1. Introduction. Let k be an algebraically closed field of any characteristic. If E is a vector bundle (= locally free sheaf) of rank 2 on $\mathbb{P}^2 = \mathbb{P}^2_k$, E is said to be stable if $\mu \leq -\frac{1}{2} c_1(E) \Rightarrow H^0(\mathbb{P}^2, E(\mu)) = 0$. Here $c_1(E)$ denotes the first Chern class of E (identified with its degree). Maruyama has showed that there exists, for each couple (c_1, c_2) of integers, a coarse moduli space $M(c_1, c_2)$ for stable rank-2 vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2 [6]. It follows from his construction that each component of $M(c_1, c_2)$ is smooth of dimension $4c_2 - c_1^2 - 3$. He has also shown that $M(c_1, c_2)$ is connected and unirational (and rational in some cases). [7] It is also known that $M(c_1, c_2)$ is a fine moduli space if and only if $4c_2 - c_1^2 \not\equiv 0 \pmod 8$ [7,5]. Barth [2] (for c_1 even) and Hulek [4] (for c_1 odd) proved irreducibility and rationality, using monads (at least in characteristic 0).

This paper is mainly devoted to the study of $M(-1,n)$. We use the number of "jumping lines" through a fixed point in $\mathbb{P}^2$ to stratify $M(-1,n)$, and obtain fairly explicit descriptions of the strata. In fact, in a later paper we shall use the stratification to compute the Picard group of $M(-1,n)$. Our construction also gives another proof of the irreducibility and rationality of $M(-1,n)$, valid in all characteristics, using only its equi-dimensionality. In the last section we indicate how similar methods can be used in the study of $M(0,n)$.

2. The standard construction (See also [1]).

Fix, once and for all, a closed point $P \in \mathbb{P}^2(k)$. Let $p : F \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ in P. Then F is a ruled surface with structure morphism $q : F \to \mathbb{P}^1$, in fact, $F = \mathbb{P}_{\mathbb{P}^1}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1))$. Denote by s the divisor class of $q^* O_{\mathbb{P}^1}(1)$, and b the divisor class of the exceptional divisor $B = p^{-1}(P) \subseteq F$. Then s and b generate the Chow ring of F over $\mathbb{Z}$ with the relations $s^2 = 0$, $b^2 = -1$, $sb = 1$. For later use we state without proof the following easy

Lemma 2.1: Let $\delta \geq -1$ be an integer.
Then $q_* O_F(\delta b) = \bigoplus_{i=0}^{\delta} O_{\mathbb{P}^1}(-i)$ and $R^1 q_* O_F(\delta b) = 0$.

Let E be a rank-2 vector bundle on $\mathbb{P}^2_T = \mathbb{P}^2 \times T$, where T is a variety. Then $(p \times 1_T)^* E$ is a vector bundle on $F_T = F \times T$, and for each $t \in T(k)$, the fiber $((p \times 1_T)^* E)_t = p^*(E_t)$ is a bundle on F, the restriction of which to B is the trivial bundle $2O_B$. We have the following criterion for a bundle D on F_T to be of the form $(p \times 1_T)^* E$:

Proposition 2.2: Let D be a 2-bundle on F_T, T any variety. Assume that for all $t \in T(k)$, the restriction of D_t to B is trivial. Then $(p \times 1_T)_* D$ is locally free on $\mathbb{P}^2_T$, and the natural map $(p \times 1_T)^*(p \times 1_T)_* D \to D$ is an isomorphism.

