DIAMETERS OF STATE SPACES OF TYPE III FACTORS

by

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1. Introduction. Let M be a von Neumann algebra and $S_0(M)$ the norm closed set of its normal states. For each $\omega \in S_0(M)$ let $[\omega]$ be the norm closure of its orbit under the action of the inner $*$-automorphisms, $\mathrm{Int}(M)$, by $\omega \to u\omega u^* = \omega \circ \mathrm{Ad}u$. The orbit space $S_0(M)/\mathrm{Int}(M)$ is a metric space with metric

$$d([\omega],[\phi]) = \inf\{\|\omega'-\phi'\|, \omega' \in [\omega], \phi' \in [\phi]\}.$$

If M is not a factor the diameter of $S_0(M)/\mathrm{Int}(M)$ is clearly equal to 2. However, if M is a factor it may be different.

Powers proved in [8] that if M is a factor of type I_n, $n<\infty$, and $\phi = \mathrm{Tr}(h\cdot)$, $\psi = \mathrm{Tr}(k\cdot)$ are states then

$$d([\phi],[\psi]) = \sum_{i=1}^{n} |\lambda_i - \mu_i|,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of h, and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ are the eigenvalues of k. From this one easily gets that

$$\mathrm{diam}(S_0(M)/\mathrm{Int}(M)) = 2(1-\frac{1}{n}).$$

The value $2(1-\frac{1}{n})$ is attained when ϕ is the tracial state and ψ is a pure state.

The arguments of Powers can be extended to the case when M is a semifinite factor with faithful normal semifinite trace τ. If $\phi = \tau(h\cdot)$, $\psi = \tau(k\cdot)$ are two positive normal functionals given by two positive operators h and k in M, which have "joint diagonalization"

$$h = \sum_{i=1}^{n} \lambda_i p_i, \quad k = \sum_{i=1}^{n} \mu_i p_i,$$

where $p_1,\ldots,p_n$ are orthogonal projections with sum 1 and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, then

$$d([\phi],[\psi]) = \sum_{i=1}^{n} |\lambda_i - \mu_i| \tau(p_i) = \|\phi - \psi\| .$$

From this one derives easily that if ϕ,ψ are two states of the form

$$\phi(x) = \frac{1}{\tau(p)} \tau(px), \quad \psi(x) = \frac{1}{\tau(q)} \tau(qx),$$

where p and q are two nonzero finite projections in M, and $p \leq q$, then

$$d([\phi],[\psi]) = 2(1 - \frac{\tau(p)}{\tau(q)}).$$

Hence for a factor of types I_∞ or II we have

$$\operatorname{diam}(S_0(M)/\operatorname{Int}(M)) = 2.$$

The main result of the present paper is a formula for the diameter when M is of type III. The result will be a characterization of factors of type III_λ, $\lambda \in [0,1]$, purely in terms of the geometry of the state space and independent of Tomita-Takesaki theory.

Theorem. Let M be a σ-finite factor of type III_λ, $\lambda \in [0,1]$.

Then

$$\mathrm{diam}(S_0(M)/\mathrm{Int}(M)) = 2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}.$$

In particular for a factor of type III_0 the diameter is 2 and for a factor of type III_1 it is 0. The last statement was previously proved by two of us in [6]. In the case when $0<\lambda<1$ it was shown by Bion-Nadal [2] that $2(1-\lambda^{\frac{1}{2}})$ is an upper bound for the diameter, a result which inspired the present work. Our proof will be divided into two parts, namely to show the inequalities $\mathrm{diam}(S_0(M)/\mathrm{Int}(M)) \gtrless 2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}$ for $\lambda\in[0,1)$.

2. Proof of the inequality $\leq$.

The number $2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}$ that gives the diameter appears as a consequence of the following function theoretic lemma.

Lemma 2.1. Let $0<a<b$ be real numbers, and let $K_{a,b}$ denote the convex set of nonnegative decreasing functions f on $[a,b]$ such that $\int_a^b f\,dt = 1$ and $af(a) = bf(b)$. Then we have

$$\sup_{f,g\in K_{a,b}} \int_a^b f\vee g\,dt = 2\,\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$$

Proof. In order to show the lemma it suffices to consider step functions in $K_{a,b}$. If $\alpha\in[0,1]$ and $f_1,f_2,\in K_{a,b}$ then we have

$$(\alpha f_1+(1-\alpha)f_2)\vee f\leq\alpha(f_1\vee f)+(1-\alpha)(f_2\vee f).$$

Hence it suffices to prove the lemma for extremal step functions in $K_{a,b}$. Let

$$f = \sum_{i=1}^{n-1} c_i \chi_{[a_i, a_{i+1})} + c_n \chi_{[a_n, a_{n+1}]} \in K_{a,b},$$

where $a = a_1 < a_2 < \ldots < a_{n+1} = b$, $c_1 > c_2 > \ldots > c_n = \frac{a}{b} c_1$. If $n \geq 3$ we can find $\varepsilon > 0$ and $\eta > 0$ such that $(1-\varepsilon)c_1 > (1+\eta)c_2$, $(1-\eta)c_2 \geq c_3$, $c_{n-1} > (1+\varepsilon)c_n$ and such that the two functions

$$f_\pm = (1\pm\varepsilon)c_1 \chi_{[a_1, a_2)} + (1\mp\eta)c_2 \chi_{[a_2, a_3)} + \sum_{i=3}^{n-1} c_i \chi_{[a_i, a_{i+1})} + (1\pm\varepsilon) c_n \chi_{[a_n, a_{n+1}]}$$

belong to $K_{a,b}$. Since $f = \frac{1}{2}(f_+ + f_-)$, f is not extremal in $K_{a,b}$. Therefore it suffices to show the lemma for step functions of the form

$$f_s = \frac{b}{s(b-a)} \chi_{[a,s)} + \frac{a}{s(b-a)} \chi_{[s,b]},$$

where $s \in (a,b]$. If $a < r < s \leq b$ we find

$$\int_a^b f_r \vee f_s \, dt = \frac{1}{b-a} \left(2b - b\frac{r}{s} - a\frac{s}{r}\right).$$

Since the maximum of this function of $\frac{s}{r}$ is obtained for $\frac{s}{r} = \left(\frac{b}{a}\right)^{\frac{1}{2}}$ the proof is complete.

