§ 1. Introduction. Statement of Problem.

A sound wave in direction n,

$$U = A \exp i(k\langle n,x\rangle - \omega t)$$

must satisfy the eigenvalue equation

$$\text{(i)} \qquad MA = \lambda A,$$

where the acoustic operator $M = M(n)$ is defined by

$$\text{(ii)} \qquad M_{jk} = \sum_{i,l} n_i C_{ij,kl} n_l$$

$C_{ij,kl}$ are the elastic coefficients. The eigenvalue λ is

$$\lambda = \rho c^2 ,$$

where ρ is the density and c the sound velocity along the propagation vector kn. M is a symmetric matrix. As function of n it possesses antipodal symmetry,

$$\text{(iii)} \qquad M(n) = M(-n) .$$

Thus, for every direction n, there are typically three sound waves with different velocities and mutually orthogonal polarizations A_1, A_2, A_3, corresponding to the three solutions of the eigenvalue problem. A direction for which two sound velocities are equal is called an <u>acoustic axis</u>, cf. [2].

In the case of no acoustic axes, i.e. no degeneracy, the triple vector field, (A_1, A_2, A_3), with the A's chosen as unit vectors, and with some initial sign convention, gives a continuous (in fact analytic, or even algebraic) field as function of n. Thus, we arrive at the notion of a continuous polarization frame field on the surface of the unit sphere $n_1^2 + n_2^2 + n_3^2 = 1$. When acoustic axes are present, discontinuities in the polarization field will result.

Recently Alshits and Lothe [1] reconsidered the theory for sound propagation in triclinic crystals. However, some problems remain.

In [1] it was shown, by an example, that linear anisotropic media without acoustic axes are possible. Topologically it turns out that there are two classes of possible polarization fields agreeing with the symmetry (iii).In this paper a complete derivation of the two classes will be given. Further, it will be shown that only one class occurs as possible solution for actual acoustic operators (ii).

The usual case includes acoustic axes. Alshits and Lothe [1] generalized the Khatkevich condition for presence of acoustic axes to an invariant form that covers all cases. However, the relation between these conditions and the discriminant of the eigenvalue problem (i) is not clear. This problem will also be dealt with in the present paper.

In order to give a smooth presentation of the main points, we delay to § 6 some of the proofs. This section also contains summary and conclusion.

§ 2. Basic Theory and Notation. Flags, Frames and Spinors.

In the following sections some elementary topological constructions play a central role. A brief introduction before the actual analysis, may be in order.

Let $S = \mathrm{Sym}(3)$ be the vector space of real symmetric 3×3-matrices and $\Delta \subset S$ the subset of matrices with multiple eigenvalues. Then Δ is a codimension 2 algebraic variety, and $S - \Delta$ is a connected open subset of S (§ 5). To every symmetric matrix M in S is associated its eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$

(always in this order) and its eigenspaces e_1, e_2, e_3 (in the corresponding order). If M is in $S-\Delta$, then $e=(e_1,e_2,e_3)$ is an ordered orthogonal system of lines through the origin in $\underline{\underline{R}}^3$, or briefly a <u>flag</u>.

The set of all flags in $\underline{\underline{R}}^3$ will be denoted Fl(3). It can be parametrized as follows. Each flag line is an element of the projective plane P^2, determined by a set of homogeneous coordinates. If $A_1=(a_{i1})$, $A_2=(a_{i2})$ and $A_3=(a_{i3})$ are sets of homogeneous coordinates of a flag (e_1,e_2,e_3), then by orthogonality

$$(1)\qquad \sum a_{i2}a_{i3} = \sum a_{i3}a_{i1} = \sum a_{i1}a_{i2} = 0 .$$

Thus Fl(3) is the subset of $P^2\times P^2\times P^2$ whose multihomogeneous coordinates (A_1,A_2,A_3) satisfy the conditions (1). In particular Fl(3) is a projective algebraic variety.

The matrix $A=(A_1,A_2,A_3)$ whose columns are A_1,A_2,A_3 is not uniquely determined by the flag $e=(e_1,e_2,e_3)$. However, if we require the A_i to be of unit length and positive orientation (i.e. $\det A=+1$), then there are only four choices of A. More precisely, the canonical mapping $\rho\colon SO(3)\to Fl(3)$, which to a rotation matrix (or frame) associates the flag determined by its column vectors, is a 4-sheeted covering of Fl(3). To see this let $I_1\in SO(3)$ be the matrix

$$I_1=\begin{Bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{Bmatrix}$$

and define I_2, I_3 similarly (by cyclic permutation of the diagonal elements). Let $I_0=I$ be the unit matrix. Then

$R = \{I_0, I_1, I_2, I_3\}$ is an abelian subgroup of $SO(3)$, and two matrices A, A' project onto the same flag if and only if $A' = AI_k$ for some k. Thus $Fl(3)$ can be identified with the homogeneous space $SO(3)/R$ and ρ with the canonical projection $SO(3) \to SO(3)/R$. Note that R is not a normal subgroup of $SO(3)$, so that $Fl(3)$ does not inherit a group structure.

Let $\rho : Spin(3) \to SO(3)$ be the connected double covering of $SO(3)$, realized by the group of unit quaternions Spin(3). Recall that the quaternions are the four-dimensional system of hypercomplex numbers

$$q = a + bi_1 + ci_2 + di_3$$

where a,b,c,d are real, and the pure quaternionic units i_k satisfy the multiplication rules

$$i_1{}^2 = i_2{}^2 = i_3{}^2 = -1$$

(2)

$$i_1 i_2 = - i_2 i_1 = i_3, \cdots$$

We denote by $\bar{q}$ the conjugate of q,

(3) $$\bar{q} = a - bi_1 - ci_2 - di_3 ,$$

and by $|q|$ its norm or absolute value,

(4) $$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2 .$$

Linearly the quaternions form a 4-dimensional real vector space $\underline{\underline{H}}$ with Euclidean metric (4). This is analogous to the complex numbers, which form a 2-dimensional space $\underline{\underline{C}}$ with the same type of metric. Also the set of unit quaternions $S^3 \subset \underline{\underline{H}}$ is a group under (quaternion) multiplication. This group is Spin(3).

Let $q = a + (bi_1 + ci_2 + di_3)$ be the splitting of q in its real and pure (or vector) part. The subset of pure quaternions ($a = 0$) is a 3-dimensional linear subspace of $\underline{\underline{H}}$, which will be identified with $\underline{\underline{R}}^3 = \underline{\underline{R}}^3_{(b,c,d)}$. Note that q is pure if and only if $q^2 = -|q|^2$.

The quaternions act linearly on $\underline{\underline{R}}^3$ by

$$(5) \qquad v \rightsquigarrow v' = qv\bar{q} \ .$$

If $q \in \text{Spin}(3)$, then $|v'| = |v|$ and so (5) is an orthogonal operation,

$$v' = A_q(v).$$

Moreover, A_q is a rotation, since $\det A_q = 1$. (Let $q(t)$ be a path from 1 to q on S^3. Then $\det A_{q(t)} \equiv \det A_{q(0)} = 1$). The mapping

$$\sigma(q) = A_q$$

is the required covering projection $\text{Spin}(3) \to SO(3)$. Note that $A_{-q} = A_q$ and $A_{\bar{q}} = A_q^{-1}$.

We shall need an explicit expression for A_q (as a matrix) in terms of the spin variables a,b,c,d of q. Up to common change of sign for θ and r we may write

$$q = \cos\theta + \sin\theta\, r \ , \qquad r^2 = -1.$$

Then r gives the axis and 2θ the angles of rotation for the orthogonal transformation A_q. We have

$$\cos\theta = a \ , \qquad \sin^2\theta = b^2 + c^2 + d^2$$

and $r = b'i_1 + c'i_2 + d'i_3$ with

$$b' = b/\sqrt{b^2 + c^2 + d^2}, \ \ldots$$

Freezing r, we see that $A_q = A_\theta$ is simply the 1-parameter group in $SO(3)$ with infinitesimal generator $2\theta S$,

$$S = \begin{Bmatrix} 0 & -d' & c' \\ d' & 0 & -b' \\ -c' & b' & 0 \end{Bmatrix}$$

Thus $A_q = \exp(2\theta S)$. Since $S = - I$, expansion in power series reduces to

$$A_q = I + \sin 2\theta S + (1-\cos 2\theta)S^2.$$

Substitution for θ and b', c', d' now yields the expression

$$(6) \qquad A_q = \begin{Bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{Bmatrix}$$

The formula shows that $\sigma(i_k) = \sigma(-i_k) = I_k$, $k = 1,2,3$, and of course that $\sigma(1) = \sigma(-1) = I$.

