Introduction

In the second of the papers in this series [8], we showed that a stochastic integral with respect to the right standard part ${}^{o}M^{+}$ of an SL^2-martingale M could be obtained from a nonstandard stochastic integral with respect to M. We did so in order to show that the standard and nonstandard theory for stochastic integration are equivalent, and we shall now complete our programme by showing that all (standard) L^2-martingales can be represented as right standard parts of SL^2-martingales.

The argument is in two steps. If $\langle Z,\{\mathcal{F}_t\},\mu\rangle$ is the stochastic basis of an L^2-martingale N, we first find a hyperfinite probability space $\langle\Omega,\mathcal{G},P\rangle$, a family $\{\mathcal{F}'_t\}_{t\in\mathbb{R}_+}$ of σ-algebras on Ω, and a measure-preserving σ-homomorphism $\Theta:\langle\Omega,\mathcal{F}'_\infty,L(P)\rangle\to\langle Z,\mathcal{F}_\infty,\mu\rangle$ which maps each $\mathcal{F}'_t$ onto $\mathcal{F}_t$. Each $\mathcal{F}'_t$ is a sub-σ-algebra of $L(\mathcal{G}_t)$ for some $\mathcal{G}_t\subset\mathcal{G}$, and we find an L^2-martingale $N^\Theta:\mathbb{R}_+\times\Omega\to\mathbb{R}$ adapted to $\langle\Omega,\{L(\mathcal{G}_t)\},L(P)\rangle$ such that each N^Θ_t is $\mathcal{F}'_t$-measurable, and $\Theta[N^\Theta_t\geq\alpha]=[N_t\geq\alpha]$ for all $t\in\mathbb{R}_+$ and all $\alpha\in\mathbb{R}$. This L^2-martingale N^Θ is called a weak Loeb-space representation of N.

The second step is to construct an SL^2-martingale M adapted to $\langle\Omega,\{\mathcal{G}_t\},P\rangle$, such that $N^\Theta={}^{o}M^{+}$. This M is called a weak hyperfinite representation of N. We shall prove that if $X\in\Lambda^2(N)$, then we may find $Y\in SL^2(M)$ such that

$$ {}^{o}\left(\int Y\,dM\right)^{+}=\left(\int X\,dN\right)^{\Theta}. $$

This will complete our programme, since we may now obtain $\int XdN$ from $\int YdM$.

The representation map Θ above will not give any correspondence between the paths of N and those of N^Θ, since it is not induced by any point-mapping $\theta:\Omega \to Z$. Working with Θ is thus a rather elusive affair. However, using the representation theorems for Radon-spaces proved by Anderson in [2], one may obtain Loeb-space representations for large classes of L^2-martingales where the representation map Θ is defined from a mapping $\theta:\Omega \to Z$ by $\Theta(\theta^{-1}(A)) = A$. Such representations we call strong Loeb-space representations, and the corresponding hyperfinite representations are called strong hyperfinite representations.

Using the Skorohod Topology on the space D of right-continuous functions with left limits (see Billingsley [4]), the following result was proved in [6]: Let $N:\mathbb{R}_+ \times Z \to \mathbb{R}$ be an L^2-martingale which is right-continuous and have left limits a.e.. Let F_t be the σ-algebra obtained from the finite dimensional sets up to time t , and let $\sim$ be the equivalence relation on Z defined by: $x \sim y$ if for all $t \in \mathbb{R}_+$, $N(t,x) = N(t,y)$. Let $Z' = Z/\sim$, $F'_t = F_t/\sim$, $\mu' = \mu/\sim$, and let $N':Z' \times \mathbb{R}_+ \to \mathbb{R}$ be the canonical process. Then N' is an L^2-martingale with respect to $\langle Z', \{F'_t\}, \mu' \rangle$, and as such has a strong hyperfinite representation. In other words, if we disregard all information not obtainable from the finite dimensional sets, all L^2-martingales have strong hyperfinite representations.

The result on strong hyperfinite representations uses Anderson's representation theorem for Radon-spaces; the methods used to construct weak hyperfinite representations seem to be new. However, similar results are announced to appear in Anderson [3].

The reader is refered to the first paper [7] for further information on the literature.

We shall use the same conventions and terminology as in the two previous papers; and we only remind the reader that we are working with polysaturated models for nonstandard analysis (see Stroyan and Luxemburg [13]); the saturation property will be used repeatedly in this paper.

We shall use the notation Theorem I-14 for Theorem 14 of the first paper [7]; and the similar convention applies to the second paper [8] and II.

1. Weak Loeb-space representations of measure-spaces

We want to represent arbitrary L^2-martingales as the right standard parts of SL^2-martingales, and to do this we first represent the corresponding measure spaces by hyperfinite Loeb-spaces.

Definition 1: Let $\langle Z, \mathcal{F}, \mu \rangle$ be a probability space. By a weak Loeb-space representation of $\langle Z, \mathcal{F}, \mu \rangle$ we shall mean a hyperfinite probability space $\langle \Omega, \mathcal{G}, P \rangle$; a sub-$\sigma$-algebra $\mathcal{F}'$ of $L(\mathcal{G})$; and a measure-preserving σ-homomorphism $\Theta: \langle \Omega, \mathcal{F}', L(P) \rangle \to \langle Z, \mathcal{F}, \mu \rangle$ which is onto $\mathcal{F}$. The representation is called a strong Loeb-space representation if there is a partial mapping $\theta: \Omega \to Z$, such that $\theta^{-1}(A) \in \mathcal{F}'$ for all $A \in \mathcal{F}$ and $\Theta(\theta^{-1}(A)) = A$ for all such A. Obviously, the domain of θ must have measure one.

Using the standard part map of the defining topology, Anderson proved in [2] that all Radon-spaces have strong Loeb-space representations. We shall prove that all probability spaces have weak Loeb-space representations. To do this we need the following theorem which we take from Sikorski's book [12] (pp. 144-145):

Proposition 2: Let Z and Z' be sets and let $\mathcal{A}$ and $\mathcal{A}'$ be σ-algebras on Z and Z' respectively. Let $\mathcal{S}$ be a generator set for $\mathcal{A}$ and let $f: \mathcal{S} \to \mathcal{A}'$ be a mapping. Then f has an extension to $\mathcal{A}$ which is a σ-homomorphism if and only if for each countable family $\{A_i\}_{i \in \mathbb{N}}$ of elements from $\mathcal{S}$:

$$\bigcap_{i \in \mathbb{N}} \varepsilon(i) A_i = \emptyset \Rightarrow \bigcap_{i \in \mathbb{N}} \varepsilon(i) f(A_i) = \emptyset ,$$

where $\varepsilon: \mathbb{N} \to \{1, -1\}$.

We may now use Proposition 2 to prove

<u>Theorem 3</u>: Each probability space has a weak Loeb-space representation.

Proof: Let $\langle Z, \mathcal{F}, \mu \rangle$ be a probability space and let $\langle {}^*Z, {}^*\mathcal{F}, {}^*\mu \rangle$ be its nonstandard version. Let $\mathcal{S} = \{{}^*F : F \in \mathcal{F}\}$, and let $\tilde{\mathcal{F}}$ be the σ-algebra generated by $\mathcal{S}$.

Define a mapping $f: \mathcal{S} \to \mathcal{F}$ by $f({}^*F) = F$. We shall use Proposition 2 to extend f to a σ-homomorphism $h: \tilde{\mathcal{F}} \to \mathcal{F}$. Since $\tilde{\mathcal{F}}$ is closed under complements, it is enough to show that for each countable family $\{{}^*F_i\}_{i \in \mathbb{N}}$ of sets from $\mathcal{F}$ such that $\bigcap_{i \in \mathbb{N}} {}^*F_i = \emptyset$, then $\bigcap_{i \in \mathbb{N}} f({}^*F_i) = \bigcap_{i \in \mathbb{N}} F_i = \emptyset$. But this is easy since saturation and $\bigcap_{i \in \mathbb{N}} {}^*F_i = \emptyset$ implies $\bigcap_{i \leq n} {}^*F_i = \emptyset$ for some $n \in \mathbb{N}$. Since ${}^*(\bigcap_{i \leq n} F_i) = \bigcap_{i \leq n} {}^*F_i = \emptyset$, we have $\bigcap_{i \leq n} F_i = \emptyset$ and consequently $\bigcap_{i \in \mathbb{N}} F_i = \emptyset$. Hence f can be extended to a σ-homomorphism $h: \tilde{\mathcal{F}} \to \mathcal{F}$.