Since for two functions f and g, $|f-g| = 2f\vee g - f - g$, we have:

<u>Corollary 2.2</u>. In the above notation, if $0<\lambda<1$ we have

$$\sup_{f,g \in K_{\lambda,1}} \int_\lambda^1 |f-g| \, dt = 2 \, \frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}.$$

<u>Lemma 2.3</u>. Let M be a σ-finite factor of type III_λ, $0<\lambda<1$, and let $T = -\frac{2\pi}{\log\lambda}$. Let ϕ_0 be a faithful normal state on M for

which $\sigma_T^{\phi_0}$ is the identity. Then for any faithful normal state ϕ on M there exists a positive operator h in the centralizer M_{ϕ_0} of ϕ_0 such that

(i) $\mathrm{Sp}\,h \subset [\lambda a, a]$ for some $a>0$,

(ii) There exists a unitary $u \in M$ such that $\phi(uxu^*) = \phi_0(hx)$, $x \in M$.

<u>Proof</u>: Put $v = (D\phi : D\phi_0)_T$, see [4]. Then for $x \in M$

$$\sigma_T^{\phi}(x) = v\sigma_T^{\phi_0}(x)v^* = vxv^*,$$

so in particular $\phi(vxv^*) = \phi(\sigma_T^{\phi}(x)) = \phi(x)$. Thus $v \in M_\phi$. By spectral theory and the Riesz representation theorem there is a unique probability measure μ on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ for which

$$\int_{\mathbb{T}} f(z)\,d\mu(z) = \phi(f(v))$$

for any Borel function f on $\mathbb{T}$. Let ν be the positive Borel measure on $\mathbb{R}$ obtained by "rewinding" μ, i.e. ν is determined by

$$\nu(B) = \mu(\exp(iB)), \quad B \subset [0, 2\pi), B \text{ Borel},$$

and

$$\nu(B+2\pi) = \nu(B), \quad B \subset \mathbb{R},\ B \text{ Borel}.$$

Note that $\nu([s, s+2\pi)) = 1$ for all $s \in \mathbb{R}$. Put

$$g(s) = \int_{[s,s+2\pi)} \exp(-\tfrac{t}{T})\,d\nu(t), \quad s \in \mathbb{R}.$$

Since $\exp(-\frac{2\pi}{T}) = \lambda$ we have

$$\int_{[s,\infty)} \exp(-\tfrac{t}{T})\,d\nu(t) = \sum_{n=0}^{\infty} \int_{[s+n2\pi,\, s+(n+1)2\pi)} \exp(-\tfrac{t}{T})\,d\nu(t)$$

$$= \left(\sum_{n=0}^{\infty} \lambda^n\right) g(s) = \frac{1}{1-\lambda}\, g(s).$$

Hence we also have

(1) $$g(s) = (1-\lambda) \int_{[s,\infty)} \exp(-\frac{t}{T}) d\nu(t).$$

This shows that g is a decreasing function on $\mathbb{R}$, continuous from left. Let $g(s+)$ (resp. $g(s-)$) denote the limits of $g(s')$ for $s' \to s$ from right (resp. left). Then

$$g(0+) = \int_{(0,2\pi]} \exp(-\frac{t}{T}) d\nu(t) < 1,$$

and

$$g((-2\pi)-) = \int_{[-2\pi,0)} \exp(-\frac{t}{T}) d\nu(t) > 1.$$

Hence we can choose $r \in [-2\pi, 0]$ such that

$$g(r+) \leq 1 \leq g(r-).$$

By (1) we have

$$g(r-)-g(r+) = (1-\lambda)\exp(-\frac{r}{T})\nu(\{r\})$$

$$= (1-\lambda)\exp(-\frac{r}{T})\mu(\{e^{ir}\}).$$

This shows that r is a point of continuity for g if and only if e^{ir} is not an eigenvalue for V.

Moreover

$$g(r-)-g(r+) = (1-\lambda)\exp(-\frac{r}{T})\phi(p),$$

where p is the projection on the eigenspace of the vectors ξ such that $v\xi = e^{ir}\xi$. There are two cases to be considered.

<u>Case 1</u>. Assume first that e^{ir} is not an eigenvalue for v. Let

$$\mathrm{Arg}_r : \mathbf{T} \setminus \{e^{ir}\} \to (r, r+2\pi)$$

be the branch of the argument functions that takes values in $(r, r+2\pi)$, and put

$$a = \mathrm{Arg}_r(v)$$

$$k = \exp(\tfrac{1}{T}a).$$

Since $v \in M_\phi$ so are a and k. Moreover, a and k are self-adjoint, and their spectra satisfy

$$\mathrm{Sp}a \subset [r, r+2\pi]$$

$$\mathrm{Sp}k \subset [\exp(\tfrac{r}{T}),\ \lambda^{-1}\exp(\tfrac{r}{T})].$$

Furthermore, since r is a continuity point for g,

$$\phi(k^{-1}) = \int_{\mathbf{T}} \exp(-\tfrac{1}{T}\,\mathrm{Arg}_r(z))d\mu(z)$$

$$= \int_r^{r+2\pi} \exp(-\tfrac{t}{T})d\nu(t)$$

$$= 1.$$

Put $\psi(x) = \phi(k^{-1}x)$, $x \in M$. Then ψ is a faithful normal state on M. Since $k^{iT} = \exp(ia) = v$, we get, see [4],

$$\sigma_T^\psi(x) = k^{-iT}\sigma_T^\phi(x)k^{iT} = v^*(vxv^*)v = x, \quad x \in M,$$

and

$$(D\psi : D\phi_0)_T = (D\psi : D\phi)_T(D\phi : D\phi_0)_T = k^{-iT}v = 1.$$

Since σ^ψ and σ^{ϕ_0} both have period T we can conclude as in the proof of [4, 4.3.2] that there exists a unitary $u \in M$ such that $\psi(uxu^*) = \phi_0(x)$ for $x \in M$. Hence, if $h = u^*ku$ we have

$$\phi(uxu^*) = \psi(kuxu^*) = \psi(uhxu^*) = \phi_0(hx).$$

Since $\mathrm{Sp}h = \mathrm{Sp}k \subset [\exp(\tfrac{r}{T}), \lambda^{-1}\exp(\tfrac{r}{T})]$, h and u satisfy the conditions in the lemma.

