§ 0. Introduction. Let k be any algebraically closed field, and denote by $M = M(-1,n)$ the fine moduli space of stable vector bundles on $\mathbb{P}^2 = \mathbb{P}^2_k$ of rank 2 with Chern classes $c_1 = -1$ and $c_2 = n$. [3, thm 7.17]. If $n \leq 0$, then $M = \emptyset$, and if $n = 1$, $M = \operatorname{Spec} k$. In this paper we prove the following

Theorem Suppose $n \geq 2$. Then $\operatorname{Pic} M$ is generated by two elements m and c with one relation $nc = 2m$. In particular, $\operatorname{Pic} M = \mathbb{Z}$ if n is odd, and $\operatorname{Pic} M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if n is even.

Remark: m and c are defined in § 2.

Remark: Le Potier [2] has computed $\operatorname{Pic} M(0,n)$ in the case $k = \mathbb{C}$, using the technique of monads.

The proof goes along the following lines: First we find a decomposition of M into the union of three locally closed subsets, M_0, M_1, and $M_{\geq 2}$ such that M_0 is open and dense in M, the closure of M_1 has codimension 1, and $M_{\geq 2}$ is closed of codimension 2. We give complete descriptions of M_0 and M_1, in particular, we compute their Picard groups. It turns out that this, together with the restriction map $\operatorname{Pic} M \to \operatorname{Pic} M_1$, is sufficient to determine $\operatorname{Pic} M$ completely.

§ 1. The stratification.

In this section we give a summary of the results in [1]. We refer to that paper for complete proofs.

Fix a closed point $P \in \mathbb{P}^2(k)$, let $p\colon F \to \mathbb{P}^2$ be the blowing up with center P, and let $q\colon F \to \mathbb{P}^1$ denote the structure morphism of the ruled surface F. Let s and b be the linear equivalence classes of a fiber of q and the exceptional divisor $B = p^{-1}(P)$. Then s and b generate the Chow ring of F with the relations $s^2 = 0$, $sb = 1$, $b^2 = -1$.

Let E be a stable rank-2 vector bundle on $\mathbb{P}^2$ with Chern classes $c_1(E) = -1$, $c_2(E) = n$. Then there exist uniquely determined integers γ and α such that $q_*p^*E(\alpha s - \gamma b) \cong \mathcal{O}_{\mathbb{P}^1}$. The pair (γ,α) is called the type of E. The isomorphism above determines a unique minimal nonzero section σ_E of $p^*E(\alpha s-\gamma b)$. Let $Z_E \subseteq F$ be the scheme of zeros of σ_E, and $I_E \subseteq \mathcal{O}_F$ denote the ideal of Z_E. There is an exact sequence

$$0 \to \mathcal{O}_F(\gamma b-\alpha s) \to p^*E \to I_E((\alpha-1)s-(\gamma+1)b) \to 0 .$$

Let $M(-1,n) = M$ be the fine moduli space for stable rank-2 vector bundles on $\mathbb{P}^2$ with Chern classes $c_1 = -1$, $c_2 = n$. In [1] the following theorem is proved:

Theorem (1.1) There is a stratification $M = \bigcup_{(\gamma,\alpha)} M_{(\gamma,\alpha)}$ into locally closed subvarieties $M_{(\gamma,\alpha)}$ **parametrizing** bundles of type (γ,α). $M_{(\gamma,\alpha)}$ is nonempty if and only if $\alpha > 0$, $\gamma \geq 0$, and $n-\alpha-2\gamma\alpha-\gamma^2 \geq 0$. If these inequalities hold, $M_{(\gamma,\alpha)}$ is an irreducible, rational, smooth and quasiprojective variety of dimension $(4n-4)-(n-\alpha+\gamma^2+2\gamma\alpha+\gamma)$. $M_{(0,n)}$ is dense in M, and

$M_{(o,n-1)}$ is the only stratum of codimension 1.

The decomposition referred to in the introduction is the following: $M_o = M_{(o,n)}$, $M_1 = M_{(o,n-1)}$, and $M_{\geq 2} = M - (M_o \cup M_1)$.

