UNIFORM SPACES IN TOPOI

by

Hans Engenes
Oslo
ABSTRACT

The present dissertation is part of the theory of categories, or more specifically - the theory of those categories that have become known as elementary topoi. As was realized by F.W. Lawvere and M. Tierney around 1969, these are precisely the categories in which the usual interpretation of equational theories can be extended to an internal interpretation of any - not only first order, but arbitrarily higher order - formal theory. This interpretation will in general not comply with the rules of classical logic, but rather with those of intuitionistic systems. This ties up with earlier uses of topological spaces in interpretations of intuitionistic theories, for the category of set-valued sheaves on a topological space is the prime example of an elementary topos.

Thus one can equip the objects of a topos with the structure of local rings, fields, well-orderings or topologies. Or uniformities, which are the subject of this dissertation.

A fairly long introduction is devoted to a detailed explanation of how one goes about interpreting a higher-order theory in a topos. Then a particular version of the theory of uniform spaces is discussed and interpreted, and basic properties established. This is used for the construction, following Bourbaki, of the completion of a given "uniform space object", and for proofs that the completion in fact has all the properties characteristic of the ordinary theory of uniform spaces.

A short appendix takes up the external approach to structured objects, which identifies a T-structure on an object A with a factorization of the functor Hom(-,A) through the underlying-set-functor T-models + Sets. This approach can be used for fairly general theories T, if only the factorizations of Hom(-,A) are required to be continuous.
A topos is a category in which one can interpret finitary higher-order theories. This can be taken as a definition of a topos, together with a suitable specification of what it is to "interpret". However, there is another way to define this class of categories, as was realized by Lawvere and Tierney around 1970, namely by extracting the elementary properties of categories of set-valued sheaves on sites (= small categories equipped with Grothendieck topologies, see [De]). The resulting set of axioms can be made extremely simple (see Ch. 1), and it must be considered a remarkable fact that one can get to "interpretability of finitary higher-order theories" in such a simple fashion.

Interpretations in topos of various concepts of ring theory were studied by Christopher Mulvey in [Mu], published in 1974. It was also in that year that Lawrence N. Stout finished his doctoral dissertation on General Topology in Topoi ([St]). The present dissertation takes up the theory of uniform spaces in the same spirit, by interpreting the basic concepts and constructing the completion of an arbitrary "uniform space object". We have not discussed the close connections, which fairly obviously must exist, between the present work and earlier works in intuitionistic and non-standard analysis. Examples of the latter are [FN], [MH] and [Lu]. Also, we have not discussed the problem of equipping uniform space objects with algebraic structure.

Finally, I hereby express gratitude to a number of people — to Fred Linton, who taught me category theory, suggested the subject
of this paper, and guided the work on it with limitless patience -
to Wesleyan University, of Middleton, Conn., for supporting me and
my family through five years of study and research - to all my
former teachers and colleagues at Wesleyan, for innumerable dis-
cussions, suggestions and encouragements - to the Séminaire de
Mathématique Supérieure, of the Université de Montréal, for
supporting my participation of their meeting in June 1974, an
occasion which concentrated my interest at topos theory, and
finally to Agnes Michelet, for typing this manuscript so flawlessly.

Oslo, April 1976

Hans Engenes
1 INTRODUCTION

1.1. The problem of structured objects. The privileged status of set theory in the mathematics of the 20th century is universally accepted, if not as desirable then at least as factual. This is so, of course, because most branches of modern mathematics are formulated as studies of structured sets. It is natural for a category theorist to view this as studies of structured objects in the category $\textbf{Set}$, of all sets and all functions between them. From that point of view it is also natural to seek an understanding of structured objects in other categories, and indeed in abstract, unspecified categories satisfying certain general conditions that may be needed.

It was early realized that in a category with finite powers (i.e. for each object $A$ and each non-negative integer $n$ there is a product of $n$ copies of $A$, denoted $A^n$) one can equip any object with finitary operations satisfying given equations (axioms). The $n$-ary operations on $A$ are then interpreted as morphisms $A^n \to A$, and satisfaction of a certain equation is interpreted as commutativity of a certain diagram. Notice that "constants" can be incorporated into this setting as 0-ary operations, and that $A^0$ is always a terminal object, usually denoted 1.

Thus one can study, for example, "group-objects" in such categories, as was done by Ekmann and Hilton in 1962 (see [EH]). It turns out, not surprisingly, that group-objects in the category of topological spaces are nothing more nor less than
topological groups, and a group-object in the category of set-valued functors on a small category $\mathcal{C}$ is just a group-valued functor on $\mathcal{C}$.

Objects structured in the manner described above, by a finitary equational theory, can also (by a simple application of the Yoneda lemma) be described as follows: First of all there is a category, $\mathcal{S}$, of sets with structure of the given type and the structure-preserving functions (homomorphisms) between them, and there is a natural faithful functor (the "forgetful" functor) $U_{\mathcal{S}} : \mathcal{S} \to \text{Set}$. Then, with $A$ being an object in a category $\mathcal{A}$ with finite powers, the various ways of equipping $A$ with a structure of the given type correspond to the liftings, over $U_{\mathcal{S}}$, of the contravariant Set-valued functor represented by $A$, as in the following diagram:

This approach suggests an obvious way of defining an "$\mathcal{S}$-structure" on an object in any category $\mathcal{A}$, even if it does not have finite powers, and everything works as it should as long as the structure we are dealing with is equational.

Going beyond this limitation, however, we immediately encounter serious problems. As an example consider the theory of fields. The axioms can be expressed in various ways, but not as a set of equations in certain operations, because the
operation of inversion is not everywhere defined. And, indeed, it is clear that a field structure on a set \( A \) does not in general correspond to a lifting of the functor 
\[
\text{Set}(-, A) : \text{Set}^{\text{op}} \to \text{Set} \quad \text{over} \quad \text{UFields} : \text{Fields} \to \text{Set}.
\]
In fact, if \( A \) has more than one element then there are no such liftings at all.

As a second example, consider the theory of partial orderings, which is also a relational, not equational, theory. Let \( \text{Pos} \) be the category of partially ordered sets and order-preserving functions. Any partial ordering on a set \( A \) clearly yields a lifting of \( \text{Set}(-, A) \) over \( \text{UPos} \) by the pointwise ordering of functions, but generally there will be more liftings than those obtained this way. To see this, let \( A \) be a two-element set, so that \( \text{Set}(-, A) \) is naturally equivalent to the contravariant power set functor \( \text{P}^*: \text{Set}^{\text{op}} \to \text{Set} \). There are precisely four different orderings on \( A \), corresponding to liftings of \( \text{P}^* \) obtained by ordering each \( \text{P}^*(B) \) discretely, indiscretely, by the inclusion relation for subsets and by the reversed inclusion relation. But there are other ways of ordering each \( \text{P}^*(B) \) functorially. One is to let \( \emptyset < B' \) for all \( B' \subset B \), and \( B' < B'' \) for non-empty \( B', B'' \) if and only if \( B' = B'' \). This is an ordering of each \( \text{P}^*(B) \), different from those already mentioned, and it is functorial, for if \( f: C \to B \) is a function between sets, and \( B' < B'' \) in \( \text{P}^*(B) \), then \( B' = \emptyset \), so \( \text{P}^*(f)(B') = f^{-1}(B') = f^{-1}(\emptyset) < \text{P}^*(f)(B'') \) in \( \text{P}^*(C) \). So this ordering defines a fifth lifting of \( \text{P}^* \) over \( \text{UPos} \).
As a third example, consider the theory of topological spaces, and let $\text{Top}$ be the category of all topological spaces and all continuous maps. Although this is a second order theory, each topology on a given set $A$ induces a lifting of $\text{Set}(-,A)$ over $U_{\text{Top}}$ by giving each hom-set $\text{Set}(B,A)$ the topology of pointwise convergence. But we can also give each $\text{Set}(B,A)$ the discrete topology, thereby defining a lifting of $\text{Set}(-,A)$ over $U_{\text{Top}}$ not induced by any topology on $A$ (for, if $B$ is infinite and $A$ has at least two points, then the topology of pointwise convergence on $\text{Set}(B,A)$ is not discrete).

In Appendix A we will give some general conditions under which this external method for structuring objects "works".

1.2 **Internal languages in categories.** The reason why we can equip sets with so many kinds of structure is that in the formal language of set theory one can express not only equational theories, but also relational and higher-order theories.

This insight makes it natural to try and interpret formal languages in other categories, in a way which makes it possible to describe models, in a given category, for theories expressed in a given formal language.

We will now describe a rudimentary beginning of such an interpretation, possible in any category. Let $\mathcal{A}$ be a given (arbitrary) category, and let $A$ be an object in $\mathcal{A}$. The class of all $\mathcal{A}$-monomorphisms with $A$ as codomain is denoted $L(\mathcal{A})/A$. We call the elements of this formulas of type $A$. The natural pre-ordering of $L(\mathcal{A})/A$ is viewed as a logical structure: For $\varphi,\psi \in L(\mathcal{A})/A$ we say "$\varphi$ implies $\psi$" or "from $\varphi$ we can infer $\psi$", if $\varphi$ factors through $\psi$: 
and \( \varphi \) and \( \psi \) are equivalent if each implies the other. The equivalence-classes in \( L(\Omega)/A \) are called subobjects of \( A \).

There is at least one formula of type \( A \) which can be inferred from every other, and that is \( \text{id}_A \), the identity map on \( A \). The formulas equivalent to \( \text{id}_A \) are said to be valid on \( A \). These are precisely the isomorphisms in \( L(\Omega)/A \).

The class \( L(\Omega) = \bigcup \{ L(\Omega)/A | A \in |\Omega| \} \) is the beginning of what we will call the internal language of \( \Omega \). In general this language will not have much expressive power, but under certain conditions on \( \Omega \), to be listed in the following, it will.

The first condition is that \( \Omega \) have pull-backs. Then, for \( \varphi, \psi \in L(\Omega)/A \), let \( \varphi \land \psi \in L(\Omega)/A \) be the formula obtained by choosing a pull-back of \( \varphi \) and \( \psi \):

\[
\begin{array}{ccc}
\text{pull-back} & \longrightarrow & \ast \\
\varphi & \downarrow & \psi \\
\varphi \land \psi & \downarrow & \ast \\
\varphi & \downarrow & A \\
\end{array}
\]

(recall that pull-backs of monics are again monic). It is easy to see that this operation (once well defined) yields binary infima in \( L(\Omega)/A \) — i.e. from \( \varphi \land \psi \) we can infer
both $\varphi$ and $\psi$, and if from $\xi \in L(\mathcal{A})/A$ we can infer both $\varphi$ and $\psi$ then from $\xi$ we can also infer $\varphi \land \psi$.

Pull-backs in $\mathcal{A}$ also make available an operation $f^{-1} : L(\mathcal{A})/B \to L(\mathcal{A})/A$, induced by a given $f \in Q(A,B)$, by simply pulling formulas back along $f$:

$$
\begin{array}{ccc}
\text{pull-back} & \rightarrow & \\
\downarrow \downarrow & & \downarrow \downarrow \\
f^{-1}(\varphi) & \rightarrow & \varphi \\
A & \rightarrow & B \\
\end{array}
$$

Notice that again we must choose pull-backs to make $f^{-1}$ a well defined operation. This is a frequently occurring phenomenon, but most often it is of little importance to mention the choosing explicitly, so we will usually simply suppress it. The operation $f^{-1}$ is called substitution along $f$. It is order-preserving and validity-preserving (i.e. $f^{-1}(\text{id}_B)$ is valid on $A$), but that is about all we can say about it without further assumptions on $\mathcal{A}$.

Now suppose that $\mathcal{A}$ has unique epic-monic factorizations. That is, for each $\mathcal{A}$-morphism $f$ there is a monomorphism $f_1$ and an epimorphism $f_2$ such that $f_1 \circ f_2 = f$, and if $f'_1$ is also a monomorphism, $f'_2$ an epimorphism, with $f'_1 \circ f'_2 = f$, then there is a unique morphism $g$ with $g \circ f_2 = f'_2$ and $f'_1 \circ g = f'_1$, and this unique $g$ is an isomorphism:
Thus each morphism has a monic part. Given $f \in A(A, B)$ and $\varphi \in L(\mathcal{A})/A$, let $\exists_f(\varphi)$ be the monic part of $f \circ \varphi$. Then $\exists_f(\varphi)$ is a formula of type $B$, and we call it the existential quantification of $\varphi$ along $f$.

It is now an easy exercise to prove that $\exists_f : L(\mathcal{A})/A \rightarrow L(\mathcal{A})/B$ is a left adjoint to $f^{-1}$, in the sense that $\exists_f(\varphi)$ implies $\psi$ if and only if $\varphi$ implies $f^{-1}(\psi)$, whenever $\varphi \in L(\mathcal{A})/A$ and $\psi \in L(\mathcal{B})/B$.

To see how these notions are connected with the logical ideas their names suggest, take $\mathcal{A}$ to be $\text{Set}$, where the formulas can simply be considered as subsets of their types. Then, given $f : A \rightarrow B$, $A' \subseteq A$ and $B' \subseteq B$, $f^{-1}(B')$ is the set of all $a \in A$ for which $f(a) \in B'$, as the notation suggests, and $\exists_f(A')$ is just the image of $A'$ under $f$, i.e. the set of all $b \in B$ for which there exists $a \in A'$ with $f(a) = b$.

In $\text{Set}$, with $f$ as above, there is also an operation $\forall_f : L(\text{Set})/A \rightarrow L(\text{Set})/B$, which is right adjoint to $f^{-1}$: $\psi$ implies $\forall_f(\varphi)$ if and only if $f^{-1}(\psi)$ implies $\varphi$, or $B' \subseteq \forall_f(A')$ if and only if $f^{-1}(B') \subseteq A'$. This makes it clear how $\forall_f(A')$ can be defined, namely as the set

$$\{b \in B \mid (\forall a \in A)(f(a) = b \rightarrow a \in A')\} = \{b \in B \mid f^{-1}(\{b\}) \subseteq A'\}.$$ 

In other categories, with pull-backs and unique epic-monic factorization there will generally not be a right adjoint to $f^{-1}$, for each morphism $f$. But whenever there is one, we will denote it by $\forall_f$ and call it universal quantification along $f$.

Assuming existence of $\forall_f$ for each $f$ makes the language $L(\mathcal{A})$ fairly powerful, but to ensure it as much expressive
power as the first-order part of $L(\text{Set})$ we must have available some further propositional connectives. We first assume existence of a minimal formula of each type: For each object $A$, there shall be $\sigma_A \in L(\mathcal{A})/A$ from which every other formula of type $A$ can be inferred. Furthermore, we assume existence of a binary operation $- \wedge -$ on $L(\mathcal{A})/A$, for each $A \in |\mathcal{A}|$, with the property that $\varphi + - : L(\mathcal{A})/A + L(\mathcal{A})/A$ is order-preserving and is right adjoint to $- \wedge \varphi : L(\mathcal{A})/A \to L(\mathcal{A})/A$, for each $\varphi \in L(\mathcal{A})/A$. That is, $\psi \wedge \varphi$ implies $\xi$ if and only if $\psi$ implies $\varphi + \xi$. Such an operation on $L(\mathcal{A})/A$ is essentially unique. Having it available, we define a unary operation, $\neg$, on $L(\mathcal{A})/A$, for each $A \in |\mathcal{A}|$, by $\neg \varphi = \varphi + \sigma_A$. Finally, we will assume that each $L(\mathcal{Q})/A$, besides having binary infima, given by the operation $- \wedge -$, also has binary suprema, given by a binary operation $- \vee -$ on $L(\mathcal{A})/A$. That is, for $\varphi$ and $\psi$ in $L(\mathcal{Q})/A$ we assume there is a formula $\varphi \vee \psi$ in $L(\mathcal{Q})/A$, which can be inferred from both $\varphi$ and $\psi$, with the property that any $\xi \in L(\mathcal{Q})/A$ which can be inferred from both $\varphi$ and $\psi$ can also be inferred from $\varphi \vee \psi$.

Assuming existence of $\sigma_A$, $\wedge$, $\vee$ and $+$ on $L(\mathcal{Q})/A$ means precisely that the partial ordering arising from the pre-ordering $L(\mathcal{Q})/A$ is a Heyting algebra, or a pseudo-Bodean algebra (see [Fr1] or [RS]).

If the product $A \times A$, exists, let $A^2$ be a chosen such, with a chosen pair of projections. Then formulas of type $A^2$ will also be called binary formulas of type $A$. Similarly, given any non-negative integer $n$, formulas of type $A^n$ will
also be called n-ary formulas of type $A$, where $A^n$ is a chosen $n^{th}$ power of $A$ in $A$. As an example take the binary formula $\Delta_A : A + A^2$ on $A$, i.e. the diagonal map, which is always a monomorphism. This particular binary formula on $A$ is called the equality predicate on $A$.

Since we want to be able to interpret in each $L(A)/A$ any finitary predicate symbol, we now make the assumption that has finite powers. Then, in particular, it has $0^{th}$ powers, i.e. it has a terminal object, and since we have already assumed it has pull-backs it follows that it has all finite limits, i.e. is finitely complete.

We say $A$ is a pre-logos (see [Fr2]) if it has the properties we have assumed so far, i.e. if it is finitely complete, has unique epic-monic factorizations, has universal quantification along any morphism, and if each $L(A)/A$ has $\sigma_A$, $\lor$ and $\land$.

1.3 Interpreting a formal language in a pre-logos. In this section $A$ will be a pre-logos. Let $L$ be a formal first-order language, as described for example in [Sh] and [CK], with logical symbols $\land$, $\lor$, $\rightarrow$, $\neg$, $\exists$ and $\forall$, variable-symbols $x_0, x_1, \ldots$, and with only relation symbols as non-logical symbols, so for each non-negative integer, $n$, $L$ has a class $R_n$ of $n$-ary relation symbols.

An interpretation of $L$ in $A$ consists of a choice of $A$-object, $A$, a choice of powers $A^n$ and projection maps $\pi_i^n : A^n \rightarrow A$ ($i \in \{0, \ldots, n-1\}$), and for each $n$ a map $I_n : R_n + L(A)/A^n$. If $\varphi = I_n(R)$ then we say $R$ is interpreted as $\varphi$, or $\varphi$ is the interpretation of $R$. 
Such an interpretation of \( \mathcal{L} \) in \( \mathcal{A} \) determines an interpretation in \( \mathcal{Q} \) of each well-formed formula \( \psi \) of \( \mathcal{L} \), defined inductively as follows. Let \( (x_{i_0}, x_{i_1}, \ldots, x_{i_{n-1}}) \) be the free variables in \( \psi \), with \( 0 \leq i_0 < i_1 < \cdots < i_{n-1} \). If \( \psi \) is atomic, then it is of the form \( R(x_{i_0}(o), x_{i_1}(1), \ldots, x_{i_{k-1}}) \) for some unique \( k \)-ary relation symbol \( R \) and some unique map \( \sigma \) of the ordinal \( k = \{0, \ldots, k-1\} \) onto the ordinal \( n = \{0, \ldots, n-1\} \). Then \( \psi \) is interpreted as \( \pi_1^{-1}(I_k(R)) \), where \( \pi_\sigma : A^n \to A^k \) is the morphism induced by \( \sigma \), i.e. \( \pi_\sigma^j o \pi_\sigma = \pi_\sigma^j(o) \) for \( j \in k \).

Next, if \( \psi \) is \( (\psi'c\psi") \), with \( c \in \{\land, \lor, \lor\} \), then there are unique \( n', n'' \leq n \) and unique strictly order-preserving maps \( \sigma' : n' \to n \) and \( \sigma'' : n'' \to n \) such that \( (x_{i_{\sigma'}(o)}, \ldots, x_{i_{\sigma'}(n'-1)}) \) and \( (x_{i_{\sigma''}(o)}, \ldots, x_{i_{\sigma''}(n''-1)}) \) are the free variables in \( \psi' \) and \( \psi'' \) respectively. If \( \psi' \) and \( \psi'' \) have been interpreted as \( \varphi' \in L(\mathcal{A})/A^{n'} \) and \( \varphi'' \in L(\mathcal{A})/A^{n''} \) respectively, then \( \psi = \psi'c\psi'' \) will be interpreted as \( \pi_\sigma^{-1}(\varphi')c\pi_\sigma^{-1}(\varphi'') \), which is in \( L(\mathcal{A})/A^n \).

Furthermore, if \( \psi \) is \( \neg\psi' \), and \( \psi' \) has been interpreted as \( \varphi' \in L(\mathcal{A})/A^n \), then \( \psi \) will be interpreted as \( \neg\varphi' = \varphi' + \sigma \in L(\mathcal{A})/A^n \).

Finally, if \( \psi \) is \( (\exists x_m)\psi' \) (or \( (\forall x_m)\psi' \)), then we consider the two cases: 1) \( x_m \) is not free in \( \psi' \), and 2) \( x_m \) is free in \( \psi' \). In case 1), \( \psi \) and \( \psi' \) have the same free variables, and \( \psi \) is given the same interpretation as \( \psi' \). In case 2), \( \psi' \) has already been given an interpretation \( \varphi' \in L(\mathcal{A})/A^{n+1} \). Moreover, there is a unique \( j \in \{0,1,\ldots,n\} \)
such that \( i_{j-1} < m < i_j \) (where "\( i_{-1} < m < i_0 \)" is read as "\( m < i_0 \)", and "\( i_{n-1} < m < i_n \)" is read as "\( i_{n-1} < m \)").

The interpretation of \( \psi \) will then be \( \exists_j (\phi') \) (or, respectively, \( \forall_j (\phi') \)), which is in \( L(\mathcal{A}/\mathcal{A}^n) \), where \( j : \mathcal{A}^{n+1} \to \mathcal{A}^n \) is "projection along the \( j \)th coordinate", i.e. \( \pi_1 \circ j = \pi_i^{n+1} \) for \( i \in \{0, \ldots, j-1\} \) and \( \pi_1 \circ j = \pi_i^{n+1} \) for \( i \in \{j, \ldots, n-1\} \).