Proof: If $T = \operatorname{Spec} k$, this is proved by Schwarzenberger [9, thm.5]. In the general case, it is therefore sufficient to show that $p_{T*}D$ commutes with base change on T, i.e. that for all $t \in T(k)$, the natural map $(p_{T*}D)_t \to p_*(D_t)$ is an isomorphism. Here $p_T = p \times 1_T$. In order to show this, let $\underline{m}_t \subseteq O_T$ be the ideal of t. Apply

p_{T*} to the short exact sequence

$$0 \to D \otimes \underline{m}_t \to D \to D_t \to 0 .$$

At the right end, we get

$$R^1 p_{T*}(D \otimes m_t) \xrightarrow{j^1} R^1 p_{T*} D \to R^1 p_*(D_t) \to 0$$

The multiplication map $(R^1 p_{T*} D) \otimes m_t \to R^1 p_{T*} D$ factors naturally through j^1. Thus the natural map $(R^1 p_{T*} D)_t \to R^1 p_*(D_t)$ is an isomorphism. By our assumption, the latter sheaf is 0. Since $R^1 p_{T*} D$ is supported on $\{P\} \times T$, it is coherent on T, so we may apply Nakayama's lemma to get $R^1 p_{T*} D = 0$. Since $R^1 p_{T*}$ is right exact, also $R^1 p_{T*}(D \otimes \underline{m}_t) = 0$, because $\underline{m}_t$ is generated by finitely many local sections. There remains an exact sequence

$$0 \to p_{T*}(D \otimes \underline{m}_t) \xrightarrow{j^0} p_{T*} D \to p_*(D_t) \to 0 .$$

Again, the image of j^0 is $(p_{T*} D) \cdot \underline{m}_t$. Thus the natural map $(p_{T*} D)_t \to p_*(D_t)$ is an isomorphism. Q.E.D.

3. <u>The invariant</u> (γ, α).

Let D be a 2-bundle on F such that $c_1(D) = -s - b$ and $c_2(D) = n$. Assume that $H^0(F, D) = 0$.

Let $\gamma = \gamma(D)$ be the rank of the coherent sheaf $R^1 q_* D$ on $\mathbb{P}^1$. This means that $D_L = O_L(\gamma) \oplus O_L(-\gamma - 1)$, where L is a general fiber of q. Now $q_* D(-\gamma b)$ is torsionfree of rank 1 on $\mathbb{P}^1$, hence invertible. Let $\alpha = \alpha(D)$ be the integer such that $q_* D(-\gamma b) = O_{\mathbb{P}^1}(-\alpha)$. Then $q_* D(\alpha s - \gamma b) = O_{\mathbb{P}^1}$. Let $\sigma \in H^0(F, D(\alpha s - \gamma b))$ be the global section induced by the canonical map $q^* q_* D(\alpha s - \gamma b) =$

$O_F \to D(\alpha s - \gamma b)$. As in [1, lemma 9] one shows that the subscheme $Z = V(\sigma) \subseteq F$ defined by the vanishing of σ has codimension 2 in F, hence is of finite length. There is an exact sequence

$$0 \to O_F \xrightarrow{\sigma} D(\alpha s - \gamma b) \to I((2\alpha - 1)s - (2\gamma + 1)b) \to 0 ,$$

where $I \subset O_F$ is the ideal of Z.

<u>Proposition</u> 3.1: Let D and Z be as above.

(i) $\alpha > 0$.

(ii) $\text{length}(Z) = n - \alpha - 2\gamma\alpha - \gamma^2$.

(iii) $q_*D = \bigoplus_{i=o}^{\gamma} O_{\mathbb{P}^1}(-i - \alpha)$.

<u>Proof</u>: (i) $H^o(F,D) = 0 \Rightarrow H^o(F,D(-\gamma b)) = 0$

$\Rightarrow H^o(\mathbb{P}^1, O_{\mathbb{P}^1}(-\alpha)) = 0 \Rightarrow \alpha > 0$.

(ii) The length of Z is the second Chern class of $D(\alpha s - \gamma b)$, which is $c_2(D) + c_1(D)(\alpha s - \gamma b) + (\alpha s - \gamma b)^2 = n - \alpha - 2\gamma\alpha - \gamma^2$.

(iii) Apply q_* to the exact sequence

$$0 \to O_F(\gamma b - \alpha s) \to D \to I((\alpha - 1)s - (\gamma + 1)b) \to 0 .$$

For L a general fiber of q, $I_L((\alpha - 1)s - (\gamma + 1)b) = O_L(-\gamma - 1)$, so $q_*I((\alpha - 1)s - (\gamma + 1)b)) = 0$, and $q_*O_F(\gamma b - \alpha s) \to q_*D$ is an isomorphism. Now apply lemma 1. Q.E.D.