Let us show that h is measure-preserving, i.e. $L({}^*\mu)(A) = \mu(h(A))$ for $A \in \tilde{\mathcal{F}}$. If $A \in \mathcal{S}$ this is obvious by definition of the Loeb-measure. Since $\nu(A) = \mu(h(A))$ defines a measure on $\tilde{\mathcal{F}}$, it follows from Caratheodory's Extension Theorem (Royden [11], page 257) that $L({}^*\mu)(A) = \mu(h(A))$ for all $A \in \tilde{\mathcal{F}}$, since $\mathcal{S}$ is an algebra of sets.

We have thus constructed an internal probability space $\langle {}^*Z, {}^*\mathcal{F}, {}^*\mu \rangle$; a sub-$\sigma$-algebra $\tilde{\mathcal{F}}$ of $L({}^*\mathcal{F})$; and a measure-preserving σ-homomorphism $h: \tilde{\mathcal{F}} \to \mathcal{F}$. It only remains to turn $\langle {}^*Z, {}^*\mathcal{F}, {}^*\mu \rangle$ into a hyperfinite probability space:

For each finite set $F_1, \ldots, F_n$ of elements of $\mathcal{F}$, there is a finite partition $\mathcal{P}$ of ${}^*Z$ such that if $P \in \mathcal{P}$ and $P \cap {}^*F_i \neq \emptyset$, then $P \subset {}^*F_i$. By saturation there exists a hyperfinite partition $\mathcal{P}$ of ${}^*Z$ such that for each $F \in \mathcal{F}$ if $P \cap {}^*F \neq \emptyset$ for a $P \subset \mathcal{P}$, then $P \subset {}^*F$. Let $\sim$ be the equivalence relation generated by $\mathcal{P}$, let

$\Omega = {}^*Z/\sim$, and let $\pi : {}^*Z \to \Omega$ be the quotient map. We may choose $\mathcal{P}$ such that the equivalence classes of $\mathcal{P}$ are elements of ${}^*\mathcal{F}$. Let $\mathcal{G} = \pi({}^*\mathcal{F})$ and let $P = \pi({}^*\mu)$, then $<\Omega, \mathcal{G}, P>$ is a hyperfinite probability space. Let $\mathcal{F}'$ be $\pi(\tilde{\mathcal{F}})$, it follows easily from the definitions of $\tilde{\mathcal{F}}$ and $\mathcal{P}$ that $\mathcal{F}' \subset L(\mathcal{G})$. Define

$$\Theta : \mathcal{F}' \to \mathcal{F} \quad \text{by} \quad \Theta = h \circ \pi^{-1} .$$

Then $\{<\Omega, \mathcal{G}, P>, \mathcal{F}', \Theta\}$ is a weak Loeb-space representation of $<Z, \mathcal{F}, \mu>$, and we have proved the theorem.

If $\{<\Omega, \mathcal{G}, P>, \mathcal{F}', \Theta\}$ is a weak Loeb-space representation of $<Z, \mathcal{F}, \mu>$, we have for each set F' in $\mathcal{F}'$ a set $\Theta(F') \in \mathcal{F}$ such that $L(P)(F') = \mu(\Theta(F'))$. On the other hand, for each $F \in \mathcal{F}$, we have a non-empty subset $\Theta^{-1}(\{F\})$ of $\mathcal{F}'$ consisting of sets that are equal to $L(P)$-a.e.. It is easy to prove that we have a similar correspondence for random variables. If $f : \Omega \to \mathbb{R}$ is $\mathcal{F}'$-measurable, then there exists a uniquely determined $f_\Theta : Z \to \mathbb{R}$ such that for each $\alpha \in \mathbb{R}$, $\Theta[f \geq \alpha] = [f_\Theta \geq \alpha]$. Conversely, if $f : Z \to \mathbb{R}$ is $\mathcal{F}$-measurable there is a nonempty set of $\mathcal{F}'$-measurable functions f^Θ such that $\Theta[f^\Theta \geq \alpha] = [f \geq \alpha]$. Two such f^Θ's are different only on a set which Θ maps on $\emptyset$, and which consequently has measure zero. Since Θ is measure-preserving we have $\int f^\Theta dL(P) = \int f d\mu$ and $\int f dL = \int f_\Theta d\mu$. If $\{f_n\}$ is a sequence of random variables on $<\Omega, \mathcal{F}', L(P)>$ which converges a.e. to f , then $\{(f_n)_\Theta\}$ converges a.e. to f_Θ . If $\{f_n\}$ is a sequence of random variables on $<Z, \mathcal{F}, \mu>$ that converges a.e. to f , then $\{f_n^\Theta\}$ converges a.e. to f^Θ . We shall use these simple facts about f_Θ and f^Θ in the sequel.

2. Weak Loeb-space representations of martingales

In this section we use the representation of measure spaces found in Theorem 3, to represent arbitrary L^2-martingales by L^2-martingales on Loeb-spaces. These are still real-valued martingales parameterized by $\mathbb{R}_+$, but in the next section we shall see how to replace them by hyperfinite martingales.

We first extend the notion of weak Loeb-space representations to stochastic bases:

Definition 4: By a weak Loeb-space representation for a stochastic basis $<Z,\{F_t\},\mu>$, we mean a sequence $<\Omega,\{G_t\}_{t\in\mathbb{R}_+},P>$ of internal probability spaces with $G_s \subset G_t$ for $s<t$; an increasing family $\{F_t'\}_{t\in\mathbb{R}_+\cup\{\infty\}}$ of σ-algebras on Ω with $F_t' \subset L(G_t)$ for $t\in\mathbb{R}_+$; and a measure-preserving σ-homomorphism $\Theta:F_\infty' \to F_\infty$ such that Θ is surjective and for each $t\in\mathbb{R}_+\cup\{\infty\}$, the tuple $\{<\Omega,G_t,P>,F_t',\Theta\restriction F_t'\}$ is a weak Loeb-space representation for $<Z,F_t,\mu>$.

Let $M:\mathbb{R}_+\times Z \to \mathbb{R}$ be an L^2-martingale adapted to the stochastic basis $<Z,\{F_t\},\mu>$, and let $\{<\Omega,\{G_t\},P>,\{F_t'\},\Theta\}$ be a weak Loeb-space representation of this basis. For each $t\in\mathbb{R}_+$ we may find a random variable $M_t^\Theta:\Omega\to\mathbb{R}$ such that $\Theta[M_t^\Theta\geq\alpha] = [M_t\geq\alpha]$ for all $\alpha\in\mathbb{R}$, and $[M_t^\Theta\geq\alpha]\in F_t'$. Since Θ is measure-preserving, the process $M^\Theta:\mathbb{R}_+\times\Omega\to\mathbb{R}$ defined by $M^\Theta(t,\omega) = M_t^\Theta(\omega)$ is an L^2-martingale adapted to the stochastic basis $<\Omega,\{F_t'\},L(P)>$.

Definition 5: Let $M:\mathbb{R}_+\times Z\to\mathbb{R}$ be an L^2-martingale adapted to the stochastic basis $<Z,\{F_t\},\mu>$, and let $\{<\Omega,\{G_t\},P>,\{F_t'\},\Theta\}$ be a weak Loeb-space representation of $<Z,\{F_t\},\mu>$. Then $\{<\Omega,\{G_t\},P>,\{F_t'\},\Theta,M^\Theta\}$ is a weak Loeb-space representation of M if M^Θ is an L^2-martingale with respect to $<\Omega,\{L(G_t)\},L(P)>$.

