1. Introduction

The ptyx was introduced in Girard [2] and it is a higher type version of the dilator introduced in Girard [1]. Girard and Ressayre [5] gave an alternative approach to the ptykes and they gave several applications.

In this paper we will investigate a category of generalized binary relations and we will see how the ptykes can be represented as objects in this category.

We will review the decomposition of a ptyx and prove a hierarchy-theorem for the corresponding decomposition trees. Employing the functorial bounding theorem from Girard and Normann [4] we will see how the recursion theorem provides us with a general notion of functorial recursion over the decomposition trees. Parts of the paper will be a review of known results or simple generalizations of such. In these cases we omit or give minor hints to the proofs. Familiarity with an introduction to dilators or denotation systems (Girard [1], [2] or Girard-Normann [3]) will be an advantage.

2. Types and Classes

We will base our study of $\Pi^1_2$-logic, the ptykes and related objects on one category, the universal type $U$. $U$ is a generalization of the category of binary relations:

2.1 Definition

a) A pair $<f,X>$ is in the universal type $U$ if $X$ is a set and $f:X^2 \rightarrow \mathbb{N}$. 
b) If \( <f,X> \) and \( <g,Y> \) are two elements of \( U \) then

\[
\phi: X \rightarrow Y
\]

is a called an imbedding if \( \phi \) is 1-1 and for all \( x_1, x_2 \in X \) we have

\[
f(x_1, x_2) = g(\phi(x_1), \phi(x_2))
\]

When no information is lost we will write \( f \) for \( <f,X> \), and if nothing else is made explicit, \( f,g \) etc. will denote elements of \( U \).

2.2 Definition

If \( f,g \in U \), then \( I(f,g) \) is the set of imbeddings from \( f \) to \( g \).

2.3 Definition

a) A **class** \( C \) is a subcollection of \( U \) such that if \( g \in C \) and \( \phi \in I(f,g) \) then \( f \in C \).

b) A **pretype** \( T \) is a class of finite objects

c) If \( T \) is a pretype, then the type \( TP(T) \) of \( T \) is the class of all objects such that each finite subfunction is in \( T \).

d) If \( C \) is a class then \( PT(C) \) is the class of all finite elements of \( C \) and \( TP(C) \), the type of \( C \), is \( TP(PT(C)) \).

As a trivial observation we get

2.4 Lemma

Let \( C \) be a class. Every \( f \in TP(C) \) is the limit of a directed system from \( PT(C) \).

We let \( U_0 \) denote the class of all finite elements of \( U \). If \( f \in U_0 \) then \( f \) is isomorphic to some \( g:n^2 \rightarrow \mathbb{N} (n=\{0,\ldots,n-1\}) \)

Using some standard enumeration of finite sequences, we may code \( g \) as a natural number.
2.5 Definition

a) For each \( f \in U_0 \), let \( D(f) \) be the \( g : n^2 \rightarrow \mathbb{N} \) isomorphic to \( f \) with the lowest number code. We call \( D(f) \) the distinguished version of \( f \).

b) Let \( U_D = \{ D(f) \mid f \in U_0 \} \)

If \( C \) is a class, let
\[
C_D = \{ D(f) \mid f \in C \cap U_0 \} = C \cap U_D
\]

c) A class \( C \) is recursively based if \( C_D \) is recursive.

2.6 Definition

a) Let \( A_0 \) be the subclass of \( U_0 \) containing all \( f \) with no nontrivial automorphisms.

b) Let \( A_D = A_0 \cap U_D \) and let \( A = TP(A_0) \).

c) A class \( C \) is an \( A \)-class if \( C \subseteq A \).

Let \( C \) be an \( A \)-class, \( f \in A \). Let \( J_f \) be the set of finite subfunctions of \( f \). For each \( g \in J_f \), there will be a unique isomorphism \( \gamma_g : D(g) \rightarrow g \). If \( g \leq h \in J_f \), let \( \gamma_{gh} = \gamma_h^{-1} \circ \gamma_g \). Then
\[
f = \lim_{\rightarrow} \langle D(g), \gamma_{gh} \rangle.
\]

We will write \( f = \lim_{\rightarrow} \langle f_i, \gamma_{ij} \rangle \) and call it the canonical limit-construction of \( f \).

Now let \( C_1 \) and \( C_2 \) be two \( A \)-classes. Let \( F : C_1 \rightarrow C_2 \) be a functor commuting to pullbacks and direct limits.