<u>Definition</u>. Let D be a 2-bundle on F such that $c_1(D) = -s - b$, $c_2(D) = n$, and $H^o(F,D) = 0$. We say that D is of <u>type</u> (γ, α) if $\gamma(D) = \gamma$, $\alpha(D) = \alpha$.

Remark. For $\mu \in \mathbb{Z}$, the Leray spectral sequence for q yields an equality of numerical polynomials:

$$\chi_{\mathbb{F}}(D(\mu s)) = \chi_{\mathbb{P}^1}(q_*D(\mu)) - \chi_{\mathbb{P}^1}(R^1q_*D(\mu)).$$

By proposition 3.1 (iii), the type of D determines and is determined by $\chi_{\mathbb{P}^1}(q_*D(\mu))$. Since $\chi_{\mathbb{F}}(D(\mu s))$ depends only on the Chern classes of D (by Riemann-Roch), we get that the type of D also determines and is determined by $\chi_{\mathbb{P}^1}(R^1q_*D(\mu))$, the Chern classes of D being fixed.

4. The stratification

Let $M = M(-1,n)$ be the fine moduli space for stable rank-2 bundles on $\mathbb{P}^2$ with Chern classes -1 and n, and let $\mathcal{E}$ be the universal bundle on $\mathbb{P}^2_M$. Let $\tilde{p} = p \times 1_M$, $\tilde{q} = q \times 1_M$, and put $\mathcal{D} = \tilde{p}^*\mathcal{E}$. Then $R^1\tilde{q}_*\mathcal{D}$ is a coherent sheaf on $\mathbb{P}^1_M$. In this situation there is a flattening stratification $\{M_P\}_{P \in \mathcal{P}}$ of M for $R^1\tilde{q}_*\mathcal{D}$ [8, lecture 8], indexed by a finite set $\mathcal{P}$ of numerical polynomials. In particular, if $m \in M(k)$, then $m \in M_P(k)$ if and only if $(R^1\tilde{q}_*\mathcal{D})_m = R^1q_*(\mathcal{D}_m)$ has Hilbert polynomial P. By the remark of section 3 we may take the set of possible types (γ,α) as the indexing set. So there is a stratification $M = \bigcup_{(\gamma,\alpha)} M_{(\gamma,\alpha)}$ of M into locally closed subschemes $M_{(\gamma,\alpha)}$ parametrizing bundles of type (γ,α). Equip $M_{(\gamma,\alpha)}$ with the reduced scheme structure. Necessary conditions for $M_{(\gamma,\alpha)}$ to be nonempty are by proposition 3.1 the inequalities $\gamma \geq 0$, $\alpha > 0$, and $n - \alpha - 2\gamma\alpha - \gamma^2 \geq 0$. We shall prove later that these conditions are also sufficient.

5. The structure of $M_{(\gamma,\alpha)}$.

Having fixed (γ,α) throughout this section (satisfying the inequalities above), put $N = M_{(\gamma,\alpha)}$, $\bar{p} = p \times 1_N$, $\bar{q} = q \times 1_N$, $\pi_N : F_N \to N$ the projection.

Lemma 5.0: $\bar{q}_* \mathcal{D}_N(\alpha s - \gamma b)$ commutes with base change on N.

Proof: For any closed point $n \in N$, the fiber $\mathcal{D}_n$ is of type (γ,α). Thus the Hilbert polynomial of $R^1 q_* \mathcal{D}_n(\alpha s - \gamma b)$ is independent of n. Since N is reduced, it follows that $R^1 \bar{q}_* \mathcal{D}_N(\alpha s - \gamma b)$ is N-flat and that $\bar{q}_* \mathcal{D}_N(\alpha s - \gamma b)$ commutes with base change. Q.E.D.