§ 2. The Picard group of $M(-1,n)$.

Let $\mathcal{E}$ be a universal bundle on $\mathbb{P}^2$, and put $\tilde{p} = p \times 1_M : F \times M = F_M \to \mathbb{P}^2_M$, $\tilde{q} = q \times 1_M : F_M \to \mathbb{P}^1_M$. Since $\{M_{(\gamma,\alpha)}\}$ is a flattening stratification for the coherent sheaf $R^1\tilde{q}_*\tilde{p}^*\mathcal{E}$ [1, §4], it follows that $\tilde{q}_*\tilde{p}^*\mathcal{E}$ commutes with base change on M_o. Therefore there exists an invertible sheaf $\mathcal{L}$ on M such that $\tilde{q}_*\tilde{p}^*\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(-n) \boxtimes \mathcal{L}$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes pr_M^*\mathcal{L}^{-1}$, we obtain another universal bundle which we will call <u>normalized</u>. The normalized universal bundle is uniquely determined by the condition $\tilde{q}_*\tilde{p}^*\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-n) \boxtimes \mathcal{O}_M$.

Since $\operatorname{Pic} \mathbb{P}^2_M$ is naturally isomorphic to $\operatorname{Pic} \mathbb{P}^2 \times \operatorname{Pic} M$, we may write $c_1(\mathcal{E}) = -t + c$, where $\mathcal{E}$ is the normalized universal bundle, $t \in \operatorname{Pic} \mathbb{P}^2$ is the class of a line, and c is some element in $\operatorname{Pic} M$.

Since M is nonsingular, $\overline{M}_1 \subseteq M$ is a Cartier divisor; let $m \in \operatorname{Pic} M$ denote its class. Then c and m are the generators of $\operatorname{Pic} M$ mentioned in the introduction.

We state the following propositions (to be proved later):

<u>Prop</u>. (2.1) Let X be an irreducible, nonsingular variety, $W \subseteq X$ a closed subset, $W_1, \dots, W_t$ the irreducible components of W of codimension 1 in X. Then the restriction map $\operatorname{Pic} X \to \operatorname{Pic}(X-W)$ is surjective, and the kernel is generated by the linear equivalence classes of the W_i, $i = 1, \dots, t$.

<u>Prop</u>. (2.2) $\operatorname{Pic} M_o \cong \mathbb{Z}/n\mathbb{Z}$ and is generated by the restriction of c.

<u>Prop</u>. (2.3) $\operatorname{Pic} M_1/\text{torsion} \cong \mathbb{Z}$.

<u>Prop.</u> (2.4) Let δ be the composed map

$$\operatorname{Pic} M \xrightarrow{\text{restriction}} \operatorname{Pic} M_1 \longrightarrow \operatorname{Pic} M_1/\text{torsion} .$$

Then, for a suitable choice of generator β for $\operatorname{Pic} M_1/\text{torsion}$, we have $\delta(m) = -\frac{n}{a}(4n-7)\beta$ and $\delta(c) = -\frac{2}{a}(4n-7)\beta$, where a = greatest common divisor of $n+2$ and 10.

<u>Proof</u> that (2.1-4) imply the theorem. By (1.1) and (2.1), there is an exact sequence $\mathbf{Z} \xrightarrow{\varphi} \operatorname{Pic} M \xrightarrow{\psi} \operatorname{Pic} M_0 \to 0$, where $\varphi(1) = m$. Since $\operatorname{Pic} M_0$ is generated by $\psi(c)$ (by (2.2)), it follows that $\operatorname{Pic} M$ is generated by c and m. By (2.2) again, there must be a relation of the form $xm = nc$ in $\operatorname{Pic} M$. Applying the map δ to this equation, we see that $x = 2$. On the other hand, if $xm = yc$ is any other relation, apply ψ to obtain $y = \lambda n$ for some integer λ, then apply δ to get $x = 2\lambda$, so the relation is just a multiple of $2m = nc$. This proves the theorem.

§ 3. Description of M_0 and M_1.