Let \( f : X^2 \rightarrow \mathbb{N} \) be in \( C_1 \) and let \( F(f) = g : Y^2 \rightarrow \mathbb{N} \).

Let \( \langle f_i, \phi_{ij} \rangle \) be the canonical limit-construction of \( f \) with imbeddings \( \phi_i : f_i \rightarrow f \). Let \( g_i = F(f_i) \) and \( F(\phi_i) = \gamma_i : g_i \rightarrow g \).

Let \( y \in Y \). Since \( F \) commutes to pullbacks there is a unique minimal \( i \) such that \( y \in \text{Im}(\gamma_i) \). Let \( Y_i \) be such that
$g_i : Y_i^2 \to \mathbb{N}$. Then there is a unique $c \in Y_i$ such that

$$y = \gamma_i(c)$$

We call $(c; \phi; f)$ a denotation for $y$. Given $F$ and $f$, this denotation will be unique.

2.7 Definition

Let $C_1, C_2$ and $F$ be as above.

a) A bone for $F$ is a pair $(c, f)$ where $f \in U_D$, $c$ is in the domain of $F(f)$ and for all $g \in U_D, \phi \in I(g, f)$, if $g \neq f$ then $c \notin \text{Im}(F(\phi))$.

b) The skeleton of $F$ is the set of bones for $F$.

c) The dimension of $F$ is the cardinality of the skeleton.

Now let $C_1$ and $C_2$ be $A$-classes and let $F : C_1 \to C_2$ be a functor commuting with pullbacks and direct limits. We will see how the skeleton supports a natural binary function.

Let $(c, f)$ and $(d, g)$ be two bones for $F$. The interesting relation between $(c, f)$ and $(d, g)$ is the set of values

$$F(h)((c; \phi; h), (d; \gamma; h))$$

seen as a function of how the imbeddings $\phi$ and $\gamma$ are related.

Recursively in $f$ and $g$ we have a list $h_1, \ldots, h_k$ from $U_D$ and imbeddings $\phi_i : f \to h_i, \gamma_i : g \to h_i$ such that for all $(c; \phi; h), (d; \gamma; h)$ there is a number $i$ and an imbedding $\phi : h_i \to h$ such that

$$\phi = \phi \circ \phi_i, \gamma = \phi \circ \gamma_i.$$  

Then $F(h)((c; \phi; h), (d; \gamma; h))$

$$= F(h_i)((c; \phi_i; h_i), (d; \gamma_i; h_i)).$$
Let $N_F((c,f),(d,g)) = \langle n_1, n_2, m_1, \ldots, m_k \rangle$ where $n_1$ and $n_2$ are numerocodes for $f, g$ resp. and

$$m_i = F(h_i)(F(\eta_i)(c), F(\gamma_i)(d)).$$

From $N_F((c,f),(d,g))$ we can recover $f$ and $g$ but not $c$ and $d$.

2.8 Definition

Let $C_1$ and $C_2$ be $A$-classes. We let $C_1 + C_2$ be the collection of all functions

$$N:Z^2 \to N$$

isomorphic to some $N_F$ as described above.

2.9 Theorem

If $C_1$ and $C_2$ are $A$-classes then $C_1 + C_2$ is an $A$-class.

Proof

Let $N_F:Z^2 \to N$ be given. We have to prove that any substructure of $N_F$ is isomorphic to some $N_G$. Let $Y \subseteq Z$. For each $h \in C_1$, let $G(h)$ be the substructure of $F(h)$ with domain the set of elements with denotations $(c; \phi; h)$ where the image of $\phi$ is in $Y$. $G$ naturally extends to a functor commuting to pullbacks and direct limits and $N_G$ is isomorphic to $N_F \upharpoonright Y^2$. This shows that $C_1 + C_2$ is a class.

It remains to show that it is an $A$-class. Let $N:Z^2 \to N$ be a finite element of $C_1 + C_2$ with corresponding functor $G_N$. Let $\phi$ be an automorphism on $N$, $h:X^2 \to N$ be a non-empty, finite element of $C_1$.