Notice that the martingale M^Θ in Definition 5 is required to be adapted to the basis $\langle\Omega,\{L(\mathcal{G}_t)\},L(P)\rangle$ and not only to $\langle\Omega,\{F_t'\},L(P)\rangle$. We shall need this to be able to replace M^Θ by a hyperfinite martingale adapted to an extension of $\langle\Omega,\{\mathcal{G}_t\},P\rangle$. But by this requirement it is no longer obvious that M has a weak Loeb-space representation; however, we shall prove in Theorem 7 that it does have one.

Let us make the connection to the setting of [8]: Let $\{\mathbf{G}_t\}_{t\in T}$ be an extension of $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ to an increasing sequence of internal algebras indexed by a hyperfinite time-line T. From Lemma II-5 and Lemma II-6, we see that

$$\mathcal{H}_t = \bigcap_{n\in\mathbb{N}} \sigma(L(\mathcal{G}_{s_n})\cup\mathcal{N})$$

for each sequence $\{s_n\}$ of elements from T such that $\{{}^o s_n\}$ decreases strictly to t. This tells us that the family $\{\mathcal{H}_t\}$ does not change if we pass to a sub-line, and that it is uniquely determined by the original family $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ and does not depend on the extension $\{\mathcal{G}_t\}_{t\in T}$.

<u>Lemma 6</u>: Let $M:\mathbb{R}_+\times Z\to\mathbb{R}$ be a right-continuous L^2-martingale with a weak Loeb-space representation $\{\langle\Omega,\{\mathcal{G}_t\},P\rangle,\{F_t\},\theta,M^\Theta\}$. Then M^Θ is an L^2-martingale adapted to $\langle\Omega,\{\mathcal{H}_t\},L(P)\rangle$.

Proof: Let $A\in\mathcal{H}_s$, then by Lemma II-6 there exists a B such that $B\in L(\mathbf{G}_{s+1/n})$ for all $n\in\mathbb{N}$ and $L(P)(A\Delta B)=0$. For $t>s$ we get:

$$\int_A(M_t^\Theta-M_s^\Theta)dL(P) = \int_B(M_t^\Theta-M_s^\Theta)dL(P) = \int_B \lim_{n\to\infty}(M_t^\Theta-M_{s+1/n}^\Theta)dL(P)$$

$$= \lim_{n\to\infty}\int_B(M_t^\Theta-M_{s+1/n}^\Theta)dL(P) = 0\,,$$ where we have used $M_{s+1/n}^\Theta\to M_s^\Theta$ since M is right-continuous, and the usual combination of Doob's inequality and Lebesgue's Convergence Theorem to get the limit outside the integral.

We now prove

Theorem 7: If M is an L^2-martingale, then M has a weak Loeb-space representation.

Proof: Let $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ be the stochastic basis of M. If $\mathcal{F}_\infty$ is finite, the result is obvious and we hence assume that $\mathcal{F}_\infty$ is infinite.

For each $t \in \mathbb{R}_+ \cup \{\infty\}$ let $\tilde{\mathcal{F}}_t$ be the σ-algebra generated by the sets $\{{}^*F : F \in \mathcal{F}_t\}$. In the proof of Theorem 3 we saw that for each t there exists a measure-preserving σ-homomorphism $h_t : \tilde{\mathcal{F}}_t \to \mathcal{F}_t$, such that $h_t({}^*F) = F$ for all $F \in \mathcal{F}_t$. If $t > s$, it is clear that $h_t | \mathcal{F}_s = h_s$ since they agree on the generator set $\{{}^*F : F \in \mathcal{F}_s\}$. Hence all h_t are obtainable from h_∞ and we shall write h for h_∞.

For each $t \in \mathbb{R}_+$ we may find an $\tilde{\mathcal{F}}_t$-measurable random variable $M'_t : {}^*Z \to \mathbb{R}$ such that $h[M'_t \geq \alpha] = h_t[M'_t \geq \alpha] = [M_t \geq \alpha]$ for all $\alpha \in \mathbb{R}$. Since $\tilde{\mathcal{F}}_t \subset L({}^*\mathcal{F}_t)$ we may find an internal $K_t : {}^*Z \to \mathbb{R}$ such that $K_t \in SL^2({}^*Z, {}^*\mathcal{F}_t, {}^*\mu)$ and ${}^oK_t = M'_t$ $L({}^*\mu)$-a.e. for all $t \in \mathbb{R}_+$.

For all $s,t \in \mathbb{R}_+$, $s < t$, and all $A \in \tilde{\mathcal{F}}_s \cap {}^*\mathcal{F}_s$, we have:

$$0 \leq \left| {}^o\int_A (K_t - K_s)\, d{}^*\mu \right| = \left| \int_A ({}^oK_t - {}^oK_s)\, dL({}^*\mu) \right| = \left| \int_A (M'_t - M'_s)\, dL({}^*\mu) \right| = \left| \int_{h(A)} (M_t - M_s)\, d\mu \right| = 0$$

Let now $B_1, B_2, \dots, B_n$ be a finite family of sets from $\mathcal{F}_\infty$, and let $\langle s_1, t_1 \rangle, \dots, \langle s_m, t_m \rangle$ be a finite set of pairs $\langle s_i, t_i \rangle \in \mathbb{R}_+^2$ such that $s_i < t_i$. Then there exists a finite partition $\mathcal{P}$ of ${}^*Z$ such that if $P \in \mathcal{P}$ and $P \cap {}^*B_i \neq \emptyset$ for some $i \in \{1, \dots, n\}$, then $P \subset {}^*B_i$; and if A is in the algebra generated by $\mathcal{P}$, and $A \in {}^*\mathcal{F}_{s_i}$ for some $i \in \{1, \dots, m\}$, then

$$\left| \int_A (K_{t_i} - K_{s_i})\, d{}^*\mu \right| \leq \frac{1}{{}^*\mathrm{card}(\mathcal{P})}$$

where ${}^*\mathrm{card}(\mathcal{P})$ denotes the internal cardinality of $\mathcal{P}$. To see that this statement is true, take P to be the partition generated by $B_1,\dots,B_n$, and use the inequality we proved above.

By polysaturation there then exists a hyperfinite partition P of $^*Z$ such that for each $B \in F_\infty$ if $P \in P$ and $P \cap {}^*B \neq \emptyset$, then $P \subset {}^*B$; and if A is in the intersection of the internal algebra generated by P, and $^*F_s$, then for all $t > s$

$$\left|\int_A (K_t - K_s)\,d^*\mu\right| \leq \frac{1}{^*\text{card } P}.$$

Since F_∞ is infinite, $^*\text{card}(P) \in {}^*\mathbb{N}\setminus\mathbb{N}$.

Let A be the internal algebra generated by P, and for each $t \in \mathbb{R}_+$ define

$$\tilde{G}_t = A \cap {}^*F_t .$$

By definition of P, we have $^*B \in A$ for all $B \in F_\infty$, and consequently $\tilde{F}_t \subset L(\tilde{G}_t)$.

If $\sim$ is the equivalence relation induced by P, let $\pi: {}^*Z \to {}^*Z/\sim$ be the quotient map. We write Ω for $^*Z/\sim$; G_t for $\pi(\tilde{G}_t)$; and P for $\pi(^*\mu)$. We also put $F'_t = \pi(\tilde{F}_t)$ for $t \in \mathbb{R}_+ \cup \{\infty\}$, and let $\Theta = h \circ \pi^{-1}$.