Since $Spin(3) = S^3$ is simply connected, $\rho\sigma : Spin(3) \to Fl(3)$ is the universal covering of $Fl(3)$ and $\sigma : Spin(3) \to SO(3)$ the universal covering of SO(3). We can therefore read off the fundamental groups of $Fl(3)$ and $SO(3)$ as the counterimages $\sigma^{-1}R$ and $\sigma^{-1}\{I\}$. By (6) they are the quaternion group

$$(7) \qquad Q = \{\pm 1, \pm i_1, \pm i_2, \pm i_3\}$$

and its subgroup $Q_0 = \{\pm 1\}$, respectively. Thus $\pi Fl(3) \simeq Q$.

There is a simple relation between $S-\Delta$ and $Fl(3)$. Let s^3 be the set of all triples of real numbers $(\lambda_1,\lambda_2,\lambda_3)$ with $\lambda_1 < \lambda_2 < \lambda_3$. This is an open contractible subset of $\mathbb{R}^3$. Let $\alpha : S-\Delta \to s^3 \times Fl(3)$ be the mapping

$$(8) \qquad \alpha(M) = (\lambda_1,\lambda_2,\lambda_3;\ e_1,e_2,e_3),$$

where $\lambda_1 < \lambda_2 < \lambda_3$ are the eigenvalues of M and $e = (e_1,e_2,e_3)$ the corresponding flag of eigenspaces. This mapping is easily seen to be a (birational) homeomorphism. Since s^3 is contractible, the <u>flag mapping</u> $\alpha_2 : S-\Delta \to Fl(3)$ is a homotopy equivalence,

$$(9) \qquad \alpha_2(M) = (e_1,e_2,e_3).$$

Consequently no homotopy invariant information is lost by passage from $S-\Delta$ to $Fl(3)$.

§ 3 Topologically Possible Polarization Fields.

We turn to the eigenvalue problem for an acoustic operator. Let $M = g(x)$ be a field of symmetric 3×3-matrices on S^2, antipodal invariant and with simple eigenvalues, i.e. a mapping $g : S^2 \to S-\Delta$ such that $g(-x) = g(x)$. *) For the present we assume only that g is continuous, and study what are then the possible eigenspaces. The further question of which of these eigenspaces are also possible eigenspaces for an actual acoustic operator with its specific analytic properties ((ii),in §1), will be treated in § 4.

From g we derive the flag field $e = \alpha_2(M) = \alpha_2(g(x))$, which to each x associates the flag of eigenspaces of M at x. Evidently this is also antipodal invariant.

More generally we may consider arbitrary flag fields $f: S^2 \to Fl(3)$, not a priori derived from symmetric matrices. Since S^2 is simply connected, f lifts to $SO(3)$ and even to $Spin(3)$, i.e. there is a frame field f' and a spinor f'' such that $f(x) = \rho(f'(x)) = \rho(\sigma(f''(x)))$, $x \in S^2$. Moreover, f' and f'' are unique up to the actions of R and Q, respectively. I.e. besides f' the lifts to $SO(3)$ are $f'\cdot I_1$, $f'\cdot I_2$ and $f'\cdot I_3$. Similarly, besides $\pm f''$ the lifts to $Spin(3)$ are $\pm f''\cdot i_1$, $\pm f''\cdot i_2$ and $\pm f''\cdot i_3$.

This much results from the general theory of covering spaces. More specifically we get

---

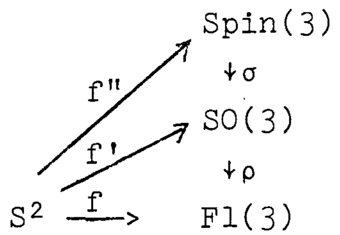
*) From now on we denote by x rather than n the variable point on S^2.

Lemma 1. *Any lift to* SO(3) *of an invariant flag field on* S^2 *is invariant.*

Before proving this lemma, let us underline that it states a real property of the flag-field itself which is best recognized in its lift. Thus, although the physical eigenspaces form a flag-field (no particular sign of polarization), it is important to introduce lifts to bring out all properties of flag field. **)

The proof of lemma 1 proceeds as follows. Consider the diagram of f and its lifts f', f''

(10)



From the identity $f(-x) = f(x)$ follows $\rho(f'(-x)) = \rho(f'(x))$, i.e. $f'(-x) = f'(x)I_k$ for some I_k in R at each point x. By the connectivity of S^2 a relation

(11) $$f'(-x) = f'(x)I_k$$

then holds identically in x for a fixed k. We claim this requires $k= 0$, i.e. I_k is the identity matrix. In fact from (11) follows

(12) $$f''(-x) = f''(x)q$$

**) This is also the reason why in ref [1] it was found convenient to *assign* a direction to polarization and to discuss in terms of vector fields of polarization rather than the physical flag fields.

up in Spin(3), where q is one of the two spinors above I_k. Replacing x by $-x$ in this identity gives $f''(x) = f''(x)q^2$ or $q^2 = 1$. Thus q is either 1 or -1 and so covers I_0. This ends the proof.

Consider the now invariant frame field f' and its spinor lift f''. By (12) we have either

(13) $$f''(-x) = f''(x)$$

or

(14) $$f''(-x) = - f''(x).$$

Thus we have still two possibilities: f'' invariant or f'' equivariant. Of course f'' is not the only lift of f', but the only other is $-f''$, which is of the same variance as f''. In fact all eight spinor lifts of the flag field f are of the same variance, as is immediate from the previous discussion.

In contrast to the situation in lemma 1 we cannot here rule out either possibility. Call f' twisted if it has equivariant spinor lifts and non-twisted if it has invariant lifts. Thus an invariant flag field has invariant frame lifts which may be twisted or non-twisted.

Because of lemma 1 emphasis will now be put on frame fields rather than flag fields. Consequently we un-prime our notation one notch and write $f, g, \cdots$ for frame fields and $f', g', \cdots$ for their spinor lifts.

Two invariant frame fields f, g on S^2 are invariantly homotopic if there is a continuous one-parameter family of invariant frame fields $H(x,t) = H_t(x)$, $0 \leq t \leq 1$, such that $H_0 = f$ and $H_1 = g$. We require H to be a continuous map of both variables $(x,t) \in S^2 \times [0,1]$. Invariant homotopy is an equivalence relation.

Proposision 2. *All twisted frame fields on* S^2 *are (mutually) invariantly homotopic, all non-twisted frame fields are invariantly homotopic, and no twisted frame field is invariantly homotopic to a non-twisted.*

Thus there are two prototypes of invariant frame fields on S^2 (up to invariant deformation). Clearly the constant field

$$(15) \qquad f(x) \equiv I$$

represents the non-twisted type. At the end of this section we give a representative of twisted type, thus exhibiting both cases.

For proof of the second claim in proposition 2 it suffices to show that any invariant mapping $f'\colon S^2 \to \mathrm{Spin}(3)$ is invariantly homotopic to the constant 1 (i.e. the mapping with value 1). This also implies the first claim, because if $f', g'\colon S^2 \to \mathrm{Spin}(3)$ are equivariant, then $f' \cdot g'^{-1}$ is invariant and by the second claim homotopic to 1. But if H_t' is an invariant homotopy connecting $f' \cdot g'^{-1}$ to 1, then $K_t' = H_t' \cdot g'$ is an equivariant homotopy connecting f' to g'. Projecting down to SO(3) we see that the twisted invariant fields f and g are connected by a (twisted) invariant homotopy K_t.

Consider the diagram

$$(16) \qquad \begin{array}{ccc} & & \mathrm{Spin}(3) \\ & \overset{f'}{\nearrow} & \downarrow \\ S^2 & \xrightarrow{f} & SO(3) \\ \downarrow & \overset{f''}{\nearrow} & \\ P^2 & & \end{array}$$

of consistent mappings f, f' and f'' and double coverings. It is immediate that f is untwisted if and only if f'' lifts to

Spin(3). But any mapping $f''' : P^2 \to \mathrm{Spin}(3)$ is homotopic to 1 since $\mathrm{Spin}(3) = S^3$ is 2-connected. And any homotopic H_t''' from f''' to 1 gives an invariant homotopy H_t' from f' to 1. Finally, if f and g are invariantly homotopic on S^2, then f'' and g'' are homotopic on P^2. By the homotopy lifting property of covering spaces, either both f'' and g'' lift to Spin(3) or neither lift. Thus f and g are of the same type.

Proposition 2 may be viewed as a statement about the set $[P^2,SO(3)]$ of (ordinary) homotopy classes of frame fields on P^2, namely that this consists of two elements. It is easily seen that restriction to "equator" $P^1 \subset P^2$ sets up a bijective correspondence $[P^2,SO(3)] \cong [P^1,SO(3)]$. (Replace $S^2 \to P^2$ with $S^1 \to P^1$ in the proof in § 6). Thus restriction to equator already determines the spin type of a field f. We use this for constructing an invariant frame field of twisted spin type on S^2.