It follows that $\bar{q}^* \bar{q}_* \mathcal{D}_N(\alpha s - \gamma b)$ is a linebundle on F_N of the form $\pi_N^* \mathcal{L}_1$, for some linebundle $\mathcal{L}_1$ on N. Let $\tilde{Z} \subseteq F_N$ be the zero-set of the canonical map $\bar{q}^* \bar{q}_* \mathcal{D}_N(\alpha s - \gamma b) \to \mathcal{D}_N(\alpha s - \gamma b)$. By (5.0) and (3.1), $\tilde{Z}$ is finite and flat over N of rank $n - \alpha - 2\gamma\alpha - \gamma^2$. Thus there is induced a morphism $\Phi : N \to H$ where H denotes the Hilbert scheme parametrizing closed subschemes of F of length $n - \alpha - 2\gamma\alpha - \gamma^2$.

Let $Z \subseteq F_H$ be the universal subscheme, and $I \subseteq O_{F_H}$ the ideal of Z. Put $A = \pi^* O_F((2\gamma + 1)b - (2\alpha - 1)s)$, where $\pi : F_H \to H$ is the projection. Note that, with $T = H$, all the assumptions (i)-(vi) of the appendix are satisfied. (To verify (ii) and (iii), use (2.1) and Leray spectral sequence for q.) Let $\mathcal{L}_2$ be the linebundle on N such that $c_1(\mathcal{D}_N) = \pi_N^* \mathcal{L}_2(-s - b)$. Now there is an exact sequence

$$0 \to O_{F_N} \to \mathcal{D}_N(\alpha s - \gamma b) \otimes \pi_N^* \mathcal{L}_1^{-1} \to I_N \otimes A_N^{-1} \otimes \pi_N^* (\mathcal{L}_2 \otimes \mathcal{L}_1^{-2}) \to 0,$$

where I_N is the ideal of $\tilde{Z}$. Suitably twisted, this becomes

$0 \to A_N \otimes \pi_N^* \mathcal{L} \to X \to I_N \to 0$, where $\mathcal{L} = \mathcal{L}_1^2 \otimes \mathcal{L}_2^{-1}$ and $X = \mathcal{O}_N((\gamma+1)b - (\alpha-1)s) \otimes \pi_N^*(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1})$. This extension splits nowhere on N, as is seen by restricting to $B \subseteq F$. Now apply proposition A.2 of the appendix to get a commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\psi} & \mathbb{P}_H(\mathrm{Ext}^1_\pi(I,A)^\vee) \\ & \underset{\Phi}{\searrow} & \downarrow f \\ & & H \end{array}$$

<u>Proposition</u> 5.1. ψ is an open dense embedding.

<u>Proof</u>: Let Y be the sheaf on F_Q corresponding to the identity map of $Q = \mathbb{P}_H(\mathrm{Ext}^1_\pi(I,A)^\vee)$ (see prop. A.2). Let $T \subseteq Q$ be the open subvariety such that, for $t \in Q(k)$, t is in $T(k)$ if and only if Y_t is locally free on F and $Y_t|B = 2\,O_B(-\gamma-\alpha)$. (Both conditions are easily seen to be open in any family of sheaves on F.) Now the sheaf $Y((\alpha-1)s + (\gamma+1)b) = D$ on F_T satisfies the conditions of prop. 2.2, and hence defines a morphism $T \to M(-1,n)$ which is obviously an inverse to ψ.

It remains only to show that T is nonempty. To do this, fix a point $h \in H(k)$ such that the corresponding subscheme $Z_h = V(I_h) \subseteq F$ is reduced and does not meet the exceptional curve B. Recall the exact sequence

$$(**) \quad 0 \to H^1(\underline{\mathrm{Hom}}(A_h,I_h)) \to \mathrm{Ext}^1_F(A_h,I_h) \to H^0(\underline{\mathrm{Ext}}^1(A_h,I_h)) \to 0.$$

An element $\xi \in \mathrm{Ext}^1_F(A_h,I_h)$ corresponds to a locally free sheaf if and only if the induced global section of $\underline{\mathrm{Ext}}^1(A_h,I_h)$ generates this sheaf, i.e. is nontrivial at each point of Z_h. In particular, the general extension is locally free.