<u>Case 2</u>. Assume next that e^{ir} is an eigenvalue for v, and let

p be the projection on the corresponding eigenspace. Clearly $p \in M_\phi$. Since

$$g(r+) \leq 1 \leq g(r-)$$

we can choose $\alpha \in [0,1]$ such that

$$1 = (1-\alpha)g(r+) + \alpha g(r-).$$

Now $\sigma_T^\phi(x) = vxv^*$ for $x \in M$ and $pv = e^{ir}p$. Thus the restriction of σ_T^ϕ to the reduced algebra pMp is trivial. Since M is σ-finite of type III, $pMp \cong M$, so is also a factor of type III_λ. Thus, as in the proof of [4, 4.2.6] the centralizer of the restriction $\phi|pMp$ is a factor of type II_1. Therefore we can choose a projection $p' \leq p$, $p' \in M_\phi$, such that $\phi(p') = \alpha\phi(p)$. Define now self-adjoint operators a and k in M_ϕ by

$$a = \mathrm{Arg}_r(v(1-p)) + rp' + (r+2\pi)(p-p')$$

$$k = \exp(\tfrac{1}{T}a).$$

The operators are well defined since e^{ir} is not in the point spectrum of $v(1-p)$. Clearly $\mathrm{Sp}(a) \subset [r, r+2\pi]$; hence

$$\mathrm{Sp}(k) \subset [\exp(\tfrac{r}{T}), \lambda^{-1}\exp(\tfrac{r}{T})].$$

Moreover, $k^{iT} = e^{ia} = v(1-p) + e^{ir}p = v$. Computing we find the following formulas:

$$\phi(k^{-1}) = \int_{(r,r+2\pi)} \exp(-\tfrac{t}{T})d\nu(t) + \alpha\phi(p)\exp(-\tfrac{r}{T}) + (1-\alpha)\phi(p)\exp(-\tfrac{r+2\pi}{T}),$$

$$g(r+) = \int_{(r,r+2\pi]} \exp(-\tfrac{t}{T})d\nu(t) = \int_{(r,r+2\pi)} \exp(-\tfrac{t}{T})d\nu(t) + \phi(p)\exp(-\tfrac{r+2\pi}{T}),$$

$$g(r-) = \int_{[r,r+2\pi)} \exp(-\tfrac{t}{T})d\nu(t) = \int_{(r,r+2\pi)} \exp(-\tfrac{t}{T})d\nu(t) + \phi(p)\exp(-\tfrac{r}{T}).$$

Adding we obtain $\phi(k^{-1}) = (1-\alpha)g(r+)+\alpha g(r-) = 1$. The proof can now be completed as in Case 1.

Proof of the inequality diam $(S_0(M)/\mathrm{Int}(M)) \leq 2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}$.

It suffices to show the inequality for faithful states. Let ϕ and ψ be faithful normal states on the factor M of type III_λ, $0<\lambda<1$. Let ϕ_0 be a faithful normal state such that $\sigma_T^{\phi_0}$ is the identity map. By Lemma 2.3 there are $\phi' \in [\phi]$, $\psi' \in [\psi]$ such that $\phi'(x) = \phi_0(hx)$, $\psi'(x) = \phi_0(kx)$, $x \in M$, where $h,k \in M_{\phi_0}$ and $\lambda a \leq h \leq a$, $\lambda b \leq k \leq b$ for some $a,b>0$.

If $\delta>0$ we can by spectral theory find an integer n and orthogonal families $\{p_1,\ldots,p_n\}, \{q_1,\ldots,q_n\}$ of projections in M_{ϕ_0} with $\phi_0(p_i) = \phi_0(q_i) = \frac{1}{n}$, $i = 1,\ldots,n$, and constants $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n = \lambda\alpha_1$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n = \lambda\beta_1$ satisfying $\sum \alpha_i = \sum \beta_i = n$ such that

$$\|h - \sum_1^n \alpha_i p_i\|_1 < \delta, \quad \|k - \sum_1^n \beta_i q_i\|_1 < \delta,$$

where $\|x\|_1 = \phi_0(|x|)$ for $x \in M_{\phi_0}$. In order to show the desired estimate we may assume h and k are of this form, i.e. $h = \sum \alpha_i p_i$, $k = \sum \beta_i q_i$. Since M_{ϕ_0} is a factor of type II_1 there is a unitary $u \in M_{\phi_0}$ such that $uq_iu^* = p_i$ for all i, hence $uku^* = \sum_1^n \beta_i p_i$. Thus the state ψ'' defined by

$$\psi''(x) = \phi_0(uku^*x) = \phi_0(ku^*xu)$$

belongs to $[\psi]$.

Let f and g be functions on the interval $[\lambda,1]$ defined by $f = (1-\lambda)^{-1} \sum_{i=1}^n \alpha_i \chi_{I_i}$, $g = (1-\lambda)^{-1} \sum_{i=1}^n \beta_i \chi_{I_i}$, where

$$I_i = \begin{cases} [\lambda+(i-1)\frac{1-\lambda}{n},\ \lambda+i\frac{1-\lambda}{n}) & \text{for } i = 1,\dots,n-1, \\ [\lambda+(n-1)\frac{1-\lambda}{n},\ 1] & \text{for } i = n. \end{cases}$$

Then f and g are decreasing step functions with integrals 1 and satifying $f(1) = \lambda f(\lambda)$, $g(1) = \lambda g(\lambda)$, i.e. f,g belong to the set $K_{\lambda,1}$ of Lemma 2.1. Thus by Corollary 2.2 we have,

$$\|\phi'-\psi''\| = \|h-uku^*\|_1 = \sum_{i=1}^{n} |\alpha_i-\beta_i|\phi(p_i) = \int_\lambda^1 |f-g|\,dt \leq 2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}},$$

completing the proof. The case $\lambda = 0$ is trivial.

3. Proof of the inequality $\geq$.

The proof of the inequality

$$\operatorname{diam}(S_0(M)/\operatorname{Int}(M)) \geq 2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}$$

for a factor of type III_λ is based on the following theorem.