Fix an integer i such that $0 \leq i < n$. (Later we will be interested only in the cases $i = 0$ and $i = 1$). Let H be the Hilbert scheme of closed subschemes of F of length i, $Z \subseteq F_H = F \times H$ the universal subscheme, $I \subseteq \mathcal{O}_{F_H}$ its ideal, and $\pi : F_H \to H$ the projection.

Put $G = \mathrm{Ext}^1_\pi(I((n-i-1)s-b), \mathcal{O}_{F_H}((-n+i)s))$, see [1, Appendix].

Then G is a locally free sheaf on H. Consider the projective bundle $Q = \mathbb{P}_H(G^\vee) \xrightarrow{g} H$, and let $\mathcal{O}_Q(\eta)$ denote the tautological linebundle on Q. Corresponding to the canonical surjection $g^*G^\vee \to \mathcal{O}_Q(\eta)$ there is a "universal" short exact sequence of sheaves on F_Q:

$$(*)_i \quad 0 \to \mathcal{O}_{F_Q}((-n+i)s+\eta) \to X \to I_Q((n-1-i))s-b) \to 0.$$

Put $M_i = M_{(0,n-i)}$. The main result of [1] is that M_i is isomorphic to the open subvariety of Q whose k-points are those $y \in Q(k)$ such that the restriction X_y of X to $F \times \{y\} \simeq F$ satisfies the following two conditions:

(a) X_y is locally free, and

(b) $X_y|B$ is the trivial bundle $2\mathcal{O}_B$.

Furthermore, under this isomorphism, $(p \times 1_{M_i})_* X_{M_i}$ is the restriction to $\mathbb{P}^2_{M_i}$ of a universal bundle on $\mathbb{P}^2_M$.

The case $i = 0$. (Proof of (2.2)). In this case, $H = \mathrm{Spec}\,k$, $Z = \emptyset$, and the condition (a) is automatically satisfied. To study condition (b), restrict $(*)_0$ to $B \times Q$:

$$0 \longrightarrow p_B^* \mathcal{O}_B(-n) \otimes p_Q^* \mathcal{O}_Q(\eta) \longrightarrow X_B \longrightarrow p_B^* \mathcal{O}_B(n) \longrightarrow 0.$$

Tensor this sequence with $p_B^* \mathcal{O}_B(-1)$ and apply $R^{\bullet} p_{Q*}$, and get

$$\mathcal{O}_Q \otimes_k H^0(\mathcal{O}_B(n-1)) \xrightarrow{\alpha} \mathcal{O}_Q(\eta) \otimes_k H^1(\mathcal{O}_B(-n-1)) \longrightarrow R^1 p_{Q*}(X_B \otimes p_B^* \mathcal{O}(-1)) \longrightarrow 0$$

Let $W \subseteq Q$ be the divisor defined by $\det(\alpha)$. It is clear that the support of W is the complement of M_o in Q, and that the class of W is $n\eta$. We want to show that W is reduced and irreducible.

Put $L = \mathrm{Ext}_B^1(\mathcal{O}_B(n), \mathcal{O}_B(-n))$. The restriction map $\rho : G \to L$ is surjective, and induces a linear projection $\rho : Q = \mathbb{P}(G^{\vee}) \dashrightarrow \mathbb{P}(L^{\vee})$. Let $W_j \subseteq \mathbb{P}(L^{\vee})$ be the locally closed subset corresponding to extensions of the form

$$0 \longrightarrow \mathcal{O}_B(-n) \longrightarrow \mathcal{O}_B(-j) \oplus \mathcal{O}_B(j) \longrightarrow \mathcal{O}_B(n) \longrightarrow 0 ,$$

and let $W' = \bigcup_{j>0} W_j$. Then W is the closure in Q of $\rho^{-1}(W')$, so if W' is irreducible, then so is W.