We need a diagram of consistent mappings

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & SO(3) \\ p\downarrow & \nearrow_{f''} & \\ P^1 & & \end{array}$$

such that f'' is not homotopic to a constant. We may identify P^1 with a copy of S^1 and parametrize it by its angular argument ν $(0 \leq \nu \leq 2\pi)$. Then the mapping

$$(17) \qquad f''(\nu) = \begin{Bmatrix} \cos\nu, & \sin\nu, & 0 \\ -\sin\nu, & \cos\nu, & 0 \\ 0 & , 0 & , 1 \end{Bmatrix}$$

is homotopically non-trivial, as is well known. Now if $\nu = p(\varphi)$ is the double covering of P^1, then p is multiplication by 2,

$$\nu = 2\varphi.$$

Substitution in (17 gives the twisted invariant frame field f $f(\varphi) = f''(2\varphi)$ on S^1. As it is homotopic to a constant (it factorizes through p), it can be extended over the upper half sphere S^2_+ and by invariance over the rest of the sphere S^2. The result is a twisted invariant frame field on S^2. Such an extension, in spherical coordinates φ, ψ $(0 \leq \varphi \leq 2\pi, -\pi/2 \leq \psi \leq \pi/2)$ is given by

$$(18) \qquad f(\varphi,\psi) = T(\varphi)S(2\psi)T(\varphi)$$

where

$$T(\varphi) = \begin{Bmatrix} \cos\varphi, & \sin\varphi, & 0 \\ -\sin\varphi, & \cos\varphi, & 0 \\ 0 & , \; 0 & , \; 1 \end{Bmatrix} \qquad S(\psi) = \begin{Bmatrix} 1 & , \; 0 & , \; 0 \\ 0 & , \; \cos\psi & , \; \sin\psi \\ 0 & , -\sin\psi & , \; \cos\psi \end{Bmatrix}$$

For $\psi = 0$ we get $f(\varphi,0) = f(\varphi)$, so that $f(\varphi,\psi)$ is in fact an extension. Moreover, $f(\varphi,\pi/2) = f(\varphi,-\pi/2) = I$, so that f is well defined on the sphere. Finally $f(\varphi+\pi,-\psi) = f(\varphi,\psi)$, showing that f is invariant.

§ 4. The polarization Field of Anisotropic Media without Acoustic Axes.

When no acoustic axes are present, the polarization field must belong to either the non-twisted or the twisted type, described in the preceeding section.

The constant field (15),the prototype of the non-twisted field, can be thought of as derived from a field of symmetric matrices $M = g(x)$ of the form

$$\begin{Bmatrix} g_{11}(x) & 0 & 0 \\ 0 & g_{22}(x) & 0 \\ 0 & 0 & g_{33}(x) \end{Bmatrix}$$

with $0<g_{11}<g_{22}<g_{33}$. Acoustic operators close to this diagonal form are possible. In fact Alshits and Lothe [1] have recently shown that a stable orthorombic material without acoustic axis is possible, and this gives an example of non-twisted polarization.

The question remains whether cases of no acoustic axis and twisted polarization field may exist.

The acoustic operator is of the form

$$g_{ij} = \sum_{k,l} B_{ij}^{kl} x_k x_l ,$$

with

$$B_{ij}^{kl} = B_{ji}^{lk} = B_{ij}^{lk} .$$

(See eq. (ii),§1) In terms of the elastic coefficients

$$B_{ij}^{kl} = \tfrac{1}{2}(C_{ik,lj}+C_{il,kj}) .$$

We shall be able to prove the following theorem.

Theorem 3. Let $M = g(x)$ be a field of symmetric matrices on S^2 with simple eigenvalues and entries which are quadratic forms. Let $A = f(x)$ be a frame field of eigenvectors for g. Then f is non-twisted.

Thus, the conclusion is that cases of no acoustic axes and twisted polarization field cannot occur.

For a proof of Theorem 3, several steps are required.

Consider an invariant frame field $A = f(x)$ and its spinor lift $q = f'(x)$. Thus $A = A_q$ in the notation of § 2. We ask when a column of A (i.e. a unit eigenvector of $M = g(x)$) lies in the particular coordinate direction

$$\begin{vmatrix} 0 \\ 0 \\ z \end{vmatrix}$$

In terms of the spin variables a,b,c,d the condition is that some one of the six equations

$$
(19)\qquad \begin{aligned} 2(bd-ac) &= \pm 1 \\ 2(cd+ab) &= \pm 1 \\ 2(a^2+d^2)-1 &= \pm 1 \end{aligned}
$$

must be satisfied. This follows from the expression (6). Applying the identity $a^2+b^2+c^2+d^2 = 1$ the equations (18) appear in the form

$$
(20)\qquad \begin{array}{ll} b-d = a+c = 0 & b+d = a-c = 0 \\ c-d = a-b = 0 & c+d = a+b = 0 \\ b = c = 0 & a = d = 0 \end{array}
$$

Given linear forms $h,l,\ldots$ in the variables a,b,c,d we denote by $S_{h,l,\ldots}$ the intersection of $Spin(3) = S^3$ with the linear subspace $h = l = \cdots = 0$ in H. Clearly z-polarization occurs whenever the spinor lift f' hits one of the six circles $S_{b-d,a+c}, S_{b+d,a-c}, \ldots, S_{a,d}$. From now on we assume f and therefore f' smooth.

<u>Lemma</u> 4. Suppose f' is equivariant. Then the solution set to any equations $h(f'(x)) = l(f'(x) = 0$ is non-empty.

We defer the proof to §6.

Applying lemma 4 to the pairs of linear forms occuring in (20) gives six different pairs of antipodal points x_+, x_- on S^2 where z-polarization takes place when $f:S^2 \to SO(3)$ is a twisted frame field. On the other hand it follows from the eigenvector equation

$$
MA_i = \lambda_i A_i \qquad (1 \le i \le 3)
$$

that $M_{13} = M_{23} = 0$ whenever A_i falls in the z-direction. Thus the quadratic forms $g_{13}(x)$ and $g_{23}(x)$ are simultanously zero along 6 different directions. This is impossible unless g_{13} and g_{23} are linearly dependent, which means that the matrix field $g:S^2 \to S-\Delta$ maps into a proper linear subspace of $S = \mathrm{Sym}(3)$.

Call the matrix field g <u>special</u> if it maps into a proper linear subspace of S and <u>ordinary</u> otherwise. We have just proved that an ordinary invariant matrix field g with entries which are quadratic forms cannot have a twisted eigenvector frame field f .

Suppose g is special. We have

$$g_{ij}(x) = \sum_{k,l} B_{ij}^{kl} x_k x_l \qquad (1 \leq i, \ j \leq 3)$$

with $B_{ij}^{kl} = B_{ij}^{lk} = B_{ji}^{lk}$. We may perturb the set of constants $B = \{B_{ij}^{kl}\}$ to a set B_1 such that the corresponding matrix field g_1 is ordinary, and such that the homotopy g_t with constants $B_t = (1-t)B + tB_1$ maps into $S-\Delta$. Then $\alpha_2 g_t$ is an invariant flag field homotopy from $\alpha_2 g$ to $\alpha_2 g_1$, where α_2 is the flag mapping (9). Since $\alpha_2 g$ lifts to f , the whole homotopy lifts to an invariant homotopy f_t from f to some f_1 covering $\alpha_2 g_1$. Since g_1 is ordinary, f_1 is non-twisted and therefore so is f (proposition 2). This completes the proof of theorem 3 .

§5. <u>The Criterion for Presence of Acoustic Axis</u>

The case of no acoustic axis discussed in the preceding, is exceptional. Usually acoustic axes are present.

In the mathematical model the inclusion of acoustic axes means that there are directions x where the matrix $M = g(x)$ has multiple eigenvalues, i.e. the mapping g into $S = \mathrm{Sym}(3)$ will meet the discriminant variety Δ . The structure of Δ

therefore plays an important role in the study of g or the polarization frame fields of g, as shown in [1]. Below we give an approach to Δ different from [1], which better displays its geometry, and we settle some points left open in [1].

Let $M = \{M_{ij}\}$ be the general symmetric 3×3-matrix. It will be convenient to denote the six free parameters of M by

$$\begin{aligned} M_{11} &= u_1 & M_{23} &= M_{32} = v_1 \\ M_{22} &= u_2 & M_{31} &= M_{13} = v_2 \\ M_{33} &= u_3 & M_{12} &= M_{21} = v_3 \end{aligned}$$

This permits us to identify M with the six-vector $(u,v) = (u_1,u_2,u_3,v_1,v_2,v_3)$ and S itself with $\underline{\underline{R}}^6 = \underline{\underline{R}}^6_{u,v}$.