The above clearly determines a unique interpretation in \( \mathcal{A} \) of each well-formed formula of \( \mathcal{L} \), induced by the given interpretation of \( \mathcal{L} \) in \( \mathcal{A} \).

An \( \mathcal{L} \)-theory is simply a set (or class) \( T \) of well-formed formulas of \( \mathcal{L} \), called the axioms of the theory \( T \). Given an \( \mathcal{L} \)-theory \( T \), a model of \( T \) in \( \mathcal{A} \) is an interpretation of \( \mathcal{L} \) in \( \mathcal{A} \) for which the induced interpretation of each axiom of \( T \) is valid, i.e. if \( \psi \in T \) is interpreted as \( \phi \in L(\mathcal{A}/\mathcal{A}^n) \) then \( \phi \) is equivalent to \( \text{id}_{\mathcal{A}^n} \).

It is desirable that models of theories have the property that any \( \mathcal{L} \)-formula which is intuitionistically derivable from the axioms also receives valid interpretations. This can be ensured by an assumption on \( \mathcal{A} \), namely that pull-backs of epimorphisms are again epimorphisms. Under this assumption the pre-logos \( \mathcal{A} \) is, by definition, a logos in the sense of Freyd (see [Fr_2]).

1.4 Interpreting higher-order languages. In the category \( \text{Set} \) there is an operation \( P \) on the objects, the power-set operation, which enables us to interpret second-order (and even higher-order) languages in this category. The interpretation
must be designed so that variable symbols, which in the language are required to "range over" n-ary predicates, under the interpretation are forced to "range over" the object $P(A^n)$, where $A$ is the chosen basic object of the interpretation. Moreover, if $R$ is a relation symbol of the language, admitting two variables, the first of which is restricted to range over elements (i.e. a first-order variable) and the second restricted to range over unary predicates, then the symbol $R$ must be interpreted as a monomorphism into $A \times P(A)$. In Set the operation $P$ has the property that, for given objects $A$ and $B$, the morphisms from $A$ to $P(B)$ correspond precisely to the subsets of $A \times B$. In particular, morphisms from $A$ to $P(1)$ correspond to subsets of $A \times 1$, and hence to subsets of of $A$. For this reason we say that $P(1)$ - which of course is a two-element set - is a subobject classifier in Set.

We can ask for an operation $P$ in any category with finite products. We can also, separately, ask for subobject classifiers in such categories. Carefully stated - and yet very simple - defining properties of a subobject classifier and an operation $P$ on objects imply not only that they both are essentially unique, but, much more importantly, their presence in a category with finite products forces that category to be a logos.

Thus, it turns out, the categories admitting interpretations of higher-order theories can be characterized by much simpler axioms than can those admitting interpretations of first-order theories.

Not every logos possesses a subobject classifier or an operation $P$ with the desired properties. For example, any
Heyting algebra, considered as a partially ordered category, is a logos, as Freyd remarked in \[Fr_2\].

We will now give the above-mentioned definitions of subobject classifiers and power-objects - as the values of the operation \( P \) will be called.

1.5 The fundamentals of topoi.

1.5.1 Definition: In a category with a terminal object \( 1 \), a subobject classifier is an object \( \Omega \) for which there exists a map \( t : 1 \rightarrow \Omega \) which, for each object \( X \), gives rise as follows to a one-to-one correspondence between maps \( X \rightarrow \Omega \) and subobjects of \( X \): For any monic \( X' \rightarrow X \) there is a unique map \( X' \rightarrow \Omega \), called the characteristic map for (the subobject represented by) \( X' \rightarrow X \), making the following diagram a pull-back diagram:

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
1 & \rightarrow & \Omega \\
\end{array}
\]

and, moreover, the map \( t \) can be pulled back along any map to \( \Omega \), thus yielding a subobject of the domain of such a map.

1.5.2 Definition: A category with a terminal object, binary products and a subobject classifier \( \Omega \) has power-objects if for each object \( X \) there is an object \( P(X) \) and a map \( \varepsilon_X : P(X) \times X \rightarrow \Omega \) such that for any map \( \varphi : Y \times X \rightarrow \Omega \) there is a unique map \( f_\varphi : Y \rightarrow P(X) \) satisfying \( \varepsilon_X(f_\varphi \times X) = \varphi \).

1.5.3 Definition: A topos is a category with a terminal object, binary products, a subobject classifier and power-objects.
1.5.4 Remarks: Any topos has equalizers, for if \( f, g \) are two maps \( X \to Y \) then these yield a map \( \langle f, g \rangle : X \times Y \to Y \), which we compose with the characteristic map for the (monic) diagonal map \( Y \to Y \times Y \). The result is a map \( X \to \Omega \), which classifies a subobject of \( X \). Any monic \( X' \to X \) representing this subobject, is an equalizer for \( f \) and \( g \).

Hence any topos has finite limits (in particular pull-backs). It is known ([Mi]) that any topos also has finite colimits (in particular, finite coproducts and an initial object 0) and is cartesian closed in the sense that any endofunctor of the form \(- \times X\) has a right adjoint, usually written \((-)^X\) ([Ko]). It is also known (see [KW] or [Fr\_1]) that topoi have unique epic-monic factorizations.

1.6 Examples of topoi:

1.6.1 The category \( \text{Set}_\Omega \) of all sets and all functions, where \( \Omega \) can be any set with two elements, and power-objects are power-sets.

1.6.2 The category \( \text{Set}_f \) of all finite sets and all their functions, with \( \Omega \) and power-objects as in \( \text{Set}_\Omega \).

1.6.3 For any small category \( \mathcal{C} \), the functor category \( \text{Set} \). Here we can construct \( \Omega \), as a functor \( \mathcal{C} \to \text{Set} \), as follows: For a \( \mathcal{C} \)-object \( A \), \( \Omega(A) \cong \text{nt.}(\mathcal{C}(A,-), \Omega) \cong \{ F : \mathcal{C} \to \text{Set} | F \) is a subfunctor of \( \mathcal{C}(A,-) \}, \) by the Yoneda lemma and the defining property of \( \Omega \). By "subfunctor" we mean that for each \( B \in |\mathcal{C}| \), \( F(B) \subseteq \mathcal{C}(A,B) \), because subfunctors in that sense are canonical representatives of the subobjects of \( \mathcal{C}(A,-) \). We write \( F \subseteq \mathcal{C}(A,-) \) to express this relation. Hence we define \( \Omega \) by \( \Omega(A) = \{ F : \mathcal{C} \to \text{Set} | F \subseteq \mathcal{C}(A,-) \} \), and its
action on a morphism \( f \in \mathcal{C}(A,B) \) is defined by
\[
\Omega(f)(F)(C) = \{ g \in \mathcal{C}(B,C) | g \circ f \in F(C) \}.
\] The map \( t : 1 \to \Omega \) is defined by \( t_A(*) = \mathcal{C}(A,-) \), where the asterisk is the universal name for the element of "the" one-element set. Then we construct the power-object of a functor \( X : \mathcal{C} \to \text{Set} \) by the definition \( P(X)(A) = \{ F : \mathcal{C} \to \text{Set} | F \subseteq \mathcal{C}(A,-) \times X \} \) for \( A \in \mathcal{C} \), and for \( f \in \mathcal{C}(A,B) : P(X)(f)(F)(C) = \{(g,x) \in \mathcal{C}(B,C) \times X(C) | (g \circ f,x) \in F(C) \} \). Finally, \( \varepsilon_X : P(X) \times X \to \Omega \) is defined by \( (\varepsilon_X)_{A}(F,x)(B) = \{ f \in \mathcal{C}(A,B) | (f,x(f)(x)) \in F(B) \} \) and \( (\varepsilon_X)_{A}(F,x)(g) = g \circ f \) for \( g \in \mathcal{C}(B,C) \). It is easy to verify that these definitions are meaningful and have the desired properties.

1.6.4 For a topological space \( T \), the category \( \text{Sh}(T) \) of Sets-valued sheaves on \( T \). Here we define \( \Omega \) and power-objects as follows: For an open subset \( U \) of \( T \), \( \Omega(U) \) is the set of all open subsets of \( U \), and for an inclusion map \( i_U^U : U' \to U \), and open \( V \subseteq U \), \( \Omega(i_U^U)(V) = V \cap U' \). The map \( t : 1 \to \Omega \) is defined by \( t_U(*) = U \). For a sheaf \( X \) on \( T \), \( P(X)(U) \) is the set of all natural transformations from \( X|_U \) to \( \Omega|_U \), and \( P(X)(i_U^U)(\lambda) = \lambda|_U = \{ \lambda|_V | V \subseteq U' \} \). Finally \( \varepsilon_X : P(X) \times X \to \Omega \) is defined by \( (\varepsilon_X)_{U}(\lambda,x) = \lambda|_U(x) \), for \( \lambda \in P(X)(U) \) and \( x \in X(U) \). Topoi of the form \( \text{Sh}(T) \) are called space-topoi.

1.7 The internal language of a topos.

When interpreting a language (first-order or higher-order) on an object \( X \) in a topos \( E \), we can - and will - replace \( L(E)/X \) by the class \( E(X,\Omega) \), the elements of the latter corresponding to the equivalence classes of the former.
We give now a brief description of how some of the logical structure in E is 'constructed. The logical connectives can all be obtained from internal operations on Ω itself, assuming that a morphism t : 1 → Ω is given, by virtue of which Ω is a subobject classifier. First of all let o : 1 + Ω be the characteristic map of the unique morphism 0 + 1 which is a monomorphism because in a topos any morphism to the initial object 0 is an isomorphism (a well known and easily established fact, see [Fr₁]). Then let ∧ : Ω × Ω + Ω be the characteristic map of the monomorphism <t, t> : 1 + Ω × Ω. Thirdly, let + : Ω × Ω + Ω be the characteristic map of the equalizer of ∧ and the first projection Ω × Ω + Ω. Finally, let v : Ω × Ω + Ω be the characteristic map of the monic part of the morphism \( \begin{pmatrix} \text{id}_{Ω} & t_{Ω} \\ t_{Ω} & \text{id}_{Ω} \end{pmatrix} : Ω + Ω → Ω × Ω \), where we have employed the notational convention that \( t_{X} : X + Ω \) is the composed morphism \( X + 1 \rightarrow Ω \).

1.7.1 Remarks: We write ≤ : Ω × Ω for the equalizer of ∧ and the first projection from Ω × Ω to Ω, and call it the internal ordering on Ω. It yields a partial ordering on any \( E(X, Ω) \), as follows: \( \varphi ≤ \psi \) if and only if \( <\varphi, \psi> : X + Ω × Ω \) factors through ≤, i.e. if and only if \( \varphi ∧ \psi = \varphi \), where \( \varphi ∧ \psi \) stands for the composed map \( X <\varphi, \psi> \rightarrow Ω × Ω \rightarrow Ω \). Similarly, we write \( \varphi \vee \psi \) for \( \varnothing o <\varphi, \psi> \), and \( \varphi + \psi \) for \( + o <\varphi, \psi> \). We also write \( o_{X} \) for the composed map \( X + 1 \rightarrow Ω \). Then, writing \( \gamma : Ω + Ω \) for the characteristic map of the unique (mono-)morphism \( 0 + Ω \), it is easy to prove that \( \gamma \varphi = \varphi + o_{X} \) for any \( \varphi ∈ E(X, Ω) \).
It is well known (see [KW] or [Fr]) that \( \Omega \), with the internal operations \( t, \circ, \land, \lor \) and \( + \), is a Heyting algebra object in \( E \), i.e. each \( E(X, \Omega) \) is a Heyting algebra with the induced operations described above.

Since any topos has unique epic-monic factorizations, it follows that it also has existential quantification along any morphism. It is well known, and shown in [KW] and [Fr\textsuperscript{1}], that it also has universal quantification along any morphism. The construction of the latter is rather involved, and can be found in [KW]. For a given morphism \( f : X \to Y \) in a topos \( E \), \( \exists_f \) and \( \forall_f \) will be considered as maps from \( E(X, \Omega) \) to \( E(Y, \Omega) \), consistently with earlier conventions.

The following proposition is easily proved:

1.7.2 Proposition: Let \( x : X' \to X \) and \( y : Y' \to Y \) be monomorphisms in a given topos, let \( \varphi \) be the characteristic map of \( x \), and let \( f : X \to Y \) be a morphism. Then the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{Y} & Y \\
\downarrow & & \downarrow \exists_f(\varphi) \\
1 & \xrightarrow{t} & \Omega
\end{array}
\]

commutes if and only if \( f \circ x : X' \to Y \) is "onto \( Y \)" in the sense that \( y \) factors through the monic of \( f \circ x \).

In the next proposition the first part is easily proved, and the second - expressing that the so-called Beck conditions, in the terminology of [La\textsubscript{3}], hold in any topos - is well known (see [KW] and [St]).
1.7.3 **Proposition**: In a topos, let

\[
\begin{array}{ccc}
\exists f' & \xrightarrow{g'} & \\
\downarrow & & \downarrow \\
\exists f & \xrightarrow{g} &
\end{array}
\]

be a commutative diagram, and let \( \phi : \text{dom}(f) \to \Omega \). Then

\[\exists f'(\phi \circ g') \leq \exists f(\phi) \circ g \quad \text{and} \quad \forall f'(\phi) \circ g \leq \forall f'(\phi \circ g').\]

If (*) is a pull-back then the above two \( \leq \) relations reduce to equalities (the Beck conditions for a topos).

The next proposition is also well known (see e.g. [KW] or [St]). It states that the internal language in a topos satisfies what in [La₃] was called Frobenius reciprocity.

1.7.4 **Proposition**: \( \exists f(\phi) \land \psi = \exists f(\phi \land \psi \circ f) \) for any diagram in a topos.

The following rule will also be needed later.

1.7.5 **Proposition**: \( \exists f(\phi) \Rightarrow \psi = \forall f(\phi \Rightarrow \psi \circ f) \) for any diagram in a topos.

**Proof**: From \( \exists f(\phi) \leq \exists f(\phi) \circ f \), hence that \( \exists f(\phi) \circ f \Rightarrow \psi \circ f \leq \phi \Rightarrow \psi \circ f \), so we have that \( \exists f(\phi) \Rightarrow \psi \leq \forall f(\phi \Rightarrow \psi \circ f) \).

Furthermore, from \( \forall f(\phi \Rightarrow \psi \circ f) \leq \forall f(\phi \Rightarrow \psi \circ f) \) we get that \( \forall f(\phi \Rightarrow \psi \circ f) \circ f \leq \phi \Rightarrow \psi \circ f \), i.e. that
\( \forall_f (\varphi \Rightarrow \psi \circ f) \circ f \land \varphi \leq \psi \circ f \), i.e. that

\( \exists_f (\forall_f (\varphi \Rightarrow \psi \circ f) \circ f \land \varphi) \leq \psi \), which by 1.7.4 is equivalent to

\( \forall_f (\varphi \Rightarrow \psi \circ f) \land \exists_f (\varphi) \leq \psi \), i.e. to \( \forall_f (\varphi \Rightarrow \psi \circ f) \leq \exists_f (\varphi) \Rightarrow \psi \).

This completes the proof.

The following rule states that one can always change the order of two consecutive quantifications, if they are of the same kind:

1.7.6 **Proposition**: For any commutative diagram

\[
\begin{array}{c}
\varphi \\
\downarrow f \\
\Omega \\
\uparrow f'
\end{array}
\quad \begin{array}{c}
g \\
\downarrow g'
\end{array}
\]

in a topos we have that \( \forall_f, (\forall_g (\varphi)) = \forall_g, (\forall_f (\varphi)) \) and \( \exists_f, (\exists_g (\varphi)) = \exists_g, (\exists_f (\varphi)) \).

**Proof**: From \( \forall_f, (\forall_g (\varphi)) \circ g' \circ f = \forall_f, (\forall_g (\varphi)) \circ f' \circ g \leq \forall_g (\varphi) \circ g \leq \varphi \) we get that \( \forall_f, (\forall_g (\varphi)) \circ g' \leq \forall_f (\varphi) \), and hence that \( \forall_f, (\forall_g (\varphi)) \leq \forall_g, (\forall_f (\varphi)) \). By symmetry we also have that \( \forall_g, (\forall_f (\varphi)) \leq \forall_f, (\forall_g (\varphi)) \). The case of existential quantifiers is proved similarly.

The following proposition will also be helpful:

1.7.7 **Proposition**: If \( f: X \rightarrow Y \) is an epimorphism in a topos, and \( \varphi: Y \rightarrow \Omega \) is given, then \( \exists_f (\varphi \circ f) = \varphi = \forall_f (\varphi \circ f) \).

**Proof**: Since \( f \) is epic it suffices to prove that
\[ \exists_f(\phi \circ f) \circ f = \phi \circ f \leq \forall_f(\phi \circ f) \circ f. \] The inequalities
\[ \exists_f(\phi \circ f) \circ f \leq \phi \circ f \leq \forall_f(\phi \circ f) \circ f \] follow from the adjointnesses
\[ \exists_f - \circ f - \leq \forall_f, \] and the converse inequalities follow from
\[ \forall_f(\phi \circ f) \leq \phi \leq \exists_f(\phi \circ f), \] which again follow from the mentioned adjointnesses.

The following fact will be needed in our later discussions of various notions of non-emptiness. It says, in effect, that the external property of having global support is equivalent to internal non-emptiness.

1.7.8 Proposition: Let \( m : X' \to X \) be a monomorphism in a topos, and let \( \Gamma m \) be the corresponding map \( 1 \to P(X) \). Then \( X' \to 1 \) is an epimorphism (i.e. \( X' \) has global support) if and only if \( \exists_p(e_X) \circ \Gamma m = t \), where \( p \) is the projection \( P(X) \times X \to P(X) \).

Proof: Let \( \Theta \to P(X) \times X \) be a monomorphism classified by \( e_X \), and look at the following diagram, where the three square cells clearly are pullbacks:

\[
\begin{array}{ccc}
1 \times X' & \xrightarrow{1 \times m} & 1 \times X & \xrightarrow{p'} & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma m \times X & \xrightarrow{\Gamma m} & P(X) \times X & \xrightarrow{\exists_p(e_X)} & P(X) \\
\downarrow & & \downarrow \varepsilon_X & & \downarrow \\
\varepsilon & \xrightarrow{t} & P(X) \times X & \xrightarrow{\exists_p(e_X)} & \varepsilon_X \\
\end{array}
\]
So by the Beck condition for \( \exists (1.7.3) \), \( \exists_p (\varepsilon_x) \circ m^! = \exists_{p'} (\varepsilon_x \circ (m^! \times X)) \). But \( 1 \times m \) is a monomorphism classified by \( \varepsilon_x \circ (m^! \times X) \), so \( \exists_{p'} (\varepsilon_x \circ (m^! \times X)) \) classifies the monic part of \( p' \circ (1 \times m) \). Hence \( \exists_{p'} (\varepsilon_x \circ (m^! \times X)) \), and hence \( \exists_p (\varepsilon_x) \circ m^! \), is t if and only if the monic part of \( p' \circ (1 \times m) \) is an isomorphism, i.e. if and only if \( p' \circ (1 \times m) \) is an epimorphism, i.e. if and only if \( 1 \times X' \) has global support. But clearly \( 1 \times X' \) has global support if and only if \( X' \) does, so the proof is complete.

1.7.9 \textbf{Remark} : We have seen, in 1.6.4, that, in a space-topos \( \text{Sh}(T) \), the morphisms from \( 1 \) to \( \Omega \) correspond to the open subsets of \( T \), and it is easy to see that this correspondence preserves the natural ordering. So \( \text{Sh}(T)(1, \Omega) \) can be identified, as a Heyting algebra, with the lattice of open subsets of \( T \). The latter is rarely a Boolean algebra, in fact it is so if and only if complements of open sets are also open in \( T \). Hence the Heyting algebra object \( \Omega \) is generally not a Boolean algebra object, i.e. one for which the Heyting algebras of morphisms into \( \Omega \) are all Boolean. Sometimes it is, however, as in \( \text{Set} \), and when that is the case in a topos \( E \) we say \( E \) is a \textbf{Boolean topos}.

The following equivalences are well known :

1.7.10 \textbf{Proposition} : For a topos \( E \) the following are equivalent :

(1) \( E \) is Boolean,

(2) \( E \) satisfies the principle of the excluded middle, in the sense that \( \forall \varphi \vDash \neg \varphi \Rightarrow t_x \) for all \( \varphi : X + \Omega \),
1.8 Set-theoretic concepts. In sets, \( \varepsilon_X : P(X) \times X + \Omega \) is the membership relation, restricted to the elements and the subsets of \( X \). For if \( \xi : 1 + P(X) \times X \) is given, corresponding to \((A, x) \in P(X) \times X\), then \( \varepsilon_X \circ \xi = t \) if and only if \( x \in A \).

We will think of the \( \varepsilon_X \)'s as local membership relations in any topos. This interpretation enables us to define, in the internal logic, most of the concepts definable in classical set theory. However, different but classically equivalent definitions (meaning that in \( \mathbf{Sets} \) they yield the same concept) will often yield different concepts in other topoi. For examples of this, see the discussions of finiteness in [KLM] and [VOL], and the discussion in [Mu] of definitions of a field object and of the real-number object.

We now define some concepts that we will make extensive use of later, using the internal logic as described above.

1.8.1 Definition: Let \( X \) be an object. The singleton map \( \{ \cdot \} : X + P(X) \), written \( \{ \cdot \}_X \) when necessary, is the map corresponding (by the defining property of the power-object) to the equality predicate \( X \times X + \Omega \). The union map \( \cup : P(P(X)) + P(X) \), written \( \cup_X \) when necessary, is the map corresponding to \( \exists_{p_2} (\varepsilon_{P(X)} \circ p_3 \wedge \varepsilon_X \circ p_1) : P(P(X)) \times X + \Omega \), where \( p_1, p_2 \) and \( p_3 \) are the obvious projections from.
\( P(P(X)) \times P(X) \times X \) to the subproducts \( P(X) \times X, P(P(X)) \times X \) and \( P(P(X)) \times P(X) \) respectively. Finally, given a map \( f : X \to Y \), \( P(f) \) is the map \( P(X) \to P(Y) \) corresponding to \( 3P(X) \times f(\varepsilon_X) : P(X) \times Y \to \Omega \).

