On spectral subspaces of automorphisms.

by

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Abstract.

It is shown that spectral subspaces of automorphisms of a von Neumann algebra can be defined by use of Stone's theorem on unitary representations.
It is well known that the theory of spectral subspaces of automorphisms as developed by Arveson [1] generalizes Stone's theorem for unitary representations. In this note we shall show a converse result, thus indicating how the theory of spectral subspaces of abelian groups of automorphisms of von Neumann algebras, see [1,2,3], can be developed from Stone's theorem. The idea is that a \( \star \)-automorphism on the bounded operators \( B(H) \) on a Hilbert space \( H \) restricts to an isometry of the Hilbert-Schmidt operators \( H_2 \) on \( H \). Thus Stone's theorem can be used on unitary representations on \( H_2 \) and then "lifted" to \( B(H) \).

Throughout this note we let \( G \) be a locally compact abelian group with Haar measure \( dt \) and dual group \( \Gamma \). \( t \to u_t \) is a strongly continuous unitary representation of \( G \) on the Hilbert space \( H \), and \( \alpha_t = Ad_{u_t} \). Then \( t \to \alpha_t \) is a continuous representation of \( G \) into the automorphism group \( \text{Aut} B(H) \) of \( B(H) \). Recall from [1,2,3] that if \( f \in L^1(G) \) and \( x \in B(H) \) then

\[
\pi_\alpha(f)(x) = \int_G f(t) \alpha_t(x) dt
\]

\[
Z(f) = \{ \gamma \in \Gamma : \hat{f}(\gamma) = 0 \}
\]

\[
\text{Sp}_\alpha(x) = \cap \{ Z(f) : f \in L^1(G), \pi_\alpha(f)(x) = 0 \}
\]

We assume \( M \) is a von Neumann algebra acting on \( H \) such that \( \alpha_t(M) = M \), \( t \in G \). Then if \( E \) is a closed subset of \( \Gamma \),

\[
M^\alpha(E) = \{ x \in M : \text{Sp}_\alpha(x) \subseteq E \}
\].
Lemma. Denote by $\tilde{\alpha}_t$ the restriction of $\alpha_t$ to the Hilbert-Schmidt operators $H_2$ on $H$. Then $t \mapsto \tilde{\alpha}_t$ is a weakly, hence strongly, continuous unitary representation of $G$ on $H_2$.

Proof: Let $x,y \in H_2$ and $\epsilon > 0$. Let $y = y_1 + y_2$ with $y_1$ of finite rank and $\|y_2\|_2 < \epsilon/\|x\|_2$, where $\|z\|_2 = \langle z, z \rangle^{1/2}$, and $\langle \cdot, \cdot \rangle$ is the inner product on $H_2$. Since $\alpha_t(x) \to x$ ultrawakly as $t \to e$, the identity in $G$, there is a neighborhood $N$ of $e$ in $G$ such that $|\langle \tilde{\alpha}_t(x) - x, y_1 \rangle| < \epsilon$ for $t \in N$. Then $|\langle \tilde{\alpha}_t(x) - x, y \rangle| \leq |\langle \tilde{\alpha}_t(x) - x, y_1 \rangle| + |\langle \tilde{\alpha}_t(x) - x, y_2 \rangle| < \epsilon + 2\|x\|_2\|y\|_2 < 3\epsilon$, proving the lemma.

By Stone's theorem applied to the continuous unitary representation $t \mapsto \tilde{\alpha}_t$ on $H_2$, there exists a projection valued measure $P_\lambda$ on $\Gamma$ with values in $B(H_2)$ such that

$$\tilde{\alpha}_t = \int_{\Gamma} \chi_t(\lambda, \omega) dP_\lambda.$$

If $E$ and $F$ are closed subsets of $\Gamma$ we denote by $E+F$ the closure of the set $\{\gamma + \lambda : \gamma \in E, \lambda \in F\}$. We denote by $P(F)$ the closed subspace of $H_2$ obtained as the range of $\int dP_\lambda$, and by $P(F)^-$ its ultraweak closure in $B(H)$.

Theorem. With the above assumptions and notation, if $E$ is a closed subset of $\Gamma$ then

$$M^\alpha(E) = \bigcap_{N} M \cap P(E+N)^-,$$

where the intersection is taken over all compact neighborhoods of the identity in $\Gamma$.