The subset $\Delta \subset \underline{\underline{R}}^6$ of matrices with multiple eigenvalues is given by a homogenous polynomial equation (27), and so it is a conical affine variety. It contains as subset the line Γ of matrices with triple eigenvalues (all matrices λI, $\lambda \in \underline{\underline{R}}$).

A real number λ is eigenvalue for M if the matrix $M - \lambda I$ has rank < 3 or, equivalently, if its determinant $D = D(\lambda, M)$ vanishes. Recall that

$$D \equiv -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3\,, \tag{21}$$

where $c_i = c_i(M)$ are the characteristic invariants of M,

$$\begin{aligned} c_1 &\equiv u_1 + u_2 + u_3 \\ c_2 &\equiv (u_2u_3 - v_1^2) + (u_3u_1 - v_2^2) + (u_1u_2 - v_3^2) \\ c_3 &\equiv u_1(u_2u_3 - v_1^2) + u_2(u_3u_1 - v_2^2) + u_3(u_1u_2 - v_3^2) + 2(v_1v_2v_3 - u_1u_2u_3) \end{aligned} \tag{22}$$

Since c_1, c_2, c_3 are homogenous polynomials of degree $1, 2, 3$, respectively, D is a cubic form in the seven variables $\lambda, u_1, u_2, u_3, v_1, v_2, v_3$.

Let $\tilde{S}$ be the eigenvalue variety in $\underline{\underline{R}}_\lambda \times \underline{\underline{R}}^6_{u,v} = \underline{\underline{R}}^7_{\lambda,u,v}$ defined by the equation $D(\lambda,M) = 0$ and $\tilde{\Delta}$ the subset defined by $D(\lambda,M) = D_\lambda(\lambda,M) = 0$. Thus (λ,M) is in $\tilde{S}$ when λ is an eigenvalue of M and in $\tilde{\Delta}$ when λ is a multiple eigenvalue. In particular $\tilde{\Delta}$ contains as subset the line $\tilde{\Gamma}$ of pairs $(\lambda,\lambda I)$, $\lambda \in \underline{\underline{R}}$. Let $\pi:\tilde{S} \to \underline{\underline{R}}_{u,v}$ be the restriction to $\tilde{S}$ of the linear projection $\underline{\underline{R}}_\lambda \times \underline{\underline{R}}^6_{u,v} \to \underline{\underline{R}}^6_{u,v}$

(23)
$$\pi(\lambda,M) = M$$

The eigenvalue variety S has a physical interpretation as well as a geometric. Geometrically S is a 3-sheeted ramified covering space of $\underline{\underline{R}}^6_{u,v}$ with ramification locus $\tilde{\Delta}$. The projection π sends $\tilde{\Delta}$ to Δ and $\tilde{\Gamma}$ to Γ, and is two-to-one on $\tilde{\Delta} - \tilde{\Gamma}$ and one-to-one on $\tilde{\Gamma}$. It should be noticed that $\tilde{\Delta}$ disconnects $\tilde{S}$. In fact $\tilde{S} - \tilde{\Delta}$ is a union of three disjoint open sets $\tilde{O}_1$, $\tilde{O}_2$, $\tilde{O}_3$ in $\tilde{S}$, each of which projects isomorphically onto $\underline{\underline{R}}^6_{u,v} - \Delta$ by π. The inverse of $\pi|\tilde{O}_i$ is $\sigma_i(M) = (\lambda_i(M),M)$. If we take the closure of $\tilde{O}_i$ in $\tilde{S}$, we obtain the <u>sheet</u> $\tilde{S}_i$. Since σ_i is defined and continous on all of $\underline{\underline{R}}^6_{u,v}$, $\tilde{S}_i = \sigma_i \underline{\underline{R}}^6_{u,v}$. Thus $\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3$, $\tilde{\Delta} = (\tilde{S}_1 \cap \tilde{S}_2) \cup (\tilde{S}_2 \cap \tilde{S}_3)$ and $\tilde{\Gamma} = \tilde{S}_1 \cap \tilde{S}_2 \cap \tilde{S}_3$. The three sets $\tilde{S}$, $\tilde{\Delta}$ and $\tilde{\Gamma}$ are conical affine varieties in $\underline{\underline{R}}^7$ given by homogenous polynomials in the variables $\lambda, u_1, \ldots, v_3$.

Physically $\tilde{S}$ may be considered the "universal slowness cone" for elastic wave propagation. The ordinary slowness cone of any particular elasticity problem $M = g(x)$ is just the counterimage $S_{\tilde{g}}$ in $\underline{\underline{R}}^4_{\tau,x}$ by the mapping $\tilde{g}: \underline{\underline{R}}^4_{\tau,x} \to \underline{\underline{R}}^7_{\lambda,u,v}$,

(24)
$$\tilde{g} \equiv \begin{cases} \lambda = \tau^2 \\ M = g(x) \end{cases},$$

i.e. the solution set to $D(\tau^2, g(x)) = 0$. The cut between $S_{\tilde{g}}$ and $\tau = 1$ in $\underline{\underline{R}}^4_{\tau,x}$ is the usual slowness surface.

From the discussion below follows that $\tilde{S}$, $\tilde{\Delta}$ and $\tilde{\Gamma}$ are subvarieties of $\underline{\underline{R}}^7$ of codimension 1 , 3 and 6 , respectively, and with singularity sets $\mathrm{sing}\,\tilde{S} = \tilde{\Delta}$ and $\mathrm{sing}\,\tilde{\Delta} = \tilde{\Gamma}$. Consequently, when $\tilde{g}$ is transverse to $\tilde{S}$ (i.e. to the smooth strata $\tilde{S}-\tilde{\Delta}$, $\tilde{\Delta}-\tilde{\Gamma}$ and $\tilde{\Gamma}$), it hits the ramification locus $\tilde{\Delta}$ in a finite number of directions in $\underline{\underline{R}}^4$ only, and the higher ramification locus $\tilde{\Gamma}$ not at all. Thus, in the transverse case we have:

(a) The slowness cone $S_{\tilde{g}}$ is a 3-dimensional conical variety in $\underline{\underline{R}}^4_{\tau,x}$ with a finite number of acoustic axes (possibly none), which form the singularity set of $S_{\tilde{g}}$.

(b) At an accoustic axis only double degeneracy occurs.

(c) The local normal structure of an acoustic axis in $S_{\tilde{g}}$ (outside the origin) is isomorphic to the local normal structure of $\tilde{\Delta}$ in $\tilde{S}$ along any direction different from $\tilde{\Gamma}$.

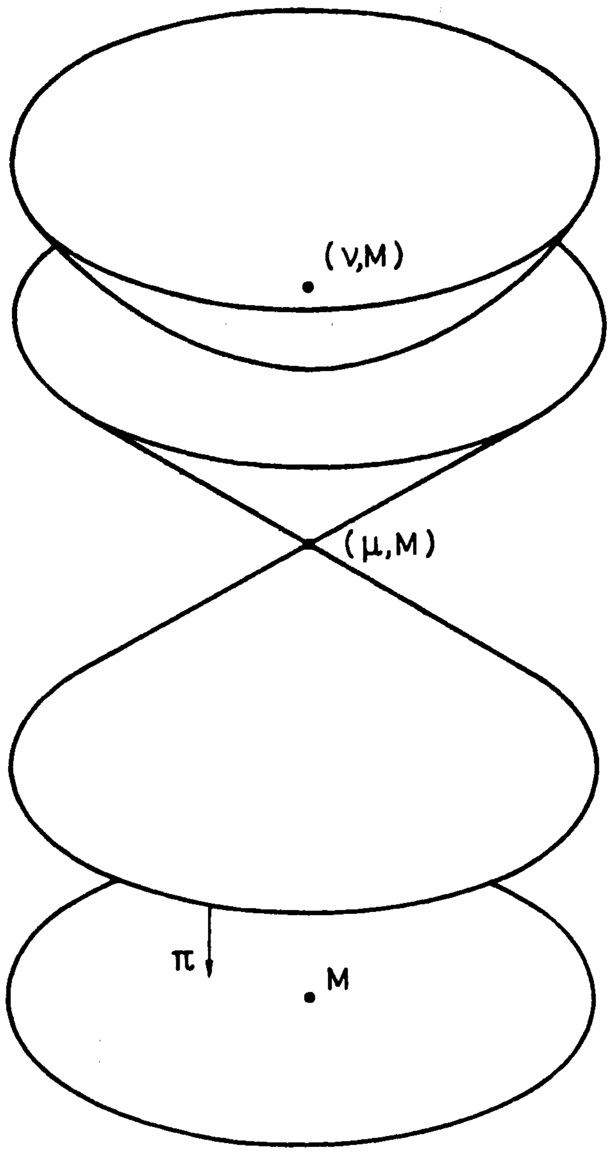
Furthermore (a), (b), (c) presents the stable situation for an anisotropic crystal, since transversality is persistent under small perturbations.