The following proposition is well known. It was proved by Kock in [Ko], and (with more detail) by Stout in [St].

1.8.2 Proposition: The operations \( X + P(X), f + P(f) \) constitute a covariant functor \( P \). The transformations \( \{\cdot\} \) and \( u \) are natural transformations to \( P \) from the identity functor and \( P \circ P \) respectively, and \( (P,\{\cdot\},U) \) is a triple (monad) on the topos, meaning that \( U_X \circ \{\cdot\}_P(X) = 1d_{P(X)} = U_X \circ P(\{\cdot\}_X) \) and \( U_X \circ U_{P(X)} = U_X \circ P(U_X) \) for any object \( X \).

1.8.3 Remark: \( P \) is called the covariant power-object functor because there is also a contravariant power-object functor \( P^* \), which acts as \( P \) on objects, and whose action on maps is such that, for \( f : X \to Y \), \( P^*(f) : P(Y) \to P(X) \) is the map corresponding to the composed formula

\[
P(Y) \times X \xrightarrow{P(X) \times f} P(Y) \times Y \xrightarrow{\varepsilon_Y} \Omega
\]

We prove now that \( P^* \) is, in fact, a contravariant functor.

Since it is clear that \( P^*(1d_X) = 1d_{P(X)} \), we prove only that \( P^*(g \circ f) = P^*(f) \circ P^*(g) \) for \( f : X \to Y \xrightarrow{f} Y \xrightarrow{\varepsilon_Y} Z \). So we must show that \( P^*(f) \circ P^*(g) \) is the map corresponding to

\[
P(Z) \times X \xrightarrow{P(Z) \times f} P(Z) \times Y \xrightarrow{P(Z) \times \varepsilon} P(Z) \times Z \xrightarrow{\varepsilon_Z} \Omega.
\]
But this means showing that
\[ \epsilon_X \circ (P^* (f) \circ P^* (g) \times X) = \epsilon_Z \circ (P(Z) \times g) \circ (P(Z) \times f). \]

Now the left-hand side in this equation is
\[ \epsilon_X \circ (P^* (f) \times X) \circ (P^* (g) \times X), \]
which, by the defining property of \( \epsilon_X \), is
\[ \epsilon_Y \circ (P(Y) \times f) \circ (P^* (g) \times X), \]
which, of course, is
\[ \epsilon_Y \circ (P^* (g) \times Y) \circ (P(Z) \times f), \]
which, by the defining property of \( \epsilon_Y \), is
\[ \epsilon_Z \circ (P(Z) \times g) \circ (P(Z) \times f), \]
as desired.

### 1.9 Interpretation in some topoi.

#### 1.9.1 In \( \text{Set} \), if we discuss formulas on \( X \) in terms of the subsets of \( X \) which they define, then \( t_X \) is \( X \) itself, \( o_X \) is the empty set, negation is complementation in \( X \), meet is intersection, join is union, and the arrow operation is described by \( X' \rightarrow X'' = (X' \cap X'') \cup (X' \setminus X'') \). For a function \( f : X \rightarrow Y \), and a subset \( X' \) of \( X \), \( \exists_f (X') \) is \( \{ y \in Y | \exists x \in X' : f(x) = y \} \), and \( \forall_f (X') \) is \( \{ y \in Y | f(x) = y \rightarrow x \in X' \} \).

Clearly, \( \text{Set} \) is a Boolean topos. In \( \text{Set}_f \), the topos of finite sets, the above discussion can be repeated verbatim.
1.9.2 In \( \text{Set} \), recall how \( \Omega \) was defined (1.6.3). The logical operations on \( \Omega \) must then be described as follows:

<table>
<thead>
<tr>
<th>Operation</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \top : 1 \to \Omega ) has already been defined,</td>
<td></td>
</tr>
<tr>
<td>( \bot : \Omega \to \Omega : \bot ) is the empty subfunctor of ( \mathcal{C}(A,-) ),</td>
<td></td>
</tr>
<tr>
<td>( \neg : \Omega \to \Omega : \neg ) ( F ) ( B ) ( \in \mathcal{C}(A,B) \to \forall C \in \mathcal{C}, \forall g \in \mathcal{C}(B,C) : g \circ f \in \mathcal{C}(C) ),</td>
<td></td>
</tr>
<tr>
<td>( \land : \Omega \times \Omega \to \Omega : \land ) ( F,G ) ( (B) = { f \in \mathcal{C}(A,B) \mid \forall C \in \mathcal{C}, \forall g \in \mathcal{C}(B,C) : g \circ f \in \mathcal{C}(C) } ),</td>
<td></td>
</tr>
<tr>
<td>( \lor : \Omega \times \Omega \to \Omega : \lor ) ( F,G ) ( (B) = { f \in \mathcal{C}(A,B) \mid ) either ( \neg (f) ), ( \neg (G) ), or both, is ( \mathcal{C}(B,-) ),</td>
<td></td>
</tr>
<tr>
<td>( \Rightarrow : \Omega \times \Omega \to \Omega : \Rightarrow ) ( F,G ) ( (B) = { f \in \mathcal{C}(A,B) \mid \forall C \in \mathcal{C}, \forall g \in \mathcal{C}(B,C) : g \circ f \in \mathcal{C}(C) } ),</td>
<td></td>
</tr>
</tbody>
</table>

and the natural ordering on \( \Omega \) is simply the subfunctor relation:

\[ \preceq \] is the subfunctor of \( \Omega \times \Omega \) defined by

\[ \preceq (A) = \{ (F,G) \in \Omega(A) \times \Omega(A) \mid F \subseteq G \}. \]

Then we describe the quantifications along a given map \( f \in \text{Set} \mathcal{C}(X,Y) \). We need the fact that epic-monic factorizations of \( \text{Set} \mathcal{C} \)-maps can be done componentwise. To see this, use the given \( f \), and for each \( A \in \mathcal{C} \) let \( X(A) \xrightarrow{f_A'} Y'(A) \xrightarrow{\iota} Y(A) \) be the epic-monic factorization of \( f_A \). Then observe that this defines a subfunctor \( Y' \) of \( Y \), with \( f' : X \xrightarrow{f'} Y' \) a natural transformation, which is clearly epic and makes \( X \xrightarrow{f'} Y' \xrightarrow{\iota} Y \) equal to \( f \).

Now let \( \varphi : X \to \Omega \) be given, defining a subfunctor \( X' \) of \( X \). Let \( Y' \subseteq Y \) be the image in \( Y \) (componentwise, by the above)
of the composed map $X' \looparrowright X \xrightarrow{f} Y$. Then $\exists_f(\varphi) : Y \rightarrow \Omega$ is the characteristic map of $Y'$. Explicitly:

$$\exists_f(\varphi)_A(y)(B) = \{ h \in \mathcal{C}(A,B) \mid \exists x \in X'(B) : Y(h)(y) = f_B(x) \}$$

for $A,B \in \mathcal{C}$ and $y \in Y(A)$. And $\forall_f(\varphi)$ can be described explicitly by

$$\forall_f(\varphi)_A(y)(B) = \{ h \in \mathcal{C}(A,B) \mid \forall x \in X(B) : Y(h)(y) = f_B(x) \Rightarrow x \in X'(B) \}$$

A well known fact (see [Fr]), which can be verified using the above description of the logical operations, is that $\mathcal{Set}$ is Boolean if and only if $\mathcal{C}$ is a groupoid, meaning that all $\mathcal{C}$-maps are isomorphisms.

Then we describe the set-theoretic constructions defined in 1.8. For $X \in \mathcal{Set}$, $P(X)$ and $\varepsilon_X$ have been described before (1.6.3), so we describe here the singleton map

$$\{\ast\} : X \rightarrow P(X) : \text{For } A \in \mathcal{C} \text{ and } x \in X(A), \{\ast\}_A(x) \text{ is the subfunctor of } \mathcal{C}(A,-) \times X \text{ defined by}$$

$$\{\ast\}(x)(B) = \{ (h,x') \in \mathcal{C}(A,B) \times X(B) \mid X(h)(x) = x' \}.$$

Then we describe the union map $U : P(P(X)) \rightarrow P(X)$, as follows (where $G \subset \mathcal{C}(A,-) \times P(X) : U_A(G)(B) = \{ (h,x') \in \mathcal{C}(A,B) \times X(B) \mid \exists F \subset \mathcal{C}(B,-) \times X : (h,F) \in G(B) \ \& \ (\forall C \in \mathcal{C})(\forall g : B \rightarrow C)(g,X(g)(x')) \in F(C) \}$.}

1.9.3 In a space topos $\text{Sh}(T)$, $\Omega$ has been defined so that $\Omega(U)$ is the set of all open subsets of $U$, for each open $U$ in $T$. The logical operations are then defined by:

$$t(U)(\ast) = U, \quad \circ_U(\ast) = \emptyset, \quad \neg_U(U') = \text{int}(U \setminus U') \quad \text{(notice here that}$$
it makes no difference whether we use "interior in $T$" or "interior in $U$"), $\land_U(U',U'') = U' \cap U''$, $\lor_U(U',U'') = U' \cup U''$, and $\neg_U(U',U'') = (U' \cap U'') \cup \text{int}(U \setminus U')$.

From this it is immediately clear that $\text{Sh}(T)$ is a Boolean topos if and only if each open subset of $T$ is also closed. This, of course, is equivalent to discreteness for spaces whose singletons are closed.

The quantifications along a given map $f \in \text{Sh}(T)(X,Y)$ can be described as follows: Given $X' \to X$, $\exists_f(X')$ is the sub-sheaf of $Y$ described by $\exists_f(X')(U) = \{y \in Y(U) \mid \exists x \in X'(U) : f_U(x) = y\}$ and $\forall_f(X')$ is the one described by $\forall_f(X')(U) = \{y \in Y(U) \mid \forall x \in X(U) : f_U(x) = y \Rightarrow x \in X'(U)\}$.

Having already described $P(X)$ and $\iota_X$ for sheaves $X$ on $T$, in 1.6.4, we describe now the singleton map \{\text{•}\} : $X \to P(X)$, as follows: \{\text{•}\}_U(x)'(x') = U\{V' \subset V \mid X(1_{V'})(x') = x'\}$
for open $U \subset T$, $x \in X(U)$, open $V \subset U$ and $x' \in X(V)$.

Finally, we describe $U : P(P(X)) \to P(X)$, as follows:

$U_U(\eta)'(x') = U\{V' \subset V \mid \forall \xi \in P(X)(V) : \xi_V(x') = V \Rightarrow \eta_V(\xi) = V\}$
for $U$ open in $T$, $\eta \in \text{n.t.}(P(X)\bigr|_U, \eta_U)$, $V$ open in $U$ and $x' \in X(V)$. 

2 NOTATION CONVENTIONS AND PRELIMINARIES.

2.1 **Notation conventions and definitions.** In this chapter the discussion is confined to one topos, $E$, whose objects are denoted $X$, $Y$, $Z$, $X'$, etc.

When we write an object explicitly as a product, as for example $X \times Z \times X \times Y$, the order in which the factors appear indicates which canonical product is referred to. Thus the example just mentioned is (the object part of) the given limit of the functor $D$ defined on the discrete category $\{1,2,3,4\}$ by $D(1) = D(3) = X$, $D(2) = Z$ and $D(4) = Y$.

If $n$ is a positive integer, and $X^n$ appears as a factor in a product: $\cdots \times X^n \times \cdots$, then that product should be read as $\cdots \times X \times \cdots \times X \times \cdots$, where $X$ is repeated $n$ times as a factor. Thus $X^2 \times Y^3$ is to be considered as a product of five (not two) factors.

So when an object is exhibited as a product of $n$ factors ($n > 0$), the notation determines which object is to be considered as the $i^{th}$ factor ($0 < i \leq n$). The given projection from such a product to its $i^{th}$ factor is denoted by a letter (usually $p, p', p''$, etc.) equipped with the subscript $1, 2, \cdots i-1, i+1, \cdots, n$. Thus the product $X \times Y$ comes equipped with a projection, $p_2$, to $X$, and another $p_1$, to $Y$. That is, the subscript lists the place-numbers of those factors along which we project the product.

This notation will prove to be very economical in the following chapters, because of the following generalization of it: If $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ ($m < n$), then the projection
map carrying the subscript \( i_1, i_2, \ldots, i_m \) and defined on a
given product of \( n \) factors is always to be interpreted as
the natural projection to the subproduct formed by the factors
whose place-numbers are not among the \( i_k \)'s. Thus, if a map
denoted \( p_{2,5} \) is said to have as its domain a given product
of at least five factors, say \( X \times Y^2 \times X \times Z^2 \), then our nota-
tion determines the function completely - in the example just
mentioned \( p_{2,5} \) would be the natural projection to the subpro-
duct formed by the \( 1^{\text{st}}, 3^{\text{rd}}, 4^{\text{th}} \) and \( 6^{\text{th}} \) factors, i.e. \( p_{2,5} \) is
the unique map \( X \times Y^2 \times X \times Z^2 \to X \times Y \times X \times Z \) determined by
the equations \( p'_{2,3,4} \circ p_{2,5} = p_{2,3,4,5,6} \), \( p'_{1,3,4} \circ p_{2,5} = p_{1,2,3,5,6} \) and \( p'_{1,2,3,4,5} \circ p_{2,5} = p_{1,2,3,4,5} \).

Sometimes (in fact quite frequently) we will omit the name
of a projection map, when it is part of a formula where it is
composed on the left with another map, in which case it will
leave its subscripts to that other map before disappearing.
Thus, if \( f : X \times Z \to Z' \) is a given map, we will write \( f_{2,3} \)
for the composed map \( X \times Y \times X \times Z \xrightarrow{\text{p2.3}} X \times Z \xrightarrow{f} Z' \).
Of course, this convention leads to ambiguities, for we would
write \( f_{2,3} \) also for the composed map
\( X \times Y' \times X' \times Z \xrightarrow{\text{p2.3}} X \times Z \xrightarrow{f} Z' \), but we will avoid con-
fusion by always making it clear what the domains are.

In the same spirit we will write \( \exists_{1,1,2,\ldots, i_m} \)
\( (\forall_{1,1,2,\ldots, i_m}) \) for \( \exists_{p_{1,1,2,\ldots, i_m}} \) \( (\forall_{p_{1,1,2,\ldots, i_m}}) \). It will
always be clear from the context along which projection maps we are quantifying. Here we can, finally, point to naturality, rather than merely to convenience and economy, as a justification for our habit of indicating which factors are left out by a projection to a subproduct, rather than indicating which of them remain. For in \( \text{Set} \), if \( A \) is a subset of \( X \times Y \) then \( \exists_1(A) \) is the subset \( \{ y \in Y | \exists x \in X : (x, y) \in A \} \) of \( Y \) and \( \forall_1(A) \) is \( \{ y \in Y | \forall x \in X : (x, y) \in A \} \). That is, the subscript of the quantification symbols indicates precisely which variables are bound by that quantification.

For \( n \geq 2 \) we will write \( P^n(X) \) for the \( n \)th iterated power-object of \( X \), i.e. \( P^n(X) = P(P^{n-1}(X)) \).

Given a map \( f \) in \( E \), we will sometimes write \( \text{dom}(f) \) for its domain, and \( \text{ran}(f) \) for its range (co-domain).

To any monomorphism \( i : \text{dom}(i) \rightarrow X \) corresponds a map \( 1 \rightarrow P(X) \), which we will denote \( \mathbb{1} \).

Given such \( i \), and an object \( Y \), we write \( \mathbb{1}_Y \), or \( \mathbb{1}(Y) \) when necessary, for the composed map \( Y \rightarrow 1 \rightarrow P(X) \).

Finally, we write \( \varepsilon_X \) for the characteristic map of the diagonal monomorphism \( \Delta_X : X \rightarrow X^2 \). See 2.2 for a further discussion of this.

2.1.1 Definitions:

(i) \( r_X : P(X)^2 \rightarrow P(X^2) \) is the map corresponding to \( \varepsilon_{2.4} \wedge \varepsilon_{1.3} : P(X)^2 \times X^2 \rightarrow \Omega \),

(ii) \( \eta_X : P(X)^2 \rightarrow P(X) \) is the map corresponding to \( \varepsilon_2 \wedge \varepsilon_1 : P(X)^2 \times X \rightarrow \Omega \),
(iii) \( \sigma_X : P(X)^2 \to \Omega \) is \( \forall_3 (\epsilon_2 \Rightarrow \epsilon_1) \), obtained from the diagram

\[
\begin{array}{ccc}
P(X)^2 \times X & \xrightarrow{p_3} & P(X)^2 \\
\downarrow \epsilon_2 \circ \epsilon_1 & & \downarrow \forall_3 (\epsilon_2 \Rightarrow \epsilon_1) \\
\end{array}
\]

(iv) \( \lambda_X : P^2(X)^2 \to \Omega \) is \( \forall_3 (\epsilon_2 \Rightarrow \exists_4 (\epsilon_1 \wedge \epsilon_3 \wedge c_{1.2}) \), obtained from the diagram

\[
\begin{array}{ccc}
P^2(X)^2 \times P^2(X)^2 & \xrightarrow{p_4} & P(X) \quad P(X) \\
\end{array}
\]

(v) \( \wedge_X : P^2(X)^2 \to P^2(X) \) is the composed map

\[
P^2(X)^2 \xrightarrow{r_{P(X)}} P(P(X)^2) \xrightarrow{P(\eta_X)} P^2(X)
\]

(vi) \( \sigma_X : P^2(X) \times X \to P^2(X) \) is the map corresponding to \( \epsilon_3 \wedge \epsilon_1 : P^2(X) \times P(X) \times X \to \Omega \) (or, more accurately, to the latter followed by the natural isomorphism

\[
P^2(X) \times X \times P(X) \to P^2(X) \times P(X) \times X
\]

(vii) \( \text{st}^X_X : P^2(X) \times X \to P(X) \) is the composed map

\[
P^2(X) \times X \xrightarrow{\sigma_X} P^2(X) \xrightarrow{U} P(X)
\]

(viii) \( \text{st}^X_X : P^2(X) \to P^2(X) \) is the map corresponding to the vertical arrow in the diagram
2.1.2 Remarks: When the situation permits it - and it often will - we omit the subscript "X" from the symbols for the maps defined above. This will make more room for other subscripts, in particular for number sequences inherited from projection maps. In fact, we made use of this convention above, in parts (iv), (viii) and (ix) of 2.1.1.

We will sometimes refer to some of the operations and relations defined in 2.1.1 by the names usually attached to them when interpreted in Set. Thus \( r_X \) is the rectangle map, because in Set it associates to each pair \((A, B)\) of subsets of \( X \) the subset \( A \times B \) of \( X^2 \).

The maps \( \cap_X \) and \( \subseteq_X \) are the intersection operation and the containment relation, respectively. The map \( \triangleleft_X \) is the refinement relation for families of subsets of \( X \), and \( \wedge_X \) is the meet operation for such. In Set, \( A \triangleleft_X B \) iff each \( A \in A \) is contained in some \( B \in B \), and \( A \wedge_X B = \{ A \cap B | A \in A \ & B \in B \} \). Finally, \( \prec_X^* \) is the star-refinement relation for families of
subsets of $X$ in Set $A \subseteq B$ iff for each $x \in X$ there is $B \in B$ such that, for all $A \in A, x \in A \Rightarrow A \subseteq B$.

2.2 Preliminaries: In 1.2 we introduced the equality predicate on an object $Y$, meaning the diagonal monomorphism $\Delta_Y : Y \to Y^2$. However, since the events are now taking place in a topos, we will refer to the characteristic function of $\Delta_Y$ - written $\theta_Y : Y^2 \to \Omega$ - as the equality predicate on $Y$. We will make use of the following properties of the equality predicate, of which part (i) is clear, and part (ii) was proved in [En]:

2.2.1 Proposition: For any object $Y$,

(i) $\theta_Y$ is a symmetric, reflexive and transitive binary relation on $Y$,
and for any map $\varphi : Y \to \Omega$,

(ii) $\varphi_2 \land \theta_Y \leq \varphi_1$ (the substitutivity property).

In formal proofs in the internal language of a topos we will make frequent use of the following lemma, whose proof is so simple that we omit it.

2.2.2 Lemma: In any category with finite products let $X_1, \cdots, X_n$ be given objects, let $J, K, L \subseteq \{1, \cdots, n\}$ be such that $L \subseteq K \cap J$, and let the maps in the following diagram be the obvious projections to subproducts:
Then this diagram commutes, and if, moreover, \( L = K \cap J \) and \( K \cup J = \{1, \ldots, n\} \), then it is a pullback diagram.

The following propositions will also be helpful.

2.2.3 Proposition: For any \( \varphi : X \to \Omega \) in a topos we have that \( \exists_1 (\varphi_2 \land \theta_X) = \varphi = \forall_2 (\theta_X \Rightarrow \varphi_1) \).