Next we prove that the general extension restricts to $2\,O_B(-\gamma-\alpha)$ on B. It is sufficient to show that the "restriction" map $\mathrm{Ext}^1_F(I_h,A_h) \to \mathrm{Ext}^1_B(I_{hB},A_{hB})$ is surjective, since $I_{hB} = O_B$, $A_{hB} = O_B(-2(\gamma+\alpha))$. But $\mathrm{Ext}^1_B(I_{hB},A_{hB}) = H^1(O_B(-2(\gamma+\alpha))$, and since $H^2(F,A_h(-b)) = 0$ by Lemma 2.1, already the map $H^1(A_h) \to H^1(A_{hB})$ is surjective. Q.E.D.

Theorem 5.2. Suppose $\gamma \geq 0$, $\alpha > 0$, and $n-\alpha-2\gamma\alpha-\gamma^2 \geq 0$. Then $M(\gamma,\alpha)$ is an irreducible, rational, smooth and quasiprojective variety of dimension $(4n-4)-(n-\alpha+\gamma^2+2\gamma\alpha+\gamma)$.

The subvariety $M_{(0,n)} \subseteq M(-1,n)$ parametrizing bundles without jumping lines through P is dense in $M(-1,n)$. In particular, M is irreducible and rational.

Proof: H is irreducible, rational, smooth and quasiprojective of dimension $2(n-\alpha-2\gamma\alpha-\gamma^2)$. [3]. An easy computation, using the sequence (**) above and Lemma 2.1, shows that rank $\mathrm{Ext}^1_\pi(I,A) = n+\gamma^2+2\gamma\alpha+3\alpha-\gamma-3$. Thus the theorem follows from 5.1 and the equidimensionality of $M(-1,n)$. Q.E.D.

6. Even first Chern class.

We sketch in this section how similar methods can be used to stratify the moduli space $M(0,n)$ for stable rank-2 vector bundles on $\mathbb{P}^2$ with Chern classes $c_1 = 0$ and $c_2 = n$. For such a bundle E, we define its type as follows: Let $\gamma \geq 0$ be the integer such that $E_L = O_L(\gamma) \oplus O_L(-\gamma)$ for the general line through P ($P \in \mathbb{P}^2(k)$ is always fixed). If $\gamma > 0$, we proceed as before, and let α be the integer such that $q_*p^*E(-\gamma b+\alpha s) = O_{\mathbb{P}^1}$.

Then (γ,α) is called the type of E. The locally closed subset $M_{(\gamma,\alpha)} \subseteq M(0,n)$ is described in the same way as before, with only trivial modifications. (The fact that $M(0,n)$ is not a fine moduli space if n is even [5] does not really matter.) If, on the other hand, $\gamma = 0$, $q_* p^* E$ is torsionfree of rank 2 on $\mathbb{P}^1$, hence of the form $O_{\mathbb{P}^1}(-\alpha) \oplus O_{\mathbb{P}^1}(-\beta)$, $0 < \alpha \leq \beta$. Then E has type $(0,\alpha,\beta)$.

To describe the corresponding subset $M_{(0,\alpha,\beta)} \subseteq M(0,n)$, we distinguish two cases:

<u>Case 1</u>: $\alpha < \beta$. Then there is a unique minimal section of $p^*E(\alpha s)$ defining a subscheme $Z \subseteq F$ of length n. (Corresponding to the n jumping lines through P.) We thus get all such bundles as extensions of I by $O_F(-2\alpha s)$, where I is the ideal of some length-n subscheme $Z \subseteq F$.

Let $\delta = n - \alpha - \beta$ = length $R^1 q_* p^* E$ = number of ≥ 2 - jumping lines for E through P. Then the length-n subscheme Z considered is contained in $n - \delta = \alpha + \beta$ fibers of q. The subset of Hilb^n_F of such Z has dimension $2n - \delta$. We eventually obtain $\dim M_{(0,\alpha,\beta)} =$ $(4n - 3) - (n - 2\alpha - 1) - (n - \alpha - \beta)$.