Then $\{\langle\Omega,\{G_t\},P\rangle,\{F'_t\},\Theta\}$ is a weak Loeb-space representation of $\langle Z,\{F_t\},\mu\rangle$. We must prove that M^Θ is a martingale with respect to the basis $\langle\Omega,L(G_t),L(P)\rangle$:

If $A \in L(G_s)$, then there exists $B \in G_s$ such that $L(P)(A\Delta B) = 0$. For $t > s$ we have

$$0 \leq \left|\int_A (M_t^\Theta - M_s^\Theta)\,dL(P)\right| = \left|\int_B (M_t^\Theta - M_s^\Theta)\,dL(P)\right|$$

$$= \left|\int_{\pi^{-1}(B)} (M'_t - M'_s)\,dL(^*\mu)\right| = \left|\int_{\pi^{-1}(B)} ({}^\circ K_t - {}^\circ K_s)\,dL(^*\mu)\right|$$

$$= \left|{}^\circ\!\int_{\pi^{-1}(B)} (K_t - K_s)\,d^*\mu\right| \leq {}^\circ\!\left(\frac{1}{^*\text{card}(P)}\right) = 0, \text{ since } \pi^{-1}(B) \in A \cap {}^*F_t .$$

But this proves that M^{Θ} is an L^2-martingale with respect to the stochastic basis $<\Omega, L(\mathcal{G}_t), L(P)>$, and hence the theorem.

If X is a predictable process with respect to $<Z,\{\mathcal{F}_t\},\mu>$, then we can find a predictable version of X^{Θ}, and if X is predictable with respect to $<\Omega,\{\mathcal{F}'_t\},L(P)>$, then X_{Θ} is predictable. If $X \in \Lambda^2(M)$, then $X^{\Theta} \in \Lambda^2(M^{\Theta})$ and for $r \in \mathbb{R}_+$:

$$\int_{[0,r]\times Z} X^2 dm_M = \int_{[0,r]\times\Omega} X^{\Theta^2} dm_{M^{\Theta}} .$$

We need these simple facts to prove the following commutation rules for Θ and stochastic integrals:

<u>Proposition 8</u>: Let M be a right-continuous L^2-martingale. If $X \in \Lambda^2(M)$, then $X^{\Theta} \in \Lambda^2(M\ )$ and $(\int XdM)^{\Theta} = \int X^{\Theta} dM^{\Theta}$.

If Y is predictable with respect to $<\Omega,\{\mathcal{F}'_t\},L(P)>$ and $Y \in \Lambda^2(M^{\Theta})$, then $Y_{\Theta} \in \Lambda^2(M)$ and $\int YdM^{\Theta} = (\int Y_{\Theta} dM)^{\Theta}$.

Proof: Since M is right-continuous, $\lim_{n\to\infty} M^{\Theta}(t+r_n) = M^{\Theta}(t)$ for each sequence $\{r_n\}$ decreasing to zero, and thus stochastic integration with respect to M^{Θ} is well-defined.

Assume first that X is of the form $X = 1_{<s,t]\times F}$ where $F \in \mathcal{F}_s$. Then $X^{\Theta} = 1_{<s,t]\times F'}$, where $F' \in \Theta^{-1}(F) \cap \mathcal{F}'_s$, and we have $\int_0^r X^{\Theta}\, dM^{\Theta} = 1_{F'}(M^{\Theta}_{t\wedge r} - M^{\Theta}_{s\wedge r}) = (\int_0^r XdM)^{\Theta}$. By linearity the assertion holds for all X of the form $X = \Sigma a_i 1_{<s_i,t_i]\times F_i}$, $F_i \in \mathcal{F}_{s_i}$.

If X is an arbitrary element of $\Lambda^2(M)$, there exists a sequence $\{X_n\}$ of elements of the above form such that

$$\lim_{n\to\infty}(L^2)(\int_0^r X_n dM) = \int_0^r XdM \quad \text{for all } r \in \mathbb{R}_+ .$$

Thus $\lim_{n\to\infty}(L^2)(\int_0^r X_n^{\Theta} dM^{\Theta}) = \lim_{n\to\infty}(L^2)(\int_0^r X_n dM)^{\Theta} = (\int_0^r XdM)^{\Theta}$.

Moreover, the elements $\{X_n\}$ are chosen such that $\lim_{n\to\infty} \int_{[0,r]\times Z} (X-X_n)^2 dm_M = 0$, but since $\int_{[0,r]\times Z} (X-X_n)^2 dm_M = \int_{[0,r]\times\Omega} (X^\Theta - X_n^\Theta)^2 dm_{M^\Theta}$ we also have $\lim_{n\to\infty} \int_{[0,r]\times\Omega} (X^\Theta - X_n^\Theta)^2 dm_{M^\Theta} = 0$. But then $\int_0^r X^\Theta dM^\Theta = \lim(L^2)\int_0^r X_n^\Theta dM^\Theta$ and hence $\int_0^r X^\Theta dM^\Theta = (\int_0^r X dM)^\Theta$.

The second part follows immediately from the first since $(Y_\Theta)^\Theta = Y$.

We have similar results for local L -martingales:

Let $F_t'' = (F_t')^+ = \bigcap_{s>t} F_s'$. Then the following result was proved in [6]:

<u>Proposition 9</u>: Let $M:\mathbb{R}_+\times Z \to \mathbb{R}$ be a right continuous local L^2-martingale adapted to the basis $<Z,\{F_t\},\mu>$, and let $X \in \Lambda(M)$. There exist a weak Loeb-space representation $\{<\Omega,\{G_t\},P>,\{F_t'\},\Theta\}$ of $<Z,\{F_t\},\mu>$ and a version of M^Θ which is a local L^2-martingale with respect to $<\Omega,\{H_t\},L(P)>$, and such that $X^\Theta \in \Lambda(M^\Theta)$. The localizing sequence of stopping times for X^Θ and M^Θ will be adapted to $<\Omega,\{F_t''\},L(P)>$, and each M_t^Θ is F_t''-measurable. Moreover,

$$(\int X dM)^\Theta = \int X^\Theta dM^\Theta .$$

Some extension of the algebras F_t' (like F_t'') is necessary since the stopping times will not be adapted to $\{F_t'\}$. We need not, however, use the whole of F_t''; it is enough to add its null-sets to F_t'.

3. L^2-martingales as right standard parts

In the last section we saw how we could replace a given martingale by a martingale adapted to a basis of Loeb-algebras having the same properties of stochastic integration. In this section we shall go one step further by showing that each martingale of the latter type may be considered as the right standard part of a hyperfinite martingale.

Before proceeding, the reader should recall the comments on the family $<\Omega,\{G_t\},P>$ preceding Lemma 6.

<u>Theorem 10</u>: Let $<\Omega,\{G_t\}_{t\in\mathbb{R}_+},P>$ be an increasing family of hyperfinite probability spaces. Let $N:\mathbb{R}_+\times\Omega\to\mathbb{R}$ be an L^2-martingale with respect to $<\Omega,\{H_t\},L(P)>$ and assume that N_0 is $\sigma(L(G_0)\cup N)$-measurable. Assume further that for each $t\in\mathbb{R}_+$ and each sequence $\{t_n\}_{n\in\mathbb{N}}$ decreasing to t we have $N(t_n)\to N(t)$ $L(P)$-a.e..

Then there exist a hyperfinite time-line T, an internal basis $<\Omega,\{G_t\}_{t\in T},P>$ extending $<\Omega,\{G_t\}_{t\in\mathbb{R}_+},P>$, and an SL^2-martingale $M:T\times\Omega\to{}^*\mathbb{R}$ adapted to $<\Omega,\{G_t\}_{t\in T},P>$ such that for each $t\in\mathbb{R}_+$

$$^oM^+(t,\omega) = N(t,\omega) \quad \text{for} \quad L(P)\text{-almost all} \quad \omega .$$

Moreover, we may take M to be well-behaved, S-right-continuous at 0, and such that for all $t\in\mathbb{R}_+$

$$^o[M]^+(t,\omega) = [N](t,\omega) \text{ for } L(P)\text{-almost all } \omega .$$

Proof: By saturation we may extend the family $\{G_t\}_{t\in\mathbb{R}_+}$ to an increasing internal family $\{G_t\}_{t\in S}$, where S is a hyperfinite time-line.