Proof: Let $F$ be a closed subset of $\Gamma$. Let $x \in M \cap P(F)^-$
and \((x_\beta)\) be a net in \(P(F)\) which converges ultraweakly to \(x\). Let \(f \in L^1(G)\) have Fourier transform vanishing in a neighborhood of \(F\). By [1, Prop. 1.6] \(\pi_\alpha(f)\) is an ultraweakly continuous linear map on \(B(H)\). Thus by [1, Remark §2] \(0 = \pi_\alpha(f)(x_\beta) \to \pi_\alpha(f)(x)\), where we have identified the operator \(\pi_\alpha(f)\) defined by \(\tilde{\alpha}\) on \(H_2\) and its extension \(\pi_\alpha(f)\) to \(B(H)\). Thus \(\pi_\alpha(f)(x) = 0\) for all such \(f\), so again by [1] \(x \in M^\alpha(F)\). Thus \(\bigcap N \cap P(E+N)^- \subset \bigcap N \cap M^\alpha(E+N) = M^\alpha(E)\).

Conversely, let \(x \in M^\alpha(E)\), so in particular \(x \in B(H)^\alpha(E)\).

If \(F\) is a closed subset of \(\Gamma\) let \(R_2^\alpha(F)\) (resp. \(R^\alpha(F)\)) denote the closed (resp. ultraweakly closed) subspace of \(H_2\) (resp. \(B(H)\)) generated by range \(\pi_\alpha(f)\) in \(H_2\) (resp. in \(B(H)\)) for all \(f \in L^1(G)\) with \(\hat{f}\) compact and contained in \(F\). By [1, Prop. 2.2], \(B(H)^\alpha(E) = \bigcap N \cap R_2^\alpha(E+N)\), where the intersection is taken over all compact neighborhoods \(N\) of the identity in \(\Gamma\). In order to prove the theorem we may therefore assume there are \(f \in L^1(G)\) such that \(\hat{f}\) is compact and contained in \(E+N\), and \(y \in B(H)\), such that \(x = \pi_\alpha(f)(y)\). Since \(H_2\) is ultraweakly dense in \(B(H)\) there is a net \((y_\beta)\) in \(H_2\) which converges ultraweakly to \(y\). Since \(\pi_\alpha(f)\) is ultraweakly continuous, \(x = \pi_\alpha(f)(y) = \lim_\beta \pi_\alpha(f)(y_\beta) \in R_2^\alpha(E+N)^-\). But from the theory of spectral subspaces applied to unitary representations, \(R_2^\alpha(F) = P(F)\) for all closed sets \(F \subset \Gamma\). Thus \(x \in P(E+N)^-\) for all \(N\), and the proof is complete.

Remark 1. We cannot sharpen the theorem to a statement like
"M^2(E) = M \cap P(E)^-". Indeed, let \( H \) be a separable Hilbert space and \( u \) a unitary operator on \( H \) such that the von Neumann algebra \( A \) generated by \( u \) is a maximal abelian subalgebra of \( B(H) \) without minimal projections. Let \( \alpha \) be the representation of the integers defined by \( \alpha_n = \text{Ad}u^n \in \text{Aut}B(H) \). Then there is no nonzero \( x \in H_2 \) such that \( \alpha_n(x) = x \) for all \( n \), so \( P(\{1\}) = \{0\} \), while \( B(H)^\alpha(\{1\}) = A \).

Remark 2. The main idea in the proof of the theorem was to consider the restriction \( \tilde{\phi} \) of a map \( \phi \in B(B(H)) \), the bounded linear maps of \( B(H) \) into itself, to \( H_2 \). If \( \tilde{\phi} \) is a bounded normal linear operator on \( H_2 \) we can do spectral theory for \( \tilde{\phi} \) in \( B(H_2) \). It is tempting to generalize the above theorem and try to "lift" spectral theory for \( \tilde{\phi} \) in \( B(H_2) \) to that of \( \phi \) in \( B(B(H)) \). This however, seems to be quite hopeless except in special cases. Indeed, while the norm \( \|\phi\| \) of \( \phi \) in \( B(B(H)) \) is never smaller than the norm \( \|\tilde{\phi}\| \) of \( \tilde{\phi} \) in \( B(H_2) \), there is no finite constant \( k > 0 \) such that \( \|\phi\| \leq k\|\tilde{\phi}\| \) for all such \( \phi \).
References.