<u>Case 2</u>: $\alpha = \beta$. In this case, there is a whole $\mathbb{P}^1$-family of minimal sections of $p^*E(\alpha s)$, allowing us to describe a certain $\mathbb{P}^1$-bundle over $M_{(0,\alpha,\alpha)}$. Again, this has dimension $(4n - 3) - 2(n - 2\alpha) + 1$.

Note that, in both cases, the maximal dimension is obtained only if $\alpha + \beta = n$, $\beta - \alpha \leq 1$.

To sum up the most interesting results, we state

<u>Theorem</u> 6.1. If $n = 2\alpha + 1$ is odd, the variety $M_{(0,\alpha,\alpha+1)} \subseteq M(0,n)$ is dense, and it is irreducible and rational.

If $n = 2\alpha$ is even, the variety $M_{(0,\alpha,\alpha)} \subseteq M(0,n)$ is dense, and there exists a $\mathbb{P}^1$-bundle over $M_{(0,\alpha,\alpha)}$ which is irreducible and rational (and carries a family).

<u>Appendix</u> Let T be any k-scheme, $\pi : F_T \to T$ the projection, and let I and A denote O_{F_T}-modules. Let $\mathrm{Hom}_\pi(I,A) = \pi_* \underline{\mathrm{Hom}}_{F_T}(I,A)$.

The functor $A \mapsto \mathrm{Hom}_\pi(I,A)$ is then left exact; let $\mathrm{Ext}^i_\pi(I,-)$ denote its i-th right derived functor. The relations $\mathrm{Hom}_\pi = \pi_* \circ \underline{\mathrm{Hom}}_{F_T}$ and $\mathrm{Hom}_{F_T} = \Gamma(T, \mathrm{Hom}_\pi)$ give spectral sequences of composite functors:

$$E_2^{p,q} = R^p\pi_*(\underline{\mathrm{Ext}}^q_{F_T}(I,A)) \Rightarrow \mathrm{Ext}^\bullet_\pi(I,A) \text{ , and}$$

$${}'E_2^{p,q} = H^p(T, \mathrm{Ext}^q_\pi(I,A)) \Rightarrow \mathrm{Ext}^\bullet_{F_T}(I,A) .$$

Throughout this section we shall make the following assumptions:

(i) I and A are coherent on F_T and flat over T.

(ii) For all $t \in T(k)$ we have

$$H^2(F, \underline{\mathrm{Hom}}_F(I_t, A_t)) = H^1(F, \underline{\mathrm{Ext}}^1_F(I_t, A_t))$$
$$= \underline{\mathrm{Ext}}^2_F(I_t, A_t) = 0.$$

In this situation, $\mathrm{Ext}^1_\pi(I,A)$ "commutes with base change":

<u>Proposition</u> A.1: Let $f : S \to T$ be any morphism, denote by I_S, A_S the pullback to F_S of I and A, and let $\pi_S : F_S \to S$ be the projection. Then the natural map $f^*\mathrm{Ext}^1_\pi(I,A) \to \mathrm{Ext}^1_{\pi_S}(I_S, A_S)$ is an isomorphism.

Proof: Consider the exact sequence of low degree terms of the spectral sequence $E_2^{p,q}$:

$$0 \to R^1\pi_* \underline{Hom}(I,A) \to Ext^1_\pi(I,A) \to \pi_* \underline{Ext}^1(I,A) \to$$
$$\to R^2\pi_* \underline{Hom}(I,A) \quad , \quad \text{(and the same for } S).$$

By the assumptions (i) and (ii) above and the usual base change theorem, $R^2\pi_*(\underline{Hom}(I,A)) = R^2\pi_{S*} \underline{Hom}(I_S,A_S) = 0$. Thus there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc} f^*R^1\pi_* \underline{Hom}(I,A) & \to & f^*Ext^1_\pi(I,A) & \to & f^*\pi_* \underline{Ext}^1(I,A) & \to & 0 \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ 0 \to R^1\pi_{S*}\underline{Hom}(I_S,A_S) & \to & Ext^1_{\pi_S}(I_S,A_S) & \to & \pi_{S*}\underline{Ext}^1(I_S,A_S) & \to & 0 \end{array}$$

By the assumptions (i) and (ii) above, α and γ are isomorphisms. Then β is an isomorphism by the five lemma. Q.E.D.

