COUNTABLE FUNCTIONALS AND THE ANALYTIC HIERARCHY

Dag Normann Oslo 1977

Abstract We work within the hierarchy of countable functionals. We prove that $2\text{-en}({}^{j+2}\psi) = \pi^1_j(h_\psi)(j \geq 1)$ where h_ψ is some function recursive in ψ. We also prove that the associates for functionals of type $j+2$ is a complete π^1_{j+1}-set $(j \geq 0)$. This generalizes work of Bergstra [1] and [2]. In the end we prove that there is a functional ψ of type $j+2$ $(j \geq 1)$ recursive in $0'$ such that $1\text{-sc}(\psi) \in \pi^1_j \setminus \Sigma^1_j$.

1. Introduction.

Kleene [5] and Kreisel [6] defined independently the notion of countable (continuous) functionals. In this paper we will use a modified version of Kleene's definition. We will be interested in sets recursive and semirecursive in a given countable functional of a certain type. For the kind of problems we deal with, the modification is essential, as will be explained through the results.

Kleene [5] defined the countable functionals of type k to be the elements of type k that when restricted to countable arguments act in a continuous way. This means that a countable functional takes both countable and non-countable arguments, they are elements of the maximal type-structure.

Kleene [5] also defines the associates for countable functionals. They are functions $\alpha : \mathbb{N} \to \mathbb{N}$ that contains intentional information about how the functional acts on countable arguments. One disadvantage in regarding countable functionals as elements of the maximal type-structure is that two different functionals may have the same associates (which means that when restricted to countable arguments, they are identical).

In more recent papers on countable functionals it is normal to regard the hierarchy of countable functionals, where a functional of type $k + 1$ is <u>only defined on countable arguments</u>. See e.g. Bergstra [1] and Gandy - Hyland [3].

We may still use the notion of associate as defined in Kleene [5].

Before turning away from Kleene's original notion of countable functionals, we take a short look at recursion in a countable functional over the maximal type-structure. Our notion of recursion and semirecursion is derived from Kleene's S1 - S9 defined in Kleene [4] (Bergstra [1] has given an equivalent definition). We let

$$\underline{\text{2-envelope of } \psi} = \text{2-en}(\psi) = \{A \subseteq \mathbb{N}^{\mathbb{N}} ; A \text{ is semirecursive in } \psi\}$$

$$\underline{\text{1-section of } \psi} = \text{1-sc}(\psi) = \{A \subseteq \mathbb{N} ; A \text{ is recursive in } \psi\}$$

For any countable functional ψ there is a function h_ψ recursive in ψ with an associate α for ψ recursive in the jump of h_ψ. We use this observation (which is valid independent of which type-hierarchy we are in) to prove

<u>Theorem 1</u> (Kleene's original notion of countable functionals)

<u>a</u> For any countable functional ψ there is an associate for

for ψ recursive in $\psi, {}^2E$.

b Let α be an associate for countable functionals of type j . Then there is a functional ψ^j recursive in $\alpha, {}^2E$ with α as an associate.

c If $j \geq 1$, then $2\text{-en}(\psi^{j+2}) = \pi^1_1 \ (h_\psi)$

d If $j \geq 1$, then $1\text{-sc}(\psi^{j+2}) \in \pi^1_1 \ (h_\psi)$

All proofs are more or less implicit in the literature, and we just indicate them. (Bergstra [1] is a good source for them all)

a follows from the observation above.

b Let α be an associate of a functional of type $j + 2$. For any φ^{j+1} define β_φ recursive in $\varphi, {}^2E$ such that if φ is countable, then β_φ is an associate for φ . If for some $n,k,\alpha(\bar{\beta}(n))=k+1$, we let $\psi(\varphi) = k$. Otherwise, $\psi(\varphi)=0$

c In this proof, let ${}^{j+2}0$ be the functional of type $j+2$ with constant value 0 . ψ is recursive in $h_\psi, {}^2E$, and every set semirecursive in ψ will be semirecursive in $h_\psi, {}^2E, {}^{j+2}0$.

By a model-theoretic argument given in Moldestad - Normann [7] we see that

$$2\text{-en}(h_\psi, {}^2E, {}^{k+2}0) = \pi^1_1(h_\psi)$$

On the other hand, every $\pi^1_1(h_\psi)$-subset of $\mathbb{N}^\mathbb{N}$ will be semi-recursive in ψ, h_ψ . Thus $2\text{-en}(\psi) = \pi^1_1(h_\psi)$

<u>Remark</u> In evaluating $2\text{-en}(\psi)$ we only use ψ on countable arguments, so by b it is fair to say that ψ is recursive in $h_\psi, {}^2E$.

d is a direct consequence of c since

$$\alpha \in 1\text{-sc}(\psi) \Longleftrightarrow \exists e \, \forall n \{e\}(n,\psi) = \alpha(n)$$

and

$$\{\langle e,n,\alpha\rangle ; \{e\}(n,\psi) = \alpha(n)\} \text{ is } \Pi^1_1(h_\psi) \text{ by } c.$$

From now on we will work within the restricted hierarchy of countable functionals. Kleene's S1 - S9 restricts smoothly to this narrowed hierarchy.

It is only natural that the notion of recursion will be somewhat different, since S8 is given a new meaning (we now regard functionals to be total when they in the old hierarchy were highly partial).

Bergstra [1] proves that if α is an associate for a countable functional ψ of type $j + 2 (j \geq 1)$, then $2\text{-en}(\psi) \subseteq \Pi^1_j(\alpha)$. In [2] Bergstra shows that if ψ is a functional of type 4, then $2\text{-en}(\psi) = \Pi^1_2(h_\psi)$. He also points out in [2] a proof for the fact that $2\text{-en}(\varphi) = \Pi^1_1(h_\varphi)$ when φ is of type 3. As an immediate corollary of Bergstra's proof we see that the set of associates for countable functionals of type 3 is a complete Π^1_2-set.