Let $\phi$ be the automorphism on $G_N(h)$ induced by $\phi$. Since $C_2$ is an $A$-class, $\phi$ is the identity. But then $\phi$ is the identity, since denotations are unique.
2.10 Remarks

a) We used that \( C_1 \) is an A-class to get a unique denotation system and that \( C_2 \) is an A-class to prove that \( C_1 \times C_2 \) is an A-class.

b) It is not in general correct that \( C_1 \times C_2 \) is recursively based when \( C_1 \) and \( C_2 \) are recursively based. The classes we will study later will be recursively based, the methods used to prove this can be found in Girard [2].

c) The construction of \( C_1 \times C_2 \) is uniform. Thus there is one class coding all partial functors from \( U_A \) to \( U_A \) with a class as the domain.

3. WD-classes

3.1 Definition

Let \( f: \mathbb{N} \to \mathbb{N} \). By \( f^- \) we mean

\[
f^-(x,y) = \begin{cases} f(x,y) & \text{if } f(x,y) > 0 \\ \text{undefined} & \text{otherwise} \end{cases}
\]

b) Let \( C \) be a class. By \( C_\omega \) we mean

\[ C_\omega = \{ f: \mathbb{N} \to \mathbb{N} \mid f^- \in C \} \]

c) A subset \( A \) on \( \mathbb{N} \) can be reduced to \( C \) if there is a recursive \( F: \mathbb{N} \to \mathbb{N} \) such that

\[ g \in A \iff F(g) \in C_\omega \]

3.2 Lemma

Let \( C, D \) be classes, let \( f \in (C \times D)_\omega \) and let \( g \in (C_\omega) \).

Uniformly recursive in \( f \) and \( g \) we may find \( h \in D_\omega \) such that

\[ h^- = f^-(g^-) \]
The proof is easy and is left for the reader.

The literature contains several theorems about effective operators from ptykes to ordinals that can be bounded by ptykes, the most recent and general is due to Kechris [6]. A standard method of proof is to functorially find well-founded trees that dominates the ordinal in question and then linearizing it by e.g. a Kleene-Brouwer ordering. The WO-classes will be a general family of classes for which these kinds of arguments works.

3.3. **Definition**

a) Let WO be the class of (characteristic functions of) well-orderings

b) A class C can be well-ordered if there is a function \( \tau : \mathbb{N} \to \mathbb{N} \) such that for all \( f \in C \), \( \tau \circ f \in WO \).

c) Assume that C can be well-ordered. We call C a WO-class if for each well-ordered family \( \{f_i\}_{i \in \beta} \) there is an \( f \in C \) and imbeddings \( \phi_i : f_i \to f \) into pairwise disjoint subsets of f such that the well ordering of f puts \( \phi_i(x) < \phi_j(y) \) if \( i < j \).

3.4 **Lemma**

If C is a class then \( C \to WO \) is a WO-class.

**Proof**

Let \( \{f_i\}_{i \in \mathbb{N}} \) be some enumeration of

\[(C \to WO) \cap U_D\]

and let \( f_\omega = \sum_{i \in \mathbb{N}} f_i \). Each \( f_i \) has a canonical imbedding

\[\gamma_i : f_i \to f_\omega\]
If \( F \in C \to \text{WO} \), let \((c,f)\) and \((d,g)\) be two bones for \( F \). We order them by the values of the denotations

\[
(c; \gamma_f; f_\omega) \quad (d; \gamma_g; f_\omega)
\]

We leave the details for the reader.

Our classes are not only closed under function-spaces but also under Cartesian products:

### 3.5 Definition

a) Let \( f_1, \ldots, f_n \) be elements of \( U \).

\[ f_i : X_i \to \mathbb{N} \]

Let \( \langle f_1, \ldots, f_n \rangle \) be the function

\[ f : (\{1\} \times X_1 \cup \ldots \cup \{n\} \times X_n)^2 \]

defined by

\[ f((i,x_1), (i,x_2)) = \langle i, f_i(x_1, x_2) \rangle + 1 \]

\[ f((i,x), (j,y)) = 0 \quad \text{if} \quad i \neq j. \]

b) If \( C_1, \ldots, C_n \) are classes, let \( C_1 \times \ldots \times C_n \) be the class of those \( f \) that are isomorphic to some \( \langle f_1, \ldots, f_n \rangle \) where \( f_i \in C_i \) for \( i = 1, \ldots, n \).

It is easily seen that this really defines a class. If \( C_1, \ldots, C_n \) are WO-classes then \( C_1 \times \ldots \times C_n \) is a WO-class, using lexicographical orderings.

We have the following:

### 3.6 Theorem

Let \( C \) be a WO-class and let \( A \) be reducible to \( C \) via the recursive function \( F \). Then the complement \( \neg A \) can be
reduced to $C + \text{WO}$ via some recursive $G$ such that if $g \in A$ there is an infinite descending sequence in $G(g)^- (F(g)^-)$ uniformly recursive in $g$.