Consider the open subspace $\mathcal{U}_j$ of $H^0(\mathcal{O}_B(n+j)) \times H^0(\mathcal{O}_B(n-j))$ consisting of pairs (f,g) such that $V(f,g) = \emptyset$. There is a map $\gamma_j : \mathcal{U}_j \longrightarrow \mathbb{P}(L^{\vee})$ such that the image of γ_j is precisely W_j. If $j > 0$, the fibers of γ_j are all isomorphic to $\left\{\begin{pmatrix} \alpha & H \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in k^*, \ H \in H^0(\mathcal{O}_B(2j))\right\}$. Thus the dimension of W_j is $(n+j+1) + (n-j+1) - (2j+3) = 2n - 1 - 2j$. It follows that W', and hence W, is irreducible. Furthermore, if x is the generic point of W, it maps to the generic point of W_1. Therefore, $R^1 p_{Q*}(X_B \otimes p_B^* \mathcal{O}_B(-1)) \otimes_{\mathcal{O}_Q} \mathcal{O}_{Q,x}$ has length 1. But $\mathcal{O}_{Q,x}$ is a discrete valuation ring, so length($\mathcal{O}_{Q,x}/\det(\alpha)$) = length((coker α) $\otimes$ $\mathcal{O}_{Q,x}$) = 1. In particular, $\det(\alpha)$ is a uniformizing parameter, thus W is reduced.

Consider once again the extension $(*)_o$. Applying $(q \times 1_Q)_*$, we see that $X(-\eta)$ restricted to M_o is the restriction to M_o of $(p \times 1_M)^* \mathcal{E}$ where $\mathcal{E}$ is the normalized universal bundle. In particular, $c|M_o = -\eta|M_o$. Using (2.1), this proves (2.2).

Q.E.D.

The case $i = 1$ (Proof of (2.3)).
In this case, $H \cong F$ and Z is the diagonal in $F_H = F \times H$. Let $\sigma, \beta \in \operatorname{Pic} H$ correspond to $s, b \in \operatorname{Pic} F$ under the isomorphism $H \cong F$. Then $\operatorname{Pic} Q$ is freely generated by σ, β and η. (We will use the canonical inclusion $g^* : \operatorname{Pic} H \longrightarrow \operatorname{Pic} Q$ to identify σ and $g^*\sigma$, β and $g^*\beta$, when no confusion is possible. The same applies for the inclusions $pr_Q^* : \operatorname{Pic} Q \longrightarrow \operatorname{Pic} F_Q$ and $pr_F^* : \operatorname{Pic} F \longrightarrow \operatorname{Pic} F_Q$.)

Put $W_a = \{y \in Q : X_y$ is not locally free$\}$, and $W_b = \{y \in Q : X_y|B \not\cong 2\mathcal{O}_B\}$.

Lemma (3.1) W_a is the support of the zero-scheme of a section of $\mathcal{O}_Q(\eta - (2n-6)\sigma + 3\beta)$. Furthermore, this scheme is reduced and irreducible.

Proof: Let $W_a' \subseteq F_Q$ be the locus where X is not locally free. Then $W_a = pr_Q(W_a')$. On the other hand, let $Z' \subseteq F_Q$ be the inverse image of $Z \subseteq F_H$. Then $W_a' \subseteq Z'$, and pr_Q maps Z' isomorphically to Q. Note also that since I has projective dimension ≤ 1 locally, $W_a' = \operatorname{Supp} \underline{\operatorname{Ext}}^1_{F_Q}(X, \mathcal{L})$, where $\mathcal{L}$ is any locally free sheaf on F_Q. The sequence $(*)_1$:

$$0 \longrightarrow \mathcal{O}_{F_Q}((-n+1)s + \eta) \longrightarrow X \longrightarrow I'((n-2)s - b) \longrightarrow 0$$

where $I' = I_Q$ is the ideal of Z', gives, when dualized, an exact sequence

$$\mathcal{O}_{F_Q} \xrightarrow{\alpha} \underline{\mathrm{Ext}}^1_{F_Q}(I'((n-2)s-b), \mathcal{O}_{F_Q}((-n+1)s+\eta)) \to \underline{\mathrm{Ext}}^1_{F_Q}(X, \mathcal{O}((-n+1)s+\eta)) \to 0.$$