In view of property (c) it is interesting to explicitize the normal space to $\tilde{\Delta}$ in $\tilde{S}$ locally at a point in $\tilde{\Delta}-\tilde{\Gamma}$.

<u>Proposition</u> 6. <u>At any point of</u> $\tilde{\Delta}$ <u>not in</u> $\tilde{\Gamma}$ <u>there is an open neighbourhood</u> $\tilde{B}$ <u>in</u> $\underline{\underline{R}}^7_{\lambda,u,v}$ <u>such that</u> $(\tilde{S}\cap\tilde{B}, \tilde{\Delta}\cap\tilde{B})$ <u>is analytically isomorphic to</u> $(C\times\underline{\underline{R}}^4, o\times\underline{\underline{R}}^4)$, <u>where</u> $C\subset\underline{\underline{R}}^3$ <u>is an elliptic cone through the origin</u> o .

Thus, let $H\subset\underline{\underline{R}}^6$ be an affine plane meeting Δ transversally at M , say, (not in Γ) and let $\tilde{H} = \pi^{-1}H$ be the lift to $\underline{\underline{R}}^7$.

This implies that $\tilde{H}$ meets $\tilde{\Delta}$ transversally at (μ, M) and $\tilde{S}$ transversally at (ν, M), where ν is the simple and μ the double eigenvalue of M. Then locally at (μ, M) $\tilde{H}$ meets $\tilde{\Delta}$ in (μ, M) only and $\tilde{S}$ in a small cone with vertex (μ, M), and locally at (ν, M) $\tilde{H}$ meets $\tilde{S}$ in a small disk.



The proof of proposition 6 is deferred to § 6.

Explicitly the equations for $\tilde{S}$ and $\tilde{\Delta}$ are

$$-\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 = 0 \tag{25}$$

and

$$\begin{aligned} -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 &= 0 \\ -3\lambda^2 + 2c_1\lambda - c_2 &= 0 \end{aligned} \tag{26}$$

Elimination of λ in (26) gives

$$c_1^2c_2^2-4c_1^3c_3-4c_2^3+18c_1c_2c_3-27c_3^2 = 0 , \tag{27}$$

which is a defining equation for Δ . Being homogenous (of degree 6) in the variables $u_1,\dots,v_3$ it defines a conical affine variety in $\underline{\underline{R}}^6_{u,v}$ as claimed. The polynomial $d = d(M)$,

$$d = c_1^2c_2^2-4c_1^3c_3-4c_2^3+18c_1c_2c_3-27c_3^2$$

is of course the discriminant of the characteristic polynomial (23).

The discriminant equation (27) does not reveal much of the geometry of Δ , nor does (25) or (26) tell much about $\tilde{\Delta}$ or $\tilde{S}$. To gain better insight we proceed as follows. For an arbitrary member (λ,M) of $\tilde{S}$ we have a unique decomposition

$$(\lambda,M) = (0,M_0) + (\lambda,\lambda I)$$

with $M_0 = M - \lambda I$. Moreover λ is a double (triple) eigenvalue for M if and only if 0 is a double (triple) eigenvalue for M_0 . Thus we have a direct sum decomposition

$$\begin{aligned} \tilde{S} &= \tilde{S}_0 + \tilde{\Gamma} \\ \tilde{\Delta} &= \tilde{\Delta}_0 + \tilde{\Gamma} \end{aligned} \tag{28}$$

where $\tilde{S}_0 \supset \tilde{\Delta}_0$ is the intersection of $\tilde{S} \supset \tilde{\Delta}$ with the hyperplane $\lambda = 0$ in $\underline{\underline{R}}^7_{\lambda,u,v}$. Identifying this with $\underline{\underline{R}}^6_{u,v}$ we have $\tilde{S}_0 = S_0$ (the set of symmetric matrices with 0 as eigenvalue) and $\tilde{\Delta}_0 = \Delta_0$ ($\cdots$ with 0 as multiple eigenvalue). In other words $\tilde{S} \supset \tilde{\Delta} \supset \tilde{\Gamma}$ is the skew cylinder over $S_0 \supset \Delta_0 \supset \{0\}$ in the direction $(1, 1,1,1,0,0,0)$ in $\mathbb{R}^7$.

To obtain equations for S_0 and Δ_0 we set $\lambda = 0$ in (25) and (26), getting

$$(29) \qquad c_3 = 0 \qquad \text{(for } S_0\text{)}$$

$$(30) \qquad c_3 = c_2 = 0 \qquad \text{(for } \Delta_0\text{)}$$

The second set of equations can be replaced by homogenous equations of 2.degree only. For the matrix M belongs to Δ_0 if and only if it is of rank ≤ 1, which means that all 2.order determinants vanish. Thus let $c_{ij} = c_{ij}(u,v)$ be the 2.order determinants of M,

$$c_{11} \equiv u_2u_3 - v_1^2 \qquad c_{22} \equiv u_3u_1 - v_2^2 \qquad c_{33} \equiv u_1u_2 - v_3^2$$
$$c_{23} \equiv v_2v_3 - u_1v_1 \qquad c_{31} \equiv v_3v_1 - u_2v_2 \qquad c_{12} \equiv v_1v_2 - u_3v_3$$

Then (30) is equivalent to

$$(31) \qquad c_{11} = c_{22} = c_{33} = c_{23} = c_{31} = c_{12} = 0$$

We may subdivide S_0 into S_{0e} and S_{0h} given by $c_3 = 0,\ c_2 \geq 0$ and $c_3 = 0,\ c_2 \leq 0$, respectively (the elliptic and the hyperbolic part of S_0). Then $S_0 = S_{0e} \cup S_{0h}$ and $\Delta_0 = S_{0e} \cap S_{0h}$. Since c_2 is a quadratic form in the variables u_i, v_j, the semialgebraic sets S_{0e} and S_{0h} are again conical. Further subdivision by means of the remainig invariant c_1 ($c_1 \geq 0$, $c_1 \leq 0$) only gives splitting into positive and negative half-cones $\Delta_0 = \Delta_{0+} \cup \Delta_{0-}$, $S_{0e} = S_{0e+} \cup S_{0e-}$, ... E.g. Δ_{0+} is the set of symmetric matrices with at most one non-zero eigenvalue and this positive.

If $(u_1,u_2,u_3,v_1,v_2,v_3)$ are taken as homogenous coordinates of the points in $P^5 = P(\mathbb{R}^6_{u,v})$, then (31) gives well known equations for the Veronese surface $V^2 \subset P^5$. The Veronese surface is the image of P^2 by an imbedding $P^2 \to P^5$ obtained as follows. A symmetric matrix $M = (u,v)$ has rank 1 precisely if $M = \pm t^* t$ for some non-zero $t = (t_1, t_2 t_3)$ in $\underline{\underline{R}}^3$. This yields

(32)
$$\begin{aligned} u_1 &= \pm t_1^2 \\ u_2 &= \pm t_2^2 \\ u_3 &= \pm t_3^2 \\ v_1 &= \pm t_2 t_3 \\ v_2 &= \pm t_3 t_1 \\ v_3 &= \pm t_1 t_2 \end{aligned}$$

which is a parametrization of Δ_0. More precisely the sign + gives a parametrization of Δ_{0+} and the sign - of Δ_{0-}. If we regard (t_1, t_2, t_3) as homogenous coordinates of a point in P^2 and $(u_1, u_2, u_3, v_1, v_2, v_3)$ as homogenous coordinates of a point in P^5, then the signs are irrelevant and (32) defines the imbedding of P^2 in P^5 called the Veronese surface. Elimination of the parameters t_1, t_2, t_3 gives back the equation (31). Thus Δ_0 is simply the affine cone in $\underline{\underline{R}}_{u,v}$ over the imbedded copy $V^2 \subset P^5_{u,v}$ of P^2. Similarly one can give parametrizations of S_{0e} and S_{0h}, from which it follows that S_{0e} and S_{0h} are affine cones over projective semialgebraic sets homeomorphic to S^4 and P^4, respectively. Finally S_0 is the affine cone in $\underline{\underline{R}}^6_{u,v}$ over a projective variety $W^4 \subset P^5_{u,v}$ given by the single determinant equation (29). It is easily checked that W^4 is a projective hypersurface in P^5 with singular locus V^2.