**Proof**

Let $g$ be given and let $f \in C$. Let $h : \mathbb{N} \to \text{dom}(f)$. Define a tree $T_g(f)$ such that $h$ is a branch in $T_g(f)$ if and only if $g' = F(g)^-$ is an element of $U$ and $h|\text{dom } g'$ is an imbedding of $g'$ into $f$. This can be done functorially. $T_g(f)$ is a tree on $f$ and using the well-ordering of $f$ it can be linearized to $O_g(f)$ which will be an element of $C + \text{WO}$ if $g \in A$. If $g \in A$ then take $f$ to be $F(g)^-$. The identity imbedding of $F(g)^-$ extended to a map from $\mathbb{N}$ to $f$ will be a branch in $T_g(f)$, so $O_g(f)$ is effectively not well-founded.

**3.7 Definition**

Let $C$ be a WO-class.

We call $A \subseteq \mathbb{N}$ a $\Pi^1_C$-set if $A$ can be reduced to $C + \text{WO}$ via a recursive $F$. $A$ is $\Pi^1_C$ if it can be reduced to $C + \text{WO}$ via a continuous $F$.

The $\Pi^1_C$-sets have many properties in common with the $\Pi^1_k$-sets. The following result is stated without proofs:

**3.8 Theorem**

a) The union of a recursively enumerated family of $\Pi^1_C$-sets is a $\Pi^1_C$-set

b) The intersection of recursively enumerated family of $\Pi^1_C$-sets is a $\Pi^1_C$-set
c) If $A$ is a $\Pi^1_C$-set, then
   $$B = \{ x \mid \forall y < x, y \in A \}$$
   is a $\Pi^1_C$-set.

d) There is a $\Pi^1_C$-set that is universal for $\Pi^1_C$-sets.

In a) we construct a tree $T(f)$ such that a branch in $T(f)$ will contain a branch in all the orderings $F_i(f)$. If no $F_i \in C \rightarrow WO$ we can take witnesses $f_i$, and since $C$ is a WO-class we can imbed all the $f_i$'s into one $f$ which will give a branch in $T(f)$. In b) we observe

$$\bigcup_{i \in \mathbb{N}} F_i \in C \rightarrow WO \iff F_i \in C \rightarrow WO \text{ for all } i.$$  

(c) is proved by a tedious but simple coding of $\forall x$ into $C \rightarrow WO$ and d) is based on the fact that the set of continuous functionals is $\Pi^1_1$ and thus reducible to WO. We are now ready to define the ptykes. The main structural properties are given inductively using theorems 3.6 and 3.8.

3.9 Definition

a) Let $Pt(0) = WO$
   $$Pt(k+1) = Pt(k) \rightarrow WO.$$  

b) The elements of $Pt(k)$ are called ptykes of pure type.

3.10 Definition

Let $\Sigma_k = \bigcup \{ F_i \mid i \in \text{element of } Pt(k) \}$

$\Sigma_0$ is essentially $\omega^\CK_1$. Girard and Ressayre [5] has shown that $\Sigma_k(\Sigma_{k-1})$ is the ordinal $\pi_k$, the supremum of all $\Pi^1_k$-well-orderings of subsets of $\mathbb{N}$. 

We can effectively decide if an ordinal $\alpha = \omega_1^{CK}$, but a similar fact does not hold for $E_k$ in general. We will find a remedy for that and indicate how it can be used.

3.11 Definition

Let $\{f_i\}_{i \in \mathbb{N}}$ be a recursive enumeration of all the partial recursive functions. Let $A_k = \{i | f_i$ is total and $f_i \in \text{Pt}(k)\}$.

a) (Precise definition)

$E_k = \sum_{i \in A_k} f_i$

b) Let $F$ be recursive such that

$i \in A_k \iff F(i) \in A_{k+1}$

Let $\theta = \sum_{k \in A_k} f_{F(i)}$.

3.12 Lemma

Let $P \in \text{Pt}(k), Q \in \text{Pt}(k+1)$. We may set-recursively decide if

$(P, Q) = (E_k, \theta_k)$

Proof

By the well-ordering we may recursively decide if two ptykes are isomorphic. Now let $P$ and $Q$ be given. Let $\{f_i\}_{i \in \mathbb{N}}$ be as in 3.11. Inductively let

$i \in A_p \text{ if } \sum_{j \leq i} f_j + f_i \in \text{Pt}$

is isomorphic to an initial segment of $P$.