Restricting this sequence to Z' and using the identity $\underline{\mathrm{Ext}}^1_{F_Q}(I', \mathcal{O}_{F_Q}) = \underline{\mathrm{Ext}}^2_{F_Q}(\mathcal{O}_{Z'}, \mathcal{O}_{F_Q}) = \omega_{Z'} \otimes \omega^{-1}_{F_Q} = \mathcal{O}_{Z'}(3\sigma + 2\beta)$, and noting that $\mathcal{O}_{F_Q}(s) \otimes \mathcal{O}_{Z'} = \mathcal{O}_{F_Q}(\sigma) \otimes \mathcal{O}_{Z'}$ (correspondingly for b and β), we finally obtain that the map α above is a section of $\mathcal{O}_{Z'}(\eta - (2n-6)\sigma + 3\beta)$. Pushing this down to Q via the isomorphism $pr_Q|Z'$, we obtain the first part of (3.1).

For the second part, note that W_a induces linear spaces on the fibers of $g : Q \to H$. To prove the lemma, it is therefore sufficient to show that W_a contains no fiber of g. This is easily checked. Q.E.D.

<u>Lemma</u> (3.2) W_b is the support of a reduced and irreducible section of $\mathcal{O}_Q((n-1)\eta + \beta)$.

<u>Proof</u>: Consider again the exact sequence $(*)_1$:

$$0 \to \mathcal{O}_{F_Q}(-(n-1)s + \eta) \to X \to I'((n-2)s - b)) \to 0$$

Restrict to $B_Q = B \times Q \subseteq F_Q$, tensor by $\mathcal{O}_{B_Q}(-s)$ and apply pr_{Q*} to get an exact sequence

$$pr_{Q*}(I' \otimes \mathcal{O}_{B_Q}((n-2)s)) \xrightarrow{\alpha} R^1 pr_{Q*}\, \mathcal{O}_{B_Q}(-ns+\eta) \to R^1 pr_{Q*}(X \otimes \mathcal{O}_{B_Q}(-s)) \to 0$$

Note that $R^1 pr_{Q*}\, \mathcal{O}_{B_Q}(-ns + \eta) = (n-1)\, \mathcal{O}_Q(\eta)$.

<u>Sublemma</u> (3.3) $pr_{Q*}(I' \otimes \mathcal{O}_{B_Q}((n-2)s))$ is locally free of rank $(n-1)$, and its $(n-1)$-th exterior power is $\mathcal{O}_Q(-\beta)$.

Granting (3.3) for a moment, we see that $\det(\alpha)$ is a section of $\mathcal{O}_Q((n-1)\eta+\beta)$, the support of which is W_b. To show that $\det(\alpha)$ is irreducible and reduced, look at the fibers of g and apply the same method as in the proof of (2.2). There remains only to show that W_b contains no fiber of g. This is straightforward to check. Q.E.D.

Proof of (3.3) Let $Z'' = Z \cap B_H \subseteq F_H$, and let $\mathcal{J}$ be the ideal of Z'' in $B_H = B\times H$. Z'' may be identified with the diagonal in $B\times B \subseteq B\times H$. In particular, it is the zeroset of a section of $\mathcal{O}_B(s) \boxtimes \mathcal{O}_B(\sigma)$ on $B\times B$. This section can be lifted to a section over B_H of $\mathcal{O}_B(s) \boxtimes \mathcal{O}_H(\sigma) = \mathcal{O}_{B_H}(s+\sigma)$, since $H^1(H, \mathcal{O}_H(\sigma-\beta)) = 0$. It follows that Z'' is a complete intersection in B_H, having the following Koszul complex:

$$0 \longrightarrow \mathcal{O}_{B_H}(-\beta-\sigma-s) \longrightarrow \mathcal{O}_{B_H}(-\sigma-s) \oplus \mathcal{O}_{B_H}(-\beta) \longrightarrow \mathcal{J} \longrightarrow 0.$$

Twist it by $(n-2)s$ to get

$$0 \to \mathcal{O}_{B_H}(-\beta-\sigma+(n-3)s) \to \mathcal{O}_{B_H}(-\sigma+(n-3)s) \oplus \mathcal{O}_{B_H}(-\beta+(n-2)s) \to \mathcal{J}((n-2)s) \to 0.$$