Theorem 3.1. Let M be a von Neumann algebra, let ϕ,ψ be two faithful normal positive functionals on M, and let $0<a<b$ be real numbers. Suppose

(i) ϕ and ψ commute and $a\phi \leq \psi \leq b\phi$,

(ii) $Sp(\Delta_\phi) \cap (\frac{a}{b},\frac{b}{a}) = \{1\}$,

where Δ_ϕ is the modular operator of ϕ.

Then $\|u\phi u^*-\psi\| \geq \|\phi-\psi\|$ for all unitary operators u in M.

The proof of the above theorem will be divided into three steps:

Step 1: M is finite,

Step 2: $T(M) = \{t : \sigma_t^\phi \in \mathrm{Int}(M)\}$ is dense in $\mathbb{R}$,

Step 3: The general case.

In order to prove Step 1 we assume M is finite and that ϕ, ψ, a, b satisfy the above conditions (i) and (ii). Since M has a faithful normal state it also has a faithful normal tracial state τ. There exist two positive operators h and k affiliated with M such that

$$\phi = \tau(h\cdot) \quad \text{and} \quad \psi = \tau(k\cdot).$$

By the usual identification of M_* and $L^1(M,\tau)$ the inequality stated in Theorem 3.1 is equivalent to

$$\|uhu^* - k\|_1 \geq \|h-k\|_1$$

for all unitary operators $u \in M$. To prove this we shall need

Lemma 3.2. Let M be a finite von Neumann algebra with a faithful normal tracial state τ and let $h, k \in M$ be two positive operators with bounded inverses such that

(i) h and k commute and $ah \leq k \leq bh$,

(ii) with $\phi = \tau(h\cdot)$, $\mathrm{Sp}(\Delta_\phi) \cap (\frac{a}{b}, \frac{b}{a}) = \{1\}$.

Then $\|uhu^* - k\|_1 \geq \|h-k\|_1$ for all unitary operators $u \in M$.

Proof. The modular automorphism group asociated with ϕ is, see [10],

$$\sigma_t^\phi(x) = h^{it} x h^{-it}, \quad x \in M.$$

Moreover M acts standardly on $L^2(M,\tau)$. Let $\mathrm{Sp}(\sigma^\phi)$ denote the Arveson spectrum of the one parameter group σ^ϕ. We shall consider $\mathrm{Sp}(\sigma^\phi)$ as a subset of the multiplicative group $\mathbb{R}_+$. Since

h is bounded and has bounded inverse, $0 \notin Sp(\Delta_\phi)$ and therefore

$$Sp(\sigma^\phi) = Sp(\Delta_\phi).$$

By [10] if J is the conjugation on $L^2(M,\tau)$ defined by σ^ϕ such that $JMJ = M'$, we have $\Delta_\phi = hJh^{-1}J$. We first assume M is a factor; then

$$Sp(\Delta_\phi) = Sp(h) \cdot Sp(h)^{-1}.$$

By condition (ii) we therefore get that if $\mu_1, \mu_2 \in Sp(h)$ and $\mu_1 > \mu_2$ then

$$\frac{\mu_2}{\mu_1} \leq \frac{a}{b}.$$

Since $Sp(h)$ is a compact subset of $(0,\infty)$ it follows that $Sp(h)$ is finite.

By (i) we have $k = mh$, where $m \in M$ commutes with h, and

$$a1 \leq m \leq b1 .$$

By continuity it is enough to prove the inequality $\|uhu^* - k\|_1 \geq \|h - k\|_1$ in the case when the spectrum of m is a finite subset of the interval $[a,b]$. In this case k also has finite spectrum, and h and k have a "joint diagonalization"

$$h = \sum_{i=1}^{n} \lambda_i p_i, \quad k = \sum_{i=1}^{n} \mu_i p_i,$$

where $p_1, \dots, p_n$ are nonzero orthogonal projections with sum 1. By permuting the indices $\{1,\dots,n\}$ we may assume that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

Let $i_1 < i_2 < \dots < i_q$ be the values of i for which $\lambda_i > \lambda_{i+1}$. By permuting the indices inside each of the $q+1$ sets on which the

λ_k's are constant we may also obtain that

$$\mu_1 \geq \ldots \geq \mu_{i_1}, \mu_{i_1+1} \geq \ldots \geq \mu_{i_2}, \ldots, \mu_{i_q+1} \geq \ldots \geq \mu_n .$$

However, since

$$\lambda_{i_k+1} \leq \frac{a}{b} \lambda_{i_k},$$

and since by (i)

$$a\mu_i \leq \lambda_i \leq b\mu_i ,$$

we also have

$$\mu_{i_1} > \mu_{i_1+1}, \mu_{i_2} > \mu_{i_2+1}, \ldots, \mu_{i_q+1} > \mu_{i_q} .$$

Hence by the extension of Powers' result mentioned in the introduction, we get

$$\| uhu^* - k \|_1 \geq \sum_{i=1}^{n} |\lambda_i - \mu_i| \tau(p_i) = \| h-k \|_1$$

for all unitary operators $u \in M$. This completes the proof in the case when M is a factor.

Let now M be general, and let $T: M \to Z$ be the center valued trace on M, where Z denotes the center of M. For every pure state ω on Z

$$\tau_\omega = \omega \circ T$$

is a (possibly nonnormal) tracial state on M. Put

$$I_\omega = \{ x \in M : \tau_\omega (x^* x) = 0 \} .$$

Then I_ω is a maximal ideal in M, and

$$M_\omega = M / I_\omega$$

is a finite factor, see [9, Ch. II]. The tracial state on M_ω will also be denoted by τ_ω. Let π_ω be the quotient map

$\pi_\omega : M \to M_\omega$, put

$$h_\omega = \pi_\omega(h), \quad k_\omega = \pi_\omega(k),$$

and put $\phi_\omega = \tau_\omega(h_\omega \cdot)$. By Arveson's definition of $Sp(\sigma^\phi)$, see [1], we have

$$\int_{-\infty}^{\infty} f(t) h^{it} x h^{-it} dt = 0 \quad \text{for every} \quad x \in M$$

if $f \in L^1(\mathbb{R})$ and $supp(\hat{f}) \cap Sp(\sigma^\phi) = \emptyset$, where the Fourier transform $\hat{f}$ of f is considered as a function on $(\mathbb{R}_+, \cdot)$. Since $t \to h^{it}$ is norm continuous it follows that under the same condition on f,

$$\int_{-\infty}^{\infty} f(t) h_\omega^{it} y h_\omega^{-it} dt = 0 \quad \text{for every} \quad y \in M_\omega .$$