We will give a complete description of semirecursion over $\mathbb{N}^{\mathbb{N}}$ relative to a countable functional by the following result:

Theorem 2 $(j \geq 1)$

a If ψ is a functional of type $j+2$, then $2\text{-en}(\psi) = \Pi^1_j(h_\psi)$

b $As(j+1)$ is a complete Π^1_j-set

where $As(k)$ is the set of associates for functionals of type k.

This theorem answers a problem raised by J. Bergstra at the Generalized Recursion Theory II-conference in Oslo -77.

As Bergstra points out in [2], if F is of type 2, then $2\text{-en}(F) = \pi^1_1(h_F)$, and if f is of type 1 , then $2\text{-en}(f) = \Sigma^0_1(h_f) = \Sigma^0_1(f)$, so we have described semirecursion at all levels.

We first observe that for the proof of Theorem 2,a it is sufficient to regard very simple functionals: Let $\underline{{}^k0}$ be the countable functional of type k that has constant value zero (This is not the same object as defined in the proof of theorem 1.c).

Lemma 1 If $2\text{-en}({}^{j+2}0) = \pi^1_j$, then $2\text{-en}(\psi) = \pi^1_j(h_\psi)$ for all ψ of type $j+2$.

Proof: $2\text{-en}(\psi) \subseteq \pi^1_j(h_\psi) \subseteq 2\text{-en}({}^{j+2}0, h_\psi) \subseteq 2\text{-en}(\psi)$.

We We also observe that it will be sufficient to prove that every Σ^1_{j-1}-set is semirecursive in ${}^{j+2}0$, since the subsets of $\mathbb{N}^{\mathbb{N}}$ semirecursive in ${}^{j+2}0$ will be closed under universal quantification over $\mathbb{N}^{\mathbb{N}}$.

Before going into any technical details, we will give a short guide to the proof of theorem 2 and discuss some of the concepts involved.

The proof is essentially by induction on j . It is known that As(2) is complete π^1_1 and we may here assume that we have proved that As(j) is complete π^1_{j-1} .

Let B be Σ^1_{j-1} . Then there is a recursive map $\alpha : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

such that

$$f \in B \iff \alpha(f) \notin As(j)$$

We find a recursive relation S such that

$$\alpha(f) \notin As(j) \iff \forall \psi^j \exists n\, S(f,\psi,n)$$

and then

$$f \in B \iff \forall \psi^j \exists n\, S(f,\psi,n)$$

Let Φ be the partial, recursive function defined by

$$\Phi(f,\psi) = \mu n\, S(f,\psi,n)$$

$\lambda\psi\Phi(f,\psi)$ will be total if and only if $f \in A$, so

$$f \in B \iff {}^{j+2}O(\lambda\psi\Phi(f,\psi)) \text{ is defined.}$$

This shows that $B \in 2\text{-en}({}^{j+2}O)$.

In order to prove part b, we let A be Π^1_j and let B be Σ^1_{j-1} such that

$$f \in A \iff \forall g\, \langle f,g\rangle \in B \iff \forall g \forall \psi^j \exists n\, S(\langle f,g\rangle,\psi,n)$$

By pairing g and ψ , we may find a recursive relation S_1 such that

$$f \in A \iff \forall \psi^j \exists n\, S_1(f,\psi,n)$$

Letting

$$\Phi(f,\psi) = \begin{cases} 0 & \text{if } \exists n\, S_1(f,\psi,n) \\ \text{undefined} & \text{otherwise} \end{cases}$$

we reduce A to $As(j+1)$ by mapping f onto an associate for $\lambda\psi\varphi(f,\psi)$

Now, let us look at the idéa behind the definition of S. To each functional ψ we define uniformly recursive in ψ the Key Set of ψ ($KS(\psi)$). By comparing $\{\sigma;\alpha(\sigma) = 0\}$ with $KS(\psi)$ for various ψ, we will be able to tell if α is an associate. An unprecise definition of $KS(\psi)$ could be:

Given σ with sequence number k. If σ may be extended to an associate, we find an extention γ_k that takes value $k+1$ as often as reasonable (here we will follow an idea from Kleene [5]). Let the functional with associate γ_k be ψ_k (or rather ψ_k^j where j is the type). We define KS by

$$\delta \in KS(\psi) \text{ if and only if } \psi(\psi_k) = k .$$

The properties of this definition that are most usefull will be

i $\sigma_1 \neq \sigma_2, \sigma_1 \in KS(\psi)$ and $\sigma_2 \in KS(\psi) \Rightarrow \psi(\psi_{\langle\sigma_1\rangle}) \neq \psi(\psi_{\langle\sigma_2\rangle})$

ii $k_1 \neq k_2 \Rightarrow \psi_{k_1} \neq \psi_{k_2}$

iii If $\psi_{\langle\sigma\rangle}(\varphi) \neq \langle\sigma\rangle$, then σ "decides" the value $\psi(\varphi)$ "within reason" when ψ has an associate extending σ.

Now S will compare to what extent $\{\sigma : \alpha(f)(\sigma) = 0\} \subseteq KS(\psi)$. We will be more precise later.

Before turning into the real proof, we need some

2. Conventions and terminology.

We identify sequences and sequence-numbers. σ,τ,δ,π will denote sequences or sequence-numbers, $i,j,k,l,m,r,s,$ will denote natural numbers, $f,g,h,\alpha,\beta,\gamma$ will denote functions; the Greek letters will be used in connection with associates. F,G will denote functionals of type 2 and φ,ψ,Φ and Ψ functionals in general. A superscript j will indicate the type.

In order to avoid confusion we may assume that all numbers are sequence-numbers, and we let $\{\sigma_k\}_{k\in\mathbb{N}}$ be the identity-enumeration of sequences (So by our convention, $\sigma_k = k$).