Proof: For the first equality consider the following diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\Delta_{X'}} & (X)^2 & \xrightarrow{X' \times \xi} & X' \times X \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\Delta_X} & X^2 & \xrightarrow{p_1} & X \\
\downarrow & & \downarrow & & \downarrow \\
X' & \xrightarrow{\xi} & X & \xrightarrow{\varphi} & \Omega \\
\end{array}
\]

where \( \xi \) is a monomorphism classified by \( \varphi \). The rectangular part is clearly a pull-back, so the monic part of \( p_1 \circ (\xi \times X) \circ (X' \times \xi) \circ \Delta_X \), which is equal to \( \xi \), hence is monic already, and, by assumption, is classified by \( \varphi \) - is classified by
For the second equality notice first that the inequality "≤" is equivalent to \( \varphi_2 \wedge \theta_X \leq \varphi_1 \), hence follows from 2.2.1 (ii). For the inequality "≥", notice that, if \( \psi \leq \forall_2 (\theta_X \Rightarrow \varphi_1) \), then \( \psi \wedge \theta_X \leq \varphi_1 \), i.e., \( \exists_1 (\psi \wedge \theta_X) \leq \varphi \), so \( \psi \leq \varphi \) by the above. Hence \( \forall_2 (\theta_X \Rightarrow \varphi_1) \leq \varphi \), and this completes the proof.

2.2.4 Corollary: Given morphisms \( X \xrightarrow{f} Y \xrightarrow{\varphi} \Omega \) in a topos, we have that \( \forall_2 (\theta_Y \circ f, p_1 \Rightarrow \varphi_1) = \psi \circ f \).

Proof: Consider the following diagram:

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f, p_1} & Y^2 \\
\downarrow p_2 \quad \quad \downarrow p_1 \quad \quad \downarrow \theta_Y \quad \quad \downarrow \varphi \\
X & \xrightarrow{f} & Y & \xrightarrow{\varphi} & \Omega
\end{array}
\]

where the rectangular part is clearly a pull-back. Moreover, \( \varphi_1 = \varphi_1 \circ f, p_1 \), so \( \forall_2 (\theta_Y \circ f, p_1 \Rightarrow \varphi_1) = \forall_2 ((\theta_Y \Rightarrow \varphi_1) \circ f, p_1) = \forall_2 (\theta_Y \Rightarrow \varphi_1) \circ f \), where we have used the Beck condition for \( \forall (1.7.3) \). The corollary now follows by 2.2.3.

The following lemma says that \( \triangleleft_X \) is a reflexive and transitive relation on \( P^2(X) \).

2.2.5 Lemma: \( \theta_{P^2(X)} \triangleleft \triangleleft_X \), and \( \triangleleft_3 \wedge \triangleleft_1 \leq \triangleleft_2 \).

Proof: The first statement is equivalent to

\[ \theta_3 \leq \varepsilon_2 \Rightarrow \exists_4 (\epsilon_{1.3} \wedge \epsilon_{1,2}) \]
i.e. to
\[ 0 \land \varepsilon \leq \exists (\varepsilon_{1.3} \land 1.2) \,.
\]
Since \( 0 < c \), it suffices to prove that
\[ 0 \land \varepsilon \leq \exists (\varepsilon_{1.3} \land 0) \,.
\]
Now \( 0 \land \varepsilon \leq \varepsilon \) by the reflexivity of \( 0 \), so it suffices to prove that
\[ \varepsilon \leq \exists (\varepsilon_{1.3} \land 1.2) \,.
\]
which follows from 2.2.1 (iii).

Then we prove the second part of the lemma, i.e. transitivity of \( < \). We will follow the pattern of the following proof in \text{Set} : Given three families, \( U, V \) and \( W \), of subsets of \( X \), assume that \( U < V \) and \( V < W \), i.e. that
\[
(\forall U \in U)(\exists V \in V)(U \subset V) \land (\forall V \in V)(\exists W \in W)(V \subset W).
\]
Then \( (\forall U \in U)(\exists V \in V)(\exists W \in W)(U \subset V \land V \subset W) \), so
\[
(\forall U \in U)(\exists V \in V)(\exists W \in W)(U \subset W) \text{ by transitivity of } \subset.
\]
That is, \( (\exists U \in U)(\exists W \in W)(U \subset W) \), or \( U < W \).

So consider the following diagram, where we have tried to indicate how the "method of elements" makes it easier to parallel - in the general case - the above proof in \text{Set}. In the diagram we have named all the projections simply by their indices, since we will not need any more specific names for them. However, some are equipped with certain marks, for the
purpose of instant recognition. Notice that all projections in the diagram are split epimorphisms, and that all parts of the diagram involving only projections commute. In the formal deduction below, "\(a\)" and "\(\varphi\)" are abbreviations for the maps \(\exists (\epsilon \wedge c_{1.2})\) and \(\epsilon \Rightarrow \exists (\epsilon \wedge c_{1.2}) = \epsilon \Rightarrow a\), respectively, from \(P^2(X)^2 \times P(X)\) to \(\Omega\).

We will need the following observation in the proof below. Some explanatory comments are given in square brackets.

\[
\exists (\epsilon \wedge c_{1.2}) \wedge \forall (\varphi) \\
= \exists ((\epsilon \wedge c_{1.2}) \wedge \forall (\varphi)) \quad \quad \text{[by 1.7.3, because the following are pullbacks by 2.2.2 :]} \\
\]

\[
P^2(X)^2 \times P(X)^2 \xrightarrow{3} P^2(X)^3 \times P(X)^2 \xrightarrow{1,4} P^2(X)^2 \times P(X) \xrightarrow{4} P^2(X)^2 \times P(X) \xrightarrow{5} P^2(X)^2 \xrightarrow{3} P^2(X)^3 \times P(X) \xrightarrow{1,4} P^2(X)^2 \\
= \exists (\epsilon \wedge c_{1.2} \wedge \varphi_{1.4}) \quad \quad \text{[by 1.7.4, because } \forall (\varphi_{1.4}) \leq \varphi_{1.4}] \\
= \exists (\epsilon \wedge c_{1.2} \wedge (\epsilon \Rightarrow a_{1.4})) \\
= \exists (\epsilon \wedge c_{1.2} \wedge a_{1.4}) \quad \quad \text{[because } a \wedge (a \Rightarrow b) \leq b \text{ always]}.\]
We are now ready for the formal deduction:

\[ \zeta_3 \land \zeta_1 \]

\[ = \forall_3 (\varphi) \land \zeta_1 \]

\[ = \forall_4 (\varphi) \land \zeta_1 \]

[by 1.7.3, because
\[
\begin{array}{ccc}
P^2(X)^2 \times P(X) & \rightarrow & P^2(X)^3 \times P(X) \\
3 & \downarrow & 4' \\
P^2(X)^2 & \leftarrow & P^2(X)^3
\end{array}
\]

is a pullback, by 2.2.2]

\[ \leq \forall_4, (\varphi \land \zeta_{1,4}) \]

\[ = \forall_4, (\epsilon \land \zeta_{1,4}) \land \forall_3 (\varphi_{1,4}) \]

[by the above observation]

\[ = \forall_4, (\epsilon \land \zeta_{1,4}) \land \forall_3 (\varphi_{1,4}) \]

[by 1.7.3 and 2.2.2 applied to the cell
\[
\begin{array}{ccc}
P^2(X)^3 \times P(X)^3 & \rightarrow & P^2(X)^2 \times P(X) \\
6 & \downarrow & 4' \\
P^2(X)^3 \times P(X)^2 & \rightarrow & P^2(X)^2 \times P(X)
\end{array}
\]
Next we will prove the internal version of the following fact in Set: \( U < V \land V < W \rightarrow U < W \). We need to observe that \( \leq 0 < st \, 2, st \rightarrow = \forall (\varepsilon \cdot (st \times X) \Rightarrow \varepsilon \cdot (st \times X)) \), by definition of "\( \leq \)" (see 2.1.1(iii)), and so that
\[ \varepsilon \cdot (st \times X) = \varepsilon \cdot ((U \cdot \sigma) \times X) = \exists \_3 ((\varepsilon \_3 \land \varepsilon \_1) \cdot (\sigma \times P(X) \times X)) = \]

\[ = \exists \_3 ((\varepsilon \cdot (\sigma \times P(X))) \land \varepsilon \_1) = \exists \_3 ((\varepsilon \_2 \land \varepsilon \_1) \land \varepsilon \_1) = \]

\[ = \exists \_3 (\varepsilon \_2 \land \varepsilon \_1 \land \varepsilon \_2), \]

where \( \varepsilon \) is the composed map \( X \times P(X) \xrightarrow{\text{twist}} P(X) \times X \xrightarrow{\varepsilon} \Omega \).

The following lemma says, in \( \text{Set} \), that

\[ U < V \Rightarrow \forall x \in X (st(x, u) \leq st(x, v)) \).

2.2.6 Lemma: \( < \leq \forall (\varepsilon \circ st, st) \).

Proof: See the following diagram for the reading of this proof (page 41), where the maps \( 6_{\text{twist}} \) and \( 5_{\text{twist}} \) are, respectively,

\[ (P^2(X)^2 \times P(X) \times X \times P(X)) \xrightarrow{\text{twist}} (P^2(X)^2 \times P(X) \times P(X))^5 \]

and

\[ (P^2(X)^2 \times P(X) \times P(X))^5 \]

The statement of the lemma is equivalent to

\[ <\_3 \leq \circ \_1 < st_2, st > \text{, i.e. to } <\_3 \leq \forall (\varepsilon \circ (st \times X) \Rightarrow \varepsilon \circ (st \times X)) \text{ by the above remarks. And, since } st_2 \times X = (st \times X) \text{ and } st_1 \times X = (st \times X)_1 \text{, it is furthermore equivalent to } (\_4 <\_3 \leq \exists^4 (\varepsilon \_2 \land \varepsilon \_1 \land \varepsilon \_2) \Rightarrow \exists^4 (\varepsilon \_2 \land \varepsilon \_1 \land \varepsilon \_2) \text{, i.e. to } \]

\[ (*): (\_4 <\_3 \leq \exists^4 (\varepsilon \_2 \land \varepsilon \_1 \land \varepsilon \_2) \leq \exists^4 (\varepsilon \_2 \land \varepsilon \_1 \land \varepsilon \_2) \leq \exists^4 (\varepsilon \_2 \land \varepsilon \_1 \land \varepsilon \_2) \text{.} \]
Now we have, by repeated applications of 1.7.3 and 2.2.2, that

\[
(<_3 4) = (\forall (e \Rightarrow 3 (e \wedge c 1.2)) \wedge 3 (e \Rightarrow 3 (e \wedge c 1.2)) = \forall (e \Rightarrow 3 (e \wedge c 1.2)) = \forall (e \Rightarrow 3 (e \wedge c 1.2)) = \forall (2.3.5 \Rightarrow 3 (e \wedge c 1.2))
\]

and also that

\[
\exists (e \wedge \bar{e} \wedge e) = \exists ((e \wedge \bar{e} \wedge e) = \exists (e \wedge \bar{e} \wedge e)
\]

and that

\[
\exists (e \wedge \bar{e} \wedge e) = \exists (e \wedge \bar{e} \wedge e)
\]

We are now ready to prove (*) above:

\[
(<_3 4) \wedge \exists (e \wedge \bar{e} \wedge e) = \forall (e \Rightarrow 3 (e \wedge c 1.2)) \wedge \exists (e \wedge \bar{e} \wedge e)
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]

\[
\leq \exists (e \Rightarrow 3 (e \wedge c 1.2)) = \exists (e \Rightarrow 3 (e \wedge c 1.2))
\]
Using 2.2.6, we can now prove the following lemma.

2.2.7 Lemma: \( <_3 \wedge <^* \leq <^*_2 \).

Proof: The statement of the lemma is equivalent to

\[
\forall_3 \left( < o st <_{2,1} st \right) \wedge \forall_3 \left( \exists_3'' \left( < o <_{1,4} st \right) \wedge < o <_{2,3} p_{1,2,4} \right)
\]

\[
< \forall_3 \left( \exists_3'' \left( < o <_{1,4} st \right) \wedge < o <_{2,3} p_{1,2,4} \right)
\]

which must be read with an eye on the diagram below.

The formal deduction of the above goes as follows:

\[
\forall_3 \left( < o st <_{2,1} st \right) \wedge \forall_3 \left( \exists_3'' \left( < o <_{1,4} st \right) \wedge < o <_{2,3} p_{1,2,4} \right)
\]

\[
= \forall_4 \left( < o st <_{2,1} st \right) \wedge \exists_4'' \left( < o <_{1,2,5} st \right) \wedge < o <_{2,3} p_{1,2,4} \right)
\]

\[
\leq \forall_4 \left( \exists_4'' \left( < o st <_{2,1} st \right) \wedge < o <_{1,2,5} st \right) \wedge < o <_{2,3} p_{1,2,4} \right)
\]

\[
= \forall_4 \left( \exists_4'' \left( < o st <_{2,1} st \right) \wedge < o <_{1,2,5} st \right)
\]
\[
\forall \alpha, (\exists \beta \in \mathbb{E} \land \varphi \in \mathbb{F})
\]

as desired.

In the above we have made use of the following two facts

1) \( \forall \alpha, (\exists \beta \in \mathbb{E} \land \varphi \in \mathbb{F}) \)

2) \( \epsilon \varphi \in \mathbb{F} \land \varphi \in \mathbb{F} \leq \varphi \)
For 1) we need the following diagram:

\[ \begin{array}{c}
\text{(U, V, U, V, x)} \\
P^2(X)^2 \times P(X)^2 \times X \\
\downarrow 4^* \\
\text{(U, V, U, x)} \\
P^2(X)^2 \times P(X) \times X \\
\downarrow 3^* \\
\text{(U, V, x)} \\
P^2(X)^2 \times P(X) \\
\end{array} \]
in which we have that

\[ \langle^* = \langle o(st \times P^2(X)) = \forall_3 (\varepsilon_2 \rightarrow \exists_4 (\varepsilon_1, 3 \land \subset_2)) o(st \times P^2(X)) \]

\[ = \forall_3 ((\varepsilon_2 \rightarrow \exists_4 (\varepsilon_1, 3 \land \subset_2)) o(st \times P^2(X) \times P(X))) \]

\[ = \forall_3 (\varepsilon_0 (st \times P(X))) \rightarrow \exists_4 ((\varepsilon_1, 3 \land \subset_2)) o(st \times P^2(X) \times P(X))) \]

\[ = \forall_3 (\exists_3 (\theta_0 <p_{1, 3}, st_{2, 2}>) \rightarrow \exists_4 (\varepsilon_1, 3 \land \subset_2)) \]

\[ = \forall_3 (\forall_4 (\theta_0 <p_{1, 3}, st_{2, 2}>) \rightarrow \exists_4 (\varepsilon_1, 3 \land \subset_2)) \] [by 1.7.5]

\[ = \forall_3 (\forall_4 (\theta_0 <p_{1, 2, 4}, st_{2, 3}) \rightarrow \exists_4 (\varepsilon_1, 3 \land \subset_2)) (\exists_4 (\varepsilon_1, 3 \land \subset_2)) \]

\[ = \forall_3 (\forall_4 (\theta_0 <st_{2, 3}, p_{1, 2, 4}) \rightarrow \exists_4 (\varepsilon_1, 3 \land \subset_2)) \] [by 1.7.6]

\[ = \forall_3 (\forall_4 (\exists_4 (\varepsilon_1, 3 \land \subset_2)) o(st_{2, 3}, p_{1, 2, 4})) \]

\[ = \forall_3 (\forall_4 (\exists_4 (\varepsilon_1, 3 \land \subset_2))) \] [by 2.2.4]

\[ = \forall_3 (\exists_4 (\varepsilon_1, 3 \land \subset_2)) \]

as desired.
For 2) we merely observe that it is an instance of the transitivity of $c : P(X)^2 \to \Omega$. This completes the proof of 2.2.7.

The following lemma will be needed later, in the description of the completion. In Set it expresses the fact that $x \in \text{st}(x, U)$ if $x \in U$, for any $x \in X$, $U \in P^2(X)$.

2.2.8 Lemma: The composed map

\[
P^2(X) \times X \xrightarrow{U \times X} P(X) \times X \xrightarrow{\varepsilon} \Omega
\]

is equal to the composed map

\[
P^2(X) \times X \xrightarrow{\text{st}, \text{Id}} P(X) \times X \xrightarrow{\varepsilon} \Omega
\]

Proof: By definitions, $\varepsilon \circ (U \times X) = \exists (\varepsilon \land \varepsilon)$ and

\[
\varepsilon \circ \text{st}, \text{Id} = \varepsilon \circ (U \times X) \circ \sigma, \text{Id} = \exists (\varepsilon \land \varepsilon) \circ \sigma, \text{Id}
\]

\[
= \exists ((\varepsilon \land \varepsilon) \circ \sigma, \text{Id}) = \exists (\varepsilon \circ \sigma, \text{Id}) = \exists (\varepsilon \land \varepsilon),
\]

where we have also used the Beck condition for $\exists (1.7.3)$, applied to the pullback diagram

2.2.9 Corollary: The composed map

\[
P^2(X) \times P(X) \times X \xrightarrow{\sigma_2, \text{Id}, \sigma_1} P^2(X) \times P(X) \times X
\]

\[
\downarrow \varepsilon
\]

\[
P^2(X) \times X \xrightarrow{\sigma, \text{Id}} P^2(X) \times X
\]

\[
\downarrow \varepsilon
\]

\[
P^2(X) \times X \xrightarrow{\varepsilon \circ \text{st}, \text{Id}} P^2(X) \times X + \Omega.
\]
Proof: Just observe that \( \Theta \leq (\Theta \land \varepsilon) \circ (U \land \text{Id}_X) \leq \varepsilon \circ (U \land \text{Id}_X) = \varepsilon \circ (U \times X) \), by comparing successive pullbacks of subobjects.

The following lemma is the general version of the easy observation in \( \text{Set} \) that \( \text{st} (x, U \land V) = \text{st} (x, U) \cap \text{st} (x, V) \).

2.2.10 Lemma: The following diagram commutes:

\[
\begin{array}{ccc}
P^2(X)^2 \times X & \overset{\land \times X}{\longrightarrow} & P^2(X) \times X \\
\downarrow <\text{st}_2,\text{st}_1> & & \downarrow \text{st} \\
P(X)^2 & \overset{\cap}{\longrightarrow} & P(X)
\end{array}
\]

Proof: One proves that the two routes around the diagram correspond to the same map \( P^2(X)^2 \times X^2 \rightarrow \Omega \), namely to

\[
\exists (\varepsilon, 2, 4, 5, 6) \overset{\varepsilon, 1, 2, 4, 5 \varepsilon, 1, 3, 5, 6 \varepsilon, 1, 2, 3, 5}{\longrightarrow} P^2(X)^2 \times X^2 .
\]

where the quantification is along the projection \( P^2(X)^2 \times P(X)^2 \times X^2 \rightarrow P^2(X)^2 \times X^2 \). This is straightforward, and we omit it.

The following lemma expresses the fact, in \( \text{Set} \), that if \( U < U' \) and \( V < V' \) then \( U \land V < U' \land V' \).
2.2.11 Lemma: The map \( \langle 3.4 \rangle \land \langle 1.2 \rangle : P^2(X)^4 \rightarrow \Omega \)
is the composed map
\[
P^2(X)^4 \xrightarrow{\langle \land_{2.4} \land_{1.3} \rangle} P^2(X)^2 \xrightarrow{\langle \rangle} \Omega.
\]
Proof: By the adjointnesses \((\ )_5 \vdash \forall_5\) and
- \(\land \varphi \vdash \varphi \Rightarrow \)
and Beck's condition for \(\forall\), one sees that the given statement is equivalent to
\[
\langle 3.4.5 \rangle \land \langle 1.2.5 \rangle \land \epsilon_0(\langle \land_{2.4} \land_{1.3} \rangle \times P(X))
\]
\[
\langle 3.4 \rangle \land \langle 1.3 \rangle \times (\langle \land_{2.4} \land_{1.3} \rangle \times P(X)),
\]
which is established by straightforward application of the standard methods.

3. UNIFORMITIES

3.1 Basic concepts and facts. The idea of a uniform structure on a set \(X\) is due to André Weil, and was first exposed in [We], as a means for discussing uniform continuity and uniform convergence in connection with spaces not necessarily metric. Weil's approach, developed further by Bourbaki ([Bo]) and Kelley ([Ke]), was made in terms of entourages, or neighborhoods, of the diagonal in \(X \times X\).

Weil showed that the family of uniformly continuous pseudometrics on a uniform space suffices to determine the uniformity on the space. This has been used by some for a different approach to uniformities, defining them as families of pseudometrics satisfying certain axioms (see, for example, [GJ]).
Finally, Weil showed that the so-called uniform coverings of a space also suffice to determine the uniformity, and he gave a list of axioms on families of coverings of a set, characterizing those such families that arise from uniformities on the set, as the families of all uniform coverings.

This last approach to uniformities, modified by Tukey ([Tu]) and developed very far by Ginsburg, Isbell, Frolik, Hager and others (see [Is]), is the one we will use in describing what a uniform structure is on an object in a topos. So in the rest of this chapter, except where otherwise stated, X will be an (arbitrary but fixed) object in a (fixed but arbitrary) topos E.

A uniformity on a set X, in the sense of Tukey, is a family u of collections of subsets of X (i.e. \( u \in P^3(X) \)), satisfying the following axioms: 1) Each \( U \in u \) covers X, 2) if \( U, V \in u \) then \( U \cup V \in u \), 3) if \( U \in u \) and \( U < V \in P^2(X) \), then \( V \in u \), 4) if \( U \in u \) then \( V \prec U \) for some \( V \in u \), and 5) u is non-empty.

When translating the above axioms into the language of a topos, we encounter a problem with axiom 5, because there are so many different (and generally inequivalent) ways of describing non-emptiness. We will handle this problem by first defining uniformities without any form of axiom 5 (thereby admitting \( \emptyset \) as a uniformity on any set - a definitely unusual admission), and then listing several (in fact three) different versions of axiom 5 as possible extra hypotheses on uniformities.
3.1.1. **Definition**: A monomorphism \( u \) into \( P^2(X) \) is a uniformity on \( X \) if it satisfies the following axioms:

\((u1)\): The diagram \( \text{dom}(u) \xrightarrow{u} P^2(X) \xrightarrow{\text{id}_X} P(X) \) commutes.

