1(S, X, O_X) \rightarrow 0 \\
0 \rightarrow H^1(S, A(d), A(d)) \rightarrow A^1(S, X'(d), O_{X'})
\end{array}
\]

which proves 4.4  
Q.E.D.

Corollary 4.5  (Negative grading of \( A(d) \))

Assume \( \text{depth}_m A \geq 1 \) and suppose there is a \( d \geq 1 \) such that

\[ H^1(X, O_X(t+1)) = 0 \quad H^1(P, \tilde{I}(t+1)) = 0 \]

\[ tH^1(S, A, A) = 0 \quad \text{for} \ t \geq d \]

Then

\[ \nu H^1(S, A(d), A(d)) = 0 \quad \text{for} \ \nu \geq 1 \]

Proof.

Use 4.3 for \( \nu = 1, 2, \ldots \)  
Q.E.D.
Corollary 4.6 (Positive grading of $A(d)$)

Assume depth $A \geq 2$. Suppose there is a $d \geq 1$ such that

$$(-t)H^1(S, A, A) = 0 \quad \text{and} \quad H^1(X, O_X(-t+1)) = 0 \quad \text{for } t \geq d$$

then

$$\forall H^1(S, A(d), A(d)) = 0 \quad \text{for } \forall < 0$$

Proof

Use 4.4 for $\forall = -1, -2, \ldots$ \hspace{1cm} Q.E.D.

Let us put 4.5 and 4.6 together in the following theorem

Theorem 4.7

Let $X = \text{Proj}(A)$

a) If $X$ is S-smooth, then there is a graded S-algebra $B$ having negative grading such that

$$X \sim \text{Proj}(B)$$

b) If depth $A \geq 2$ and if there is an integer $n$ such that

$$H^1(X, O_X(t)) = 0 \quad \text{for } t \leq n$$

then there is an S-algebra $B$ having positive grading such that

$$X \cong \text{Proj}(B)$$

c) If $X$ satisfies the conditions of a) and b) then

$$X \cong \text{Proj}(B)$$

for an S-algebra $B$ which has both positive and negative grading

Proof

If $X$ is S-smooth, then
\[ \forall H^1(S,A,A) = 0 \]

for large \( \nu \). In fact the sequence

\[ \cdots \to H^0(S,A,H^1_m(A)) \to H^1(S,A,A) \to \Lambda'(S,X',O_{X'}) \to \]

is exact and

\[ \forall H^0(S,A,H^1_m(A)) = \forall \text{Der}_S(A, H^1(I(t))) = 0 \]
\[ \forall H^1(S,A',O_{X'}) = \forall H^1(X',\theta_{X'}) = \forall H^1(X,\pi_*\theta_{X'}) = 0 \]

for large \( \nu \). Thus (4.5) proves a). (4.6) proves b) since

\[ \forall H^1(S,A,A) = 0 \]

for small \( \nu \). This follows from the surjection

\[ H^1(R,A,A) \to H^1(S,A,A) \]

and from the fact that

\[ \forall H^1(R,A,\Lambda) = \forall \text{Hom}_A(I/I^2,\Lambda) = 0 \]

for small \( \nu \).

Q.E.D.

For similar results, see [S3] and [M].
CHAPTER 5

The existence of a \( k \)-algebra which is unliftable to characteristic zero.

In [Se] Serre gives an example of a \( k \)-smooth projective variety \( X \) in characteristic \( p \) which cannot be lifted to characteristic zero. This means that for any complete local ring \( \Lambda \) of characteristic zero such that \( \Lambda / \mathfrak{m} \Lambda = k \), it is impossible to lift \( X \) to \( \Lambda \). His variety is of the form

\[ X = Y / G \]

when \( Y \) is a complete intersection of dimension 3 and \( G \) is a finite group operating on \( Y \) without fixpoints. Furthermore the order of \( G \) divides \( p \).