From this one easily deduces that $R^1 pr_{H*}\mathcal{J}((n-2)s) = 0$, and that $pr_{H*}\mathcal{J}((n-2)s)$ is locally free of rank $(n-1)$ and commutes with base change on H. In particular, applying the base change $g : Q \longrightarrow H$, we get the following resolution of $pr_{Q*}(I' \otimes \mathcal{O}_{B_Q}((n-2)s))$:

$$0 \to (n-2)\mathcal{O}_Q(-\sigma-\beta) \to (n-2)\mathcal{O}_Q(-\sigma) \oplus (n-1)\mathcal{O}_Q(-\beta) \to pr_{Q*}(I' \otimes \mathcal{O}_{B_Q}((n-2)s)) \to 0.$$

From this one computes the $(n-1)$-th exterior power, and finds the formula of (3.3). Q.E.D.

Proof of (2.3) By (2.1), (3.1) and (3.2), $\operatorname{Pic} M_1$ is generated by σ, β and η, with the two relations

$$\eta - (2n-6)\sigma + 3\beta = 0$$

$$(n-1)\eta + \beta = 0.$$

Eliminating β, we get the single relation

$$(2n-6)\sigma + (3n-4)\eta = 0$$

So we have proved the following, which easily implies (2.3):

(3.4) $\operatorname{Pic} M_1$ is generated by σ and η with one relation $(2n-6)\sigma + (3n-4)\eta$. In particular, if $a = (2n-6, 3n-4) = (n+2, 10)$ then $\operatorname{Pic} M_1 = \mathbf{Z} \oplus \mathbf{Z}/a\mathbf{Z}$.

Proof of (2.1) Since X is nonsingular, the closure in X of any divisor on $X - W$ is a (Cartier) divisor on X. This proves the surjectivity. For the second statement, let $\mathcal{L}$ be an invertible sheaf on X which restricts to $\mathcal{O}_X$ on $X - W$. Then $\mathcal{L}$ admits a rational section which is defined and nowhere vanishing on $X - W$. It follows that the associated divisor is a linear combination of the W_i. Q.E.D.

§ 4. Proof of (2.4)

Lemma (4.1) $c|M_1 = 2\sigma - \eta$.

Proof: Consider the cartesian square

$$\begin{array}{ccc} F_{M_1} & \xrightarrow{i} & F_M \\ {\scriptstyle q\times 1_{M_1}}\downarrow & & \downarrow {\scriptstyle q\times 1_M} \\ \mathbb{P}^1_{M_1} & \longrightarrow & \mathbb{P}^1_M \end{array}$$

Put $\mathcal{D} = (p\times 1_M)^*\mathcal{E}$, where $\mathcal{E}$ is the normalized universal bundle. Now $(q\times 1_M)_*\mathcal{D} = \mathcal{O}_{\mathbb{P}^1}(-n) \boxtimes \mathcal{O}_M$, and $(q\times 1_{M_1})_* i^*\mathcal{D} = \mathcal{O}_{\mathbb{P}^1}(-(n-1)) \boxtimes \mathcal{L}$ for some linebundle $\mathcal{L}$ on M_1. The natural base-change map β gives an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n) \boxtimes \mathcal{O}_{M_1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}(-(n-1)) \boxtimes \mathcal{L} \longrightarrow \operatorname{coker}\beta \longrightarrow 0$$

In order to determine $\mathcal{L}$, restrict to $\{\xi\}\times M_1$ for a point $\xi \in \mathbb{P}^1(k)$.

$$\operatorname{Supp}(\operatorname{coker}\beta) \cap \{\xi\}\times M_1 = \operatorname{Supp}(R^1(q\times 1_M)_*\mathcal{D}) \cap \{\xi\}\times M_1$$

= {vector bundles E of type $(0,n-1)$ such that the length-1 subscheme $Z_E \subseteq F$ lies on $q^{-1}(\xi)$}.