If $P \neq \sum_{i \in A_p} f_i$ then $P = E_k$.

If $P = \sum_{i \in A_p} f_i$, let $Q_p = \sum_{i \in A_p} f_{F(i)}$.

If $Q_p = Q$ then we have $P = E_k, Q = \theta_k$. 
otherwise not.

Set-recursion is defined in Normann [7]

3.13 Corollary

If $\succ$ is a well-ordering of $\mathbb{N}$ recursive in a complete
$\Pi^1_{k+1}$-set, then there is a recursive functor
$F \in \text{Pt}(k) \times \text{Pt}(k+1) + \text{WO}$ such that \( \| F(\varepsilon_k, \theta_k) \| > 1 \succ 1. \)

Proof

This follows from 3.12 and well-known bounding theorems.

4. Decomposition of a ptyx

The ptykes of type 1 are also called dilators. One of the
main structural properties of dilators is the well-founded decom-
position of a dilator into "smaller" dilators. A principal tool is
recursion over this decomposition and a principal obstacle is the
need of getting functorial operators out of these recursions.

Girard constructed a similar decomposition of ptykes. We will
review this decomposition and prove a hierarchy-theorem for it. In
the next section we will combine it with the functorial bounding
theorem from Girard-Normann [4] to give a general method for
recursion over the decomposition.

1. Definition

a) If $\{ P_i \}_{i < \alpha}$ is a family from $\text{Pt}(k)$ we define $\bigvee_{i < \alpha} P_i$ in the
   usual way.

b) $P \in \text{Pt}(k)$ is called connected if $P$ is not a nontrivial sum
   $P = P_1 + P_2$.

We have
4.2 Lemma

All ptykes is the unique sum of a well-ordered family of connected ptykes.

The proof is essentially as in the dilator case.

From now on assume that $k > 1$.

4.3 Definition

Let $P \in \text{Pt}(k)$ and let $(c, f)$ be a bone for $P$. Let

$$f = \sum_{i \in \mathbb{N}} f_i$$

where $f_i$ is connected.

Let $h = \sum_{i \in \mathbb{N}} (f_i + f_i)$ and let $\phi_i : f + h$ be the imbedding that for $j \neq i$ sends $f_j$ on the first corresponding occurrence in $h$ while it sends $f_i$ on the second.

We say that $f_i$ is **more important** than $f_j$ if

$$(c; \phi_i; h) > (c; \phi_j; h).$$

Note that it is the indexed occurrence of $f_i$ that is more important than the ditto of $f_j$.

As in the dilator-case we will slow down a connected Ptyx by laying restrictions on its most important part.

4.4 Definition

a) Let $P \in \text{Pt}(k)$ be connected, $P \neq 1$. Let $h_0 \in \text{Pt}(k-1)$. Let $h_0^0(h_1)$

be the order-type of the subset of $P(h_0 + h_1)$ given by the denotations

$$(c; \phi + \psi; h_0 + h_1)$$

where $(c, f)$ is a bone for $P$, $i$ is the most important index,
\[ \psi : \sum_{j \leq i} f_j + h_0 \]
\[ \psi : \sum_{i < j < n} f_j + h_1 \]

b) If \( P \in \text{Pt}(k) \) and \( P = \sum_{i < \beta} P_i \) where \( \beta > 1 \) and each \( P_i \) is connected, then each \( P_i \) is a \textit{component} of \( P \).

c) If \( P \in \text{Pt}(k) \) is connected and \( h \in \text{Pt}(k-1) \) then \( P^h \) is a \textit{component} of \( P \) if \( P^h \neq 0 \).

d) A component of a component of \( P \) is itself a component of \( P \).

4.5 \textbf{Theorem} (Girard, unpublished)

The decomposition tree of \( P \), i.e. the tree of sequences \((P_1, \ldots, P_n)\) where \( P_1 = P \) and each \( P_{i+1} \) is a component of \( P_i \), is well-founded.

The proof is an elaboration of the proof in the dilator case, and is based on a sequence of lemmas leading up to that result.

We would not gain much if the decomposition trees of a Ptyx could be dominated by that of a dilator. Our next task will be to show that this is not the case.