For each $t\in\mathbb{R}_+$, the random variable $N_t:\Omega\to\mathbb{R}$ is H_t-measurable and thus it is $\sigma(L(G_s)\cup N)$-measurable for all $s>t$, $s\in\mathbb{R}_+$. Consequently there is an $L_t^{(s)}\in SL^2(\Omega,G_s,P)$ such that $^oL_t^{(s)} = N_t$ $L(P)$-a.e.. We may extend the sequence $\{L_t^{(t+1/n)}\}_{n\in\mathbb{N}}$ by saturation to an internal sequence $\{L_t^{(t+1/\gamma)}\}_{\gamma\leq\eta}$, $\eta\in{}^*\mathbb{N}\setminus\mathbb{N}$, where $L_t^{(t+1/\gamma)}$ is $G_{\overline{(t+1/\gamma)}}$-measurable (recall that $\bar{r}$ is the least element in S larger than r). There must be an internal initial segment $\{L_t^{(t+1/\gamma)}\}_{\gamma\leq\xi}$ such that

$$\|L_t^{(t+1/\gamma)} - L_t^{(t+1)}\|_2 \leq 1/\gamma ,$$

for all $\gamma \leq \xi$.. By construction ξ must be infinite and it follows that ${}^oL_t^{(t+1/\xi)} = N_t$ $L(P)$-a.e.; $L_t^{(t+1/\xi)} \in SL^2(\Omega, G_{(\overline{t+1/\xi})}, P)$ and $\overline{t+1/\xi} \approx t$.

We denote $(\overline{t+1/\xi})$ by $\hat{t}$ and choose $\hat{0} = 0$; this is possible since we have assumed that N_0 is $\sigma(L(G_0)\cup N)$-measurable.

Given a <u>finite</u> set $\hat{S} = \{\hat{t}_1,\ldots,\hat{t}_n\}$ of such elements we show how to turn the process $L_{t_i}^{(\hat{t}_i)}$ into an SL^2-martingale with respect to the basis $\langle\Omega,\{G_t\}_{t\in\hat{S}},P\rangle$; i.e. we construct an SL^2-martingale $\hat{S}M:\hat{S}\times\Omega \to \mathbb{R}$ adapted to $\langle\Omega,\{G_t\}_{t\in\hat{S}},P\rangle$ such that $\|\hat{S}M(\hat{t}_i) - L_{t_i}^{(\hat{t}_i)}\|_2 \approx 0$ for each $\hat{t}_i \in \hat{S}$:

Assume that $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_n$; we define $\hat{S}M(\hat{t}_1) = L_{t_1}^{(\hat{t}_1)}$. If we have constructed $\hat{S}M(\hat{t}_j)$ for $j \leq i$, we define $\hat{S}M(\hat{t}_{i+1})$ in the following way: For each $\omega \in \Omega$, let

$$[\omega]_i = \cap\{A \in G_{\hat{t}_i} : \omega \in A\} ,$$

and define B by:

$$B = \{\gamma \in {}^*\mathbb{N} : P\{\omega : |\int_{[\omega]_i} (L_{t_{i+1}}^{(\hat{t}_{i+1})} - \hat{S}M(\hat{t}_i))dP| > 1/\gamma P([\omega]_i)\} < 1/\gamma\} .$$

Then B is an internal set and we prove that $\mathbb{N} \subset B$:
Assume $n \in \mathbb{N}$, but $n \notin B$: Then either

$$P\{\omega : \int_{[\omega]_i} (L_{t_{i+1}}^{(\hat{t}_{i+1})} - \hat{S}M(\hat{t}_i))dP > 1/nP([\omega]_i)\} \geq 1/2n$$

or

$$P\{\omega : \int_{[\omega]_i} (L_{t_{i+1}}^{(\hat{t}_{i+1})} - \hat{S}M(\hat{t}_i))dP < -1/nP([\omega]_i)\} \geq 1/2n .$$

Assume the first; the argument in the second case is similar. The set $C = \{\omega: \int_{[\omega]_i} (L_{t_{i+1}}^{(\hat{t}_{i+1})} - {}^{\hat{S}}M(\hat{t}_i))dP > 1/nP[\omega]_i\}$ is in $G_{\hat{t}_i}$ and $P(C) \geq 1/2n$. We have

$$1/2n^2 \leq {}^{o}\int_C (L_{t_{i+1}}^{(\hat{t}_{i+1})} - {}^{\hat{S}}M(\hat{t}_i))dP = \int_C ({}^{o}L_{t_{i+1}}^{(\hat{t}_{i+1})} - {}^{o\hat{S}}M(\hat{t}_i))dL(P)$$

$= \int_C (N_{t_{i+1}} - N_{t_i})dL(P) = 0$, which is impossible. Hence $\mathbb{N} \subset B$, and since B is internal we may find an infinite $\gamma \in B$.

Let D be the set of internal measure less than $1/\gamma$ such that $|\int_{[\omega]_i} (L_{t_{i+1}}^{(\hat{t}_{i+1})} - {}^{\hat{S}}M(\hat{t}_i))dP| \geq 1/\gamma P[\omega]_i$ for $\omega \in D$; then $D \in G_{\hat{t}_i}$.

For $\omega \in D$, we define ${}^{\hat{S}}M(\hat{t}_{i+1}) = {}^{\hat{S}}M(\hat{t}_i)$. For $\omega \in \Omega \setminus D$, let

$${}^{\hat{S}}M(\hat{t}_{i+1},\omega) = L_{t_{i+1}}^{(\hat{t}_{i+1})}(\omega) - \int_{[\omega]_i} (L_{t_{i+1}}^{(\hat{t}_{i+1})} - {}^{\hat{S}}M(\hat{t}_i))dP/P[\omega_i].$$

It follows immediately that for each $\omega \in \Omega$, $\int_{[\omega]_i} ({}^{\hat{S}}M(\hat{t}_{i+1}) - {}^{\hat{S}}M(\hat{t}_i))dP = 0$, and hence ${}^{\hat{S}}M$ is a martingale. Since ${}^{\hat{S}}M(\hat{t}_i) \in SL^2(\Omega, G_{\hat{t}_i}, P)$ by induction hypothesis, we have $\int_D {}^{\hat{S}}M(\hat{t}_{i+1})^2 dP = \int_D {}^{\hat{S}}M(\hat{t}_i)^2 dP \approx 0$. We also have $\int_{\Omega \setminus D} |{}^{\hat{S}}M(\hat{t}_{i+1}) - L_{t_{i+1}}^{(\hat{t}_{i+1})}|^2 dP \leq 1/\gamma^2$, and thus ${}^{\hat{S}}M(\hat{t}_{i+1}) \in SL^2(\Omega, G_{\hat{t}_{i+1}}, P)$.

Since obviously ${}^{o\hat{S}}M(\hat{t}_{i+1}) = {}^{o}L_{t_{i+1}}^{(\hat{t}_{i+1})} = N_{t_{i+1}}$ a.e., we have constructed the martingale ${}^{\hat{S}}M$ by induction.

We now turn to the next step of the proof: So far we have only constructed approximating martingales on finite time-lines, we must extend this to hyperfinite time-lines: Let $\{q_n\}_{n \in \mathbb{N}}$ be an enumeration of the non-negative rationals, and let $\hat{S}_n = \{\hat{q}_1, \ldots, \hat{q}_n\}$. We may extend the sequence $\{\hat{S}_n\}_{n \in \mathbb{N}}$ to an increasing internal sequence $\{\hat{S}_n\}_{n \leq \nu}$ of hyperfinite subsets of S, and the sequence $\{{}^{\hat{S}_n}M\}_{n \in \mathbb{N}}$ to a sequence of internal martingales adapted to $\langle \Omega, \{G_t\}_{t \in \hat{S}_n}, P \rangle$.