It follows that if β is reduced, then $\mathcal{L} = \mathcal{O}_{M_1}(\sigma)$. But β is reduced, since if y is a generic point of $\operatorname{Supp} V(\beta)$, then because $R^1(q\times 1)_*\mathcal{D}$ is, by definition of M_1, locally free of rank 1 over M_1 we have

$$1 = l_{\mathcal{O}_y}(R^1(q\times 1)_*\mathcal{D}_y) = l_{\mathcal{O}_y}(\operatorname{Tor}_1^{\mathcal{O}_y}(k(y), R^1(q\times 1)_*\mathcal{D}_y)) = l_{\mathcal{O}_y}(\operatorname{coker}\beta_y).$$

Consider the universal exact sequence $(*)_1$ restricted to $M_1 \subseteq Q$

$$0 \longrightarrow \mathcal{O}_{F_{M_1}}((-n+1)s+\eta)) \longrightarrow X \longrightarrow I_{M_1}((n-2)s-b) \longrightarrow 0.$$

Applying $(q \times 1_{M_1})_*$, we see that $(q \times 1_{M_1})_* X = \mathcal{O}_{\mathbb{P}^1}(-(n-1)s) \boxtimes \mathcal{O}_{M_1}(\eta)$. Therefore, noting that $\mathcal{L} = \mathcal{O}_{M_1}(\sigma)$, $X(\sigma-\eta)$ is the restriction to F_{M_1} of $\mathcal{D}$. It follows that

$$c|M_1 = c_1(X(\sigma-\eta)) + (s+b) = 2\sigma - \eta. \qquad \text{Q.E.D.}$$

<u>Lemma</u> (4.2) $m|M_1 = (2n-3)\sigma + (n-2)\eta$.

<u>Proof</u>: Let $R \in \mathbb{P}^2(k)$ be a point different from P. Denote by F' the blowing up of F at the point $p^{-1}(R)$, and let $q' : F' \longrightarrow \mathbb{P}^1$ be the morphism induced by the linear system of lines passing through R.

Let M_1' be the codimension one stratum in the stratification of M defined by the point R. An automorphism of $\mathbb{P}^2$ taking P to R moves M_1 to M_1', hence the divisors $\overline{M}_1$ and $\overline{M}_1'$ are linearly equivalent. Furthermore, it is easily verified by deformation theory that M_1 and M_1' intersect transversally. Hence $m|M_1$ is defined by the divisor $M_1 \cap \overline{M}_1'$ in M_1.

Pulling back the sequence $(*)_1$ to $M_1 \times F'$, and using that $\mathcal{D}|M_1 \times F = X(\sigma-\eta)$, where $\mathcal{D}$ is as in the proof of (2.4), we deduce the sequence

$$0 \to \mathcal{O}_{M_1}(\sigma) \boxtimes \mathcal{O}_{F'}(-(n-1)s) \to \mathcal{D}' \to \mathcal{O}_{M_1}(\sigma-\eta) \boxtimes \mathcal{O}_{F'}((n-2)s-b) \to \mathcal{O}_{Z'}((n-2)s-b+\sigma-\eta) \to 0,$$

where $\mathcal{D}'$ is the pullback of $\mathcal{D}$ to $M_1 \times F'$ and Z' is the pullback of Z.

Let $r : M_1 \times F' \longrightarrow M_1$ be the projection. It can be factored $M_1 \times F' \xrightarrow{1\times q'} M_1 \times \mathbb{P}^1 \xrightarrow{p_1} M_1$. Now it is easily checked that as sets, $M_1 \cap \overline{M}_1' = \mathrm{Supp}(p_{1*}R^1(1\times q')_* \mathcal{D}')$, and since $M_1 \cap \overline{M}_1'$ is reduced and the rank of $p_{1*}R^1(1\times q')_*\mathcal{D}'$ is 1 generically on its support, it follows that $m|M_1 = c_1(p_{1*}R^1(1\times q')_* \mathcal{D}')$.