If $\sigma = \langle k_0,\ldots,k_{n-1}\rangle$ we will without mentioning use that $\sigma > k_i$ for all $i = 0,\ldots,n-1$ and $\sigma > n = \mathrm{lh}(\sigma)$. We write $\sigma(i)$ for k_i, and for a function f, we write $\bar{f}(n)$ for $\langle f(0),\ldots,f(n-1)\rangle$.

By $\underline{\sigma(f) = n}$ we mean that for some m $\sigma(\bar{f}(m)) = n+1$. If for no $n, \sigma(f) = n$ we say that $\underline{\sigma(f)}$ $\underline{\text{is undefined}}$. We will only use this terminology when n is unique.

All functionals we construct will have canonical associates, and if φ is a functional of type j we write $\sigma(\varphi) = n$ if for the cannonical associate γ for $\varphi, \sigma(\gamma) = n$.

We will use terminology from Kleene [5] and it will be an advantage to know §§1-2 from that paper. In particular we will use

$\mathrm{Con}^j(\pi_1,\pi_2) \Longleftrightarrow$	There is an element φ of type j such that both π_1 and π_2 may be extended to associates for φ.
$\mathrm{ext}^j(n,\pi_1,\pi_2)$	is defined if $\mathrm{Con}^j(\pi_1,\pi_2)$, and then $\lambda n\ \mathrm{ext}^j(n,\pi_1,\pi_2)$ defines an associate extending π_1 for the functional φ

mentioned in the definition of Con^j .

We let

$\mathrm{Ext}^j(\pi_1,\pi_2)$ be the functional with associate $\lambda n\ \mathrm{ext}^j(n,\pi_1,\pi_2)$.

It follows from the definition in Kleene [5] that $\mathrm{Ext}^j(\pi_1,\pi_2) = \mathrm{Ext}^j(\pi_2,\pi_1)$. Kleene proves that the relations Con^j and the functions ext^j are primitive recursive, and it is implicit in his proof that the functionals $\mathrm{Ext}^j(\pi_1,\pi_2)$ are uniformly recursive in π_1,π_2 .

We will now make some of the concepts from the introduction more precise.

Definiton 1 For superscript j we assume that $\mathrm{Con}^j(\pi)$.

a $j = 1$

$$f_{\pi,k}(t) = \psi^1_{\pi,k}(t) = \begin{cases} \pi(t) & \text{if } \pi(t) \text{ is defined} \\ k & \text{otherwise} \end{cases}$$

$j = 2$

$$F_{\pi,k}(f) = \psi^2_{\pi,k}(f) = \begin{cases} \pi(f) & \text{if } \pi(f) \text{ is defined} \\ k & \text{otherwise} \end{cases}$$

$j > 2$

$$\psi^j_{\pi,k}(\varphi) = \begin{cases} s & \text{if for some } \tau \text{ we have } \mathrm{Con}^{j-1}(\tau)\ , \\ & \pi(\tau)=s+1 \text{ and for all } \sigma,\sigma^1\ , \text{ if} \\ & \mathrm{Con}^{j-2}(\sigma,\sigma^1), \tau(\sigma) > 0 \text{ and } \sigma^1 \leq \mathrm{lh}(\pi) \\ & \text{then } \varphi(\mathrm{Ext}^{j-2}(\sigma,\sigma^1)=\tau(\sigma)-1 \\ k & \text{otherwise} \end{cases}$$

Remark All these functionals will be uniformly recursive in π, k . The functional $\psi^j_{\pi,k}$ is based on the same idea as Kleene's $\mathrm{Ext}^j(\pi)$. It is fairly stright-forward to prove that $\psi^j_{\pi,k}(\varphi)$ is uniquely defined. A sufficient argument is found in the proof of lemma 5.

b Let $\psi^j_k = \psi^j_{\sigma_k,k}$. These are the functionals discussed in the introduction.

c If Φ is of type $j+1$, we let the Key set of Φ be

$$KS(\Phi) = \{\sigma_k \ ; \ \Phi(\psi^j_k) = k\}$$

d If φ is a functional of type $j-1$, we say

$\pi(\varphi) = s$ (potentially)

if for all $k \in \mathbb{N}$, $\psi^j_{\pi,k}(\varphi) = s$.

Remark $\pi(\varphi)$ will take s as a potential value if we never used the "otherwise"-case in defining $\psi^j_{\pi,k}(\varphi)$. The motivation for this definition is that we will need to foretell the value $\psi(\varphi)$ from π when φ has an associate that extends π . If φ has an associate α such that $\pi(\alpha)$ is defined, this foretelling should be correct, and moreover it should be recursive. If $\pi(\varphi) = s$ (potentially) we foretell in a recursive way that $\psi(\varphi) = s$. (There is no guarantee for stating $\psi(\varphi) = s$ from $\pi(\varphi) = s$ (potentially), unless $\pi(\varphi)$ is defined as above).

If $\psi^j_{\pi,k}(\varphi) \neq k$, then $\pi(\varphi)$ has a potential value, namely $\psi^j_{\pi,k}(\varphi)$. $\pi(\varphi)$ will never have more than one potential value.

We say that $\alpha \in As(j)$ (locally) if $\mathrm{Con}^j(\bar{\alpha}(n))$ for each n .

Remark We may now define h_ψ by

If ψ is of type $j + 1$, let

$$h_\psi(\sigma) = \begin{cases} \psi(\psi^j_{\sigma,0}) & \text{if } \mathrm{Con}^j(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

3. Proof of theorem 2.

By induction on $j \geq 1$ we are going to prove that

Induction hypothesis

For each π^1_j set $A \subseteq \mathbb{N}^{\mathbb{N}}$ there is a recursive relation R such that

i $f \notin A \iff \forall \psi^{j+1} \exists n\ (R(f,n)\ \&\ \sigma_n \notin KS(\psi))$

ii For each f,n ; $R(f,n) \Rightarrow \mathrm{Con}^j(\sigma_n)$

iii For each f there are $n_1 \neq n_2$ such that $R(f,n_1)$ and $R(f,n_2)$

We showed in the introduction how to prove Theorem 2.a from part i of the induction hypothesis. Theorem 2.b will be proved as a corollary at each induction step (see lemma 4). ii and iii are technical parts of the induction hypothesis, and will only be used in lemma 3.