\((u2)\): The composed map \( \text{dom}(u)^2 \xrightarrow{u^2} P^2(X)^2 \xrightarrow{\wedge} P^2(X) \) factors through \( u \),

\((u3)\): The intersection (pullback) of

\[ \text{dom}(u) \times P^2(X) \xrightarrow{u \times P^2(X)} P^2(X)^2 \]

factors through

\[ \bigodot \xrightarrow{} P^2(X)^2 \]

and

\[ P^2(X) \times \text{dom}(u) \xrightarrow{P^2(X) \times u} P^2(X)^2 \],

\((u4)\): The diagram

\[ \text{dom}(u) \xrightarrow{u} P^2(X) \leftarrow \pi_1 \text{dom}(u) \times P^2(X) \]

\[ \xrightarrow{} \exists_1(\bigodot \circ (u \times P^2(X))) \]

\[ 1 \xrightarrow{t} \Omega \]

commutes.

Moreover, \( u \) is an \( \{1\}-\text{uniformity} \), \( i \in \{1,2,3\} \), on \( X \) if it satisfies \((u_{5i})\) below:

\((u_{51})\): \( \text{dom}(u) \) is not an initial object,

\((u_{52})\): \( \text{dom}(u) \) has global support, i.e. \( \text{dom}(u) \to 1 \) is epi,

\((u_{53})\): \( \text{id}_{P^2(X)} : 1 \to P^2(X) \) factors through \( u \).
Obviously, \((u5_3) \Rightarrow (u5_2) \Rightarrow (u5_1)\) in a non-degenerate topos, i.e. one where 1 is not initial, for in that case there are no morphisms from 1 to 0 (see [Fr_1]). Moreover, these are the only implications that hold in general between the \((u5_i)\)'s.

3.1.2 Definition: If \(u\) is a uniformity on \(X\), we will write \(uX\) for the pair \((X,u)\), and call it a uniform space object (uso) in \(E\).

3.1.3 Definition: If \(uX\) and \(vY\) are uso's in \(E\), and \(f \in E(X,Y)\), we say \(f\) is uniformly continuous \((u.c.)\) from \(uX\) to \(vY\) if \(P(P^*(f))v\) factors through \(u\).

The following proposition is an immediate consequence of the functoriality of \(P_0P^*\):

3.1.4 Proposition: The uso's in \(E\), with all the u.c. morphisms, form a category \(Unif(E)\), which has a canonical forgetful functor to \(E\) acting as the identity on morphisms.

3.2 Bases for uniformities. Classically, i.e. in the topos \(Set\), a basis for a uniformity \(u\) on a set \(X\) is a subfamily \(b\) of \(u\) with the property that for each \(U \in u\) there is \(V \in b\) with \(V < U\). This property can be formalized in the language of \(E\), as in the following definition:

3.2.1 Definition: Let \(uX\) be a uso, and \(b\) some monomorphism into \(P^2(X)\). We say that \(b\) is a basis for \(u\) if
it factors through $u$ and makes the following diagram commute:

$$
\begin{array}{ccc}
\text{dom}(u) & \xrightarrow{u} & P^2(X) \\
\downarrow & & \downarrow \exists_1(<o(b \times P^2(X))) \\
1 & \xrightarrow{t} & \Omega \\
\end{array}
$$

3.2.2 Proposition: Any uniformity is a basis for itself. If a basis $b$ for a uniformity $u$ on $X$ factors through a monomorphism $b'$ into $P^2(X)$, and $b'$ factors through $u$, then $b'$ is also a basis for $u$. If $b$ is a basis for two uniformities, $u$ and $u'$, on $X$, then $u$ and $u'$ are equivalent (meaning that they factor through each other).

Proof: By 2.2.3, $<$ is a reflexive relation, so $< o(u \times P^2(X))$ is greater than or equal to the characteristic map of $\text{dom}(u) \times u : \text{dom}(u)^2 \rightarrow \text{dom}(u) \times P^2(X)$. Now the latter - and hence also the former - of these two maps is "true on $\text{dom}(u)$" when existentially quantified along $p_1 : \text{dom}(u) \times P^2(X) \rightarrow P^2(X)^2$, because $\text{dom}(u)^2 \xrightarrow{p_1} \text{dom}(u) \xrightarrow{u} P^2(X)$ is the epic-monic factorization of

$$
\text{dom}(u)^2 \xrightarrow{\text{dom}(u) \times u} \text{dom}(u) \times P^2(X) \xrightarrow{p_1} P^2(X)
$$

(notice that, in any category, projection maps of the form $A^2 \rightarrow A$ are epimorphisms).

The second statement of the proposition is a triviality.

For the third statement, it clearly suffices to show that if $b$ is a basis for $u$, and also factors through $u'$, the $u$
factors through \( u' \). So we assume that \( \exists_1(\langle o(b \times P^2(X)) \rangle) \) is "true on \( u \), which by 1.7.2 means that \( u \) factors through the monic part, \( k : K \rightarrow P^2(X) \), of the composed map

\[
\exists \odot (\text{dom}(b) \times P^2(X)) \rightarrow P^2(X)^2 \xrightarrow{p_1} P^2(X).
\]

So it suffices to show that \( k \) factors through \( u' \). And it does, because \( \exists \odot (\text{dom}(b) \times P^2(X)) \rightarrow P^2(X)^2 \) factors through \( \exists \odot (\text{dom}(u') \times P^2(X)) \rightarrow P^2(X)^2 \), by assumption (see the diagram below), so \( k \) factors through the monic part, \( k' : K' \rightarrow P^2(X) \), of the composed map \( \exists \odot (\text{dom}(u') \times P^2(X)) \rightarrow P^2(X)^2 \xrightarrow{p_1} P^2(X) \), and \( k' \) factors through \( u' \) because \( \exists \odot (\text{dom}(u') \times P^2(X)) \rightarrow P^2(X)^2 \) factors through \( P^2(X) \times \text{dom}(u') \rightarrow P^2(X) \), by axiom \((u_3)\), and \( p_1 \circ (P^2(X) \times u') = u' \circ \pi \), by naturality of projections.

This completes the proof of the proposition.

\[
\begin{array}{ccccccccc}
\odot \odot & (\text{dom}(b) \times P^2(X)) & \rightarrow & P^2(X) \times \text{dom}(u') & \rightarrow & P^2(X) \\
\downarrow & & & & & & & \downarrow \text{p}_1 \\
\odot \odot & (\text{dom}(b) \times P^2(X)) & \rightarrow & P^2(X)^2 & \rightarrow & P^2(X) \\
\downarrow & & & & & & & \downarrow \text{p}_1 \\
\text{dom}(u) & \rightarrow & u & \rightarrow & P^2(X) \\
\downarrow & & & & & & & \downarrow \text{p}_1 \\
K & \rightarrow & \text{dom}(u') & \rightarrow & P^2(X) \\
\downarrow & & & & & & & \downarrow \text{p}_1 \\
k & \rightarrow & K' & \rightarrow & \text{dom}(u') & \rightarrow & P^2(X) \\
\end{array}
\]
3.2.3 Corollary: If \( u \) is a uniformity on \( X \), and \( b \) is a basis for \( u \), then \( u \) is the smallest uniformity on \( X \) through which \( b \) factors. Hence, if \( \text{dom}(b) \) is an initial object, then so is \( \text{dom}(u) \).

3.2.4 Proposition: A monic \( b \) into \( P^2(X) \) is a basis for some uniformity if and only if the following two diagrams commute:

\[
\begin{array}{ccc}
\text{dom}(b) & \xrightarrow{b} & P^2(X) \\
\downarrow & & \downarrow u \\
1 & \xrightarrow{\text{id}_X} & P(X),
\end{array}
\]

\( (b1) \) : 

\[
\begin{array}{ccc}
dom(b)^2 & \xrightarrow{b^2} & P^2(X)^2 \xrightarrow{\Delta} P^2(X) \xleftarrow{\pi_1} \text{dom}(b) \times P^2(X) \\
\downarrow & & \downarrow \exists_1(\lhd \circ (b \times P^2(X))) \\
1 & \xrightarrow{t} & \Omega
\end{array}
\]

\( (b2) \) : 

in which case that uniformity, unique by last proposition, is the subobject of \( P^2(X) \) classified by the vertical map in the diagram.
Moreover, that uniformity satisfies \((u_5^1)\) if and only if

\((b_3^1)\): \(\text{dom}(b)\) is not an initial object, it satisfies \((u_5^2)\) if and only if
\[(b_3^2)\): \(\text{dom}(b)\) has global support, and, finally, it satisfies \((u_5^3)\) if (and, provided the topos satisfies AC, only if)

\[(b_3^3)\): there exists a mpa from 1 to \(\text{dom}(b)\).

**Proof**: First suppose \(b\) is a basis for a uniformity \(u\) on \(X\). Since \(b\) then factors through \(u\), \((b1)\) commutes by axiom \((u_1)\) for \(u\), and, since \(b^2\) factors through \(u^2\), for \((b2)\) it suffices to prove that the following diagram commutes:

\[
\begin{array}{ccc}
\text{dom}(u)^2 & -\stackrel{u^2}{\longrightarrow} & P^2(X)^2 \\
\downarrow & & \downarrow \\
1 & -\stackrel{t}{\longrightarrow} & \Omega \\
\end{array}
\]

so by axiom \((u_2)\) it suffices to prove that \(\exists_1 \left( \triangleright o(b \times P^2(X)) \right)\) is "true on \(u\". By assumption, \(\exists_1 \left( \triangleright o(b \times P^2(X)) \right)\) is "true on \(u\"", so the desired result now follows with the help of \((u_4)\) and 2.2.7.

Now suppose only that \(b\) makes \((b1)\) and \((b2)\) commute, and let \(u(b)\) be a monomorphism into \(P^2(X)\) with characteristic map \(\exists_1 \left( \triangleright o(b \times P^2(X)) \right) : P^2(X) \rightarrow \Omega \). We must then show that 1) \(u(b)\) is a uniformity on \(X\) and 2) \(b\) is a basis for it.

To prove 2), we show first that \(b\) factors through \(u(b)\), or, in other words, that \(\exists_1 \left( \triangleright o(b \times P^2(X)) \right)\) is "true on \(b\". This,
by 1.7.2 is equivalent to \( b \) factoring through the monic part of \( A \xrightarrow{a} \text{dom}(b) \times \mathbb{P}^2(X) \xrightarrow{\text{dom}(b) \times b} \text{dom}(b) \times \mathbb{P}^2(X) \), where \( a \) represents the subobject of \( \text{dom}(b) \times \mathbb{P}^2(X) \) classified by \( \varphi \circ (b \times \mathbb{P}^2(X)) \). Notice that \( a \) can be obtained as the pullback of \( \bigotimes \rightarrow \mathbb{P}^2(X)^2 \) along \( b \times \mathbb{P}^2(X) \). Now we show more than is strictly needed, namely that \( b \) factors through \( p_1 \circ a \) itself. First,

\[
\text{dom}(b) \xrightarrow{\Delta \text{dom}(b)} \text{dom}(b)^2 \xrightarrow{\text{dom}(b) \times b} \text{dom}(b) \times \mathbb{P}^2(X)
\]

is a factoring of \( b \) through \( p_1 \). Second,

\[
(b \times \mathbb{P}^2(X)) \circ (\text{dom}(b) \times b) \circ \Delta \text{dom}(b) = b^2 \circ \Delta \text{dom}(b)
\]

factors through \( \Delta \mathbb{P}^2(X) \), and hence through \( \bigotimes \rightarrow \mathbb{P}^2(X)^2 \) (by reflexivity of \( \ll ; \) see 2.2.5). So \((\text{dom}(b) \times b) \circ \Delta \text{dom}(b)\) factors through the pullback of \( \bigotimes \rightarrow \mathbb{P}^2(X)^2 \) along \( b \times \mathbb{P}^2(X) \), yielding a factorization of \( b \) through \( p_1 \circ a \).

Then, before completing the proof of 2), we prove 1), and first of all we observe that \( u(b) \) satisfies \((u1)\), by the above and by axiom \((b1)\) for \( b \).

To show that \( u(b) \) satisfies \((u2)\), notice that the subobject \( u(b)^2 \) of \( \mathbb{P}^2(X)^2 \) can be obtained as the intersection of the pullbacks of \( u(b) \) along the two projections \( \mathbb{P}^2(X)^2 \rightarrow \mathbb{P}^2(X) \). So the characteristic map for \( u(b)^2 \) is

\[
\exists_1 (\ll \circ (b \times \mathbb{P}^2(X))) \land \exists_1 (\ll \circ (b \times \mathbb{P}^2(X))) , \text{ which is equal to }
\]

\[
\exists_{1,2} (\ll_{3,4} \land \ll_{1,2} \circ (b^2 \times \mathbb{P}^2(X)^2)) . \text{ Now, by 2.2.11, the subobject classified by this (i.e. } u(b)^2 \text{) factors through the subobject classified by }
\]

\[
\exists_{1,2} (\ll \circ (\land_{3,4}, \land_{1,2} \circ (b^2 \times \mathbb{P}^2(X)^2)) = \exists_{1,2} (\ll \circ ((\land_{3,4} \circ b^2) \times \land_{1,2} ))
\]

\[
\leq \exists_1 (\ll \circ (b \times \land_1 )) \leq \exists_1 (\ll \circ (b \times \land_1 )) .
\]
Hence $\wedge_0 u(b)^2 : \text{dom}(u(b))^2 \to P^2(X)$ factors through the subobject classified by $\exists_1 (< o(b \times P^2(X)))$, i.e. through $u(b)$, as desired.

Now, to show that $u(b)$ satisfies $(u3)$, observe that the characteristic map of $u(b) \times P^2(X)$ is $\exists_1 (< o(b \times P^2(X)))_2$, and that of $P^2(X) \times u(b)$ is $\exists_1 (< o(b \times P^2(X)))_1$, so we must prove that $\exists_1 (< o(b \times P^2(X)))_2 \wedge < \leq \exists_1 (< o(b \times P^2(X)))_1$.

This follows from transitivity of $<$, as follows:

\[
\exists_1 (< o(b \times P^2(X)))_2 \wedge < = \exists_1 (< o(b \times P^2(X)^2)) \wedge < \\
= \exists_1 ( <_3 o(b \times P^2(X)^2) \wedge <_1 ) \quad \text{(by Frobenius reciprocity)} \\
= \exists_1 ( <_3 \wedge <_1 ) o(b \times P^2(X)^2)) \\
\leq \exists_1 ( <_2 o(b \times P^2(X)^2)) \quad \text{(by transitivity of $<$)} \\
= \exists_1 ( < o(b \times P^2(X)))_1 , \quad \text{as desired.}
\]

The following diagram where $i \in \{1,2\}$ illustrates the above proof:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$\text{dom}(b) \times P^2(X)^2$};
\node (b) at (3,0) {$P^2(X)^2$};
\node (c) at (3,-3) {$P^2(X)$};
\node (d) at (6,-3) {$P^2(X)\times \cdot$};
\node (e) at (-3,-3) {$b \times P^2(X)^2$};
\node (f) at (0,-6) {$P^2(X)^3$};
\node (g) at (-3,-6) {$P^2(X)$};
\node (h) at (3,-6) {$P^2(X)^2$};

\draw[->] (a) to node [above] {$p_1$} (b);
\draw[->] (a) to node [below] {$p_3$} (e);
\draw[->] (b) to node [right] {$p_1$} (c);
\draw[->] (b) to node [below] {$p_3$} (d);
\draw[->] (c) to node [right] {$p_1$} (d);
\draw[->] (d) to node [above] {$\pi_3$} (h);
\draw[->] (d) to node [below] {$\pi_3$} (g);
\draw[->] (e) to node [left] {$\wedge_0 u(b)^2$} (f);
\draw[->] (f) to node [right] {$\wedge_3$} (h);
\draw[->] (g) to node [left] {$<_1$} (c);
\draw[->] (g) to node [right] {$<_3$} (f);
\draw[->] (h) to node [above] {$<_3$} (e);
\draw[->] (h) to node [below] {$<_1$} (a);
\end{tikzpicture}
\end{center}

So $u(b)$ satisfies $(u3)$. 
For \((u^4)\) it suffices to observe that, by \((b^2)\),
\[ \exists_1 (\langle^*o(b \times P^2(X))) \leq \exists_1 (\langle^*o(u(b) \times P^2(X))) \]
so \[ \exists_1 (\langle^*o(u(b) \times P^2(X))) \circ u(b) \geq \exists_1 (\langle o(b \times P^2(X))) \circ u(b) , \]
which factors through \(1 \rightarrow \Omega\) by construction of \(u(b)\).

This completes the proof of 1). Now we observe that the rest of the proof of 2) - that \(b\) in fact is a basis for \(u(b)\) - is a triviality, by the way we defined bases in general.

To complete the proof of 3.2.4, we show that \(u(b)\) satisfies \((u^5_1)\) if and, assuming \(AC\) when \(i = 3\), only if \(b\) satisfies \((b^3_1)\), \(i \in \{1,2,3\}\).

For \(i = 1\), this is a triviality, since \(0 \times P^2(X)\) is a uniformity on \(X\) always.

For \(i = 2\), "if" is trivial, and for "only if" it suffices to show that \(\text{dom}(b) \times P^2(X)\) has global support, which it has because a subobject of it, namely \(\bigotimes n(\text{dom}(b) \times P^2(X))\), maps by an epimorphism to \(\text{dom}(u(b))\), which we assume has global support.

This is illustrated in the diagram

```
\[ \bigotimes n(\text{dom}(b) \times P^2(X)) \rightarrow \text{dom}(b) \times P^2(X) \rightarrow 1 \]
\[ \bigotimes \rightarrow \text{dom}(b) \times P^2(X) \rightarrow 2 \]
\[ b \times P^2(X) \rightarrow \]
\[ \bigotimes \rightarrow P^2(X)^2 \rightarrow \]
\[ \bigotimes \rightarrow \Omega \rightarrow \]
```

\(\bigotimes\)
For \( i = 3 \), assume first that there is a map \( 1 \to \text{dom}(b) \).

Then there is one \( 1 \to \text{dom}(u(b)) \). Now notice that the following diagram commutes:

\[
\begin{array}{ccc}
1 & \xrightarrow{(u(b) \circ c, \text{Id}_{\text{P}(X)})} & \text{P}^2(X)^2 \\
\downarrow_{t} & & \downarrow_{\Omega} \\
\end{array}
\]

so \((u_5)_i\) follows from \((u_3)\) for \(u(b)\). Then assume there is a map \( c : 1 \to \text{dom}(u(b)) \), and that the topos satisfies AC. Then the epimorphism \( \bowtie \cap (\text{dom}(b) \times \text{P}^2(X)) \to \text{dom}(u(b)) \) splits, and we get a map \( 1 \to \text{dom}(b) \) as follows:

\[
\begin{array}{ccc}
1 & \xrightarrow{c} & \text{dom}(u(b)) \\
\xleftarrow{\bowtie \cap (\text{dom}(b) \times \text{P}^2(X))} & & \text{dom}(b) \times \text{P}^2(X) \\
\downarrow & & \downarrow^{2} \\
\text{dom}(b) & \to & \text{dom}(b). \\
\end{array}
\]

This completes the proof of 3.2.4.
4. THE COMPLETION

4.1 Filters. Traditionally, a filter on a set $X$ is a non-empty set of non-empty subsets of $X$, closed under enlargements and binary intersections.

These properties can be expressed in the language of any topos, for any object $X$ and any monomorphism $F : \text{dom}(F) \rightarrow \mathcal{P}(X)$. However, we must then again deal with the ambiguity of non-emptiness, only this time we have it two-fold. For the filter itself, and for its members, we can interpret non-emptiness as (a) being non-initial, (b) having global support, and (c) having a global section (i.e. a point or a morphism from $1$). For the members of the filter we may also consider a fourth, possibly natural, non-emptiness-condition, namely (d) "$F$ is disjoint from $\emptyset$", in the sense that

\[
\begin{array}{ccc}
0 & \rightarrow & \text{dom}(F) \\
\downarrow & & \downarrow \\
1 & \overset{\circ X}{\rightarrow} & \mathcal{P}(X)
\end{array}
\]

is a pullback.

Clearly, (c) $\Rightarrow$ (b) $\Rightarrow$ (a), and it is easy to see that also (d) $\Rightarrow$ (a).

It is instructive to interpret the above four conditions on $F$ in a space-topos, $\text{Sh}(T)$, so let $X$ be a given sheaf on the space $T$, and let $\text{e.e.}(X) \rightarrow T$ be the associated espace étalé over $T$. Then a monomorphism $F$ into $\mathcal{P}(X)$ yields a local homeomorphism $f : \text{e.e.}(F) \rightarrow \text{e.e.}(\mathcal{P}(X))$, which, being one-to-one, is the embedding of an open subset of $\text{e.e.}(\mathcal{P}(X))$. Now, as far as
F itself is concerned, condition (a) above means simply that e.e.(F) is nonempty, condition (b) means that the composed map e.e.(F) \( \xrightarrow{f} \) e.e.(P(X)) \( \xrightarrow{P(x)} \) T is onto, and condition (c) means that there is a continuous map (a global section) \( s : T \to e.e.(F) \) satisfying \( P(x) \circ s = \text{id}_T \).

For an interpretation of conditions (a)-(d) on the "members" of F, consider the family \( \mathcal{F} \) of those open subsets of e.e.(P(X)) that correspond to global sections of P(x)off. Then condition (a) means that \( \emptyset \notin \mathcal{F} \), condition (b) means that each \( U \in \mathcal{F} \) is mapped onto T by P(x), (c) means that for each \( U \in \mathcal{F} \) there is a global section of P(x) mapping T into U, and (d) means that no open subset of e.e.(P(X)) has image in T disjoint from the image of any member of \( \mathcal{F} \), which is equivalent to the condition that each \( U \in \mathcal{F} \) is mapped by P(x) onto a dense subset of T. The latter condition is what some call "\( \tau \)-dense" (see, for example [Fr1]).