Hence $Sp(\sigma^{\phi_\omega}) \subset Sp(\sigma^\phi)$. Therefore h_ω and k_ω satisfy the conditions of Lemma 3.2, so by the first part of the proof

$$\| v h_\omega v^* - k_\omega \|_1 \geq \| h_\omega - k_\omega \|_1$$

for every unitary $v \in M_\omega$. By the spectral theorem $Z \cong C(\hat{Z})$. Thus if ν is the probability measure on $\hat{Z}$ which corresponds to the restriction of τ to Z, we have for $x \in M$:

$$\tau(x) = \tau \circ T(x) = \int_{\hat{Z}} \tau_\omega \circ T(x) d\nu(\omega) = \int_{\hat{Z}} \tau_\omega(x) d\nu(\omega).$$

Hence for any unitary operator $u \in M$,

$$\begin{aligned} \| uhu^* - k \|_1 &= \int_{\hat{Z}} \| \pi_\omega(u) h_\omega \pi_\omega(u)^* - k_\omega \|_1 d\nu(\omega) \\ &\geq \int_{\hat{Z}} \| h_\omega - k_\omega \|_1 d\nu(\omega) \\ &= \| h - k \|_1 . \end{aligned}$$

This completes the proof of Lemma 3.2.

<u>Completion of Step 1</u>. To complete the proof of Theorem 3.1 in the case when M is finite we need to extend Lemma 3.2 to the case when h and k are (possibly unbounded) positive operators in $L^1(M,\tau)$ with trivial nullspaces.

Let p_n be the spectral projection of h corresponding to the interval $[\frac{1}{n},n]$, $n\in\mathbb{N}$. Then $h_n = p_n h$ and $k_n = p_n k$ satisfy the conditions of Lemma 3.2 with respect to the von Neumann algebra $p_n M p_n$. For every unitary $u\in M$ we can find a sequence of partial isometries $u_n \in M$ with support and range projections equal to p_n such that $u_n \to u$ in the strong-* topology (for instance write u in the form $u = \exp(ia)$ and put $u_n = p_n \exp(ip_n a p_n)$). Then

$$\|uhu^* - k\|_1 = \lim_{n\to\infty} \|u_n h_n u_n^* - k_n\|_1$$

$$\geq \lim_{n\to\infty} \|h_n - k_n\|_1 = \|h-k\|_1 .$$

This completes the proof of Step 1.

<u>Step 2</u>. For any faithful normal positive functional ϕ on a von Neumann algebra M we let $\|\cdot\|_\phi^\#$ be the norm

$$\|x\|_\phi^\# = \phi(\tfrac{1}{2}(x^* x + x x^*))^{\frac{1}{2}}.$$

Note that if ϕ is a state and u is unitary then $\|u\|_\phi^\# = 1$.

<u>Lemma 3.3</u>. Let M be a von Neumann algebra for which $T(M)$ is dense in $\mathbb{R}$. Let ϕ be a faithful normal state on M, and let u be a unitary operator in M. For every $\varepsilon > 0$ there exist a faithful normal state ω on M and a unitary operator $v\in M$ such that

(a) ϕ and ω commute,

(b) $M_\phi \subset M_\omega$,

(c) $v \in M_\omega$ and $\|u-v\|_\phi^\# < \varepsilon$.

Proof. Let $\delta>0$. Since the function $t \to \sigma_t^\phi(u)$ is strong-* continuous there is $t_1>0$ such that

$$\|\sigma_t^\phi(u)-u\|_\phi^\# < \delta \quad \text{for} \quad |t| \leq t_1 .$$

Since $T(M)$ is dense in $\mathbb{R}$ we can therefore choose $t_0>0$, $t_0 \in T(M)$ such that

$$\|\sigma_t^\phi(u)-u\|_\phi^\# < \delta \quad \text{for} \quad |t| \leq t_0 .$$

Let $w \in M$ be a unitary operator such that

$$\sigma_{t_0}^\phi(x) = wxw^*, \quad x \in M.$$

By [4, 1.3.2] w belongs to the center of M_ϕ. Hence

$$\|uw-wu\|_\phi^\# = \|u-wuw^*\|_\phi^\# < \delta .$$

Let Arg be the branch of the argument function on $\mathbb{C} \setminus \{0\}$ that takes values in the half-open interval $[0,2\pi)$. Then for $\theta \in \mathbb{R}$

$$\mathrm{Arg}_\theta(z) = \mathrm{Arg}(e^{-i\theta}z)+\theta$$

is the branch of the argument function that takes values in $[\theta, 2\pi+\theta)$. Put

$$a_\theta = \mathrm{Arg}_\theta(w), \quad \theta \in \mathbb{R}.$$

We shall show that θ can be chosen such that

$$\|ua_\theta - a_\theta u\|_\phi^\# < (2\pi\delta)^{\frac{1}{2}}.$$

Let H_ϕ denote the completion of M with respect to the norm $\| \ \|_\phi^\#$. Let

$$\langle x,y\rangle_\phi^\# = \tfrac{1}{2}\phi(y^* x + xy^*)$$

be the corresponding inner product on M. Define a unitary representation π of $\mathbf{Z}^2$ on H_ϕ by

$$\pi(n,m)x = w^n x w^m$$

(the representation is unitary since $w \in M_\phi$). By Bochner's theorem there exists a probability measure μ on $\mathbf{T}^2 = (\mathbf{Z}^2)^\wedge$ such that

$$\langle w^n u w^m, u \rangle_\phi^\# = \iint_{\mathbf{T}^2} \alpha^n \beta^m d\mu(\alpha,\beta).$$

Hence for any pair of bounded Borel functions f and g on $\mathbf{T}$

$$\langle f(w) u g(w), u \rangle_\phi^\# = \iint_{\mathbf{T}^2} f(\alpha) g(\beta) d\mu(\alpha,\beta).$$

From this equality we obtain that

(1) $$(\| f(w)u - uf(w) \|_\phi^\#)^2 = \iint_{\mathbf{T}^2} |f(\alpha) - g(\beta)|^2 d\mu(\alpha,\beta)$$

for every bounded Borel function f on $\mathbf{T}$ (compare with the proof of Proposition 1.1 in [5]). In particular

$$\iint_{\mathbf{T}^2} |\alpha - \beta|^2 d\mu(\alpha,\beta) = (\| wu - uw \|_\phi^\#)^2 < \delta^2.$$