Induction basis $j = 1$ This proof is just a reformulation of the argument given by Bergstra [2]. We could have used his proof directly as an induction basis, but that would require a nonuniform formulation of the induction hypothesis.

Lemma 2 Let $\alpha \in As(2)$ (locally). Then the following two statements are equivalent.

<u>i</u> For some F , $\{\sigma ; \alpha(\sigma) = 0\} \subseteq KS(F)$

<u>ii</u> $\alpha \in As(2)$

<u>Proof</u> Assume $\{\sigma ; \alpha(\sigma) = 0\} \subseteq KS(F)$. Let f be a function. We prove that for some $m, \alpha(\bar{f}(m)) > 0$ by showing that for some $m, \bar{f}(m) \notin KS(F)$.

Let β be an associate for F and choose m_o, k such that $\beta(\bar{f}(m_o)) = k+1$. Now, if $m > m_o$ and $\bar{f}(m) \in KS(F)$, then $F(f_{\bar{f}(m)}) = \bar{f}(m)$ (as a sequence-number). By definition of $f_{\bar{f}(m)}$ as an extention of $\bar{f}(m)$, $\bar{f}_{\bar{f}(m)}(m_o) = \bar{f}(m_o)$

Thus $F(f_{\bar{f}(m)}) = k$, so $\bar{f}(m) = k$.

There is atmost one $m_1 > m_o$ such that $\bar{f}(m_1) = k$. Then for $m > m_1$, $\bar{f}(m) \notin KS(F)$, so $\alpha(\bar{f}(m)) > 0$. This proves <u>i</u> $\Rightarrow$ <u>ii</u> .

Now, assume that $\alpha \in As(2)$. We let $\sigma \prec \tau \prec f$ mean that σ is a subsequence of τ which again is of the form $\bar{f}(m)$. Define

$$F(f) = \begin{cases} k & \text{if } \alpha(\sigma_k) = 0 \text{ and for some } \tau \ \alpha(\tau) > 0 , \\ & \sigma_k \prec \tau \prec f \text{ and } \tau \prec f_k \\ 0 & \text{otherwise} \end{cases}$$

i.e. we check for each $\sigma_k \prec f$ such that $\alpha(\sigma_k) = 0$ if f looks like f_k up to the least n such that $\alpha(f(n)) > 0$. Then we let $F(f) = k$.

We must prove that F is well defined. If $F(f) = k_1$ and $F(f) = k_2$, choose n minimal such that $\alpha(\bar{f}(n)) > 0$. $\bar{f}(n)$ is an extention of σ_{k_1} by k_1 and an extention of

σ_{k_2} by k_2 , and since both extentions are proper, we see that $f(n) = k_1$ and $f(n) = k_2$. Then $k_1 = k_2$, which proves that F is well defined.

F is countable since it is recursive in α .

If $\alpha(\sigma_k) = 0$ it is clear that $F(f_k) = k$, so $\sigma_k \in KS(f)$.

This proves ii > i.

We may now prove the induction basis:

Let $A \in \pi^1_1$. Let S be a recursive tree such that

$$f \in A \iff \forall g \; \exists n \; \neg S(\bar{f}(n), \bar{g}(n), n)$$

For each $f \in \mathbb{N}^{\mathbb{N}}$, let

$$\alpha_f(\sigma) = \begin{cases} 0 & \text{if } lh(\sigma) \leq 1 \text{ or } S(\bar{f}(lh(\sigma)), \sigma, lh(\sigma)) \\ 1 & \text{otherwise} \end{cases}$$

$\alpha_f \in As(2)$ (locally) and

$$f \in A \iff \alpha_f \in As(2)$$

By lemma 2

$$f \notin A \iff \alpha_f \notin As(2) \iff \forall G \exists \sigma (\alpha_f(\sigma) = 0 \text{ and } \sigma \notin KS(G))$$

Let

$$R(f,n) \stackrel{d}{\iff} \alpha_f(\sigma_n) = 0$$

i and ii in the induction hypothesis are clearly satisfied.

iii is satisfied since $\alpha_j(\sigma) = 0$ if $lh(\sigma) \leq 1$.

This ends the proof of the induction basis.

The induction stop.

We now assume that the theorem is proved for all π^1_j sets. Let $A \in \pi^1_{j+1}$ and let B be Σ^1_j such that

$$f \in A \iff \forall g \, \langle f,g \rangle \in B$$

By the induction hypothesis there is a recursive relation R_1, satisfying ii and iii, such that

$$h \in B \iff \forall \psi^{j+1} \, \exists n (R_1(h,n) \, \& \, \sigma_n \notin KS(\psi))$$

so

$$f \in A \iff \forall g \forall \psi^{j+1} \, \exists n (R_1(\langle f,g \rangle,n) \, \& \, \sigma_n \notin KS(\psi))$$

If we let $\psi_1(\varphi) = \psi(\langle {}^j0,\varphi \rangle)$ and $g_\psi(n) = \psi({}^j(n+1))$ (where ${}^jk$ is the type j functional with constant value k) we see that

(1) $$f \in A \iff \forall \psi^{j+1} \exists n (R_1(\langle f,g_\psi \rangle,n) \, \& \, \sigma_n \notin KS(\psi_1))$$

We will now for each f try to define an "associate" for the functional that to each ψ^{j+1} gives 0 if there is an n such that

$$R_1(\langle f,g_\psi \rangle,n) \, \& \, \sigma_n \notin KS(\psi_1)$$

If $Con^{j+1}(\sigma)$ we say that σ contains sufficient information about ψ to compute $\psi(\varphi)$ when there is a canonical associate γ for φ and $\sigma(\gamma)$ is defined.