We will follow Stout ([St]) in adopting condition (c) for members of filters, but the nonemptiness of the filters themselves will be dealt with as that of uniformities and uniformity-bases, by allowing consideration of three different possibilities.

Thus we have the following.

4.1.1 **Definition**: Let X be a given object in a topos. A **filter** on X is a monomorphism F into P(X) satisfying the condition that the diagram
commutes for each of the following three choices of \( \psi \):

(f1) : \( \forall_{2,3} (\varepsilon_3 \land \varepsilon_2 \Rightarrow \varepsilon \circ (P^2(X) \times \Omega)) \), where the quantification is along \( P^2(X) \times P(X)^2 \xrightarrow{2.3} P^2(X) \),

(f2) : \( \forall_{2,3} (\varepsilon_3 \land c \Rightarrow \varepsilon_2) \), where remark to (f1) applies,

(f3) : \( \forall_2 (\varepsilon_3 \Rightarrow \exists_3 (\varepsilon_1)) \), where the quantifications are along \( P^2(X) \times P(X) \times X \xrightarrow{3} P^2(X) \times P(X) \xrightarrow{2} P^2(X) \).

Moreover, a filter \( F \) on \( X \) is an \( i \)-filter on \( X \) if it satisfies \((f4_1)\) below:

\( (f4_1) : \) \( \text{dom}(F) \) is not an initial object,

\( (f4_2) : \) \( \text{dom}(F) \) has global support,

\( (f4_3) : \) \( \text{id}_X \) : \( 1 \rightarrow P(X) \) factors through \( F \).

4.1.2 Remarks : Axiom (f1) means that "\( F \) is closed under binary intersections", (f2) means that "\( F \) is closed under enlargements", and (f3) means "there is something in each member of \( F \)". Moreover, \((f4_1)\) means that "\( F \) is not empty", \((f4_2)\) means that "there is something in \( F \)", and \((f4_3)\) means that "\( F \) has an element (a "point")".

We will need equivalent versions of \((f4_i), \ i \in \{1,2,3\}\), which are statements of the same form as (f1) - (f3). This is provided by the following proposition.
4.1.3 Proposition: A monomorphism $F$ into $P(X)$ satisfies $(f^4_1)$ if (provided, when $i = 1$, that the topos is two-valued) and only if the following diagram commutes

$$
\begin{array}{ccc}
1 & \xrightarrow{t} & \Omega \\
\downarrow & & \downarrow \psi \\
P^2(X) & \xrightarrow{\text{for } i = 1} & \Omega
\end{array}
$$

where $\psi$ - for $i = 1$ - is the map $\neg \circ \text{char}(\neg \circ P(X)^{-1})$, for $i = 2$ $\psi$ is $\exists (\varepsilon)$, where the quantification is along $P^2(X) \times P(X) \xrightarrow{2} P^2(X)$, and for $i = 3$ $\psi$ is the composed map

$$
P^2(X) \xrightarrow{\langle \text{id, id}_X \rangle} P^2(X) \times P(X) \xrightarrow{\varepsilon P(X)} \Omega.
$$

Proof: For $i = 1$, assume first that $\text{dom}(F)$ is an initial object. Then $\psi \circ F = \neg \circ \text{char}(\neg \circ P(X)^{-1}) \circ F = \neg \circ \text{char}(\neg \circ P(X)^{-1}) \circ P(X)$ $= \neg \circ t \neq t$, so the diagram does not commute.

Conversely, assuming the diagram does not commute, and that the topos is two-valued, we have that $\text{char}(\neg \circ P(X)^{-1}) \circ F = t$, since $\neg \circ \text{char}(\neg \circ P(X)^{-1}) \circ F \neq t$. So, in fact, $F$ is (equivalent to) $\circ P(X)$, i.e. $\text{dom}(F)$ is initial.

For $i = 2$ the proposition is simply an instance of 1.7.8.

Finally, for $i = 3$, it suffices to observe that the following diagram commutes:

$$
\begin{array}{ccc}
1 & \xrightarrow{\langle \text{id, id}_X \rangle} & P^2(X) \\
\downarrow & & \downarrow \text{char}(F) \\
1 \times P(X) & \xrightarrow{\text{char}(F)} & \Omega
\end{array}
$$
4.2 **Cauchy filters.** Given a uniform space $uX$ in $\text{Set}$, and a filter $F$ on $X$, $F$ is called a Cauchy filter on $uX$ if it contains a member of each uniform cover of $uX$, i.e. if it intersects each $U \in u$ (see [Is]). In attempting to internalize this definition, to make it meaningful in any topos, we encounter the same problem as the one we discussed in connection with the filter axioms, namely the uncertainty of which degree of non-emptiness is correct, useful or in other ways desirable.

For the sake of compatibility, we will use the same kind of non-emptiness as the one we chose for axiom $(f3)$ in 4.1.1.

Thus we have the following definition.

**4.2.1 Definition**: Let $uX$ be a uso in a topos, $F$ a filter on $X$. Then $F$ is a Cauchy filter on $uX$ if the following diagram commutes:

\[
\begin{array}{ccc}
1 & \xrightarrow{FP} & P^2(X) \\
\downarrow & & \downarrow \\
\forall_2(\varepsilon \circ \bar{u}, \text{id}_{P^2(X)})_1 & \Rightarrow & \exists_3(\varepsilon_2 \land \varepsilon_1)
\end{array}
\]

where the quantifications are made along the maps

\[
P^2(X)^2 \times P(X) \xrightarrow{3} P^2(X)^2 \xrightarrow{2} P^2(X), \quad \text{and} \quad \bar{u} \text{ is the composed map}
\]

\[
P^2(X) \xrightarrow{1} \bar{u} \xrightarrow{3} P^3(X).
\]

A Cauchy $i$-filter on $uX$ is a Cauchy filter on $uX$, which is also an $i$-filter on $X$, $i \in \{1, 2, 3\}$. \[
\]
4.2.2 Notation: In the next section we will need the notion of a minimal Cauchy filter, by which is meant, in Set, a Cauchy filter \( F \) on a uniform space \( uX \), with the property that if \( F' \) is also a Cauchy filter on \( uX \), and \( F' \subseteq F \), then \( F' = F \).

For the purpose of internalizing this definition, let us write \( (Cf) \) for the conjunction of the four defining formulas for Cauchy filters, and \( (Cf_1) \) for the conjunction of \( (Cf) \) with the remaining defining formula for \( i \)-filters, as given in 4.1.3. Thus \( (Cf_1) \) is available only in two-valued topos, while \( (Cf_2) \) and \( (Cf_3) \) are always available.

Explicitly, \( (Cf) \) is the morphism

\[
\forall_{2,3} (\varepsilon_3 \land \varepsilon_2 \Rightarrow \varepsilon \circ (P^2(X) \times \Omega))
\]

\[
\land \forall_{2,3} (\varepsilon_3 \land \varepsilon_1 \Rightarrow \varepsilon_2)
\]

\[
\land \forall_2 (\varepsilon \Rightarrow \exists (\varepsilon_3))
\]

\[
\land \forall_2 (\varepsilon \circ <\bar{u}, \text{id}_{P^2(X)}>_1 \Rightarrow \exists (\varepsilon_3 \land \varepsilon_1)) : P^2(X) \rightarrow \Omega,
\]

\( (Cf_1) \) - in two-valued topos - is

\( (Cf) \land \neg \circ \text{char}(\circ_{P(X)}) \), \( (Cf_2) \) is \( (Cf) \land \exists_2 (\varepsilon) \) and

\( (Cf_3) \) is \( (Cf) \land \varepsilon \circ <\text{id}_{P^2(X)}, \text{id}_X> \).

With this notation established we are ready to define minimal Cauchy filters.

4.2.3 Definition: A Cauchy filter \( F \) on a uso \( uX \) in a topos is a minimal Cauchy filter on \( uX \) if the following diagram
commutes:

\[
\begin{array}{c}
1 \\
\downarrow \\
\forall_1((\text{Cf})_2 \land \varepsilon \Rightarrow \Theta_{P^2(X)}) \\
\downarrow \\
\Omega
\end{array}
\]

where the quantification is along the projection \( P^2(X)^2 \rightarrow P^2(X) \).

For \( i \in \{1,2,3\} \), a Cauchy \( i \)-filter \( F \) on \( uX \) is a **minimal Cauchy \( i \)-filter** on \( uX \) if it makes commutative the diagram obtained from the above by replacing "(Cf)" with "(Cf\(_i\))".

The **object of minimal Cauchy filters on \( uX \)** is the subobject \( \gamma_{uX} : P^2(X) \rightarrow P^2(X) \) classified by \( (\text{Cf}) \land \forall_1((\text{Cf})_2 \land \varepsilon \Rightarrow \Theta) : P^2(X) \rightarrow \Omega \) (notice that, for each \( uX \), we have to choose such a monomorphism \( \alpha_{uX} \) from its equivalence class).

The **object of minimal Cauchy \( i \)-filters on \( uX \)** is the subobject \( \gamma_{i,uX} : P^2(X) \rightarrow P^2(X) \) classified by \( (\text{Cf}_i) \land \forall_1((\text{Cf}_i)_2 \land \varepsilon \Rightarrow \Theta) : P^2(X) \rightarrow \Omega \).

As shown in [Bo] for the topos \( \text{Set} \), every Cauchy filter on a uniform space contains a unique minimal one. This is true in general, and to express it we need the following definition:

4.2.4 **Definition**: \( St : P^2(X) \times P(X) \rightarrow P(X) \) is the map corresponding to the formula

\[
\exists_{2,4}(\varepsilon_{3,4,5} \land \varepsilon_{1,3,5} \land \varepsilon_{1,2,5} \land \varepsilon_{1,2,4}) : P^2(X) \times P(X) \times X \rightarrow \Omega
\]

where the quantification is along the projection

\[
P^2(X) \times P(X)^n \times X^2 \rightarrow P^2(X) \times P(X) \times X.
\]

(In \( \text{Set} \), \( St(U,V) \) is
the set of all $x$ for which there are $x'$ and $U$ with $U \in U$ and $x' \in U$ and $x' \in V$ and $x \in V$, i.e. it is $U \{U \in U \mid U \cap V \neq \emptyset \}$.)

4.2.5 Proposition: If $F$ is a Cauchy filter on a uX, let $F_m$ be a subobject of $\mathcal{P}(X)$ classified by the formula

$\exists_{1,2}(c \circ ((\text{St} \circ (u \times F)) \times \mathcal{P}(X))) : \mathcal{P}(X) \rightarrow \Omega$

where the quantification is along the projection

$\text{dom}(u) \times \text{dom}(F) \times \mathcal{P}(X) \xrightarrow{1,2} \mathcal{P}(X)$. Then $F_m$ is also a Cauchy filter on uX, and it is contained in very Cauchy filter on uX contained in $F$. Moreover, if $u$ is an $i$-uniformity, and $F$ is an $i$-filter, then $F_m$ is also an $i$-filter ($i \in \{1,2,3\}$).

Proof: (sketch): Axiom (f2) is trivially satisfied by $F_m$, and axiom (f3) follows from the formalization of the fact that "each member of $F_m$ contains a member of $F$", i.e. from the fact that

$\exists_{1}(c \circ (F \times F_m)) = t_{\text{dom}(F_m)}$, which itself follows from the fact that the composed map

$\mathcal{P}^2(X) \times \mathcal{P}(X) \xrightarrow{<p, \text{St}>} \mathcal{P}(X)^2 \xrightarrow{\subseteq} \Omega$

is $t_{\mathcal{P}^2(X) \times \mathcal{P}(X)}$. Axiom (f1) follows from the fact that the map

$\mathcal{P}^2(X)^2 \times \mathcal{P}(X)^2 \xrightarrow{\wedge \times \text{n}} \mathcal{P}^2(X) \times \mathcal{P}(X) \xrightarrow{\text{St}} \mathcal{P}(X)$

is $<$ the map

$\mathcal{P}^2(X)^2 \times \mathcal{P}(X)^2 \xrightarrow{<\text{St}_{2,4}, \text{St}_{1,3}>} \mathcal{P}(X)^2 \xrightarrow{\text{n}} \mathcal{P}(X)$. The fact that $F_m$ is Cauchy on uX is easy to prove in Set (see [Bo]), and essentially - except for the time and space needed -
just as easy in general. The same holds for minimality of $F_m$ — see [Bo].

Finally, for the statement about i-filters, just notice that there exists a morphism from $\text{dom}(u) \times \text{dom}(F)$ to $\text{dom}(F_m)$, because $\text{dom}(u) \times \text{dom}(F) \xrightarrow{u \times F} P^2(X) \times P(X) \xrightarrow{\text{st}} P(X)$ factors through $F_m$.

4.2.4 Notation : We write $\eta_{ux}$ for the composed map

$$ X \xrightarrow{\langle \bar{u}, \{\cdot\} \rangle} P^3(X) \times P(X) \xrightarrow{r} P(P^2(X) \times X) \xrightarrow{P(\text{st})} P^2(X). $$

4.2.5 Proposition : For any uso $uX$, $\eta_{ux}$ corresponds to the map

$$ \exists_{X \times \text{st}}(\text{char}(X \times u) \land \theta_2) : X \times P(X) \to \Omega $$

where the quantification is along the map

$$ X \times \text{st} : X \times P^2(X) \times X \to X \times P(X). $$

Proof : Consider the diagram on page 70, where the top and right-hand edges form the map corresponding to $\eta_{ux}$. The two square cells are clearly pull-backs, and it is also easy to check that the three triangular cells below them commute — the left-hand one by definition of $\bar{u}$ and $\{\cdot\}$, the middle one trivially, and the right-hand one by definition of $r$ (2.1.1(i)). The desired statement now follows by the Beck condition for $\exists$ (1.7.3).
4.2.6 Remark: In Set, for a given point \( x \) in a uniform space \( uX \), \( \eta_{uX}(x) \) is the family \( \{ \text{st}(x, U) | U \in u \} \) of subsets of \( X \). We prove now in Set that \( \eta_{uX}(x) \) is a minimal Cauchy filter on \( uX \), and that it is a minimal Cauchy i-filter if \( u \) is an i-uniformity on \( X \) \((i \in \{1, 2, 3\})\).

Firstly, \( \eta_{iX}(x) \) is closed under binary intersections, because \( u \) is closed under binary meets, and \( \text{st}(x, U) \cap \text{st}(x, V) = \text{st}(x, U \land V) \) always (see 2.2.8). Secondly, if \( A \supseteq \text{st}(x, U) \in \eta_{uX}(x) \), then \( A = \text{st}(x, U \cup \{A\}) \) is in \( \eta_{uX}(x) \), because \( U \cup \{A\} \) is in \( u \) if \( U \) is. Thirdly, \( \text{st}(x, U) \) is non-empty for each \( U \in u \), because \( x \in \text{st}(x, U) \) for any non-empty family \( U \) of subsets of \( X \), and \( U \) is non-empty since it covers \( X \). Fourthly, assuming \( u \) is a 1-uniformity on \( X \), \( u \) contains a covering \( U_o \), so \( \eta_{uX}(x) \) contains \( \text{st}(x, U_o) \), i.e. \( \eta_{uX}(x) \) is also non-empty, i.e. \( \eta_{uX}(x) \) is a
1-filter on $X$. The cases $i = 2$ and $i = 3$ coincide with the case $i = 1$ in $\text{Set}$. Fifthly, given $U_0 \in u$, we must show that $\eta_{uX}(x) \cap U_0 \neq \emptyset$. Let $V \in u$ be a star-refinement of $U_0$, and let $U_0 \in U_0$ be such that $\text{st}(x, V) \subseteq U_0$. Then, with $U = V \cup \{ U_0 \}$, we have that $U \in u$ and $U_0 = \text{st}(x, V) \cup U_0 = \text{st}(x, U) \in \eta_{uX}(x)$.

Finally, let $F$ be a Cauchy filter on $uX$, contained in $\eta_{uX}(x)$. If $u$ is empty then so is $\eta_{uX}(x)$, in which case $F = \eta_{uX}(x)$. If $u$ is non-empty, let $U_0 \in u$, and let $U_0 \in F \cap U_0$. Then $x \in U_0$, since $U_0$ is of the form $\text{st}(x, V)$ with $V \in u$. Hence $U_0 \subseteq \text{st}(x, U_0)$, so $\text{st}(x, U_0) \in F$ since $F$ is closed under enlargements. Hence $F = \eta_{uX}(x)$ also in this case, so in each case $\eta_{uX}(x)$ is minimal among the Cauchy filters on $uX$. And then, of course, when $u$ is a 1-uniformity, in which case we have seen that $\eta_{uX}(x)$ is a Cauchy 1-filter, it is clearly a minimal such. This completes the proof, in $\text{Set}$.

The following proposition states that the fact we just proved in $\text{Set}$ holds in any topos, i.e. if $uX$ is a uso in a topos then $\eta_{uX}$ factors through $a_{uX}$, i.e. $\eta_{uX}$ takes its values among the minimal Cauchy filters on $uX$.

4.2.7 Proposition: If $uX$ is a uso in a topos, then $\eta_{uX}$ factors through $a_{uX}$. If, moreover, $u$ is an $i$-uniformity on $X$, and - in the case $i = 1$ - the topos is two-valued, $i \in \{1, 2, 3\}$, then $\eta_{uX}$ factors through $a_{1, uX}$.

Proof: We must prove that $\psi \circ \eta_{uX} = t_X$ for each $\psi$ in the set of defining axioms for minimal Cauchy $(i-)\text{filters}$ on $uX$. 
As in \textit{Set}, axiom (f1) is true on $u_{\text{ux}}$ by 2.2.8. We omit the detailed verification.

For axiom (f2), we again imitate the proof in \textit{Set}. By 4.2.5,

\[ \forall_{2.3} (\epsilon_3 \subset_1 \Rightarrow \epsilon_2) \circ u_{\text{ux}} = t_X \text{ if and only if the pullback of } u_{\text{ux}} \]

along $P^2(X) \times P(X)^2 \overset{2,3}{\rightarrow} P^2(X)$, when composed with $\epsilon_3 \subset_1 \Rightarrow \epsilon_2$, yields "true". And that pullback, of course, is simply

\[ u_{\text{ux}} \times P(X)^2 : X \times P(X)^2 \rightarrow P^2(X) \times P(X)^2, \]

so we must show that

\[ (\epsilon_3 \subset_1 \Rightarrow \epsilon_2) \circ (u_{\text{ux}} \times P(X)^2) = t_X \times P(X)^2. \]

This is equivalent, by the usual manipulations, to $\subseteq \circ ((\text{st} \circ (u \times X)) \times P(X))$

\[ \subseteq \exists_{(\theta \circ P(X)) \circ ((u_2 \circ (u' \times X)) \times P(X))) \]

in the lattice of maps

\[ \text{dom}(u) \times X \times P(X) \rightarrow \mathbb{u}, \]

where the existential quantification is along the projection map $\text{dom}(u)^2 \times X \times P(X) \overset{1}{\rightarrow} \text{dom}(u) \times X \times P(X)$.

The last statement above follows from the fact that the subobject of $\text{dom}(u) \times X \times P(X)$ classified by $\subseteq \circ ((\text{st} \circ (u \times X)) \times P(X))$ is contained in (i.e. factors through) the equalizer of $p_{1,2}$ and

\[ \text{st} \circ \langle u_2, p_{1,3} \rangle \circ (u \times X \times \{\cdot\} \circ P(X)) : \text{dom}(u) \times X \times P(X) \rightarrow P(X). \]

This is the general version of the statement in \textit{Set} that

\[ \text{st}(x, U \cup \{A\}) = A \text{ provided } \text{st}(x, U) \subseteq A \text{, for elements } x, \text{ subsets } A \text{ and coverings } U \text{ of the set } X. \]

In \textit{Set} we would prove this by first observing that $x \in \text{st}(x, U)$ for any covering $U$, so that

\[ \text{st}(x, U) \subseteq A \Rightarrow x \in A, \text{ and hence } \text{st}(x, U \cup \{A\}) = \text{st}(x, U) \cup A = A. \]

Moreover, $U \cup \{A\} \in u$ by axiom (u3) of 3.1.1.

In an arbitrary topos, the first observation has been made already, in 2.2.9, and for the second one we need only show the general version of the fact that $U(U \cup \{A\}) = U U \cup A$ for any
subset $A$ and family $\bigcup$ of subsets of a set $X$. By 1.8.2, $\bigcup(A) = A$ is valid generally, so it suffices to prove distributivity of internal union over binary union: $\bigcup(\bigcup_U \bigcup_V) = \bigcup_U \bigcup_V$. But this follows immediately from the fact that, for any morphism $f : X \to Y$, $\exists f : E(X, \Omega) \to E(Y, \Omega)$ has a right adjoint and hence preserves $- \vee -$ the (binary) coproduct in the partially ordered category $E(X, \Omega)$.