Moreover,

(2) $$(\| a_\theta u - u a_\theta \|_\phi^\#)^2 = \iint_{\mathbf{T}^2} |\mathrm{Arg}(e^{-i\theta}\alpha) - \mathrm{Arg}(e^{-i\theta}\beta)|^2 d\mu(\alpha,\beta).$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} (\| a_\theta u - u a_\theta \|_\phi^\#)^2 d\theta = \iint_{\mathbf{T}^2} h(\alpha,\beta) d\mu(\alpha,\beta),$$

where

$$h(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Arg}(e^{-i\theta}\alpha) - \mathrm{Arg}(e^{-i\theta}\beta)|^2 d\theta.$$

For $\alpha = 1$ and $\beta = e^{i\sigma}$, $0 \leq \sigma < 2\pi$, we have

$$\mathrm{Arg}(e^{-i\theta}\alpha) = 2\pi-\theta, \qquad 0<\theta\leqslant 2\pi,$$

$$\mathrm{Arg}(e^{-i\theta}\beta) = \begin{cases} \sigma-\theta, & 0<\theta\leqslant\sigma \\ \sigma-\theta+2\pi, & \sigma<\theta\leqslant 2\pi \end{cases}.$$

Now the function

$$f(\sigma) = 4\pi \sin\frac{\sigma}{2} - \sigma(2\pi-\sigma)$$

is continuous on the interval $[0,2\pi]$ and $f(0) = f(2\pi) = 0$. Moreover, its derivative

$$f'(\sigma) = 2\pi(\cos\frac{\sigma}{2} - (1-\frac{\sigma}{\pi}))$$

is positive for $0<\sigma<\pi$ and negative for $\pi<\sigma<2\pi$, because $\cos\frac{\sigma}{2}$ is concave on $[0,\pi]$ and convex on $[\pi,2\pi]$. Hence

$$4\pi \sin\frac{\sigma}{2} - \sigma(2\pi-\sigma)>0 \quad \text{for} \quad 0<\sigma<2\pi.$$

We therefore find

$$\begin{aligned} h(1,e^{i\sigma}) &= \frac{1}{2\pi}\left(\int_0^\sigma (2\pi-\sigma)^2 d\theta + \int_\sigma^{2\pi} \sigma^2 d\theta\right) \\ &= \sigma(2\pi-\sigma) \\ &\leqslant 4\pi \sin\frac{\sigma}{2} \\ &= 2\pi|1-e^{i\sigma}|. \end{aligned}$$

Thus

$$h(1,\beta)\leqslant 2\pi|1-\beta|, \quad \beta\in\mathbf{T}.$$

It is clear that $h(e^{it}\alpha,e^{it}\beta) = h(\alpha,\beta)$, $t\in\mathbf{R}$. Therefore

$$h(\alpha,\beta) = h(1,\frac{\beta}{\alpha})\leqslant 2\pi|1-\frac{\beta}{\alpha}| = 2\pi|\alpha-\beta|, \quad \alpha,\beta\in\mathbf{T}.$$

Using that $\mu(1) = 1$ we therefore get

$$\frac{1}{2\pi}\int_0^{2\pi}(\| a_\theta u - ua_\theta \|_\phi^{\#})^2 d\theta \leq 2\pi \iint_{\mathbf{T}^2} |\alpha-\beta|\, d\mu(\alpha,\beta)$$

$$\leq 2\pi \Big(\iint_{\mathbf{T}^2} |\alpha-\beta|^2 d\mu(\alpha,\beta)\Big)^{\frac{1}{2}}$$

$$= 2\pi \| wu-uw\|_\phi^{\#}$$

$$< 2\pi\delta .$$

Hence we can choose $\theta\in[0,2\pi)$ such that with $a = a_\theta$

$$(\| au-ua\|_\phi^{\#})^2 < 2\pi\delta .$$

For $\sigma_1,\sigma_2\in\mathbf{R}$, $|e^{i\sigma_1}-e^{i\sigma_2}|\leq|\sigma_1-\sigma_2|$. Using formulas (1), (2) and the fact that $a = \mathrm{Arg}_\theta(w)$ we therefore have

$$\| \exp(isa)u-u\exp(isa)\|_\phi^{\#} =$$

$$= \Big(\iint_{\mathbf{T}^2} |\exp(is\mathrm{Arg}_\theta(\alpha))-\exp(is\mathrm{Arg}_\theta(\beta))|^2 d\mu(\alpha,\beta)\Big)^{\frac{1}{2}}$$

$$\leq |s|\Big(\iint_{\mathbf{T}^2} |\mathrm{Arg}_\theta(\alpha)-\mathrm{Arg}_\theta(\beta)|^2 d\mu(\alpha,\beta)\Big)^{\frac{1}{2}}$$

$$= |s|\,\| au-ua\|_\phi^{\#},$$

for all $s\in\mathbf{R}$.

Put $h = \exp(\frac{1}{t_0}a)$ and

$$\omega(x) = \frac{1}{\phi(h^{-1})}\,\phi(h^{-1}x), \quad x\in M.$$

Since w belongs to the center of M_ϕ so does h. Therefore ω is a faithful normal state on M, ω commutes with ϕ, and

$$M_\phi \subset M_\omega .$$

Moreover, we have

$$\sigma_t^\omega(x) = h^{-it}\sigma_t^\phi(x)h^{it} = \sigma_t^\phi(h^{-it}xh^{it}), \quad x\in M.$$

Since $h^{it_0} = w$ we get in particular

$$\sigma^{\omega}_{t_0}(x) = x, \quad x \in M.$$

Therefore we can define a conditional expectation E_ω of M onto M_ω by

$$E_\omega(x) = \frac{1}{t_0}\int_0^{t_0} \sigma^{\omega}_t(x)\,dt, \quad x \in M.$$