Remark. For each n, when we check if $R_1(\langle f,g_\psi \rangle,n)$ & $\sigma_n \notin KS(\psi_1)$ holds, we only apply ψ to functionals of the type ${}^jk, \psi^j_k$ or variations over these, for which we have canonical associates.

Definition 2. If $Con^{j+1}(\sigma)$ we use the following recursive procedure to find $\alpha_j(\sigma)$.

Go to the least n such that $Con^j(\sigma_n)$ and i or ii below hold:

i σ does not contain enough information about ψ to evaluate the truth of $R_1(\langle f, g_\psi \rangle, n)$ or to compute $\psi_1(\psi_n^j)$

ii σ does contain this information, $R_1(\langle f, g_\psi \rangle, n)$ and $\psi_1(\psi_n^j) \neq n$.

If this least n satisfies i, let $\alpha_f(\sigma) = 0$, if it satisfies ii, let $\alpha_f(\sigma) = 1$.

Remark α_f will be the associate for the partial functional we described above. If $\sigma < \tau$ and $\alpha_f(\sigma) = 1$, then $\alpha_f(\tau) = 1$ by the same verification.

We will prove that α_f is total for all f, and thus locally an associate. α_f takes values 0 and 1 only, so there are no conflicting positive values. If $\neg Con^{j+1}(\sigma)$ we may let $\alpha_f(\sigma) = 0$.

Lemma 3 For each f and σ, if $Con^{j+1}(\sigma)$, then $\alpha_f(\sigma)$ is defined.

Proof. Let σ_1 be the part of an associate for ψ_1 that we may extract from σ. There are two cases:

i $\sigma_1(\langle \ \rangle) > 0$. Pick ψ such that ψ has an associate extending σ. ψ_1 will be constant, say $\psi_1 = {}^{j+1}k$.

Choose $n_1 \neq n_2$ such that

$$R(\langle f, g_\psi\rangle, n_1) \quad \text{and} \quad R(\langle f, g_\psi\rangle, n_2)$$

We may assume that $n_1 \neq k$. Then $\sigma_{n_1} \notin KS(\psi_1)$ since $\psi_1(\psi^j_{n_1}) = k \neq n_1$, and σ contains sufficient information to state this. By ii in the induction hypothesis, $Con^j(\sigma_{n_1})$, so if for no $n \leq n_1$, n satisfies i or ii, certainly n_1 will satisfy ii.

ii $\sigma_1(\langle\ \rangle) = 0$ (or is undefined)

It is sufficient to prove that $\sigma_1(\psi^j_n)$ is undefined for some n, since then i will hold for this n.

Let $\sigma_{n_k} = \langle k+1\rangle$. Any functional with an associate extending σ_{n_k} will be ${}^jk$, so in particular $\psi^j_{n_k} = {}^jk$ (constant value k). Now $\sigma_1(\langle k+1\rangle)$ is undefined for some k since σ_1 is finite, so $\sigma_1(\psi^j_{n_k})$ will be undefined.

We prove theorem 2.b and end the first part of the induction step by:

Lemma 4. $\quad f \in A \Longleftrightarrow \alpha_f \in As(j+2)$

Proof. $\Rightarrow$: Let $f \in A, \beta \in As(j+1)$ be an associate for a functional ψ. To establish $\Rightarrow$ we will show that $\alpha_f(\bar{\beta}(n)) = 1$ for some n.

Since $f \in A$ it follows from (1) that

$$\exists n(R_1(\langle f, g_\psi\rangle, n) \ \&\ \sigma_n \notin KS(\psi_1))$$

Let n_0 be the least such n. For a number of functionals $\varphi_1, \ldots, \varphi_l$ we compute $\psi(\varphi_i)$ $(i = 1, \ldots, l)$ in order to verify this property of n_o . Let $\gamma_1,\dots,\gamma_l$ be canonical assiciates for $\varphi_1,\dots,\varphi_l$.

Pick m so large that $\bar{\beta}(m)(\gamma_i)$ are defined for $i = 1,\dots,l$ This is possible since $\beta \in As(j+1)$. But then $\alpha_f(\bar{\beta}(m)) = 1$ since $\bar{\beta}(m)$ contains sufficient information about ψ to compute $\psi(\varphi_1),\dots,\psi(\varphi_l)$ and thereby verify the property of n_o .

$\Leftarrow$ Assume that $f \notin A$. Then by (1) there is a functional ψ for which

$$\forall n(R_1(\langle f,g_\psi\rangle,n) \Rightarrow \sigma_n \in KS(\psi_1))$$

Let β be an associate for ψ . If $\alpha_f(\bar{\beta}(m)) = 1$ for some m , this means that $\bar{\beta}(m)$ contains enough information about ψ to ensure that for some n

$$R_1(\langle f,g_\psi\rangle,n) \;\&\; \sigma_n \notin KS(\psi_1)$$

which is impossible. Thus $\alpha_f(\bar{\beta}(m)) = 0$ for all m , and α_f is not an associate.

To end the proof of the induction step we will give a higher type version of Lemma2 for the α_f's . There are some obstacles that prevent us from giving the same argument without modifications. One of these is that we may have a pair $\sigma_k \prec \tau$ such that $\alpha_f(\sigma_k) = 0, \alpha_f(\tau) = 1, \tau$ may be extended to an associate for ψ_k^{j+1} but τ never takes the value $k+1$. (See the proof that F is well defined in Lemma 2). If this is the case, σ_k will contain enough potential information to indicate that we do not need the value k from ψ_k^{j+1} to ensure that

$$\exists n(R_1(\langle f, g_{\psi_k^{j+1}}\rangle, n) \ \& \ \sigma_n \notin KS((\psi_k^{j+1})_1)$$

(Potential computations are defined in Definition 1.d).