We first define a family of projections \( \pi_k : \text{Pt}(k) \to \text{Pt}(k-1) \) and inverses \( v_k : \text{Pt}(k-1) \to \text{Pt}(k) \):

4.6 \textbf{Definition}

a) Let \( \pi_1(D) \) be the collapse of a dilator \( D \) to a well-ordering. Let \( v_1(\alpha) \) be the constant \( \alpha \) dilator.

b) Assume that \( \pi_k \) and \( v_k \) are defined for some \( k > 1 \). Let

\[ \pi_{k+1}(P)(h) = P(v_k(h)) \]

where \( P \in \text{Pt}(k+1) \), \( h \in \text{Pt}(k-1) \). Let \( v_k(D)(E) = D(\pi_k(E)) \). These maps are extended to functors in the canonical way.
4.7 **Lemma**

If $\mathcal{D} \in \text{Pt}(k)$ and $k > 0$ then

$$\pi_{k+1}(\nu_{k+1}(\mathcal{D})) = \mathcal{D}$$

**Proof**

Use induction on $k$. The induction start is obvious and the induction step is standard.

4.8 **Remark**

Observe that $\pi_k$ and $\nu_k$ will commute with sums, with pull-backs and with direct limits.

4.9 **Definition**

Let $\mathcal{D} \in \text{Pt}(k)$. Let

$$P_\mathcal{D}(\mathcal{E}) = D(\pi_k(\mathcal{E}))$$

4.10 **Lemma**

$P_\mathcal{D} \in \text{Pt}(k+1)$ and the decomposition tree of $\mathcal{D}$ can be imbedded into the decomposition tree of $\mathcal{P}_\mathcal{D}$.

**Proof**

We use induction on the decomposition of $\mathcal{D}$.

1. If $\mathcal{D} = \prod_k \mathcal{D}$ then $P_\mathcal{D} = \prod_{k+1} \mathcal{D}$ and the decomposition trees are isomorphic.

2. If $\mathcal{D} = \sum_{i \leq x} \mathcal{D}_i$ then $P_\mathcal{D} = \sum_{i \leq x} P_{\mathcal{D}_i}$ and the induction is trivial.

3. Let $\mathcal{D} = \prod_k \mathcal{D}$ be connected, let $h \in \text{Pt}(k-1)$. Let $h_1 = \nu_k(h)$.

We will show that $P_{\mathcal{D}} h_1$ can be imbedded into $P_{\mathcal{D}} h$, and by the induction-hypothesis the lemma will follow.

Let us consider
\[ P_D h(E) = D^h(\pi_k(E)). \]
The value is a subset of
\[ D(h+\pi_k(E)) \]
where the most important part of each denotation is taken from \( h \),
the rest from \( \pi_k(E) \).

Now \( P_D^1(E) \subseteq P_D(h_1+E) \) where the most important part of the
denotation is taken from \( h_1 \) and the rest from \( E \). Since
\[ P_D(h_1+E) = D(\pi_k(h_1+E)) = D(h+\pi_k(E)), \]
to each denotation for \( P_D \) we find a denotation for \( D \) using
\( \pi_k \), so \( v_k \) gives us the desired imbedding.

4.11 Theorem

There is an element \( P \in \text{Pt}(k+1) \) that is connected and such
that \( P_D \) is isomorphic to \( P^D \) for all \( D \in \text{Pt}(k) \).

Proof

We will describe the denotation-system for \( P_D \). Each bone
\((c,f)\) will be on the form \((c,f_1+f_2)\) where \( f_1 \) is any singleton
\((d,g)\) that may serve as a possible bone for some element in
\( \text{Pt}(k-1) \), and \( f_2 \) is minimal such that \( g \subseteq \pi_k(f_2) \).

To be more precise, for each \( g \in U_D \cap \text{Pt}(k-1) \) take any
\( f_2 \in \text{Pt}(k) \) such that \( g \subseteq \pi_k(f_2) \) while for no proper subfunction
\( f_3 \preceq f_2 \) we have \( g \subseteq \pi_k(f_3) \). Let \( f_1 \) be any element of \( \text{Pt}(k) \)
with exactly one bone of the form \((d,g)\). Let \( f \in U_D \) be isomor-
phic to \( f_1 + f_2 \).

For each such choice of \( f \), let \((c,f)\) be a bone for \( P \).
So far \( c \) can be anything, its canonical value will be determined
when we have described the ordering between denotations.

Let \((c_1; \phi_1+\phi_2; h)\) and \((c_2; \phi_3+\phi_4; h)\) be two denotations based on

\((c_1, f_1+f_2), g_2\) and \((c_2, f_3+f_4), g_4\)

as above.