It follows that Δ_0 is a 3-dimensional cone in $\underline{\underline{R}}^6_{u,v}$ which is smooth (i.e. a manifold) except the origin, and that S_0 is a 5-dimensional cone which is smooth except along Δ_0. And so $\tilde{\Delta}$

is a 4-dimensional cone in $\underline{\underline{R}}^7_{\lambda,u,v}$ which is smooth except along the generatrice $\tilde{\Gamma}$, and $\tilde{S}$ is 6-dimensional and smooth except along $\tilde{\Delta}$. Since $\pi:\tilde{S}\to R^6_{u,v}$ is a finite algebraic map, the projection $\Delta = \pi\tilde{\Delta}$ is also 4-dimensional. Thus $\tilde{\Delta}$ and Δ are of codimension 2 in $\tilde{S}$ and $\underline{\underline{R}}^6_{u,v}$.

From (32) and (28) we get the parametrization $(\lambda,t) \sim (\lambda,u,v)$ of $\tilde{\Delta}$, where

$$
\begin{aligned}
u_1 &= \pm t_1^2+\lambda \\
u_2 &= \pm t_2^2+\lambda \\
u_3 &= \pm t_3^2+\lambda \\
v_1 &= \pm t_2 t_3 \\
v_2 &= \pm t_3 t_1 \\
v_3 &= \pm t_1 t_2
\end{aligned}
\tag{33}
$$

and (33), as it stands, is a parametrization of $\Delta = \Delta_+ \cup \Delta_-$. Thus we have the members of Δ (or $\tilde{\Delta}$) parametrized by their unique multiple eigenvalue and their one unique eigendirection, i.e. to (λ,t_1,t_2,t_3) corresponds the matrix $M = (u,v)$ with multiple eigenvalue λ and corresponding eigenspace $x \perp t$, $t_1x_1+t_2x_2+t_3x_3 = 0$ (and the remaining eigenvalue either $\geq\lambda$ or $\leq\lambda$ depending on the sign in (33)).

From (33) we obtain

$$u_j - u_k = \frac{1}{v_j} v_k v_i - \frac{1}{v_k} v_i v_j ,$$

describing the matrices of Δ with no off diagonal zeroes. To capture the rest of Δ we consider the corresponding expressions for the higher order terms $(u_k-u_i)(u_i-u_j)$ (three equations) and $(u_2-u_3)(u_3-u_1)(u_1-u_2)$ (one equation). Written out in homogenous form they are

$$(u_2-u_3)v_2v_3 + v_1(v_2^2-v_3^2) = 0$$

(34a) $$(u_3-u_1)v_3v_1 + v_2(v_3^2-v_1^2) = 0$$

$$(u_1-u_2)v_1v_2 + v_3(v_1^2-v_2^2) = 0$$

$$(u_3-u_1)(u_1-u_2)v_1 + (u_1-u_2)v_2v_3 + v_1(v_1^2-v_2^2) = 0$$

(34b) $$(u_1-u_2)(u_2-u_3)v_2 + (u_2-u_3)v_3v_1 + v_2(v_2^2-v_3^2) = 0$$

$$(u_2-u_3)(u_3-u_1)v_3 + (u_3-u_1)v_1v_2 + v_3(v_3^2-v_1^2) = 0$$

(34c) $$(u_2-u_3)(u_3-u_1)(u_1-u_2)+(u_2-u_3)v_1^2+(u_3-u_1)v_2^2+(u_1-u_2)v_3^2 = 0$$

Apart from linear rearrangement these are the seven conditions put up by Alshits and Lothe as an invariant generalization of the two Khatkevich conditions. Locally at any particular M at most two of these are independant, the others being redundant or collapsed to the trivial identity $0 = 0$. E.g. let V_1 be the subset of matrices with at least one v_i zero. Then (34b), (34c) and one of (34a) are redundant on $\Delta - V_1$, and one is left with the two Khatkevich conditions. On the other hand (34a) collapses on V_1, and then the remaining conditions are important, discussed as special cases by Khatkevich.

To exhibit the structure of Δ completely (as an algebraic variety) the seven conditions (34) are necessary, as well as sufficient. We shall make this claim precise.

For this purpose it is convenient to symmetrize somewhat the expressions in (34b). This can be done by a simple linear rearrangement. Consider

$$R_1 \equiv (u_2-u_3)v_2v_3 + v_1(v_2^2-v_3^2) = 0$$

(35a) $$R_2 \equiv (u_3-u_1)v_3v_1 + v_2(v_3^2-v_1^2) = 0$$

$$R_3 \equiv (u_1-u_2)v_1v_2 + v_3(v_1^2-v_2^2) = 0$$

$$R_4 \equiv 2(u_3-u_1)(u_1-u_2)v_1 + (2u_1-u_2-u_3)v_2v_3+v_1(2v_1^2-v_2^2-v_3^2) = 0$$

(35b) $$R_5 \equiv 2(u_1-u_2)(u_2-u_3)v_2 + (2u_2-u_3-u_1)v_3v_1 + v_2(2v_2^2-v_3^2-v_1^2) = 0$$

$$R_6 \equiv 2(u_2-u_3)(u_3-u_1)v_3 + (2u_3-u_1-u_2)v_1v_2 + v_3(2v_3^2-v_1^2-v_2^2) = 0$$

(35c) $$R_7 \equiv (u_2-u_3)(u_3-u_1)(u_1-u_2) + (u_2-u_3)v_1^2 + (u_3-u_1)v_2^2 + (u_1-u_2)v_3^2 = 0$$

Then (35) is equivalent to (34), since (35a) = (34a), (35b) = (34a) + 2(34b), and (35c) = (34c).

Let $P = R[u,v]$ be the polynomial algebra in the six variables u_1,u_2,u_3,v_1,v_2,v_3. The set of polynomials vanishing on Δ form an ideal $I = I(\Delta)$ in P, which is in fact generated by $R_1,R_2,\ldots,R_7$.

<u>Theorem</u> 5. <u>Every polynomial</u> Q <u>vanishing on</u> Δ <u>can be written</u> $Q = q_1R_1 + \cdots + q_7R_7$ <u>for suitable</u> $q_1,\ldots,q_7$ <u>in</u> $R[u,v]$.

The proof is deferred to §6.

For any nonnegative integer n, let P_n be the real vector space of polynomials homogenous of degree n. Then $P = \Sigma_{n\geq 0}P_n$ (direct sum), and for multiplication we have $P_m \cdot P_n \subset P_{m+n}$. This splitting is inherited on the ideal I, so that $I = \Sigma_{n\geq 0}I_n$ and $P_m \cdot I_n \subset I_{m+n}$. By theorem 5 there are no non-zero polynomials vanishing on Δ below degree 3. For $F = q_1R_1 + \cdots + q_7R_7$ homogenous of degree 3 the coefficients q_i must be constants. Hence $R_1,\ldots,R_7$ form a set of generators for the real vector space I_3. As they are linearly independant (over $\mathbb{R}$), they even give a basis, i.e. $I_3 \simeq \mathbb{R}^7$.

<u>Corollary</u> 6. $R_1,\ldots,R_7$ <u>is a linear basis for</u> $I_3(\Delta)$ <u>and a minimal generator set for the full ideal</u> $I(\Delta)$. $I_2(\Delta) = I_1(\Delta) = I_0(\Delta) = \{0\}$.

It should perhaps be mentioned that theorem 5 and corollary 6 holds equally well over the complex numbers, i.e. with $\mathbb{C}[u,v]$ in stead of $R[u,v]$ and $\Delta_{\mathbb{C}}$ in stead of Δ, provided $\Delta_{\mathbb{C}} \subset \mathbb{C}^6_{u,v}$ is defined by the same seven equations (35). Over the complex numbers, however, (35) is no longer equivalent to (27). The equivalence between (35) and (27) is a real phenomenon and is explained in proposition 8 below.

The natural operation of the rotation group $SO(3)$ on $Sym(3) = \mathbb{R}^6_{u,v}$

$$M \rightsquigarrow M' \qquad (M' = CMC^*)$$

can be extended to $P = \mathbb{R}[u,v]$ by

$$F \rightsquigarrow F' \qquad (F'(M) = F(M'))$$

This operation is linear and multiplicative, preserves degree and leaves $I(\Delta)$ invariant. In this way we get a sequence of representations $I_3(\Delta), I_4(\Delta), \ldots$ of $SO(3)$. Moreover we can choose an inner product in P such that $P = \Sigma P_n$ is an orthogonal decomposition and $SO(3)$ operates orthogonally on P.

Corollary 7. $I_3(\Delta)$ is the irreducible 7-dimensional representation of $SO(3)$.