We will see that it is sufficient to regard those σ_k for which we need the value k from ψ_k^{j+1} . We make this property precise in the following definition:

Definition 3 The function α_f^p (α_f potentially) is defined as α_f with the difference that we demand $\sigma(\varphi)$ (potentially) where we in the definition of α_f would demand that σ contains sufficient information to compute $\psi(\varphi)$.

Remark. By the argument of lemma 3 we may also show that α_f^p is total.

If $\alpha(\varphi) = k$, then $\sigma(\varphi) = k$ (potentially). This shows that if $\alpha_f(\sigma) = 1$, then $\alpha_f^p(\sigma) = 1$. Both functions α_f and α_f^p will be uniformly recursive in f .

In general $\{\sigma : \alpha_f^p(\sigma) = 0\}$ will not be a tree, and α_f^p is not an associate even locally.

We are now going to prove that

(2) α_f is an associate $\Leftrightarrow \exists \Phi^{j+2} \forall n(Con^{j+1}(\sigma_n) \ \& \ \alpha_f^p(\sigma_n) = 0 \Rightarrow \sigma_n \in KS(\Phi))$.

If we let $R(f,n) \Leftrightarrow (Con^{j+1}(\sigma_n)$ and $\alpha_f^p(\sigma_n) = 0)$ we see that i and ii in the induction hypothesis is established. To see part iii, note that if $\sigma_n = \langle 0,\ldots,0\rangle$ then $Con^{j+1}(\sigma_n)$ and $\alpha_f^p(\sigma_n) = 0$ (σ_n contains no information at all).

$\Leftarrow$ in (2) will be established through lemmas 5 and 6, while $\Rightarrow$ is proved in lemma 7. For later use, lemma 7 will be a bit too strong for this proof.

Lemma 5. Let φ be a functional of type j and let $\beta \in As(j+1)$. If for some n, k_1 and k_2

$$\bar{\beta}(n)(\varphi) = k_1 \ \text{(potentially) and } \bar{\beta}(n+1)(\varphi) = k_2 \ \text{(potentially)}$$

then $k_1 = k_2$.

Proof. If $j=1$, this is trivial since potential and real computations are the same. So assume that $j > 1$.

According to definitions 1.a and d, choose τ_1 such that $\bar{\beta}(n)(\tau_1) = k_1+1$ and such that when $\tau_1(\delta) > 0$, $Con^{j-1}(\delta, \delta')$ and $\delta' \leq n$, then

$$\varphi(Ext^{j-1}(\delta, \delta')) = \tau_1(\delta) - 1$$

Choose τ_2 for $\bar{\beta}(n+1), n+1$ and k_2 in a similar way. It is sufficient to prove $Con^j(\tau_1, \tau_2)$, since

$$Con^{j+1}(\bar{\beta}(n+1)) \ \& \ \bar{\beta}(n+1)(\tau_1) = k_1+1 \ \& \ \bar{\beta}(n+1)(\tau_2) = k_2+1$$
$$\& \ Con^j(\tau_1, \tau_2) \Rightarrow k_1 = k_2$$

by Kleene's definition of Con^{j+1} from [5].

So choose π_1, π_2 such that $Con^{j-1}(\pi_1, \pi_2), \tau_1(\pi_1) > 0$ and $\tau_2(\pi_2) > 0$. According to Kleene [5], we must prove that $\tau_1(\pi_1) = \tau_2(\pi_2)$.

$\bar{\beta}(n+1)(\tau_2)$ is defined, so $\tau_2 \leq n$ and $\pi_2 \leq n$. Clearly $\pi_1 \leq n$. Then by choice of τ_1 and τ_2

$$\tau_1(\pi_1) - 1 = \varphi(Ext^{j-1}(\pi_1, \pi_2)) = \varphi(Ext^{j-1}(\pi_2, \pi_1)) = \tau_2(\pi_2) - 1$$

This proves the lemma.

<u>Lemma 6</u> Let Φ be of type $j+2$ and assume that

$$\{\sigma_n \ ; \ \mathrm{Con}^{j+1}(\sigma_n) \ \& \ \alpha_f^p(\sigma_n) = 0\} \subseteq \mathrm{KS}(\Phi)$$

Then α_f is an associate.

<u>Proof</u> Let $\beta \in \mathrm{As}(j+1)$. As in the parallell part of lemma 2 we may conclude that there is an n such that for no $m \geq n$ $\bar{\beta}(m) \in \mathrm{KS}(\Phi)$.

By assumption

$$m \geq n \Rightarrow \alpha_f^p \ (\bar{\beta}(m)) = 1 \ .$$

In order to obtain a contradiction, assume that for no $m \geq n, \alpha_f(\bar{\beta}(m)) = 1$.

In the verification of $\alpha_f^p(\bar{\beta}(n)) = 1$ we demand potential values $\bar{\beta}(n)(\varphi)$ for certain functionals φ . Since β is an associate, $\bar{\beta}(m)(\varphi)$ will be defined for some m . But the real and potential values cannot be the same for all φ occuring in the verification of $\alpha_f^p(\bar{\beta}(n)) = 1$, since otherwise there would be an $m \geq n$ such that $\alpha_f(\bar{\beta}(m)) = 1$.

Thus there must be a first place in the verification of $\alpha_f^p(\bar{\beta}(n)) = 1$ where for some φ , $\bar{\beta}(n)(\varphi)$ is demanded, $\bar{\beta}(n)(\varphi)$ has a potential value s , but for some $m > n$ $\bar{\beta}(m)(\varphi)$ does not have potential value s .

By lemma 5, for the least such m , $\bar{\beta}(m)(\varphi)$ has no potential value at all. By the choice of φ we would demand the potential value of $\bar{\beta}(m)(\varphi)$ in order to compute $\alpha_f^p(\bar{\beta}(m))$, and would get $\alpha_f^p(\bar{\beta}(m)) = 0$. This contradicts the assumption, and the lemma is proved.