If $m,n \in \mathbb{N}$, $m < n$, then for all $\hat{q}_k$, $k \le m$, we have:

$$\| {}^{\hat{S}}\eta_M(\hat{q}_k) - {}^{\hat{S}}m_M(\hat{q}_k) \|_2 < 1/m .$$

By saturation there exists an infinite η such that

$$\| {}^{\hat{S}}\eta_M(\hat{q}_k) - {}^{\hat{S}}n_M(\hat{q}_k) \|_2 < 1/\eta$$

for all $n \in \mathbb{N}$, $k \le n$. It follows that ${}^{\hat{S}}\eta_M(\hat{q}_k) \in SL^2(\Omega, G_{\hat{q}_k}, P)$ and ${}^{o\hat{S}}\eta_M(\hat{q}_k) = N(q_k)$, $L(P)$-a.e., for $q_k \in \mathbb{Q}$.

Using this and the assumption that N is right continuous we have ${}^{o}({}^{\hat{S}}\eta_M)^+ = N$ a.e.. ${}^{\hat{S}}\eta_M$ is right continuous at 0, since $\hat{0} = 0$.

By letting T be a suitable sub-line of $\hat{S}_\eta$, we can make the restriction of ${}^{\hat{S}}\eta_M$ to T well-behaved by Theorem II-23. Let M be this restriction, then by Theorem II-21 ${}^{o}[M]^+ = [N]$. This proves the theorem.

The assumption that N_o is $\sigma(L(G_o) \cup N)$-measurable was made in Theorem 10 to ensure us that we could choose M right-continuous at 0. We saw in Theorem II-17 that this was a necessary condition when we compared stochastic integration with respect to M with stochastic integration with respect to ${}^{o}M^+$.

Using Theorem 10 one may prove the following analogous statement for local L^2-martingales:

<u>Theorem 11</u>: Let $<\Omega, \{G_t\}_{t \in \mathbb{R}_+}, P>$ be an increasing sequence of hyperfinite probability spaces. Let $N: \mathbb{R}_+ \times \Omega$ be a right continuous local L^2-martingale with respect to $<\Omega, \{H_t\}, L(P)>$, such that N_o is $\sigma(L(G_o) \cup N)$-measurable. Then there exists a local SL^2-martingale $M: T \times \Omega \to {}^*\mathbb{R}$ adapted to an extension $<\Omega, \{G_t\}_{t \in T}, P>$ of $<\Omega, \{G_t\}_{t \in \mathbb{R}_+}, P>$ such that for all $t \in \mathbb{R}_+$:

$$^{o}M^{+}(t,\omega) = N(t,\omega) \quad \text{for} \quad L(P)\text{-almost all} \quad \omega .$$

Moreover, we may assume M to be right continuous at 0 and such that for all $t \in \mathbb{R}_{+}$:

$$^{o}[M]^{+}(t,\omega) = [N](t,\omega) .$$

I don't know if we can make the martingale M in Theorem 11 well-behaved; application of Theorem II-23 as in the proof of Theorem 10 is not possible since the restriction may fail to be a local SL^2-martingale (see Example II-10 and the comments following the proof of Theorem II-23). The equality $^{o}[M]^{+} = [N]$ can be proved directly without using Theorem II-21.

4. Hyperfinite representation of martingales

Time has come to gather our results. In Section 2 we found a representation of arbitrary L^2-martingales as L^2-martingales on Loeb-spaces, and in the last section we saw how we could represent martingales of the latter type as right standard parts of SL^2-martingales. From Theorem II-17 we know the relationship between stochastic integration with respect to an SL^2-martingale and stochastic integration with respect to its right standard part. The following definition seems natural:

Definition 12: By a weak hyperfinite representation of a martingale N we mean a weak Loeb-space representation $\{<\Omega,\{G_t\},P>,\{F'_t\},\theta,N^{\theta}\}$ and a hyperfinite martingale M adapted to some extension $<\Omega,\{G_t\}_{t\in T},P>$ of $<\Omega,\{G_t\}_{t\in\mathbb{R}_+},P>$ such that $^{o}M^{+}$ is equivalent to N^{θ} .

Combining the results mentioned above, we get:

<u>Theorem 13</u>: Let N be a right continuous L^2-martingale. Then N has a weak hyperfinite representation M which is a well-behaved SL^2-martingale S-right continuous at 0, such that ${}^\circ[M]^+ = [N]^\Theta$. If $X \in \Lambda^2(N)$ then there exists a $Y \in SL^2(M)$ such that $\int Y dM$ is a hyperfinite representation of $\int X dN$.

Proof: By Theorem 7, N has a weak Loeb-space representation $\{\langle\Omega,\{G_t\},P\rangle, \{F'_t\}, \Theta, N^\Theta\}$. By Lemma 6, N^Θ satisfies the conditions of Theorem 10, and by that theorem we obtain a weak hyperfinite representation M of N, which is well-behaved, S-right continuous at 0, and such that ${}^\circ[M]^+ = [N^\Theta] = [N]^\Theta$.

If $X \in \Lambda^2(N)$, then $X^\Theta \in \Lambda^2(N^\Theta)$ and

$$(\int X dN)^\Theta = \int X^\Theta dN^\Theta$$

by Proposition 8. By Lemma II-15 and Theorem II-17, X^Θ has a 2-lifting Y with respect to M such that

$${}^\circ(\int Y dM)^+ = \int X^\Theta d{}^\circ M^+ = \int X^\Theta dN^\Theta = (\int X dN)^\Theta .$$

Thus $\int Y dM$ is a hyperfinite representation of $\int X dN$, and the theorem is proved.

Using Corollary II-18 and Theorem 11, one may prove the corresponding result for local L^2-martingales:

<u>Theorem 14</u>: Let N be a right continuous local L^2-martingale and let $X \in \Lambda(N)$. Then there exists a weak hyperfinite representation M of N which is a local SL^2-martingale, S-right continuous at 0, and with ${}^\circ[M]^+ = [N]^\Theta$. Moreover, there is a $Y \in SL(M)$ such that $\int Y dM$ is a hyperfinite representation of $\int X dN$.

Our purpose in this paper has been to show that everything that can be done by the standard theory of stochastic integration can also be done by the nonstandard theory. Theorems 13 and 14 seem to indicate this, since by using them we can define stochastic integration with respect to any right continuous local L^2-martingale from the nonstandard stochastic integral with respect to a certain local SL^2-martingale. The construction is complicated and cumbersome and it is not my intention that it should be used to construct nonstandard martingales from standard ones in a concrete situation; indeed, one of the nicest and most interesting aspects of the nonstandard approach to stochastic analysis is the simple and natural constructions it gives for certain classes of processes; see e.g. Loeb's construction of Poisson processes in [9], and Anderson's of a Brownian motion in [1] (see Examples I-1, I-15 and II-24). Also consult Keisler's lifting results in [5]. My intention has rather been once and for all to show that by using nonstandard methods we do not condemn our theory to less generality than what we could obtain by sticking to the standard methods.