Since we know that $R_1,\ldots,R_7$ span the invariant space $I_3(\Delta)$, it suffices to check on the 1-parameter subgroup $SO(2) \subset SO(3)$ describing rotations around the z-axis in $\mathbb{R}^3$. By standard methods we find a set of unitary eigenfunctions

$$\psi_0 = R_3 , \quad \psi_{\pm 1} = -3R_1 - R_4 \pm i(3R_2 - R_5) ,$$
$$\psi_{\pm 2} = -R_6 \pm iR_7 , \quad \psi_{\pm 3} = -5R_1 + R_4 \pm i(5R_2 + R_5)$$

This shows that $I_3(\Delta)$ is the unique 7-dimensional irreducible orthogonal representation of $SO(3)$.

Since the discriminant variety Δ is given by the equations $R_1 = \cdots = R_7 = 0$ and also by the single equation $d = 0$ (27), one suspects a simple relation between d and $R_1, \ldots, R_7$. Of course, by theorem 7 $d = q_1 R_1 + \cdots + q_7 R_7$ for suitable q_i, homogenous of degree 3. But in addition d is nonnegative and vanishes only when all R_i vanish.

<u>Proposition</u> 8. <u>The discriminant</u> d <u>is a positive definite quadratic form in the polynomials</u> $R_1, \ldots, R_7$,

$$d = \sum_i a_{ii} R_i^2 + 2 \sum_{i<j} a_{ij} R_i R_j \tag{36}$$

<u>The following coefficients are uniquely determined</u>

$$\begin{aligned} a_{44} &= a_{55} = a_{66} = a_{77} = 1 \\ a_{47} &= a_{57} = a_{67} = 0\,. \end{aligned} \tag{37}$$

<u>The remaining are only determined up to relations</u>

$$\begin{aligned} {}^1/_3\, a_{23} &= a_{26} = a_{56} = -a_{35} = -{}^1/_4\, a_{17} \\ {}^1/_3\, a_{13} &= a_{34} = a_{46} = -a_{16} = -{}^1/_4\, a_{27} \\ {}^1/_3\, a_{12} &= a_{15} = a_{45} = -a_{24} = -{}^1/_4\, a_{37} \\ a_{14} + a_{25} + a_{36} &= 0 \\ a_{11} - 2a_{25} + 2a_{36} &= 15 \\ a_{22} - 2a_{36} + 2a_{14} &= 15 \\ a_{33} - 2a_{14} + 2a_{25} &= 15\,. \end{aligned} \tag{38}$$

<u>However, only one of the expressions (36) is invariant under rotation</u>:

$$d = 15R_1^2 + 15R_2^2 + 15R_3^2 + R_4^2 + \cdots + R_7^2\,. \tag{39}$$

Proposition 8 (once it is stated) can be proved by comparing coefficients in (36), using (35) and (27). The condition

$\partial d/\partial\theta = 0$ (cf. sec 1) singles out the unique diagonal form (37). Since this is positive definite all the expressions (36) are positive definite. We give no further details.

By (38) there are precisely 5 degrees of freedom in the expression for d. This gives rise to interesting relations between the R_i's. Namely set $a_{23} = w_1$, $a_{13} = w_2$, $a_{12} = w_3$, $a_{14} = w_4$ and $a_{25} = w_5$. Then (36) takes for form

$$d = 15R_1^2 + 15R_2^2 + 15R_3^2 + R_4^2 + \cdots + R_7^2 + w_1P_1 + w_2P_2 + w_3P_3 + w_4P_4 + w_5P_5$$

with

$$(40)\qquad \begin{aligned} P_1 &= 3R_2R_3 + R_2R_6 - R_3R_5 + R_5R_6 - 4R_1R_7 \\ P_2 &= 3R_3R_1 + R_3R_4 - R_1R_6 + R_6R_4 - 4R_2R_7 \\ P_3 &= 3R_1R_2 + R_1R_5 - R_2R_4 + R_4R_5 - 4R_3R_7 \\ P_4 &= 3R_1^2 - 3R_2^2 - 2R_3R_6 + R_1R_4 + R_2R_5 \\ P_5 &= 3R_1^2 - 3R_3^2 + 2R_2R_5 - R_3R_6 - R_1R_4 \end{aligned}$$

But by (39) $w_1P_1 + \cdots + w_5P_5$ must be zero for all values of $w_1,\ldots,w_5$. This means that all P_i vanish identically.

<u>Corollary</u> 9. <u>The generators</u> $R_1,\ldots,R_7$ <u>satisfy the quadratic relations</u> $P_1 = \cdots = P_5 = 0$.

It can be checked that $P_1,\ldots,P_5$ span an invariant subspace of $I_6(\Delta)$, the 5-dimensional irreducible representation of $SO(3)$.

An interesting consequence of corollary 9 is that not all equations $R_1 = \cdots = R_7 = 0$ are needed to define Δ set theoretically. Requiring the four conditions

$$(41)\qquad R_4 = R_5 = R_6 = R_7 = 0$$

one deduces from corollary 9 and the expressions (40) that $R_1 = R_2 = R_3 = 0$. Thus (41) is sufficient to give $\Delta \subset \underline{\underline{R}}^6_{u,v}$.

The number of equations cannot be further reduced and still give all of Δ. On the other hand it should be borne in mind that while (41) and even (27) give Δ, neither can give the full set of polynomials vanishing on Δ. The "paradox" disappear if we pass to the complexification $\underline{\underline{C}}^6_{u,v}$, since here the equations (27), (35) and (41) are no longer equivalent.

§ 6. Delayed Proofs. Summary and Conclusion

We return to lemma 4. Consider again the frame field $f:S^2 \to SO(3)$ and its spin lift $f':S^2 \to Spin(3)$. For the proof we need

Lemma 10. *Suppose f' is equivariant and transverse to S_h. Then the solution set to the equation $h(f'(x)) = 0$ consists of an odd number of simple closed smooth curves in S^2, of which precisely one is antipodal invariant.*

The solution set $S_{hf'}$ is the counterimage $f'^{-1}S_h$, and transversality (which means that the gradient of hf' never vanishes on $S_{hf'}$) ensures that $S_{hf'}$ is either empty or a finite number of simple closed curves ([3] p. 208). Since hf' is equivariant, it must be 0 somewhere on S^2, so $S_{hf'}$ is not empty. The set $S_{hf'}$ is stable under antipodal involution on S^2, and therefore so is the complement $S^2 - S_{hf'}$. This means the complement consists of a number of antipodal connected open zones on which hf' has constant sign. Since hf' is equivariant, there is a negative zone for each positive and vice versa. Thus there is an even number of zones and an odd number of curves. But then at least one curve C is mapped to itself by the antipodal mapping. Its complement $S^2 - C$ consists of two antipodal connected sets O_+, O_-, and therefore the remaining curves must lie either in O_+ or in O_-. But evidently no curve in O_+ or O_- can be antipodal invariant. This proves lemma 10.

To prove lemma 4 we have to show that the equations $h(f'(x)) = l(f'(x)) = 0$ have a common solution when f' is equivariant. Suppose there is no solution. Then a suitable small perturbation of f will yield a twisted invariant frame field f_1 with a spinor lift f_1' transversal to S_h and S_l and such that $h(f_1'(x)) = l(f_1'(x)) = 0$ has no solution. But this is impossible, because by lemma 10 the sets $S_{hf_1'}$ and $S_{lf_1'}$ contain antipodal invariant curves C and D, which necessarily meet in antipodal points x_+, x_-. Clearly x_+, x_- are solutions to the perturbed equation. The contradiction proves lemma 4. The perturbation to f_1 is best performed at the level of f'' (cf. diagram (16)). First project S_h, S_l and $S_{h,l}$ to submanifolds P_h, P_l and $P_{h,l}$ in $SL(3)$. Then perturb f'' into a homotopic mapping f_1'' which is transverse to P_h and P_l and avoids $P_{h,l}$. This defines a twisted invariant mapping f_1 on S^2 and an invariant homotopy from f to f_1. Since f lifts, so does the homotopy, thereby defining f_1'.

Next we turn to proposition 6. To prove proposition 6 consider the action of the rotation group $SO(3)$ on $S = \mathbb{R}^6_{u,v}$,

$$M \sim CMC^* \qquad C \in SO(3)$$

The orbit of M has a tangent space T_M at M, which as linear subspace of S is simply $[o(3), M]$,

$$T_M = [o(3), M] .$$

Here $o(3)$ is the Lie algebra of skewsymmetric matrices, and $[o(3),M]$ is the set of all Lie products $[L,M]$ with L in $o(3)$.