Lemma 7 If α_f is an associate then there is a functional Φ uniformly recursive in f such that

$$\{\sigma_n : Con^{j+1}(\sigma_n) \ \& \ \alpha_f^p(\sigma_n) = 0\} \subseteq KS(\Phi)$$

Proof: We give the following algorithm for computing $\Phi(\psi)$:

Find the least n such that $R_1(\langle f, g_\psi\rangle, n) \ \& \ \sigma_n \notin KS(\psi_1)$. For a finite sequence of functionals $\varphi_1, \ldots, \varphi_l$ we evaluate $\psi(\varphi_1), \ldots, \psi(\varphi_l)$ in order to verify this property of n . Now, if for some $\sigma_k, \alpha_f^p(\sigma_k) = 0$ and ψ and ψ_k^{j+1} agree on $\varphi_1, \ldots, \varphi_l$, we let $\Phi(\psi) = k$. Otherwise we let $\Phi(\psi) = 0$.

Through claims 1-3 we will prove that Φ is well defined, recursive in f and satisfies the inclusion above.

Claim 1. If ψ agrees with ψ_k^{j+1} on $\varphi_1, \ldots, \varphi_l$ and $\alpha_f^p(\sigma_k) = 0$, then for some $i \leq l$ $\psi(\varphi_i) = k$.

Proof: As we pointed out after Definition 1, if $\psi_k^{j+1}(\varphi) = s \neq k$, then $\sigma_k(\varphi) = s$ (potentially). If for no $i \leq l$, $\psi_k^{j+1}(\varphi_i) = k$, we obtain potential values $\sigma_k(\varphi_i)$ for all $i \leq l$. By choice of $\varphi_1, \ldots, \varphi_l$ we see that $\alpha_f^p(\sigma_k) = 1$, contradicting the assumption. Thus $\psi(\varphi_i) = \psi_k^{j+1}(\varphi_i) = k$ for some $i \leq n$.

By this claim we see that to compute $\Phi(\psi)$ we only need to check if $\alpha_f^p(\sigma_k) = 0$ and ψ agrees with ψ_k^{j+1} on $\varphi_1, \ldots, \varphi_l$ for $k \in \{\psi(\varphi_1), \ldots, \psi(\varphi_l)\}$. Thus $\Phi(\psi)$ is defined by a recursive instruction.

Claim 2 If $\Phi(\psi) = k_1$ and $\Phi(\psi) = k_2$, then $k_1 = k_2$.

Proof. We may assume that $\Phi(\psi)$ is not defined by the otherwise case. Assume $k_1 < k_2$. By claim 1, $\psi(\varphi_i) = k_2$ for some $i \leq l$. But $\psi(\varphi_i) = \psi_{k_1}^{j+1}(\varphi_i) \leq k_1$ for all $i \leq l$. This gives a contradiction.

Claim 3 If $\alpha_f^p(\sigma_k) = 0$ then $\sigma_k \in KS(\Phi)$

Proof: By the algorithm for $\Phi(\psi_k^{j+1})$, the value is k since $\alpha_f^p(\sigma_k) = 0$ and ψ_k^{j+1} agrees with itself on $\varphi_1,\ldots,\varphi_l$.

This ends the proof of lemma 7, and Theorem 2 is established.

4 On the 1-section of a countable functional.

We recall that the 1-section of ψ will be

$$1\text{-sc}(\psi) = \{f \ ; \ f \text{ is recursive in } \psi\} .$$

As in the proof of Theorem 1.d. we see that

$$1\text{-sc}(\psi^{j+2}) \in \pi_j^1(h_\psi)$$

Also $1\text{-sc}(F) \in \pi_1^1(h_F)$.

In Normann [8] we proved that there is a countable functional F of type 2 recursive in $0'$ such that $1\text{-sc}(F) \in \pi_1^1 \setminus \Sigma_1^1$. In this section we extend this result to arbitrary types by proving

Theorem 3 $(j \geq 1)$

There is a countable functional Φ of type $j+2$ recursive in $0'$ such that

$$1\text{-sc}(\Phi) \in \pi_j^1 \setminus \Sigma_j^1 .$$

Corollary

a For j 2 there is a 1-obtainable functional ψ^{j+1} such that for all Φ^j

$$1\text{-sc}(\Phi) \neq 1\text{-sc}(\psi)$$

b There is no "plus-one" theorem for countable functionals.

c (Bergstra [1]) For each $j = 2$ or $j > 3$ there is a non-reducible 1-obtainable functional of type j. (Bergstra proved this for $j = 3$ as well).

d There are functionals ψ of any type such that

$$\mathbb{N}^{\mathbb{N}} \setminus 1\text{-sc}(\psi)$$

is not semirecursive in ψ.

Remark. A functional is 1-obtainable if it is recursive in a function.

A "plus-one theorem" would say that for all $k \geq 2$ and all ψ^k there is an F such that $1\text{-sc}(F) = 1\text{-sc}(\psi^k)$. Sacks [11] and [12] proved plus-one theorems for normal functionals in the maximal type-structure.

d indicates that in general there is no gap in complexity between the 1-section and the semirecursive relations (There certainly is a gap in complexity between the recursive and semirecursive relations). As a curiosity we mention that adding ^2E will alter this picture completely. (It makes good sense to do recursion in a countable functional and ^2E as long as we do not apply the functional on discontinuous arguments). By arguments from Theorem 1 we can prove that $1\text{-sc}(\psi^j, {}^2E) = 1\text{-sc}(h_\psi, {}^2E) \in \pi^1_1(h_\psi)$.

$$2\text{-en}(\psi^{j+2}, {}^2E) = \pi^1_j(h_\psi) \quad \text{as before.}$$

In Normann [8] we constructed the functional by letting it "climb" a long wellorded set of r.e. degrees. The production of the 1-section was a rather slow process. The functional we will construct to prove Theorem 3 will produce it's 1-section in a more hastily way.

