A NOTE ON THE BIDUAL OF A JB-ALGEBRA

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In [1] Alfsen, Shultz and Størmer defined the concept of a JB-algebra and proved a generalized Gelfand-Neumark theorem, stating that the study of JB-algebras can be reduced to the study of JC-algebras (norm closed Jordan algebras of self-adjoint operators on a Hilbert space) and the "exceptional" algebra $M_3^S$ (of all self-adjoint $3 \times 3$-matrices over the Cayley numbers). An important technical tool is the "enveloping" JB-algebra $\tilde{A}$, which was initially defined as a certain monotone complete JB-algebra contained in the bidual $A^{**}$ of the given JB-algebra $A$, [1;§3]. The authors of [1] conjectured that $A^{**}$ is itself a JB-algebra, and that, in fact, $\tilde{A} = A^{**}$. Later on, this was proved by Shultz [3], but only via an a posteriori verification based on the final results of [1].

This note contains a short direct proof that $\tilde{A} = A^{**}$, which may replace the material of [1;§3] in the development leading up to the generalized Gelfand-Neumark theorem.

We adopt the notation and terminology established in §§ 1-2 of [1]. In particular, $A$ will denote a JB-algebra and $K$ its state space. It is well known that the bidual $A^{**}$ may be identified, both in order and norm, with the space $A^b(K)$ of all bounded affine functions on $K$. We shall use the term "weak topology" for the $\sigma(A^{**},A^*)$-topology of $A^{**}$. 
Theorem 1  If $A$ is a JB-algebra, then $A^{**}$ is a JB-algebra with a (necessarily unique) product which extends the original product on $A$ and which is weakly continuous in each variable separately.

Proof  For each $\rho \in K$, consider the seminorm $\| \cdot \|_\rho$ on $A$, defined by

$$\|a\|_\rho = \langle a^2, \rho \rangle^{\frac{1}{2}}$$

Factoring by the kernel of this norm and completing, we obtain a real Hilbert space $H_\rho$ and a linear mapping $\eta_\rho$ of $A$ onto a dense subspace of $H_\rho$ such that

$$(\eta_\rho(a)|\eta_\rho(b)) = \langle a^*b, \rho \rangle \quad (a, b \in A).$$

The mapping $\eta_\rho$ has norm 1, and its bidual $\eta^{**}: A^{**} \rightarrow H^{**} = H_\rho$ is a weakly continuous extension of $\eta_\rho$. For any pair $a, b \in A^{**}$ we define

$$f_{a, b}(\rho) = (\eta^{**}(a)|\eta^{**}(b)) \quad (\rho \in K)$$

From (1) it follows that $f_{a, b}$ is an affine function on $K$, whenever $a, b \in A$. However, by weak continuity of $\eta^{**}$, $f_{a, b}(\rho)$ is a weakly continuous function of each parameter $a, b$. By weak density of $A$ in $A^{**}$ we conclude that $f_{a, b}$ is affine for any pair $a, b \in A^{**}$. Obviously, $f_{a, b}$ is bounded on $K$, and so may be identified with an element $a^*b$ of $A^{**}$. Thus, by the definition (2):

$$\langle a^*b, \rho \rangle = (\eta^{**}(a)|\eta^{**}(b)) \quad (a, b \in A^{**}).$$

It follows readily that this product on $A^{**}$ is bilinear, commutative, and weakly continuous in each variable separately.
The defining Jordan identity $a^2(x \cdot a) = (a^2 \cdot x) \cdot a$ can be written $[L_a, L_{a^2}] = 0$. The latter can be "linearized" to the equivalent formula (cf. [2,p.34]):

\[(3) \quad [L_a, L_{b \cdot c}] + [L_b, L_{c \cdot a}] + [L_c, L_{a \cdot b}] = 0\]

(Substitute $a + \lambda b + \mu c$ for $a$ and compute the $\lambda \mu$-term in the resulting polynomial in $\lambda, \mu$). Applying the operator in (3) to an element $d \in A$ we obtain a quadrilinear identity which holds for $a, b, c, d \in A$. Taking a weak limit in each variable separately, we find that the same identity holds for $a, b, c, d \in A^{**}$. Thus $A^{**}$ is a Jordan algebra.

If $a \in A^{**}$ and $-1 \leq a \leq 1$, then $\langle a^2, \rho \rangle = \|\eta_\rho(a)\|^2 \leq \|a\|^2 \leq 1$, so $0 \leq a^2 \leq 1$. By [1;Thm.2.1] $A^{**}$ is a JB-algebra.

The strong topology on $A^{**}$ is the topology defined by the seminorms $a \rightarrow \langle a^2, \rho \rangle^{\frac{1}{2}}$, where $\rho \in K$. By the Cauchy-Schwarz inequality, $|\langle a, \rho \rangle| \leq \langle a^2, \rho \rangle^{\frac{1}{2}}$ whenever $\rho \in K$, $a \in A^{**}$. Since $K$ generates $A^*$ linearly, this shows that the strong topology is stronger than the weak topology.

**Proposition 2** [3;Lemma 1.3] The weak and strong topologies on $A^{**}$ admit the same continuous linear functionals.

**Proof:** Let $\varphi$ be a strongly continuous linear functional on $A^{**}$. By definition of the strong topology,

$$|\varphi(a)| \leq M \max_{1 \leq i \leq n} \langle a^2, \rho_i \rangle^{\frac{1}{2}} \quad (a \in A^{**}).$$

Here $\rho_1, \ldots, \rho_n \in K$. Putting $\rho = \frac{1}{n} \sum \rho_i$ and $M' = M \cdot n^{\frac{1}{2}}$, we find

$$|\varphi(a)| \leq M' \langle a^2, \rho \rangle^{\frac{1}{2}} = M' \|\eta^{**}_\rho(a)\| \quad (a \in A^{**}).$$
Thus, \( \varphi \) defines a continuous linear functional on \( H_p \):

\[
\varphi(a) = (\eta_p^*(a)|\xi) \quad (a \in A^{**})
\]

By weak continuity of \( \eta_p^* \), this implies that \( \varphi \) is weakly continuous.

The following Corollary, which corresponds to \([1;\text{Prop. 3.9}]\), shows that \( \tilde{A} = A^{**} \).

**Corollary 3.** The unit ball \( A_1 \) of \( A \) is strongly dense in the unit ball \( A_1^{**} \) of \( A^{**} \).

**Proof.** By the bipolar theorem, \( A_1 \) is weakly dense in \( A_1^{**} \). By Proposition 2 and Hahn-Banach separation, the weak and strong topologies admit the same closed convex sets. This completes the proof.

As a final remark, we note that the proof of \([1;\text{Prop. 3.7}]\) is applicable in our setting. Thus, multiplication is jointly strongly continuous on bounded subsets of \( A^{**} \).

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References