Denote by $\chi_r(A)$ the formal sum $\sum_{i\geq 0}(-1)^i[R^i r_* A]$, for any sheaf A on $M_1 \times F'$. Applying χ_r to the exact sequence above and using the easily verified formulas:

$$\chi_r(\mathcal{O}_{M_1}(\sigma) \boxtimes \mathcal{O}_{F'}(-(n-1)s)) = -[(n-2)\,\mathcal{O}_{M_1}(\sigma)]$$

$$\chi_r(\mathcal{O}_{M_1}(\sigma-\eta) \boxtimes \mathcal{O}_{F'}((n-2)s - b)) = 0, \text{ and}$$

$$\chi_r(\mathcal{O}_{Z'}((n-2)s - b + \sigma - \eta)) = [\mathcal{O}_{M_1}((n-1)\sigma + (n-2)\eta)],$$

(recall that $\beta = -(n-1)\eta$ in $\mathrm{Pic}\, M_1$), we get the expression

$$\chi_r(\mathcal{D}') = -[(n-2)\,\mathcal{O}_{M_1}(\sigma)] - [\,\mathcal{O}_{M_1}((n-1)\sigma + (n-2)\eta)].$$

On the other hand, since M_1 and M_1' intersect transversally, $(1\times q')_* \mathcal{D}' = \mathcal{O}_{M_1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-n)$. Furthermore, $r_* \mathcal{D}' = R^2 r_* \mathcal{D}' = 0$, since the bundles induced by $\mathcal{D}'$ on the fibers of r are pull-backs of stable bundles on $\mathbb{P}^2$, hence have no H^0 or H^2. By the Leray spectral sequence for the composition $r = p_1 \circ (1\times q')$, we get the expression

$$[p_{1*}R^1(1\times q')_* \mathcal{D}'] = -[\chi_r \mathcal{D}'] + [r_* \mathcal{D}'] + [R^2 r_* \mathcal{D}']$$

$$-[R^1 p_{1*}(1\times q')_* \mathcal{D}'] = -[\chi_r \mathcal{D}'] - [(n-1)\,\mathcal{O}_{M_1}].$$

Using the expression above for $\chi_r(\mathcal{D}')$ and taking first Chern class, we finally obtain

$$c_1(p_{1*}R^1(1\times q')_* \mathcal{D}') = (n-2)\sigma + (n-1)\sigma + (n-2)\eta = (2n-3)\sigma + (n-2)\eta.$$

Q.E.D.

<u>Proof</u> of (2.4). Choose integers a, a_i, b_i such that $2n-6=a_1a$, $3n-4=a_2a$, and $a_1b_1+a_2b_2=1$. Use the invertible matrix $\begin{bmatrix} b_1 & b_2 \\ -a_2 & a_1 \end{bmatrix}$ to get a new basis $\{\alpha,\beta\}$ for the free abelian group generated by σ and η, such that $\sigma = b_1\alpha - a_2\beta$, $\eta = b_2\alpha + a_1\beta$.

Then $(2n-6)\sigma + (3n-4)\eta = a(a_1\sigma + a_2\eta) = a((a_1b_1+a_2b_2)\alpha + (a_1\cdot(-a_2) + a_2\cdot a_1)\beta) = a\alpha$. In particular, by (3.4) one sees that $\operatorname{Pic} M_1 = (\mathbb{Z}/a\mathbb{Z})\alpha \oplus \mathbb{Z}\beta$, and that β generates $\operatorname{Pic} M_1/\text{torsion}$.

Now, by (4.1):

$$c|M_1 = 2\sigma - \eta = (2b_1-b_2)\alpha + (-2a_2-a_1)\beta$$
$$\equiv \frac{1}{a}(-2(3n-4)-(2n-6))\beta = -\frac{2}{a}(4n-7)\beta \pmod{\alpha}.$$

Similarly, by (4.2):

$$m|M_1 = (2n-3)\sigma + (n-2)\eta \equiv ((2n-3)(-a_2) + (n-2)a_1)\beta$$
$$= -\frac{1}{a}((2n-3)(3n-4)-(n-2)(2n-6))\beta = -\frac{n}{a}(4n-7)\beta \pmod{\alpha}.$$ Q.E.D.

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