Note that for any linebundle $\mathcal{L}$ on T, there are canonical isomorphisms $Ext^i_\pi(I,A) \otimes \mathcal{L} \to Ext^i_\pi(I,A \otimes \pi^*\mathcal{L})$.

For the rest of this appendix we will make the additional assumptions:

(iii) $\dim_k Ext^1_F(I_t,A_t)$ is independent of $t \in T(k)$.

(iv) T is reduced.

(v) $Hom_F(I_t,A_t) = 0$ for all $t \in T(k)$.

Then (iii), (iv) and the proposition implies that $Ext^1_\pi(I,A)$ is a locally free O_T-module. By (v), $Hom_{\pi_S}(I_S,A_S) = 0$ for all $S \to T$. Therefore the spectral sequence $'E^{p,q}$ (for S) yields an isomorphism $Ext^1_{F_S}(I_S,A_S \otimes \pi_S^*\mathcal{L}) \to H^0(S, Ext^1_{\pi_S}(I_S,A_S) \otimes \mathcal{L})$ for any linebundle $\mathcal{L}$ on S.

Consider the projective bundle $Q = \mathbb{P}_T(\mathrm{Ext}^1_\pi(I,A)^\vee) \xrightarrow{f} T$. Let S be a T-scheme, then a T-morphism $\lambda : S \to Q$ is given by a linebundle $\mathcal{L}$ on S and a surjection $f^*\mathrm{Ext}^1_\pi(I,A)^\vee \to \mathcal{L}$. This is equivalent to a locally split map $O_S \to f^*\mathrm{Ext}^1_\pi(I,A) \otimes \mathcal{L} = \mathrm{Ext}^1_{\pi_S}(I_S,A_S) \otimes \mathcal{L}$, i.e. a nowhere vanishing section $\xi(\lambda) \in H^0(S,\mathrm{Ext}^1_{\pi_S}(I_S,A_S) \otimes \mathcal{L}) = \mathrm{Ext}^1_{F_S}(I_S,A_S \otimes \pi_S^*\mathcal{L})$. This section gives rise to an extension

$$(*) \qquad 0 \to A_S \otimes \pi_S^* \mathcal{L} \to X \to I_S \to 0 \quad \text{on} \quad F_S.$$

The condition "$\xi(\lambda)$ nowhere vanishing" is equivalent to "$(*) \otimes_{O_S} k(s)$ does not split for any $s \in S(k)$".

The section $\xi(\lambda)$ is determined by λ only up to $\mathrm{Aut}(\mathcal{L}) = H^0(S,O_S)^*$, but the sheaf X itself is uniquely determined, up to isomorphism. Conversely, any extension $(*)$, splitting over no $s \in S(k)$, gives rise to a unique T-morphism $\lambda : S \to Q$.

<u>Proposition</u> A.2: Assume that

(vi) $\mathrm{Hom}_F(A_t,I_t) = 0$ for all $t \in T(k)$, and that $\underline{\mathrm{Hom}}(I,I) = O_{F_T}$. Then the assignment $\lambda \mapsto X$ described above is a 1-1 correspondence from $\mathrm{Mor}_T(S,Q)$ to {isomorphism classes of sheaves X on F_S such that there exists a linebundle $\mathcal{L}$ on S and an exact, nowhere on S splitting sequence

$$0 \to A_S \otimes \pi_S^* \mathcal{L} \to I_S \to 0\} .$$

<u>Proof</u>: The assumption (vi) implies that $\mathrm{Hom}_{F_S}(A_S \otimes \pi_S^* \mathcal{L}) = 0$ for any $\mathcal{L}$. Therefore, given X, the map $X \to I_S$, and thus the whole extension, is uniquely determined by X up to automorphisms of I_S. The rest is clear from the discussion above. Q.E.D.

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