Since $\sigma^{\omega}_t(u)-u = \sigma^{\phi}_t(h^{-it}uh^{it}-u)+\sigma^{\phi}_t(u)-u$, and since $h^{-it} = \exp(-i\frac{t}{t_0}a)$, we get for $0 \leq t \leq t_0$,

$$\begin{aligned}\|\sigma^{\omega}_t(u)-u\|^{\#}_{\phi} &\leq \|h^{-it}uh^{it}-u\|^{\#}_{\phi}+\|\sigma^{\phi}_t(u)-u\|^{\#}_{\phi} \\ &= \|h^{-it}u-uh^{-it}\|^{\#}_{\phi}+\|\sigma^{\phi}_t(u)-u\|^{\#}_{\phi} \\ &\leq \frac{t}{t_0}\|au-ua\|^{\#}_{\phi}+\delta \\ &< (2\pi\delta)^{\frac{1}{2}}+\delta .\end{aligned}$$

Therefore we also have

$$\|E_\omega(u)-u\|^{\#}_{\phi} < (2\pi\delta)^{\frac{1}{2}}+\delta .$$

Put $y = E_\omega(u)$ and $\delta' = (2\pi\delta)^{\frac{1}{2}}+\delta$. Since M_ω is a finite von Neumann algebra the partial isometry in the polar decomposition of y can be extended to a unitary operator $v \in M_\omega$. Clearly $y = v|y| = |y^*|v$. Using the inequality $(1-t)^2 \leq 1-t^2$ for $t \in [0,1]$ we get

$$\phi((v-y)^*(v-y)) = \phi((1-|y|)^2) \leq \phi(1-|y|^2),$$

and

$$\phi((v-y)(v-y)^*) = \phi((1-|y^*|)^2) \leq \phi(1-|y^*|^2).$$

Hence

$$(\|v-y\|^{\#}_{\phi})^2 \leq \tfrac{1}{2}\phi(2-y^*y-yy^*) = 1-(\|y\|^{\#}_{\phi})^2 .$$

On the other hand

$$\|y\|_\phi^\# \geq \|u\|_\phi^\# - \|u-y\|_\phi^\# > 1-\delta'.$$

Thus

$$(\|v-y\|_\phi^\#)^2 < 1-(1-\delta')^2 \leq 2\delta'.$$

Therefore

$$\|u-v\|_\phi^\# \leq \|u-y\|_\phi^\# + \|y-v\|_\phi^\# < \delta' + (2\delta')^{\frac{1}{2}}.$$

Since δ was arbitrary we have proved Lemma 3.3.

Completion of step 2. Assume that $T(M)$ is dense in $\mathbb{R}$. Let ϕ and ψ be commuting faithful normal positive functionals on M such that there are positive real numbers a and b with

$$a\phi \leq \psi \leq b\phi,$$

and such that

$$Sp(\Delta_\phi) \cap (\frac{a}{b}, \frac{b}{a}) = \{1\}.$$

We shall prove that

$$\|u\phi u^* - \psi\| \geq \|\phi - \psi\|$$

for every unitary operator $u \in M$. Clearly it is enough to prove the inequality for a strongly dense set of unitaries. Hence by Lemma 3.3 we may assume that there exists a faithful normal state ω on M, ϕ and ω commute, $M_\phi \subset M_\omega$, and such that $u \in M_\omega$. Let ϕ_1 and ψ_1 be the restrictions of ϕ and ψ to M_ω. Since $\omega \circ \sigma_t^\phi = \omega$, M_ω is a σ_t^ϕ-invariant subalgebra of M, and therefore $\sigma_t^{\phi_1}$ is simply the restriction of σ_t^ϕ to M_ω. In particular

$$Sp(\Delta_{\phi_1}) \subset Sp(\Delta_\phi),$$

hence

$$Sp(\Delta_{\phi_1}) \cap (\frac{a}{b}, \frac{b}{a}) = \{1\}.$$

We have $\psi = \phi(m\cdot)$ for some positive operator $m \in M_\phi$. Since $M_\phi \subset M_\omega$, $\psi_1 = \phi_1(m\cdot)$, so ϕ_1 and ψ_1 also commute. Clearly $a\phi_1 \leq \psi_1 \leq b\phi_1$, so by step 1

$$\| u\phi_1 u^* - \psi_1 \| \geq \| \phi_1 - \psi_1 \| .$$

Let $E_\omega : M \to M_\omega$ be the conditional expectation for which $\omega \circ E_\omega = \omega$. Since ϕ and ψ can be written in the form

$$\phi = \omega(h\cdot), \qquad \psi = \omega(k\cdot),$$

where h and k are positive operators affiliated with M_ω, we have

$$\phi = \phi_1 \circ E_\omega, \qquad \psi = \psi_1 \circ E_\omega .$$

Therefore

$$\| \phi - \psi \| = \| (\phi_1 - \psi_1) \circ E_\omega \| = \| \phi_1 - \psi_1 \| ,$$

which implies that

$$\| u\phi u^* - \psi \| \geq \| \phi - \psi \| .$$

This completes the proof of step 2.

Step 3. Let now M be an arbitrary von Neumann algebra and let ϕ and ψ be normal positive functionals on M which satisfy the condition of Theorem 3.1. We can assume that M acts on a Hilbert space H with a separating and cyclic vector ξ_0 such that $\phi(x) = (x\xi_0, \xi_0)$, $x \in M$. Let G be a countable dense subgroup of $\mathbb{R}$ and let

$$N = M \times_{\sigma^\phi} G$$

be the crossed product of M with the discrete group $\{\sigma_t^\phi : t \in G\}$ of automorphisms. N is the von Neumann algebra on $\ell^2(G, H)$ generated by $\pi(M)$ and $\lambda(G)$, where

$$(\pi(x)\xi)(t) = \sigma^{\phi}_{-t}(x)\xi(t), \quad x\in M, \quad \xi\in\ell^2(G,H),$$

$$(\lambda(s)\xi)(t) = \xi(t-s), \qquad s\in G, \quad \xi\in\ell^2(G,H).$$

For this and the following the reader may consult [7] and [3], see also [11]. Since G is discrete there is a faithful normal conditional expectation ε of N onto $\pi(M)$ such that

$$\varepsilon(\lambda(s)\pi(x)) = \begin{cases}\pi(x) & \text{if } s = 0\\ 0 & \text{if } s \neq 0\end{cases}.$$