Define an inner product on S by $\langle A,B\rangle = \mathrm{trace}(AB)$. (This gives a metric on $S = \mathbb{R}^6_{u,v}$ equivalent to the euclidean metric.) With respect to this inner product the orthogonal complement N_M to T_M is seen to be the centralizer to M in S ,

$$N_M = \{A \in S \mid AM = MA\} .$$

Suppose M has eigenvalues $\lambda_1 = \lambda_2 = \mu,\ \lambda_3 = \nu\ \ (\mu<\nu)$. Since the orbit of M pass through the diagonalization of M , we may as well suppose M is diagonalized,

$$M = \begin{Bmatrix} \mu, & 0, & 0 \\ 0, & \mu, & 0 \\ 0, & 0, & \nu \end{Bmatrix} .$$

The general matrix of the centralizer is then

$$A = \begin{Bmatrix} u_1, & v, & 0 \\ v, & u_2, & 0 \\ 0, & 0, & u_3 \end{Bmatrix} ,$$

hence N_A is 4-dimensional (and the orbit 2-dimensional). By adding the normal field $N = N(u_1,u_2,u_3,v)$ to M we get an expression for the matrices M' in a normal slice to the orbit depending on four parameters (the minimal number),

$$M' = \begin{Bmatrix} \mu+u_1, & v, & 0 \\ v, & \mu+u_2, & 0 \\ 0, & 0, & \nu+u_3 \end{Bmatrix} .$$

We use this to determine the local structure of $\widetilde{S}$ at (μ,M) .

Since $(\widetilde{S},\widetilde{\Delta})$ is a cylinder over $(\widetilde{S}_0,\widetilde{\Delta}_0)$, we may consentrate on the particular case $\mu = 0$ and determine the local structure of $\widetilde{S}_0$ at $(0,M)$, i.e. of S_0 at M . The general matrix from the normal slice (in S) now looks like

$$M' = \begin{Bmatrix} u_1 & , v & , 0 \\ v & , u_2 & , 0 \\ 0 & , 0 & , \nu + u_3 \end{Bmatrix}$$

The intersection of S_0 with the normal slice is clearly given by the equation $c_3(M') = 0$, or

$$(\nu+u_3)(u_1u_2-v^2) = 0 .$$

For $u = (u_1,u_2,u_3)$ small, this is equivalent to

$$u_1u_2 - v^2 = 0 ,$$

which exhibits an elliptic cone C in $\underline{\underline{R}}_u^3$ and a cylinder on that cone, $C \times \underline{\underline{R}}_v$, in $\underline{\underline{R}}_u^3 \times \underline{\underline{R}}_v$. Thus the trace of S_0 in a normal slice to the orbit is analytically of the form $C \times \underline{\underline{R}}$, and so S_0 is, locally at M , of the form $C \times \underline{\underline{R}}^3$ (the orbit being 2-dimensional). Thus $\tilde{S}$ is, locally at (0,M) , of the form $C \times \underline{\underline{R}}^4$. The singularity stratum $o \times \underline{\underline{R}}^4$ corresponds to Δ and shows that Δ is smooth locally around M , as already established.

Finally we turn to the proof of theorem 5. Consider a polynomial F of degree r in the variables u_1,u_2,u_3,v_1,v_2,v_3 , and let $F = F_{(0)} + \cdots + F_{(r)}$ be its decomposition in homogenous components.

It is easily verified that F vanishes on Δ if and only if all $F_{(i)}$ vanishes. Therefore it suffices to prove the theorem for <u>homogenous</u> polynomials Q . We proceed by induction. Obviously the claim holds for polynomials of degree 0 . Assume it has been verified for homogenous polynomials of degree $\leq k-1$, and let $Q = Q(u,v)$ be homogenous of degree $k \geq 1$.

We make a linear change of coordinates in $\underline{\underline{R}}_{u,v}^6$. More precisely change the u-coordinates to

$$\begin{cases} z = u_1 + u_2 + u_3 \\ w_1 = u_2 - u_3 \\ w_2 = u_3 - u_1 \end{cases}$$

and leave the v_i unchanged. Also, for convenience, introduce $w_3 = -(w_1+w_2) = u_1 - u_2$. In the new coordinates the equations (35) take the form $R_i(w,v) = 0$, and the lack of the z-coordinate shows that $\Delta \subset \mathbb{R}^6_{z,w,v}$ is a cylinder over $\Delta \cap \mathbb{R}^5_{w,v}$ along the z-axis. Expanding $Q = Q(z,w,v)$ in powers of z,

$$Q \equiv Q_0 + Q_1 z + \cdots + Q_k z^k ,$$

where $Q_i = Q_i(w,v)$ does not depend on z, it follows that Q_i vanishes on Δ for $0 \le i \le k$. By our induction assumption $Q_1,\ldots,Q_k$ can be expressed in terms of $R_1,\ldots,R_7$. It remains to see that Q_0 can be so expressed.

When $v_1 = v_2 = v_3 = 0$, $R_1,\ldots,R_6$ collapse and $R_7 = w_1w_2w_3$. Thus $Q_0(w,0)$ vanishes on $\{w_1w_2w_3 = 0\}$ in $\mathbb{R}^2_w$, showing that $Q_0(w,0) = w_1w_2w_3h_1(w)$ for some polynomial h_1. It follows that $Q_0 = w_1w_2w_3h_1 + v_1f_1 + v_2f_2 + v_3f_3$. But by (35c) $w_1w_2w_3 = R_7 - v_1w_1 - v_2w_2 - v_3w_3$, thus

$$Q_0 = h_1R_7 + v_1f_1 + v_2f_2 + v_3f_3$$

(with new f_i).

When $v_1 = v_2 = 0$, $R_1,\ldots,R_5$ collapse and $R_6 = 2v_3(w_1w_2 + v_3^2)$, $R_7 = (w_1w_2+v_3)w_3$. Thus $v_3f_3(w,0,0,v_3)$ vanishes on $\{w_1w_2 - v_3^2 = 0\}$ in $\mathbb{R}^3_{w,v_3}$, and we would like to conclude that $f_3(w,0,0,v_3) = (w_1w_2+v_3^2)h(w,v_3)$ for some h. In fact $w_1w_2 + v_3^2 = 0$ defines a non-degenerate quadric Δ' in $\mathbb{R}_{w,v}$, and it is easily seen that its vanishing ideal is the prime ideal I' generated by the single polynomial $R' \equiv w_1w_2 - v_3^2$, Then $f_3(w,0,0,v_3)$ is in I', since $v_3 f_3(w,0,0,v_3)$ is and v_3 is not. Therefore $f_3(w,0,0,v_3)$ is a multiple of R' and so $v_3 f_3(w,0,0,v_3)$ is a multiple of $v_3 R' = \frac{1}{2}(R_6+v_1^2 v_3+v_2^2 v_3+(w_1-w_2)v_1 v_2)$. It follows that $v_3 f_3(w,v)$ can be written

$$v_3 f_3 = R_6 h_2 + v_2 v_3 k_1 + v_3 v_1 k_2 + v_1 v_2 k_3$$

for suitable k_1, k_2, k_3 . Similarly by taking restrictions to $v_3 = v_1 = 0$ and $v_2 = v_3 = 0$ we find

$$v_2 f_2 = R_5 h_3 + \cdots , \quad v_1 f_1 = R_4 h_4 + \cdots ,$$

so that altogether

$$Q_0 = h_1 R_7 + h_2 R_6 + h_3 R_5 + h_4 R_4 + g_1 v_2 v_3 + g_2 v_3 v_1 + g_3 v_1 v_2 ,$$

$g_i = g_i(w,v)$ being polynomials in w_1, w_2, v_1, v_2, v_3 .

Consider $g_3 v_1 v_2$, This term is made up of monomials containing $w_1 v_1 v_2$, $w_2 v_1 v_2$, $v_1^2 v_2$, $v_1 v_2^2$ or $v_1 v_2 v_3$. But by (35) each of these are congruent (mod $R_1, \ldots, R_6$) to a multiple of v_3 . Thus we have

$$Q_0 = h_1 R_7 + \cdots + h_7 R_1 + q v_3 .$$

It follows that q vanish on Δ . Since q is homogenous of degree $k-1$, by assumption it can be expressed in terms of the R_i . Hence so can Q_0 . This completes the proof of theorem 5.

Summary and Conclusion: Crystals without acoustic axes always have a nontwisted polarization field. The twisted type, although topologically possible, cannot be a solution for sound propagation in anisotropic materials.

Usually crystals have acoustic axes. The relation between the Khathevich condition for acoustic axis and the discriminant of the eigenvalue problem, has been elucidated. Also the geometry of acoustic operator space, in the vicinity of acoustic axes, has been derived in a general way.

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