5. The transformationformula revisited

As an application of the theory for weak hyperfinite representations, we shall use the nonstandard version of the transformationformula (Theorem I-22) to prove the standard version. Let us first prove it for the right standard part ${}^{o}M^{+}$ of a well-behaved SL^2-martingale M which is S-right continuous in 0 and such that ${}^{o}[M]^{+} = [{}^{o}M^{+}]$, and then use Theorem 13 to prove it in general: If $\varphi: \mathbb{R} \to \mathbb{R}$ is in $C^2(\mathbb{R})$, we get from Theorem I-22:

$$\varphi({}^oM^+(t)) - \varphi({}^oM^+(0)) = \lim_{n\to\infty}{}^o(*\varphi(M(t+1/n)) - *\varphi(M_0))$$

$$= \lim_{n\to\infty}{}^o\Big(\int_0^{t+1/n} *\varphi'(M)dM + \tfrac{1}{2}\int_0^{t+1/n} *\varphi''(M)d[M]$$

$$+ \sum_{s\le t+1/n} {}^o(*\varphi(M_s) - *\varphi(M_{s^-}) - *\varphi'(M_{s^-})(M_s-M_{s^-}) - \tfrac{1}{2}*\varphi''(M_{s^-})(M_s-M_{s^-})^2)\Big)$$

By Lemma II-20, $*\varphi'(M)$ is a lifting of $\varphi'({}^oM^-)$ (where ${}^oM^-$ is the left standard part of M), and consequently $\lim_{n\to\infty}{}^o\int_0^{t+1/n} *\varphi'(M)dM = \int_0^t \varphi'({}^oM^-)d{}^oM^+$. Using that M is well-behaved, S-right continuous at 0 and such that ${}^o[M]^+ = [{}^oM^+]$, it is not difficult to see that $\lim_{n\to\infty}{}^o\int_0^{t+1/n} *\varphi''(M)d[M] = \int_0^t \varphi''({}^oM^-)d{}^o[M]^+$. Since M is well-behaved, the jumps of ${}^oM^+$ are exactly the noninfinitesimal jumps of M, and hence

$$\lim_{n\to\infty} \sum_{s\le t+1/n} {}^o(*\varphi(M_s) - *\varphi(M_{s^-}) - *\varphi'(M_{s^-})(M_s-M_{s^-}) - \tfrac{1}{2}*\varphi''(M_{s^-})(M_s-M_{s^-})^2)$$

$$= \sum_{s\le t} (\varphi({}^oM_s^+) - \varphi({}^oM_s^-) - \varphi'({}^oM_s^-)({}^oM_s^+-{}^oM_s^-) - \tfrac{1}{2}\varphi''({}^oM_s^-)({}^oM_s^+-{}^oM_s^-)^2) .$$

Combining these results we have:

$$\varphi({}^oM^+(t)) - \varphi({}^oM^+(0)) = \int_0^t \varphi'({}^oM^-)d{}^oM^+ + \tfrac{1}{2}\int_0^t \varphi''({}^oM^-)d[{}^oM^+]$$

$$+ \sum_{s\le t} (\varphi({}^oM_s^+) - \varphi({}^oM_s^-) - \varphi'({}^oM_s^-)({}^oM_s^+ - {}^oM_s^-) - \tfrac{1}{2}\varphi''({}^oM_s^-)({}^oM_s^+-{}^oM_s^-)^2)$$

which is the transformationformula for ${}^oM^+$.

Let now N be a right-continuous L^2-martingale with left limits. To prove the transformationformula for N , we find a weak hyperfinite representation M of N as in Theorem 13. If N^- denotes the left limit of N , N^- is predictable, and we know from Theorem 13 that $(\int_0^t \varphi'({}^oM^-)d{}^oM^+)_\Theta = \int_0^t \varphi'(N^-)dN$.

It is easy to see that $(\int_0^t \varphi''({}^oM^-)d[{}^oM^+])_\Theta = \int_0^t \varphi''(N^-)d[N]$ by approximating $\varphi''({}^oM^-)$ by the predictable processes $X_n = \sum_{k\ge 0} 1_{\langle k/2^n,k+1/2^n]}\varphi''({}^oM^-_{k/2^n})$, and $\varphi''(N^-)$ by the processes

$Y_n = \sum_{k\geq 0} 1_{<k/2^n,k+1/2^n]}\varphi''(N^-_{k/2^n})$: Since $\varphi''({}^oM^-)$ is left-continuous, $X_n \to \varphi''({}^oM^-)$ everywhere, and similarly $Y_n \to \varphi''(N^-)$. Now $(\int_0^t \varphi''({}^oM^-)d[{}^oM^+])_\Theta = (\lim_{n\to\infty} \int_0^t X_n d[{}^oM^+])_\Theta = \lim_{n\to\infty}(\int_0^t X_n d[{}^oM^+])_\Theta$
$= \lim_{n\to\infty}(\int_0^t Y_n d[N]) = \int_0^t \varphi''(N^-)d[N]$.

It remains to show that

$$(\sum_{s\leq t} (\varphi({}^oM_s^+) - \varphi({}^oM_s^-) - \varphi'({}^oM_s^-)({}^oM_s^+ - {}^oM_s^-) - \tfrac{1}{2}\varphi''({}^oM_s^-)({}^oM_s^+ - {}^oM_s^-)^2))_\Theta$$
$$= \sum_{s\leq t} (\varphi(N_s) - \varphi(N_s^-) - \varphi'(N_s^-)(N_s - N_s^-) - \tfrac{1}{2}\varphi''(N_s^-)(N_s - N_s^-)^2) .$$

To do this we need two lemmas:

Lemma 15: Let N be a right continuous L^2-martingale with left limits adapted to a basis $<Z,\{F_t\},\mu>$. Then there exist a sequence $\{f_n\}_{n\in\mathbb{N}}$ of F_∞-measurable functions and a set $Z' \subset Z$ of measure one such that if $z \in Z'$ and $N(z,t) \neq N^-(z,t)$, then $t = f_n(z)$ for some $n \in \mathbb{N}$. Moreover, if $z \in Z'$ and $n \neq m$, then $f_n(z) \neq f_m(z)$.

Proof: Let $\tilde{F}_\infty$ be the completion of F_∞ with respect to μ, and let N be the null-sets of $\tilde{F}_\infty$. Define $\tilde{F}_t = \bigcap_{s>t} \sigma(F_s \cup N)$. Then $<Z,\{F_t\},\mu>$ satisfies Meyer's "usual conditions", and since N is right continuous, N is a martingale with respect to this basis. By Satz 17.5 of Metivier [10], there is a disjoint sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of stopping-times enumerating the jumps of N, each τ_n adapted to $<Z,\{\tilde{F}_t\},\mu>$. Let $f_n = E(\tau_n | F_\infty)$. Then $f_n = \tau_n$ a.e., and the lemma follows.

Lemma 16: Let N be a right continuous L^2-martingale with left limits, and let $\varphi: \mathbb{R} \to \mathbb{R}$ be in $C^2(\mathbb{R})$. Let $\tilde{N}$ be a weak Loeb--space representation of N which also is right-continuous with left limits. Then for all $r \in \mathbb{R}_+$

$$\sum_{s\le t} (\varphi(N_s) - \varphi(N_s^-) - \varphi'(N_s^-)(N_s-N_s^-) - \tfrac{1}{2}\varphi''(N_s^-)(N_s-N_s^-)^2)$$
$$= (\sum_{s\le t} (\varphi(\tilde{N}_s) - \varphi(\tilde{N}_s^-) - \varphi'(\tilde{N}_s^-)(\tilde{N}_s-\tilde{N}_s^-) - \tfrac{1}{2}\varphi''(\tilde{N}_s^-)(\tilde{N}_s-\tilde{N}_s^-)^2))_\Theta .$$

Proof: Applying Lemma 15 to $\tilde{N}$, we get a sequence $\{f_n\}_{n\in\mathbb{N}}$ of F'_∞-measurable functions enumerating the jumps of $\tilde{N}$, and hence the second sum above equals:

$$\sum_{\substack{n\in\mathbb{N}\\ f_n(\omega)\le t}} (\varphi(\tilde{N}_{f_n}) - \varphi(\tilde{N}_{f_n}^-) - \varphi'(\tilde{N}_{f_n}^-)(\tilde{N}_{f_n}-\tilde{N}_{f_n}^-) - \tfrac{1}{2}\varphi''(\tilde{N}_{f_n}^-)(\tilde{N}_{f_n}-\tilde{N}_{f_n}^-)^2)$$

Approximating f_n from above and below with simple functions and using the right continuity of $\tilde{N}$ and N, we see that:

$$[\omega \to \tilde{N}_{f_n(\omega)}(\omega)]_\Theta = [\omega \to N_{(f_n)_\Theta(\omega)}(\omega)] \quad \text{a.e., and}$$
$$[\omega \to \tilde{N}^-_{f_n(\omega)}(\omega)]_\Theta = [\omega \to N^-_{(f_n)_\Theta(\omega)}(\omega)] \quad \text{a.e..}$$