For axiom (f3), we must show that

$$\forall (\varepsilon \Rightarrow \exists (\varepsilon \_)) \circ \eta_{uX} = t_X, \text{ i.e. that } \forall, ((\varepsilon \Rightarrow \exists (\varepsilon \_)) \circ (\eta_{uX} \times P(X))) = t_X,$$

because the following diagram is a pullback:

```
\begin{array}{cccc}
X & \longrightarrow & \eta_{uX} & \longrightarrow & P^2(X) \\
\downarrow & & \downarrow & & \downarrow \\
X \times P(X) & \longrightarrow & \eta_{uX} \times P(X) & \longrightarrow & P^2(X) \times P(X) .
\end{array}
```

But the latter statement above is equivalent to

$$\varepsilon \circ (\eta_{uX} \times P(X)) \leq \exists (\varepsilon \_ \circ (\eta_{uX} \times P(X))), \text{ where the left-hand side is equal to}
$$

$$\varepsilon \circ (P(st) \times P(X)) \circ ((r \circ \tilde{u}, \{\_\}) \circ (P(X)) =
$$

$$= \exists (P^2(X) \times X) \times st (\varepsilon P^2(X) \times X) \circ ((r \circ \tilde{u}, \{\_\}) \times P(X))
$$

$$= \exists (X \times P(X) \times st (\varepsilon P^2(X) \times X \circ (r \times P^2(X) \times X)) \circ (\tilde{u}, \{\_\}) \times P(X))
$$

$$= \exists (X \times st (\varepsilon_{2, 4} \land \varepsilon_{1, 3}) \circ (\tilde{u}, \{\_\}) \times P^2(X) \times X))
$$

$$= \exists (X \times st (\varepsilon_{2, 4} \land \varepsilon_{1, 3}) \circ (\tilde{u}, \{\_\}) \times P^2(X) \times X))
$$

$$= \exists (X \times st (\varepsilon P^2(X) \circ (\tilde{u}, P_{1, 3} \land (\theta_X)_2)).
$$
so the statement to be proved is equivalent to:

\[ \varepsilon_{P^2(X)} \circ \langle \bar{u}, p_{1,3} \rangle \land (\theta_X)_2 \leq \exists_3 (\varepsilon) \circ (\eta_{uX} \times P(X)) \circ (X \times st), \]

where the right-hand side is equal to

\[
= \exists_4 (\varepsilon \circ (P^2(X) \times st \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\varepsilon \circ (st \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\varepsilon \circ (u \times X) \circ (\sigma_X \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\exists (\varepsilon_3 \land \varepsilon_1) \circ (\sigma \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\exists (\varepsilon \circ (\sigma \times P(X)) \land \varepsilon_1) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\exists (\varepsilon \land \varepsilon_1 \land \varepsilon_2) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\exists (\varepsilon_1 \land \varepsilon_2 \land \varepsilon_3) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\varepsilon_1 \land \varepsilon_2 \land \varepsilon_3 \circ (\eta_{uX} \times P^2(X) \times X) \circ (u \times P(X) \times X) \circ (\sigma_X \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\varepsilon_1 \land \varepsilon_2 \land \varepsilon_3 \circ (\eta_{uX} \times P^2(X) \times X) \circ (u \times (P(X) \times X) \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\varepsilon_1 \land \varepsilon_2 \land \varepsilon_3 \circ (\eta_{uX} \times P^2(X) \times X) \circ (u \times P(X) \times X) \circ (\sigma_X \times X)) \circ (\eta_{uX} \times P^2(X) \times X)
= \exists_4 (\varepsilon_1 \land \varepsilon_2 \land \varepsilon_3 \circ (\eta_{uX} \times P^2(X) \times X) \circ (u \times (P(X) \times X) \times X)) \circ (\eta_{uX} \times P^2(X) \times X)

so the statement to be proved is equivalent to

\[ \varepsilon_{P^2(X)} \circ \langle \bar{u}, p_{1,3} \rangle \land (\theta_X)_2 \leq \exists_4.5 (\varepsilon_1 \land \varepsilon_2 \land \varepsilon_3 \circ (\eta_{uX} \times P^2(X) \times X) \circ (u \times P(X) \times X) \circ (\sigma_X \times X) \circ (\eta_{uX} \times P^2(X) \times X)) \]

Now, by 2.2.1, we have that

\[ \varepsilon_{P^2(X)} \circ \langle \bar{u}, p_{1,3} \rangle \land \theta_2 \leq \varepsilon_3 \circ (u \times P^2(X) \times X) \]

and by axiom (u1) for uniformities we have

\[ \varepsilon_3 \circ (u \times P^2(X) \times X) \leq (\theta_{P(X)})_3 \circ (\text{id}_X \times u \times X), \]
and the latter is \( \exists (\epsilon \land \bar{\epsilon}_{1.2}) \), where the quantification is along \( X \times P^2(X) \times X \times P(X) \to X \times P^2(X) \times X \). But clearly

\[
\exists (\epsilon \land \bar{\epsilon}_{1.2}) \leq \exists (\epsilon \land \bar{\epsilon}_{1.2} \land \epsilon_{1.3.5}) \leq \exists (\epsilon \land \bar{\epsilon}_{1.2} \land \epsilon_{1.2.3}),
\]

so we are done with axiom (f3).

Next we look at the axiom for Cauchy filters, i.e. we show that

\[
\forall (\epsilon o <\bar{u}, id_{P^2(X)}> \Rightarrow \exists (\epsilon_2 \land \epsilon_1)) = \tau_X.
\]

Imitating the proof in Set, we use axiom (u4) for uniformities, and then use the fact, for which a proof was sketched above, that

\[
\text{st}(x,\mathcal{V}) \subseteq U \Rightarrow \text{st}(x,\mathcal{V} \cup \{U\}) = U
\]
generalizes to any topos. We have that

\[
\forall (\epsilon o <\bar{u}, id_{P^2(X)}> \Rightarrow \exists (\epsilon_2 \land \epsilon_1)) \circ u_X
\]

\[
= \forall (\epsilon o (\bar{u} \times P^2(X)) \Rightarrow \exists ((\epsilon_2 \land \epsilon_1) \circ (u_X \times P^2(X))))
\]

\[
= \forall (\epsilon o (\bar{u} \times P^2(X)) \Rightarrow \exists ((\epsilon_2 \land \epsilon_1) \circ (u_X \times P(X))))
\]

\[
= \forall (\epsilon o (\bar{u} \times P^2(X)) \Rightarrow \exists (\epsilon_1 \land \exists_{X \times \text{st}}(\text{char}(X \times u) \land \theta_2))
\]

Now, from axiom (u4) it follows that

\[
\epsilon o (\bar{u} \times P^2(X)) \leq \exists (\epsilon o <\bar{u}, P_{1.3} > \land \theta_1^{*}) ,
\]

where the quantification is along the projection \( X \times P^2(X)^2 \to P^2(X)^2 \). Moreover, from the above mentioned fact we have that

\[
\exists (\epsilon o <\bar{u}, P_{1.3} > \land \theta_1^{*}) \leq \exists (\epsilon(P^2(X)) o <\bar{u}, P_{1.3} > \land \epsilon(P)(o(P^2(X) \times \text{st}) \circ \text{twist}^{1.3}),
\]

so it suffices for us to show that the right-hand side above is
\[ \exists \exists (e_1 \land \exists_{X^2 (\text{char}(X \times u) \land \theta_2)}) . \]

This is elementary, and we omit further details.

Now we turn to the axiom for minimal Cauchy filters, i.e. we show that

\[ \forall_1 ((Cf) \land \subset \Rightarrow \theta_{P^2(X)} \circ \eta_{uX} = t_X . \]

We start with the following facts, the proofs of which are straightforward, and omitted:
\[ \forall (\epsilon \circ \langle \bar{u}, \text{id} \rangle \text{p}^2 (X) >_{1.3.4} \Rightarrow \exists (\epsilon \wedge \overline{\epsilon}) \wedge (\epsilon \Rightarrow \overline{\epsilon}) \wedge \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \]
\[ \leq \exists (\epsilon \circ \langle \bar{u}, \text{id} \rangle \text{p}^2 (X) >_{1.3.4} \wedge \exists (\epsilon \wedge \overline{\epsilon}) \wedge (\epsilon \Rightarrow \overline{\epsilon}) \]

and \[ \forall (\epsilon \wedge \epsilon \Rightarrow \epsilon) \wedge \exists (\epsilon \circ \langle \bar{u}, \text{id} \rangle \text{p}^2 (X) >_{1.3.4} \wedge \exists (\epsilon \wedge \overline{\epsilon}) \wedge (\epsilon \Rightarrow \overline{\epsilon}) \]
\[ \leq \epsilon_2 , \]
both illustrated by the diagram on page 79. Hence
\[ \forall (\epsilon \wedge \epsilon \Rightarrow \epsilon) \wedge \forall (\epsilon \circ \langle \bar{u}, \text{id} \rangle \Rightarrow \exists (\epsilon \wedge \overline{\epsilon}) \wedge (\epsilon \Rightarrow \overline{\epsilon}) \]
\[ \wedge \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \leq \epsilon_2 , \]
i.e. \[ \forall (\epsilon \wedge \epsilon \Rightarrow \epsilon) \wedge \forall (\epsilon \circ \langle \bar{u}, \text{id} \rangle \Rightarrow \exists (\epsilon \wedge \overline{\epsilon}) \wedge (\epsilon \Rightarrow \overline{\epsilon}) \]
\[ \wedge \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \leq \epsilon_2 , \]
and hence \( (Cf) \wedge (\epsilon \Rightarrow \overline{\epsilon}) \wedge \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \leq \epsilon_2 . \)

Since \( \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \leq \epsilon_X , \) by an easy application of axiom (u1) for \( u \) and 2.2.9, it follows that
\[ (Cf) \wedge (\epsilon \Rightarrow \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta)) \wedge \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \leq \epsilon_2 , \]
and hence that
\[ (Cf) \wedge \forall (\epsilon \Rightarrow \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta)) \leq \exists_{X \times \text{st}} (\text{char}(X \times u) \wedge \theta) \wedge \epsilon_2 , \]
i.e. that \( (Cf) \wedge \forall (\epsilon \Rightarrow \epsilon \circ (\eta_{uX} \times P(X))) \leq \epsilon \circ (\eta_{uX} \times P(X)) \Rightarrow \epsilon_2 , \)
by 4.2.5.
Hence

\[(\text{Cf})_2 \land \forall_3 (\varepsilon_2 \Rightarrow \varepsilon \circ (\eta_{uX} \times P(X))) < \forall_3 (\varepsilon \circ (\eta_{uX} \times P(X))) \Rightarrow \varepsilon_2 \),

i.e. \[(\text{Cf})_2 \land \forall_3 (\varepsilon_2 \Rightarrow \varepsilon_1) \circ (P^2(X) \times \eta_{uX} \times P(X)) < \forall_3 (\varepsilon_1 \Rightarrow \varepsilon_2) \circ (P^2(X) \times \eta_{uX} \times P(X))\),

i.e. \[(\text{Cf})_2 \land \forall_3 (\varepsilon_2 \Rightarrow \varepsilon_1) \circ (P^2(X) \times \eta_{uX}) < \forall_3 (\varepsilon_1 \Rightarrow \varepsilon_2) \circ (P^2(X) \times \eta_{uX})\),

or, in short: \[(\text{Cf})_2 \land \circ (P^2(X) \times \eta_{uX}) < \circ (P^2(X) \times \eta_{uX})\).

Since \(\varepsilon = \epsilon \land \varepsilon\), it follows that

\[(\text{Cf})_2 \land \circ (P^2(X) \times \eta_{uX}) < \circ (P^2(X) \times \eta_{uX})\),
which is equivalent to

\[(\text{Cf}_2 \land \circ (P^2(X) \times \eta_X) \leq \circ (P^2(X) \times \eta_X)),\]

i.e.

\[(\text{Cf}_2 \land \circ (P^2(X) \times \eta_X)) = t_{P^2(X) \times X},\]

i.e.

\[\forall_1 ((\text{Cf}_2 \land \circ (P^2(X) \times \eta_X))) = t_X,\]

i.e.

\[\forall_1 ((\text{Cf}_2 \land \circ \eta_X) \circ \eta_X = t_X,\]

as desired.

We now turn to the axioms (f4_1), i.e. we show that

\[(f_4_1) \circ \eta_X = t_X\]

provided \(u\) is an \(i\)-uniformity on \(X\). In fact,

\[(f_4_1) \circ \eta_X = t_X\]

for any uniformity \(u\) on \(X\), i.e. it is always the case that

\[\circ \text{char}([\overline{P}(X)]) \circ \eta_X = t_X,\]

which is equivalent to

\[\circ \text{char}([\overline{P}(X)]) \circ \eta_X = \text{char}(c_X),\]

because in the following diagram all four squares are pullbacks:

\[
\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\circ X & \downarrow & \downarrow <\overline{P}(X), \overline{P}(X)> \\
X & \downarrow <u, {\ast}> & P^3(X) \times P(X) \\
\end{array}
\begin{array}{ccc}
& \longrightarrow & 1 \\
& \downarrow \overline{P}(X) \times X & \downarrow \overline{P}(X) \times X \\
& \downarrow \overline{P}(X) \times X & \downarrow \circ \text{char}([\overline{P}(X)]) \\
\end{array}
\]

In the case \(i = 2\) we must show that

\[\exists_2 (\varepsilon) \circ \eta_X = t_X,\]

assuming that \(u\) is a 2-uniformity on \(X\). We have that

\[\exists_2 (\varepsilon) \circ \eta_X = \exists_2 (\varepsilon \circ (\eta_X \times P(X))) \circ \exists_2 (\exists_{X \times \text{st}} (\text{char}(X \times u) \land \theta)) = \]

\[= \exists_{2, 3} (\text{char}(X \times u) \land \theta),\]

because the triangle at the top of the
Now the subobject of $X \times P^2(X) \times X$ classified by $\text{char}(X \times u)_3 \wedge \Theta_2$ is $<p_2, u, p_2>: X \times \text{dom}(u) \rightarrow X \times P^2(X) \times X$, and

$p'_{2,3} \circ <p_2, u, p_2>: X \times \text{dom}(u) \rightarrow X$ is simply $p'_2: X \times \text{dom}(u) \rightarrow X$, the monic part of which is the subobject of $X$ classified by $\exists_{2,3} (\text{char}(X \times u)_3 \wedge \Theta_2)$. But $p'_2$ is epic, since $\text{dom}(u)$ by assumption has global support, and since $p'_2$ can be written as a composed map $X \times \text{dom}(u) \xrightarrow{id \times \text{dom}(u)} X \times 1 \xrightarrow{=} X$. So the monic part of $p'_2$ is $\text{id}_X$, so $\exists_{2,3} (\text{char}(X \times u)_3 \wedge \Theta_2) = t_X$, as desired.

Finally, we consider the case of axiom $(f^4_3)$, where we are to show that

$$
\varepsilon_{P(X)} \circ <\text{id}_{P^2(X)}, \text{id}_X> \circ \eta_{uX} = t_X
$$

under the assumption that $u$ is a 3-uniformity on $X$. Consider the following diagram, where the cells are numbered for easy
reference. It will clearly suffice to show that each cell – except 
(x) – commutes, and that in cell (x) we have that 
\( \{\cdot\}_P(X) \circ \text{id}_X \circ !_X \leq \sigma \circ \langle \text{id}_P(X), \text{id}_X \rangle \). We prove this latter statement 
first. It is equivalent to \( \theta \circ (\langle \text{id}_X \circ !_X \times P(X) \rangle \leq \varepsilon \circ \varepsilon \circ \langle \text{id}_P(X), P_1, P_2 \rangle, \)

by exponential adjointness and definition of \( \sigma \). We prove first 
that \( \theta \circ (\langle \text{id}_X \circ !_X \times P(X) \rangle \leq \varepsilon \circ \varepsilon \circ \langle \text{id}_P(X), P_1, P_2 \rangle, \) i.e. that
\[ \theta \circ (\text{id}_X \times P(X)) \circ \langle \text{id}_X, P_2 \rangle \leq \varepsilon_{P(X)}. \]

The following squares are easily seen to be pullbacks:

\[ \begin{array}{cccc}
1 \times X & \longrightarrow & 1 & \longrightarrow & P(X) & \longrightarrow & 1 \\
\downarrow \text{id}_X \times X & & \downarrow \langle \text{id}_X, P_2 \rangle & & \downarrow \Delta_{P(X)} & & \downarrow t \\
P(X) \times X & \longrightarrow & \langle P(X), P_2 \rangle & \longrightarrow & 1 \times P(X) & \longrightarrow & \Gamma \end{array} \]

so \[ \text{id}_X \times X : 1 \times X \rightarrow P(X) \times X \] represents the subobject classified by \[ \theta \circ (\text{id}_X \times P(X)) \circ \langle \text{id}_X, P_2 \rangle, \] so it suffices to prove that \[ \varepsilon_{P(X)} \circ (\text{id}_X \times X) = t_{1 \times X}, \] which is easy. Then we prove that \[ \theta \circ ((\text{id}_X \circ \text{id}_X) \times P(X)) \leq \varepsilon_3 \circ \langle \text{id}_{P(X)}, P_1, P_2 \rangle. \] By the above we need only show that

\[ \varepsilon_3 \circ \langle \text{id}_{P(X)}, P_1, P_2 \rangle \circ (X \times \text{id}_X) = t_X \times 1, \]

i.e. that \[ \varepsilon \circ \langle \text{id}_{P(X)}, P_1, P_2 \rangle \circ (X \times \text{id}_X) = t_X \times 1, \]

which also is easy.

It remains to comment on the commutativity of the remaining cells of the last diagram. Cell (i) is easily checked. For cell (ii), observe that

\[ \theta_{P^2(X) \times X} = \theta_{2.4} \wedge \theta_{1.3} = \]
\[ = \varepsilon \circ (\{\cdot\} \times P^2(X) \times \text{id}_X) \]
\[ = \varepsilon_{2.4} \circ ((\{\cdot\} \times \{\cdot\}) \times (P^2(X) \times X))  \]
\[ = (\varepsilon_{2.4} \wedge \varepsilon_{1.3}) \circ ((\{\cdot\} \times \{\cdot\}) \times (P^2(X) \times X)), \]
where we have used an elementary property of the equality predicate (see [En]), the adjointness relation between \{•\} and \(\theta\), and naturality of projections.

Cell (iii) commutes by naturality of the transformation \{•\} : \text{id} \to P, cell (iv) by definition of \(s_t\), cell (v) by one of the triple identities for the triple \((P,\{•\},U)\) (see 1.8.2), cell (vi) is trivial, cell (vii) is easy to check, cell (viii) commutes by adjointness relation between \{•\} and \(\theta\), and cell (ix) commutes because \(\langle \text{id}_X, \text{id}_X \rangle : 1 \to P(X)^2\) factors through \(\Delta_P(X) : P(X) \to P(X)^2\).

This completes the proof of 4.2.7.

4.3 The completion. Classically, a uniform space is said to be complete if each Cauchy filter on it has a limit, a definition which makes sense only after a discussion of the topology associated with a given uniformity, and of limits of filters on topological spaces. Our approach here will be a different one. We first describe (what will be called) the completion of any given space, and then use that to define not only completeness for the original space, but also separatedness, or the Hausdorff property.

4.3.1 Notation : For any uso \(u_X\), the unique factorization of \(\eta_{u_X} : X \to P^2(X)\) through \(\alpha_{u_X} : \gamma_{u_X} \to P^2(X)\) is denoted \(\gamma_{u_X} : X \to \gamma_{u_X}\). The unique (when available) factorization of \(\eta_{u_X}\) through \(\alpha_{1,u_X}\), \(i \in \{1,2,3\}\), is denoted \(\gamma_{1,u_X} : X \to \gamma_{1,u_X}\).

4.3.2 Definition : A uso \(u_X\) is complete if \(\gamma_{u_X} : X \to \gamma_{u_X}\) is an epimorphism, and is \(1\)-complete, \(i \in \{1,2,3\}\), if \(\gamma_{1,u_X}\) exists and is an epimorphism. Moreover, \(u_X\) is separated if \(\gamma_{u_X}\)
is a monomorphism (notice that there is no reason to define "i-separatedness", because $\gamma_1, uX$ is a monomorphism if and only if $\gamma_{uX}$ is).

4.3.3 Remarks: We will define uniformities, $\tilde{u}$ and $\tilde{u}_1$, on the objects $\gamma uX$ and $\gamma_1 uX$ respectively, and we will depart from established conventions by writing simply $\gamma uX$ and $\gamma_1 uX$ for the resulting uso's, the uniformities $\tilde{u}$ and $\tilde{u}_1$ being understood. We will then prove that $\gamma_{uX}(\gamma_1, uX)$ is u.c. from $uX$ to $\gamma uX$ (to $\gamma_1 uX$), that $\gamma uX(\gamma_1 uX)$ is a complete (i-complete) and separated uso, and that $\gamma_{uX}(\gamma_1, uX)$ is a reflection of $uX$ into the full subcategory of $U_E$ whose objects are the complete (i-complete) separated uso's. This reflection, or sometimes, somewhat inaccurately, the uso $\gamma uX(\gamma_1 uX)$ itself, is what we will call the completion (i-completion) of $uX$.

In order to describe the uniformity $\tilde{u}$ in $\text{Set}$, we make two preliminary definitions: For $U \in u$ and $F \in \gamma uX$, $A_{U,F}$ is the subset $\{F' \in \gamma uX| F' \cap F \cap U \neq \emptyset\}$ of $\gamma uX$, and $A_U$ is the family $\{A_{U,F}| F \in \gamma uX\}$ of subsets of $\gamma uX$. We prove now that for any given basis $b$ for $u$, $\{A_{U}| U \in b\}$ is a basis for a uniformity on $\gamma uX$. To see that each $A_{U}$ covers $\gamma uX$, simply observe that $F \in A_{U,F}$ for all $U \in u$, $F \in \gamma uX$, by definition of Cauchy filter. Then, given $U$ and $V$ in $b$, let $W \in b$ be such that $W \prec U \wedge V$, and observe that $A_{W} \prec A_{U} \wedge A_{V}$ follows easily from the following two lemmas: 1) For any filter $F$ on $X$ and coverings $U$ and $V$ of $X$, if $U \prec V$ then $A_{U,F} \subseteq A_{V,F}$. Proof: Given $F' \in A_{U,F}$, let $U \in F' \cap F \cap U$. Then let $V \in V$ contain $U$. Clearly, $V \in F' \cap F \cap V$, so $F' \cap F \cap V \neq \emptyset$ so $F' \in A_{V,F}$. 2) If $U \prec V$ then $A_{U} \prec A_{V}$. 
Proof: Let $F_0 \in \gamma uX$, and suppose $F_0 \in A_{u,F}$. We claim that $A_{u,F} \subseteq A_{u,F}$. Let $F' \in A_{u,F}$. Let $U \in F_0 \cap F \cap U$ and $U' \in F' \cap F \cap U$. Then $U \cup U'$ is in $F$, and hence is non-empty, so $U \cup U'$ is contained in some $V \in V$. Such $V$ is clearly in $F' \cap F_0 \cap V \neq \emptyset$, so $F' \in A_{u,F_0}$ as desired. This completes the proof of 2) above.