Proof of theorem 3

We will use a result from ordinary degree-theory, proved by a finite priority argument. The method used in the proof of this proposition is found in most introductions to degree-theory, and it is a slight improvement of the way. Friedberg and Muchnic solved Post's problem (see e.g. Rogers [1], Sacks [10] or Shoenfield [13])

Proposition

There is an r.e. set $B \subseteq \mathbb{N} \times \mathbb{N}$ such that for no $m \in \mathbb{N}$

$$B_m \leq_T B_{-m}$$

where

$$B_m = \{k \; ; \; (m,k) \in B\}$$

$$B_{-m} = \{(n,k) \in B \; ; \; m \neq n\}$$

Now, let $A \subseteq \mathbb{N}$ be π^1_j. We will construct a functional Φ of type $j+2$ such that

$$m \in A \iff B_m \in 1\text{-sc}(\Phi)$$

If $1\text{-sc}(\Phi) \in \Delta^1_j$, we see that $A \in \Delta^1_j$, so we may choose A not to be Δ^1_j and thereby prove theorem 3.

By the proof of Theorem 2 there is a recursive relation

R such that

$$m \in A \iff \exists \psi^{j+1} \forall n (R(m,n) \ > \sigma_n \in KS(\psi))$$

where we by lemma 7 may choose ψ recursive when $m \in A$.
Let S be recursive such that

$$(m,k) \in B \iff \exists s (s,m,k) \in S$$

Let Φ be the functional of type $j+2$ defined by

$$\Phi(\psi,m,k) = \begin{cases} 1 & \text{if } \exists s[(s,m,k) \in S \ \& \ \forall n \leq s(R(m,n) \Rightarrow \sigma_n \in KS(\psi))] \\ 0 & \text{otherwise} \end{cases}$$

<u>Lemma 8</u> Φ is recursive in B and thus countable.

<u>Proof:</u> To compute $\Phi(\psi,m,k)$ from B, ask

$$\exists s (s,m,k) \in S \quad (\text{i.e. } (m,k) \in B\ ?)$$

If no, $\Phi(\psi,m,k) = 0$. If yes, let s_o be the least such s and ask if the following hold:

$$\forall n \leq s_o(R(m,n) \Rightarrow \sigma_n \in KS(\psi))$$

If no, $\Phi(\psi,m,k) = 0$, if yes $\Phi(\psi,m,k) = 1$.

<u>Lemma 9.</u> If $m \in A$, then B_m is recursive in Φ.

<u>Proof:</u> Let ψ be recursive such that

$$R(m,n) \Rightarrow \sigma_n \in KS(\psi)$$

Then

$$\Phi(\psi,m,k) = 1 \iff \exists s (s,m,k) \in S \iff (m,k) \in B$$

and $\lambda k\Phi(\psi,m,k)$ is the characteristic function of B_m .

Lemma 10. If $m_o \notin A$, then Φ is recursive in B_{-m_o} .

Proof: We show how to compute $\Phi(\psi,m,k)$ from B_{-m_o} .
If $m \neq m_o$ we act as in lemma 8.
If $m = m_o$, then $m \notin A$ and

$$\exists n(R(m,n) \ \& \ \sigma_n \notin KS(\psi))$$

Let n_o be the least such n . Now

$$\Phi(\psi,m,k) = \begin{cases} 1 & \text{if } \exists s < n_o \ (s,m,k) \in S \\ 0 & \text{if } \forall s < n_o \ (s,m,k) \notin S \end{cases}$$

This proves the lemma.

Now, if $m \notin A, B_m$ cannot be recursive in Φ since it then will be recursive in B_{-m} by lemma 10.

By choice of B this is not the case. This ends the proof of theorem 3.

Remark. By turning the argument from theorem 3 upside down, we see that Φ is recursive in B_{-m} if and only if $m \notin A$. This shows that the relation

$$\Phi^{j+2} \text{ is recursive in } f$$

is not a π^1_j-relation. This should be no surprise since the most natural definition of this relation, given an associate for Φ , will be by a π^1_{j+1}-formula.

R E F E R E N C E S

1. J. Bergstra, Computability and continuity in finite types, Disertation, Utrecht 1976.

2. J. Bergstra, 1-Envelopes of Continuous Functionals, Recursive Function Theory: Newsletter No 16 (1977) Item 196.

3. R.O. Gandy and J.M.E. Hyland, Computable and recursively countable functions of higher type, in R.O. Gandy and J.M.E. Hyland (eds) Logic Colloquium 76 407-438, North-Holland 1977.

4. S.C. Kleene, Recursive Functionals and Quantifiers of Finite Types I, Tran.Amer.Math.Soc. 91 (1959) 1-52; and II 108 (1963) 106-142.

5. S.C. Kleene, Countable functionals, in A. Heyting (ed): Constructivity in Mathematics, North-Holland (1959) 87-100.

6. G. Kreisel, Interpretation of Analysis by means of Constructive Functionals of Finite Types, in A. Heyting (ed): Co structivity in Mathematics, North-Holland (1959) 10 -128.

7. J. Moldestad and D. Normann, Models for recursion theory, Journal of Symbolic Logic 47 (1976) 719-729.

8. D. Normann, On a problem of S. Wainer, Oslo Preprint Series No 13, 1976. To appear in Journal of Symbolic Logic under title:

9. H. Rogers Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill 1967.

10. G.E. Sacks, Degrees of Unsolvability, Annals of Mathematical studies, Princeton University Press, 1963.

11. G.E. Sacks, The 1-section of a type n object, in J.E. Fenstad and P.G. Hinman (eds) Generalized Recursion Theory (North-Holland,Amsterdam 1974) 81-93.

12. G.E. Sacks, The k-section of a type-n object, to appear.

13. J.R. Shoenfield, Degrees of Unsolvability, Mathematics Studies 2, North-Holland/American Elsevier 1971.