Put $\tilde{\phi} = \phi\circ\pi^{-1}\circ\varepsilon$. Then $\tilde{\phi}$ is the "dual weight" of ϕ, so we have

$$\sigma^{\tilde{\phi}}_t(\pi(x)) = \pi(\sigma^{\phi}_t(x)), \quad x\in M,$$

$$\sigma^{\tilde{\phi}}_t(\lambda(s)) = \lambda(s), \qquad s\in G.$$

Moreover, the vector $\tilde{\xi}_0\in\ell^2(G,H)$ given by

$$\tilde{\xi}_0(t) = \begin{cases}\xi_0 & \text{if } t = 0\\ 0 & \text{if } t \neq 0\end{cases},$$

is cyclic and separating for N,

$$\tilde{\phi}(y) = (y\tilde{\xi}_0,\tilde{\xi}_0), \quad y\in N,$$

and

$$(\Delta^{it}_{\tilde{\phi}}\xi)(t) = \Delta^{it}_{\phi}\xi(t), \quad \xi\in\ell^2(G,H),$$

where $\Delta^{it}_{\tilde{\phi}}$ is computed with respect to $\tilde{\xi}_0$.

From the above formulas it follows that

$$\sigma^{\tilde{\phi}}_t(y) = \lambda(t)y\lambda(t)^*, \quad t\in G, \quad y\in N.$$

Hence $G\subset T(N)$, whence $T(N)$ is dense in $\mathbb{R}$, and step 2 is applicable. Since $\Delta_{\tilde{\phi}}$ is just an amplification of Δ_{ϕ} it is clear that $sp(\Delta_{\tilde{\phi}}) = sp(\Delta_{\phi})$, so also

$$sp(\Delta_{\tilde{\phi}}) \cap (\frac{a}{b},\frac{b}{a}) = \{1\}.$$

Put $\tilde{\psi} = \psi \circ \pi^{-1} \circ \varepsilon$. Then clearly $a\tilde{\phi} \leq \tilde{\psi} \leq b\tilde{\phi}$. Moreover one verifies easily that

$$\pi^{-1} \circ \varepsilon \circ \sigma_t^{\tilde{\phi}} = \sigma_t^{\phi} \circ \pi^{-1} \circ \varepsilon .$$

Indeed, it is easily checked that the formula holds on elements in N of the form $\lambda(s)\pi(x)$, $s \in G$, $x \in M$. Since $\psi \circ \sigma_t^{\phi} = \psi$ it follows that $\tilde{\psi} \circ \sigma_t^{\tilde{\phi}} = \tilde{\psi}$, i.e. $\tilde{\phi}$ and $\tilde{\psi}$ commute. Therefore $\tilde{\phi}$ and $\tilde{\psi}$ also satisfy the conditions of the theorem, whence by step 2 we have

$$\| v\tilde{\phi} v^* - \tilde{\psi} \| \geq \| \tilde{\phi} - \tilde{\psi} \|$$

for all unitaries $v \in N$.

Let $u \in M$ be a unitary operator. Then

$$\pi(u)\tilde{\phi}\pi(u)^* - \tilde{\psi} = (u\phi u^* - \psi) \circ \pi^{-1} \circ \varepsilon .$$

Thus

$$\| u\phi u^* - \psi \| \geq \| \pi(u)\tilde{\phi}\pi(u)^* - \tilde{\psi} \| \geq \| \tilde{\phi} - \tilde{\psi} \| = \| \phi - \psi \| .$$

This completes the proof of Theorem 3.1.

The proof of the main theorem follows from section 2 and the following result.

Corollary 3.4. Let M be a σ-finite factor of type III_{λ}, $0 \leq \lambda \leq 1$. Then

$$\mathrm{diam}(S_0(M)/\mathrm{Int}(M)) \geq 2\,\frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}}.$$

Proof. For $\lambda = 1$ there is nothing to prove.

Suppose $0 < \lambda < 1$. Then we can choose a faithful normal state

ϕ on M such that

$$Sp(\Delta_\phi) = \{\lambda^n : n \in Z\} \cup \{0\}.$$

Thus $Sp(\Delta_\phi) \cap (\lambda, \lambda^{-1}) = 1$. Moreover, the centralizer M_ϕ of ϕ is a type II_1 factor [4, 4.2.6]. Hence we can choose a projection $p \in M_\phi$ such that

$$\phi(p) = \frac{1}{1+\lambda^{\frac{1}{2}}}.$$

Put $m = \lambda^{\frac{1}{2}} p + \lambda^{-\frac{1}{2}}(1-p) \in M_\phi$. Then $\lambda^{\frac{1}{2}} 1 \leq m \leq \lambda^{-\frac{1}{2}} 1$, and $\phi(m) = 1$. Thus

$$\psi(x) = \phi(mx), \quad x \in M$$

defines a normal state on M such that ϕ and ψ commute, and $\lambda^{\frac{1}{2}} \phi \leq \psi \leq \lambda^{-\frac{1}{2}}$. By Theorem 3.1 it follows that

$$\| u\phi u^* - \psi \| \geq \| \phi - \psi \|$$

for every unitary operator u in M. Let ϕ_1 and ψ_1 be the restrictions of ϕ and ψ to M_ϕ. Since ϕ is a trace on M_ϕ we can identify $(M_\phi)_*$ with $L^1(M_\phi, \phi_1)$. Therefore

$$\| \phi - \psi \| \geq \| \phi_1 - \psi_1 \| = \phi_1(|1-m|) = 2\, \frac{1-\lambda^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}},$$

proving the corollary when $0 < \lambda < 1$.

Finally if $\lambda = 0$ we can for every $\mu \in (0,1)$ choose a faithful normal state ϕ such that

$$Sp(\Delta_\phi) \cap (\mu, \mu^{-1}) = \{1\}.$$

As in [4, 3.2.7] one gets that the centralizer of ϕ is a type II_1 von Neumann algebra with diffuse center. Hence we can choose a projection $p \in M_\phi$ such that

$$\phi(p) = \frac{1}{1+\lambda^{\frac{1}{2}}}.$$

Arguing as above we get that

$$\mathrm{diam}(S_0(M)/\mathrm{Int}(M)) \geq 2\,\frac{1-\mu^{\frac{1}{2}}}{1+\mu^{\frac{1}{2}}},$$

so in the limit as $\mu \to 0$ we find that the diameter is (at least) 2. The proof is complete.

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