If \(\phi_1\) and \(\phi_3\) sends \(f_1\) and \(f_3\) into different addends of \(h\), then we order the denotations by the order of the addends (This makes \(f_1\) resp. \(f_3\) to the most important parts). Now assume that \(h_1\) is an addend of \(h\) and essentially \(\phi_1: f_1 \rightarrow h_1, \phi_3: f_3 \rightarrow h_1\). Let \(h_2\) be the part of \(h\) that is above \(h_1\). The maps \(\phi_1\) and \(\phi_3\) give us two \(h_1\)-bones \((d_1, g_1)\) and \((d_3, g_3)\). Let

\[\psi_2 = \pi_k(\phi_2) \upharpoonright g_1, \quad \psi_4 = \pi_k(\phi_4) \upharpoonright g_3\]

Then

\[(d_1; \psi_2; \pi_k(h_2))\) and \((d_3; \psi_4; \pi_k(h_2))\)

are two \(h_1(\pi_k(h_2))\)-denotations. We order \((c_1; \phi_1+\phi_2; h)\) and \((c_2; \phi_3+\phi_4; h)\) by the value of these denotations.

It is now easy to see that \(P_D\) and \(P^D\) will be isomorphic for all \(D\).

4.12 Corollary

For each \(k\) there is a recursive ptyx \(P\) of type \(k+1\) such that the decomposition tree of \(D\) can be imbedded into the decomposition tree of \(P\) for all \(D \in \text{Pt}(k)\).
5. The functional recursion scheme

One important aspect of the decomposition of a dilator is the functorial recursion one may define over it. One problem is to arrange the definitions in such away that the result is a functor. The \( \Lambda \)-operator of Girard [1] is an example of a successful inductive definition.

Let us make a crude attempt to generalize \( \Lambda \). We will define \( \Lambda(P) \) to be an operator from \( \text{Pt}(k-1) \) to \( \text{Pt}(k-1) \), where \( P \in \text{Pt}(k) \), and we will use induction on \( P \):

\[
\begin{align*}
\Lambda(l_{-k})(h) &= h+h \quad \text{for } h \in \text{Pt}(k-1) \\
\Lambda(P+Q)(h) &= \Lambda(P)(\Lambda(Q)(h)) \\
& \quad \text{if } Q \text{ is connected}
\end{align*}
\]

\[
\Lambda(\sum_{i<\beta} P_i)(h) = \sum_{i<\beta} \Lambda(\sum_{j<i} (P_j))(h)
\]

if \( \beta \) is a limit ordinal and each \( P_i \) are connected.

\[
\Lambda(P)(h) = \Lambda(P^h)(h)(\Lambda(P^h)(h))
\]

If \( P \doteq_{-k} \) is connected.

This is of course nonsense, this \( \Lambda \) is not going to be functorial, and the recursive definition will break down because if won't make any sense.

The problem is the equalities, but for most applications it will be satisfactory just to find a \( \Lambda \) such that the right-hand side can be imbedded in the left-hand side, and we will show that there is indeed a recursive \( \Lambda \) satisfying this. The method is quite general and will be called the functorial recursion scheme.

The construction is based on the following result from Girard-Normann[4]:
5.1 Proposition

Let \( k, n \) be given. Let \( F \) be a partial set-recursive function. Then uniformly recursive in an index for \( F \) there is a functor \( P \) commuting to pullbacks and direct limits such that for all \( D \in \text{Pt}(k) \), if \( F(E) \in \text{Pt}(n) \) for all \( E \) that can be imbedded into \( D \) then \( P(D) \in \text{Pt}(n) \) and \( F(D) \) can be imbedded into \( P(D) \). This will also hold if we replace \( \text{Pt}(k) \) with a mixed type of ptykes.

Now assume that we have a partial recursive functor \( \Lambda \).

Uniformly in the index for \( \Lambda \) we define the set-recursive function \( \Phi_\Lambda : \text{Pt}(k) \times \text{Pt}(k-1) \to \text{Pt}(k-1) \) by

\[
\Phi_\Lambda(P, h) = h + h \quad \text{if} \quad P = \frac{1}{-k}
\]

\[
\Phi_\Lambda(P, h) = \Lambda(P_1)(\Lambda(P_2)(h))
\]

if \( P \preceq_\Lambda 0 \) is connected.