The uniformity on $\gamma uX$ generated by the basis $\{A_u\}_{u \in u}$ is denoted $\mathfrak{u}$. It is clear that, if $b$ is a basis for $u$, then $\{A_u\}_{u \in b}$ is a basis for $\mathfrak{u}$.

We now prove (still in $\text{Set}$) that $\gamma uX : uX \to \gamma uX$ is u.c. It will suffice to show that $\{\gamma uX^{-1}(A_{u,F})\}_{F} \in \gamma uX$ is in $u$, for each $U \in u$. So let $U_o \in u$ be arbitrary, and let $V_o,W_o \in u$ be such that $W_o <^* V_o <^* U_o$. We claim that $W_o < \{\gamma uX^{-1}(A_{u,F})\}_{F} \in \gamma uX$ - in fact, for any $W_o \in W_o$ and any $x_o \in W_o$ we claim that $W_o \subseteq \gamma uX^{-1}(A_{U_o,YuX(x_o)}), \text{i.e. that } x \in W_o \Rightarrow \gamma uX(x) \cap \gamma uX(x_o) \cap U_o \neq \emptyset$.

For, if $V_o \in V_o$ is such that $st(x,W_o) \subseteq V_o$, and $U_o \in U_o$ is such that $st(x_o,W_o) \subseteq U_o$, (and by assumption there is such a $U_o$), then $U_o \in \gamma uX(x) \cap \gamma uX(x_o)$, as we now show. Let $W_o'$ be the cover $W_o \cup \{U_o\}$, which is in $u$ since $W_o$ refines it. Then $st(x,W_o') = st(x,W_o') \cup U_o = U_o$, where we have used that $x \in U_o$, which follows from: $x \in W_o \subseteq V_o \Rightarrow x \in V_o \subseteq st(x,W_o') \subseteq U_o$, and also that $st(x,W_o') \subseteq U_o$, which is clear since $V_o \subseteq U_o$. Hence $U_o \in \gamma uX(x)$. Moreover, $st(x,W_o') = st(x,W_o') \cup U_o = U_o$, where we have used that $x \in U_o$, which follows from $x \in st(x,\mathfrak{u})$, and that $st(x,W_o') \subseteq U_o$, which follows from $st(x,W_o') \subseteq st(x,W_o')$. Hence $U_o \in \gamma uX(x_o)$, and we have proved that $\gamma uX$ is u.c.

Next we prove that $\gamma uX$ is a complete and separated uniform space, i.e. that $\gamma uX : uX \to \gamma uX$ is an isomorphism, which we show...
by producing a u.c. two-sided inverse for it. Appropriately, the inverse of \( \gamma_{uX} \) will be denoted \( L \), for "limit".

So let \( F \) be a minimal Cauchy filter on \( \gamma_{uX} \). We show that \( \{\gamma_{uX}^{-1}(A)\}_{A \in F} \) is a basis for a Cauchy filter on \( uX \). Firstly, each \( \gamma_{uX}^{-1}(A) \) is nonempty. To see this, observe that \( F \), being Cauchy, has an element of the form \( A_{U^*,F} \), with \( F \in \gamma_{uX} \), for each \( U \in u \). Since \( \{U^*|U \in u\} \) is a basis for \( u \), these elements of \( F \) clearly form a basis for a Cauchy filter on \( \gamma_{uX} \), so by minimality of \( F \) they form a basis for \( F \). Hence it suffices to show that \( \gamma_{uX}^{-1}(A_{U^*,F}) \) is non-empty, for each \( U \in u \) and \( F \in \gamma_{uX} \). That is, for such \( U \) and \( F \) we must find \( x \in X \) such that \( \gamma_{uX}(x) \cap F \cap U^* \neq \emptyset \).

Well, \( F \) is Cauchy, so there is some \( U \in F \cap U^* \). Such \( U \) is of the form \( st(x,U) \), for some \( x \in X \), so for this \( x \), \( U \in \gamma_{uX}(x) \).

This proves that \( \gamma_{uX}^{-1}(A) \neq \emptyset \) for \( A \in F \). And since \( \gamma_{uX}^{-1} \) preserves containment and (binary) intersections, it is clear that \( \{\gamma_{uX}^{-1}(A)\}_{A \in F} \) is (a basis for) a filter on \( X \). To show it is a Cauchy filter on \( uX \), let \( U \in u \). Since \( F \) is Cauchy on \( \gamma_{uX} \), \( F \cap A_{U} \neq \emptyset \), so let \( F \in \gamma_{uX} \) be such that \( A_{U,F} \in F \). We claim that \( \gamma_{uX}^{-1}(A_{U,F}) \) is contained in some member of \( U^{**} \). Now \( \gamma_{uX}^{-1}(A_{U,F}) = \{x|\gamma_{uX}(x) \cap F \cap U^* \neq \emptyset\} = \{x|\exists \emptyset \in u: st(x,\emptyset) \in F \cap U\} \).

Let \( x_0 \in U_0 \in F \cap U \). We claim that \( \gamma_{uX}^{-1}(A_{U,F}) \subseteq st(x_0,U^*) \). For, given \( x \in \gamma_{uX}^{-1}(A_{U,F}) \), let \( \emptyset \in u \) be such that \( st(x,\emptyset) \in F \cap U \).

Let \( x_1 \in U_0 \cap st(x,\emptyset) \), where the latter is nonempty because \( F \) is a filter. Then \( st(x_1,U) \in U^* \), \( x_0 \in U_0 \subseteq st(x_1,U) \), and \( x \in st(x,U) \subseteq st(x_1,U) \). Hence \( x \in st(x_0,U^*) \), as desired. And since \( U^{**} \) is a basic uniform cover of \( uX \), this proves that the filter on \( X \), constructed above from \( F \), is Cauchy on \( uX \). Let \( L(F) \) be the unique minimal Cauchy filter contained in the one just constructed.
Next we show that $L(\gamma yuX(F_o)) = F_o$, and that $\gamma yuX(L(F_o)) = F_o$ always. For the first of these, it suffices to show that each member of $F_o$ contains a set of the form $\gamma yuX^{-1}(A_U, F)$, with $U \in u$ and $F \in yuX$. Since $F_o$ is minimal, each of its members is of the form $St(B, U)$ for some $B \in F_o$ and $U \in u$ (see 4.2.5). We show that $\gamma yuX^{-1}(A_U, F) \subseteq St(B, U)$. If $x \in \gamma yuX^{-1}(A_U, F)$, there is $V \in u$ such that $st(x, V) \in F_o \cup U$. Since $F_o$ is a filter, $B \cap st(x, V) \neq \emptyset$, so $x \in st(x, V) \subseteq St(B, U)$.

To see that $\gamma yuX(L(F_o)) = F_o$, it suffices to see that $L(F_o)$ is a member of each $A \in F_o$. And since, for each such $A$, there is $U \in u$ with $A_U, L(F_o) \subseteq A$, it suffices to observe that $L(F_o) \in A_U, L(F_o)$, which is clear since $L(F_o)$ is Cauchy.

We now use the above discussion in $Set$ to sketch a proof of the following theorem.

4.3.4 Theorem: Let $u$ be a uniformity $[i$-uniformity, $i \in \{1, 2, 3\}]$ on an object $X$. Then there is a uniformity $[i$-uniformity] on the object $yuX[\gamma_1 uX]$, with respect to which $\gamma yuX[\gamma_1 uX]$ is a complete $[i$-complete], separated $u$ [provided, when $i = 3$, that the topos has AC].

Proof (sketch): Let $a : P^2(X) \times P^2(yuX)$ be the map corresponding to the formula

$$\exists \left( \forall (e \quad \iff \quad \exists \left( e_3 \quad 4 \quad 1, 3 \quad 5 \quad 2, 3, 4 \quad e_1, 2, 4 \quad e_1, 2, 3 \right) \right) : P^2(X) \times P(yuX) \to \Omega,$$

where the "bi-implication", $\iff$, has the obvious meaning, and the quantifications are along the three projections.
Furthermore, let $a_1 : P^2(X) + P^2(\gamma_1 uX)$ correspond to the formula $P^2(X) \times P(\gamma_1 uX) + \Omega$ which is written just as the one given above.

We claim that, if $b$ is a basis for $u$, then the monic part of $a \circ b : \text{dom}(b) + P^2(\gamma_1 uX)$ is a basis for a uniformity, $\tilde{u}$, on $\gamma_1 uX$. And if $u$ is an $i$-uniformity on $X$, then the monic part of $a_1 \circ b : \text{dom}(b) + P^2(\gamma_1 uX)$ is a basis for a uniformity, $\tilde{u}_1$, on $\gamma_1 uX$. This is straightforward use of 3.2.4 and the above discussion in $\text{Set}$. It is easy to see that $\tilde{u}_1$ is an $i$-uniformity on $\gamma_1 uX$, simply because there exists, by construction, a morphism $\text{dom}(u) + \text{dom}(\tilde{u}_1)$ compatible with $u$ and $\tilde{u}_1$.

To show that $\gamma uX[\gamma_1 uX]$ is a complete [$i$-complete] and separated uso, we construct an inverse, $L [L_1]$, to $\gamma uX [\gamma_1, \gamma_1 uX]$, just as we did in $\text{Set}$ above. First we define $L' : \gamma uX + P^2(X)$ as the map corresponding to the formula

$$\exists((\varepsilon_{P^3(X)})_3 \circ (\alpha_{\gamma uX} \times P^3(X) \times P(X)) \wedge \circ (P^*(\gamma_{ux}) \times (P(X)))) : \gamma uX \times P(X) + \Omega$$

[and, quite similarly, $L'_1 : \gamma_1 \gamma_1 uX + P^2(X)$], obtained from the diagram on next page.

Let $M_u : P^2(X) + P^2(X)$ be the map corresponding to the formula

$$\exists((\varepsilon_{P^3(X)})_2, \wedge \circ (\alpha_{P^3(X) \times (P(X)}) \times (P(X)))) : P^2(X) \times P(X) + \Omega$$
where the quantification is along the projection
\[ P^2(X) \times \text{dom}(u) \times P(X)^2 \xrightarrow{2,3} P^2(X) \times P(X). \]
Then let
\[ L : \gamma uX \to P^2(X) \quad [L_1 : \gamma_1 \gamma_1 uX \to P^2(X)] \]
be the composed map
\[ M_u \circ L' \quad [M_u \circ L'_1]. \]

It is now straightforward (and lengthy) to verify that \( L \ [L_1] \) factors through \( a_{uX} : \gamma uX \to P^2(X) \quad [a_{1,1} uX : \gamma_1 uX \to P^2(X)] \), provided \( u \) is an \( i \)-uniformity, and - when \( i = 3 \) - the topos has \( \text{AC} \),
and that this factorization is a two-sided inverse to \( \gamma_{\gamma uX} \quad [\gamma_1 \gamma_1 uX] \).

This completes our sketch of the proof of Theorem 4.3.4.
APPENDIX : More on structured objects.

In order to avoid the set-theoretic difficulties often encountered in attempts to construct too large categories, we will in the following use a category $\mathcal{S}$ of (some, but possibly not all) sets, instead of the category $\text{Set}$ of all sets. We assume only that $\mathcal{S}(n,m) = \text{Set}(n,m) \in |\mathcal{S}|$ whenever $n, m \in |\mathcal{S}|$, and that $\emptyset \in |\mathcal{S}|$. If the set-theoretically inclined reader wishes to assume that $|\mathcal{S}|$ is a set, then this will not necessitate any changes in the following.

Now let $U : A \rightarrow \mathcal{S}$ be a given functor, and let $\mathcal{C}$ be an $\mathcal{S}$-category (i.e. $\mathcal{C}(A,B) \in |\mathcal{S}|$ for all $A, B \in |\mathcal{C}|$). Then an $A$-object - or, more precisely, an $(A,U)$-object - in $\mathcal{C}$, in the external sense, is a pair $(C, G)$ where $C \in |\mathcal{C}|$, $G : \mathcal{C}^{\text{op}} \rightarrow A$ and $U \circ G = \mathcal{C}(-, C)$. An $A$-homomorphism between two such pairs, $(C, G) \rightarrow (C', G')$, is a pair $(f, n)$ where $f \in \mathcal{C}(C, C')$ and $n$ is a natural transformation $G \Rightarrow G'$ satisfying $U n = \mathcal{C}(-, f)$, i.e. for $B \in |\mathcal{C}|$ and $g \in \mathcal{C}(B, C)$ we insist that $U(n_B)(g) = \mathcal{C}(B, f)(g) = f \circ g : B \rightarrow C'$. It is easy to see that we get a category of $A$-objects in $\mathcal{C}$, and we denote it $\mathcal{C} \otimes_{\mathcal{S}} A$. It has canonical functors (projections) to $\mathcal{C}$ and $A^{\text{op}}$, making the following a pull-back diagram (where $Y_{\mathcal{C}}$ is the Yoneda embedding):

```
\begin{array}{ccc}
\mathcal{C} \otimes_{\mathcal{S}} A & \rightarrow & A^{\text{op}} \\
\downarrow & & \downarrow U_{\mathcal{C}}^{\text{op}} = "U \circ -" \\
\mathcal{C} & \rightarrow & S^{\text{op}} \\
\end{array}
```
There is one obvious criterion for judging whether this, for given $U: A \to S$ and for all $S$-categories $Q$, is a good definition, and that is to ask if the category of $A$-objects in $S$ itself is what it should be, namely $A$ itself.

In fact, we will answer the following precise question:

For which $(A, U)$ is there an equivalence of categories $A \to S \otimes_S A$ making the following diagram commute?

\[
\begin{array}{ccc}
A & \rightarrow & S \otimes_S A \\
\downarrow U & & \downarrow \ \ \\
S & \rightarrow & S \otimes_S A
\end{array}
\]

\[
(*)
\]

First of all, the functors $A \to S \otimes_S A$ making $(*)$ commute correspond to functors $E: A \to A^{\text{op}}$ satisfying $U^{\text{op}} \circ E = Y_S \circ U$, i.e. satisfying for all $A$-objects $A$ and $S$-objects $n$:

$U(E(A)(n)) = S(n, U(A))$. That is, $E(A)(n)$ must be an $A$-object with "underlying set" equal to the $n$th power of the set $U(A)$.

Moreover, with $1$ being a one-element $S$-object (take, for example, $1 = S(\emptyset, \emptyset)$), the $S$-maps $i: 1 \to n$ must satisfy:

$U(E(A)(1)) = S(1, U(A))$, i.e. for each $x \in S(n, U(A))$ - and this corresponds to an element $(a_j)_{j \in \mathbb{N}} \in U(A)^n$ - $U(E(A))(1)(x)$ shall equal $S(1, U(A))(x) = x \circ i \in S(1, U(A))$ - corresponding to $a_i \in U(A)$. In other words, $U(E(A)(1))$ shall be the projection of the product set $U(A)^n$ to its $i$th factor.

We give now conditions under which all this can be accomplished.

We say $A$ has $S$-powers if for each $A \in |A|$ and $n \in |S|$
there is an object $A^n$ in $A$ and a map $p : n \to A(A^n, A)$ with the usual universal property: any $q : n \to A(B, A)$ yields a unique $f \in A(B, A^n)$ with $p(i) \circ f = q(i)$ for each $i \in n$.

We say $U$ preserves $S$-powers if, for each $(A^n, p)$ as above, $(U(A^n), U \circ p)$ has the universal property of an $n$th power of $U(A)$ in $S$.

And we say $A$ has $U$-canonical $S$-powers if, for any $A$ and $n$, $(A^n, p)$ can be chosen so that $U(A^n) = S(n, U(A))$ and $U(p(i))(g) = g(i)$ for $i \in n$ and $g \in S(n, U(A))$.

This condition is satisfied if $U$ creates $S$-powers, and only if $U$ preserves $S$-powers.

Notice also that, if $A$ has $S$-powers, then any choice of them yields a unique functor $E : A \to A^S\text{op}$, defined on objects by $E(A)(n) = A^n$ and on morphisms by making use of the universal property of powers in the obvious way.

We can now answer part of our earlier question:

A 1. Proposition : Let $U : A \to S$ be a functor such that $A$ has $U$-canonical $S$-powers. Then $E : A \to A^S\text{op}$ can be chosen so that $U^S\text{op} \circ E = Y_S \circ U$. Each such choice of $E$ yields a functor $A \to A \otimes_S A$ making $(\ast)$ commute.

Proof : Straightforward.

To state conditions under which the above produced functor $A \to A \otimes_S A$ is an equivalence of categories, we need another definition:
We say \( U : A \to S \) reflects \( S \)-powers if, given any map \( q \) from some \( n \in |S| \) into some \( A(B, A) \) such that \((U(B), U \circ q)\) has the universal property of an \( n \)th power of \( U(A) \) in \( S \), \((B, q)\) must already have had the universal property of an \( n \)th power of \( A \) in \( A \).

A 2. Proposition: Suppose that \( U : A \to S \) is a functor reflecting \( S \)-powers, and that \( A \) has \( U \)-canonical powers. Then there is an equivalence of categories \( A \to S \otimes_A A \) making (*) commute.

Proof: The functor \( A \to S \otimes_A A \) described in A 1 takes an object \( A \) in \( A \) to \((U(A), E(A)) \in |S \otimes_A A|\), where \( E(A) \) is a functor \( S^{op} \to A \) such that \( E(A)(n) = A^n \), and a morphism \( f \in A(A,B) \) to the morphism

\[
(U(f), E(f)) \in (S \otimes_A A)((U(A), E(A)), (U(B), E(B))),
\]

where \( E(f) \) is the natural transformation \( E(A) \to E(B) \) described by the formula \( E(f)_n = f^n : A^n \to B^n \). If \( f \not\subseteq g : A \to B \) then \( f^1 \not\subseteq g^1 : A^1 \to B^1 \), so \( E(f) \not\subseteq E(g) : E(A) \to E(B) \). Hence \( A \to S \otimes_A A \) is a faithful functor. It is also full, for given a morphism \((h, \eta) : (U(A), E(A)) \to (U(B), E(B)), \eta_1 : E(A)(1) \to E(B)(1)\) corresponds to a morphism \( f : A \to B \), satisfying \( U(f) = h \) and \( E(f) = \eta \). Hence it remains only to show that each \((n, G) \in |S \otimes_A A|\) is isomorphic to some \((U(A), E(A))\). This can be done with \( A = G(1) \). Then \( U(A) = U(G(1)) = S(1, n) \). For each \( m \in |S| \) let \( e_m : m \to S(1, m) \) be the obvious one-to-one- and onto map. Then let \( \eta : G \to E(A) \) be the natural transformation defined as follows: For \( m \in |S| \) let \( \eta_m : G(m) \to E(A)(m) = A^m \) be the unique map resulting from the family of \( A \)-morphisms.
\[ G(m) + G(1) = A \quad (i \in m). \]

Now \( U(G(e_m(i))) = S(e_m(i),n) = - \cdot e_m(i) : S(m,n) + S(1,n) = U(A), \)
for all \( i \in m. \) These form a family of projections by virtue of
which \( S(m,n) \) is \( U(A)^m. \) And since \( U \) reflects \( S \)-powers this
means that \( (G(m), G(e_m(i)))_{i \in m} \) has the universal property of
\( A^m, \) so \( \eta_m - \) and hence \( \eta - \) is an isomorphism.

Hence \( e_n : (n,G) \to (U(G(1)),E(G(1))) \) is an isomorphism in
\( S \otimes A, \) as desired.

This completes the proof.

Next, given a functor \( U : A \to S, \) and an \( S \)-category \( C, \)
let \( C \otimes S A \) be the full subcategory of \( C \otimes S A \) with objects
those \( (C,G) \in |C \otimes S A| \) where \( G \) preserves \( S \)-powers. Again we
seek conditions on \( U \) that will ensure that \( C \otimes S A \) is a good
candidate for a category of \( A \)-objects in \( C, \) and the criterion
is as before.: When is \( S \otimes S A \approx A \) over \( S \)? More specifically,
since \( E(A) : S^{op} + A \) preserves \( S \)-powers for all \( A \in |A| : \)
When is \( A \otimes (U(A),E(A)) : A + S \otimes S A \) an equivalence of
categories ?

As in the proof of Prop. A 2 we need only show that, under
some suitable conditions on \( U, \) each \( (n,G) \in |S \otimes S A| \) is iso-
morphic to some \( (U(A),E(A)). \) Again, for given \( (n,G), \) we set
\( A = G(1). \) Then \( G(m) = G(1)^m = A^m, \) since any set \( m \) is the \( m \)th
power of \( 1 \) in \( S^{op}, \) and \( G \) preserves \( m \)th powers. So in this
case \( n : G \to E(A) \) is an isomorphism without any assumption that
\( U \) reflects \( S \)-powers. Hence we have the following:
A 3. Proposition: Let $U : A \to S$ be a functor such that $A$ has $U$-canonical $S$-powers. Then there is an equivalence of categories making the following diagram commute:

\[
\begin{array}{ccc}
A & \longrightarrow & S \times_S A \\
\downarrow U & & \downarrow U \\
S & & \\
\end{array}
\]

We note here that related statements were made by John Isbell in [Is$_2$]. We note also that if $U : A \to S$ creates $S$-powers then the above described equivalence $A \to S \times_S A$ is an isomorphism of categories.
BIBLIOGRAPHY


MacLane, S., Categories for the Working Mathematician, Graduate Texts in Mathematics no. 5, Springer-Verlag, 1971.

Mikkelsen, C.J., Doctorial dissertation, to appear at Aarhus University, Aarhus, Denmark.


Volger, H., Ultrafilters, Ultrapowers and Finiteness in a Topos, Preprint, April 1974.