By (4.7 a) there exists a graded \( k \)-algebra \( B \) with negative grading such that

\[ X = \text{Proj}(B) \]

Hence (2.6) proves that \( B \) cannot be lifted to any \( \text{noetherian} \) complete local ring \( \Lambda \) of characteristic zero. In fact the example of Serre satisfies even (4.7 c), thus proving the existence of a graded \( k \)-algebra \( C \) satisfying \( \nu \mathcal{H}^1(k, C, C) = 0 \) for \( \nu \neq 0 \), such that \( X = \text{Proj}(\Lambda) \). (2.6) reduces to the almost trivial result

\[ R^0(C) \sim R(C) \]

Clearly \( C \) is unliftable to any complete local ring \( \Lambda \) of characteristic zero.

The reason why Serre's example works is obviously that \( p \), the characteristic of \( k \), divides the order of \( G \). To see
this, let us prove

Theorem 5.1

Let \( B \rightarrow A \) be an \( S \)-algebra homomorphism having a \( B \)-linear retraction. Let \( I \subseteq A \) be an ideal such that the composed morphism

\[
U = \text{Spec}(A) - V(I) \hookrightarrow \text{Spec}(A) \rightarrow \text{Spec}(B)
\]

is étale. If \( \text{depth}_IA \geq n+2 \), then there is an injection

\[
H^i(S, B, B) \hookrightarrow H^i(S, A, A)
\]

for \( i \leq n \).

Proof

By étaleness \( A^i(B, U, 0_U) = 0 \) for all \( i \), and the depth condition implies

\[
H^i_I(B, A, A) = 0 \quad \text{for } i \leq n+1
\]

Using the exact sequence

\[
\rightarrow H^i_I(B, A, A) \rightarrow H^i(B, A, A) \rightarrow A^i(B, U, 0_U) \rightarrow
\]

we conclude

\[
H^i(B, A, A) = 0 \quad \text{for } i \leq n+1
\]

However, there is an exact sequence

\[
\rightarrow H^i(B, A, A) \rightarrow H^i(S, A, A) \rightarrow H^i(S, B, A) \rightarrow H^{i+1}(B, A, A) \rightarrow
\]

Hence

\[
H^i(S, A, A) \rightarrow H^i(S, B, A) \quad i \leq n
\]

Since the injection \( B \rightarrow A \) has a \( B \)-linear retraction

\[
H^i(S, B, B) \rightarrow H^i(S, B, A)
\]

is injective for any \( i \)

Q.E.D.
Apply (5.1) to the following situation. Let

\[ Y = \text{Proj}(A) \]

be a projective \( k \)-scheme, and let \( G \) be a finite group acting on \( A \) such that the graded injection

\[ A^G \rightarrow A \]

induces \( Y \rightarrow Y/G = X \). Assume \( Y \rightarrow X \) étale and suppose that the order of \( G \) does not divide the characteristic of the field \( k \).

**Corollary 5.2**

a) If \( \text{depth}_mA \geq 3 \) then

\[ H^1(k, A, A) = 0 \quad \text{implies} \quad H^1(k, A^G, A^G) = 0 \]

b) If \( \text{depth}_mA \geq 4 \) then

\[ H^2(k, A, A) = 0 \quad \text{implies} \quad H^2(k, A^G, A^G) = 0 \]

**Proof**

Clearly \( A^G \rightarrow A \) has a retraction by the assumption on \( \text{ord}(G) \). Moreover the morphism

\[ \text{Spec}(A) - \mathcal{V}(m) \rightarrow \text{Spec}(A^G) \]

is étale. We use (5.1) \hspace{1cm} Q.E.D.

Assume \( Y \) to be a complete intersection

\[ Y = \text{Proj}(A) \]

with \( \text{depth}_mA \geq 4 \). Under the same conditions as in (5.2) we deduce

\[ H^2(k, A^G, A^G) = 0 \]

Clearly \( X = \text{Proj}(A^G) \) behaves as the example of Serre except
for the condition on $\text{ord}(G)$.

Remark

Clearly (5.2) is true not only for graded $k$-algebras $A$. In this case we suppose the condition on $\text{ord}(G)$ and that the morphism

$$\text{Spec}(A^G) - V(m) \longrightarrow \text{Spec}(A)$$

is étale. For (5.2 a), see [S2].
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