If \( P = \sum_{i \in \beta} P_i \), \( \beta \) is a limit ordinal

and each \( P_i \) are connected, then

\[
\Phi_\Lambda(P, h) = \sum_{i \in \beta} \Lambda(\sum_{j \leq i} (P_j))(h)
\]

If \( P \preceq_\Lambda 0 \) is connected, let

\[
\Phi_\Lambda(P, h) = \Lambda(\Lambda(P^h)(h))(\Lambda(P^h)(h)).
\]

We then use proposition 5.1 to find a functor \( \Psi_\Lambda(P) \) such that

\[\lambda h \Phi_\Lambda(P, h) \text{ can be imbedded into } \Psi_\Lambda(P).\]

By the recursion theorem there will be an index \( e \) for a partial operator \( \Lambda \) such that \( \Psi_\Lambda = \Lambda \). It is then not difficult to see by induction on \( P \) that \( \Lambda \) will be defined everywhere and \( \Lambda \) will be functorial since \( \Psi_\Lambda \) is functorial.
5.2 **Remark**

We will not state the functorial recursion scheme as a precise result since we have not found a good optimal formulation. Any combination of the recursion theorem and a functorial bounding principle in order to bound an operator recursively defined over the decomposition of a ptyx will be an instance of the scheme.

We will end this paper by showing that the decomposition of ptykes is optimal in a certain sense.

As a trivial observation we see

5.3 **Lemma**

Let $P$ be recursive Ptyx of type $k > 1$. Then the decomposition tree of $P$ restricted to countable elements of $Pt(k-1)$ can be realized as a well-founded $\Pi^1_k$-relation.

**Proof**

The decomposition is $\Delta^1_1$ and the restriction to elements of $Pt(k-1)$ is $\Pi^1_k$.

We will show that any $\Pi^1_k$-well-founded relation can be imbedded into the decomposition-tree of a ptyx. We need the following.

5.4 **Lemma**

Let $K : Pt(k-1) \to Pt(k)$ be functorial. Then there is a connected functor $P \in Pt(k)$ such that for all $h \in Pt(k-1)$

$$K(h) \text{ can be imbedded into } P^{1+h}$$

The proof is simple and is left for the reader.

5.5 **Theorem**

Let $<$ be a well-founded $\Pi^1_k$-relation. Then there is a
recursive $P \in \text{Pt}(k)$ and an order preserving map $\phi: \text{dom}(\prec) \to$ countable part of the decomposition tree of $P$.

Proof

We replace $\prec$ by its tree $T$ of descending sequences.

Uniformly recursive in each node $\sigma$ in $T$ we will define $P_\sigma$ such that the tree $T_\sigma$ of nodes below $\sigma$ can be mapped into the decomposition-tree of $P_\sigma$.

The definition will be by induction on $\sigma$ and we will use the recursion theorem to tie the whole definition together.

Without loss of generality we may assume

1) The domain of $\prec$ is $\Pi^1_k$.

2) There is a fixed recursive least element $a_0$ in $\prec$.

We are then ready to give the definition: If $\tau$ ends with $a_0$ we let $P_\tau = 1^\perp_k$.

Now let $\sigma \in T$. $T_\sigma = \{ \tau \mid \tau \in T \land \tau \text{ extends } \sigma \}$ is $\Pi^1_k$. Thus there is a continuous function $F_\sigma$ uniformly recursive in $\sigma$ such that

$$F_\sigma(\tau) \in \text{Pt}(k-1) \iff \tau \in T_\sigma$$

By a version of the functorical bounding theorem from Girard-Normann [4], there is a functor $Q_\sigma: \text{Pt}(k-1) \to \text{Pt}(k)$ such that for all $\tau$ and $h \in P(k-1)$:

If $F_\sigma(\tau)$ can be imbedded into $h$ then $P_\tau$ can be imbedded into $Q_\sigma(h)$.

Let $P_\sigma$ be such that $Q_\sigma(h)$ can be imbedded into $P_\sigma^{1+h}$, by lemma 5.4. Then $T_\sigma$ can be imbedded into the decomposition tree of $P_\sigma$.

Finally let $P = P_\prec$. Then $\prec$ can be mapped into the decomposition-tree of $P$. 

5.6 Remark

When we reduce a $\Pi^1_k$-set to $\operatorname{Pt}(k-1)$ we can make the reduction 1-1 by coding the real into the ptyx $(k>2)$ and then this proof really gives an imbedding.

References