Thus:

$$(\sum_{\substack{n\in\mathbb{N}\\ f_n(\omega)\le t}} (\varphi(\tilde{N}_{f_n}) - \varphi(\tilde{N}_{f_n}^-) - \varphi'(\tilde{N}_{f_n}^-)(\tilde{N}_{f_n}-\tilde{N}_{f_n}^-) - \tfrac{1}{2}\varphi''(\tilde{N}_{f_n}^-)(\tilde{N}_{f_n}-\tilde{N}_{f_n}^-)^2))_\Theta$$
$$= \sum_{\substack{n\in\mathbb{N}\\ f_n(\omega)\le t}} (\varphi(N_{(f_n)_\Theta}) - \varphi(N^-_{(f_n)_\Theta}) - \varphi'(N^-_{(f_n)_\Theta})(N_{(f_n)_\Theta} - N^-_{(f_n)_\Theta}) - \tfrac{1}{2}\varphi''(N^-_{(f_n)_\Theta})(N_{(f_n)_\Theta} - N^-_{(f_n)_\Theta})^2)$$

To prove the lemma it only remains to prove that the sequence $\{(f_n)_\Theta\}$ enumerates all the jumps of N; by Lemma 15 there exists a sequence $\{g_n\}$ of F_∞-measurable functions which does this. Let $A_m = \Omega \setminus \bigcup_n [g_m=(f_n)_\Theta]$. By approximating the restriction of g_m to A_m from above and below by simple functions, we get $(N_{g_m}-N^-_{g_m})^\Theta = (\tilde{N}_{g_m^\Theta}-\tilde{N}^-_{g_m^\Theta})$ on A_m^Θ. But $\tilde{N}_{g_m^\Theta} = \tilde{N}^-_{g_m^\Theta}$ on A_m^Θ since $\{f_n\}$ enumerates the jumps of N. Consequently $N_{g_m} = N^-_{g_m}$ a.e. on A_m, and $\{(f_n)_\Theta\}$ enumerates the jumps of N. As we have already observed, this proves the lemma.

Theorem 17: (The transformationformula): Let N be a right continuous local L^2-martingale with left limits, and let $\varphi:\mathbb{R}\to\mathbb{R}$ be in $C^2(\mathbb{R})$. Then for all $t\in\mathbb{R}_+$:

$$\varphi(N_t)-\varphi(N_0) = \int_0^t \varphi'(N^-)dN + \tfrac{1}{2}\int_0^t \varphi''(N^-)d[N]$$

$$+\sum_{s\le t}(\varphi(N_s)-\varphi(N_s^-)-\varphi'(N_s^-)(N_s-N_s^-)-\tfrac{1}{2}\varphi''(N_s^-)(N_s-N_s^-)^2) \quad \text{a.e.}$$

where the sum is absolutely convergent.

Proof: It is clearly enough to prove the theorem for L^2-martingales. Let M be a weak hyperfinite representation of N as in Theorem 13. By the transformationformula for ${}^oM^+$ and the observations $(\int_0^t\varphi'({}^oM^-)d{}^oM^+)_\Theta = \int_0^t\varphi'(N^-)dN$ and $(\int_0^t\varphi''({}^oM^-)d[{}^oM^+])_\Theta = \int_0^t\varphi''(N^-)d[N]$:

$$\varphi(N_t)-\varphi(N_0) = (\varphi({}^oM_t^+)-\varphi({}^oM_0^+))_\Theta = (\int_0^t\varphi'({}^oM^-)d{}^oM^+)_\Theta + \tfrac{1}{2}(\int_0^t\varphi''({}^oM^-)d[{}^oM^+])_\Theta +$$

$$\sum_{s\le t}(\varphi({}^oM_s^+)-\varphi({}^oM_s^-)-\varphi'({}^oM_s^-)({}^oM_s^+-{}^oM_s^-)-\tfrac{1}{2}\varphi''({}^oM_s^-)({}^oM_s^+-{}^oM_s^-)^2))_\Theta =$$

$$\int_0^t\varphi'(N^-)dN+\tfrac{1}{2}\int_0^t\varphi''(N^-)d[N]+\sum_{s\le t}(\varphi(N_s)-\varphi(N_s^-)-\varphi'(N_s^-)(N_s-N_s^-)-\tfrac{1}{2}\varphi''(N_s^-)(N_s-N_s^-)^2),$$

where we have used the following argument for the equality of the sums: Let $\hat{N}$ be the weak Loeb-space representation corresponding to N. By Satz 20.7 of Métivier, $\hat{N}$ has an equivalent L^2-martingale $\tilde{N}$ adapted to $\langle\Omega,\{\mathcal{F}_t^{!+}\},L(P)\rangle$ which is right-continuous and have left limits. There must be a set $\Omega'\subset\Omega$ of Loeb-measure one where $\tilde{N} = {}^oM^+$ for all t. Applying Lemma 16 to $\tilde{N}$, we get the equality above.

That the sum is absolutely convergent follows from the fact that the sum in Theorem I-22 is absolutely convergent.

Corollary 18: (Itô's formula): Let N be a continuous local L^2-martingale, and let $\varphi:\mathbb{R}\to\mathbb{R}$ be in $C^2(\mathbb{R})$. Then:

$$\varphi(N_t) - \varphi(N_o) = \int_0^t \varphi'(N)dN + \tfrac{1}{2}\int_0^t \varphi''(N)d[N] .$$

We may now prove the result we used in the proof of Theorem II-21:

Corollary 19: Let N be a right continuous local L^2-martingale with left limits. Then for $t \in \mathbb{R}_+$:

$$[N](t) = N(t)^2 - N(0)^2 - 2\int_0^t N^- dN \quad \text{a.e..}$$

Proof: Apply Theorem 17 with $\varphi : x \to x^2$:

$$N_t^2 - N_o^2 = \int_0^t 2N^- dN + \tfrac{1}{2}\int_0^t 2d[N] + \sum_{s\leq t} (N_s^2 - N_s^{-2} - 2N_s^-(N_s - N_s^-) - \tfrac{1}{2}\cdot 2(N_s - N_s^-)^2)$$

Performing the multiplications in each term inside the sum we get zero, and hence we are left with $N_t^2 - N_o^2 = 2\int_0^t N^- dN + [N](t)$, which proves the corollary.

Notice that the proof we have given of Corollary 19 is circular; we used Corollary 19 to prove Theorem II-21 which we used to prove Theorem 10 which we used to prove Theorem 13 which we used to prove Theorem 17 and hence Corollary 19. This also makes the proof of the transformationformula circular, since Métivier used this formula to prove Corollary 19. Luckily, it is not difficult to avoid this circularity since we can prove Theorem 10 without using Theorem II-21: The argument would proceed as in the proof we have given up to the use of Theorem II-21; we thus have a well-behaved SL^2-martingale M which is S-continuous at 0 , such that ${}^oM^+ = N$. Using the difinition of $[N]$ as the L^1-limit of $\sum_{t_i \in \pi} (N(t_{i+1}) - N(t_i))^2$ as the partition π gets finer, it is easy to find a restriction $\tilde{M}$ of M to a subline such that ${}^o[\tilde{M}]^+ = [N]$. This martingale $\tilde{M}$ would then satisfy Theorem 10.

As we have now proved Corollary 19 (and thereby Theorem II-21), the theory developed in these three papers should be reasonably self--contained. In addition to some simple facts about the paths of real--valued martingales (i.e. Satz 17.5 and Satz 20.7 of Métivier [10]), we have only used the most basic results from the standard theory for stochastic integration. We should therefore be able to develop by purely nonstandard methods a theory for stochastic integration just as powerful as the standard theory. One may also hope that the discreteness of the hyperfinite time-line and the simplicity of the nonstandard definition of the stochastic integral will make the hyperfinite theory easier and more intuitive to work with; this seems indeed to be the case in Keisler's theory for stochastic differential equations.

Statements left unproved in the present paper were proved in [6].

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