Quasi-symmetric Domains and Derivative of Curvature.

§ 1. Introduction.

In [4] we characterized quasi-symmetric domains among bounded homogeneous domains using socalled j-algebras associated to bounded homogeneous domains. The conditions were translated into curvature conditions in [5]. In this paper we translate (some of) the conditions in another way into curvature conditions: Symmetric domains are quasi-symmetric, and they are characterized by the vanishing of ∇R, the covariant derivative of the curvature. Now quasi-symmetric domains are "almost" symmetric, hence we look at ∇R. It turns out that some of the quasi-symmetry conditions in [4] are equivalent to the vanishing of ∇R on certain subspaces. In all of these papers a bounded homogeneous domain is called quasi-symmetric if it is biholomorphic to a quasi-symmetric Siegel domain in the sense of Satake [3]. For notation, facts and terminology we refer the reader to [4] and [5], of which this paper is a sequal.

§ 2. Derivative of Curvature.

Let the bounded homogeneous domain $\mathfrak{D}$ be described by the normal j-algebra $\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{k}_\alpha = \mathfrak{t} + j\mathfrak{t} + \mathfrak{u}$ ([2], [4], [5]). Here $\mathfrak{g}$ is identified with the tangent space $T_o\mathfrak{D}$ to $\mathfrak{D}$ at a (chosen) base point o. The covariant derivative ∇ and the curvature R were computed in [5]. We had ([5], lemma 2)

$$1) \qquad R(X,H) = \alpha(H)\nabla_X \quad \text{for} \quad H \in \mathfrak{h}, \ X \in \mathfrak{k}_\alpha,$$

where ∇_X operates on $\mathfrak{g}$ as $\nabla_X Y = \frac{1}{2}\{[X,Y] - (\operatorname{ad} X)'Y - (\operatorname{ad} Y)'X\}$, $'$ being the adjoint with respect to the Bergman metric $\langle , \rangle$

on $\mathfrak{g} \cong T_o\mathcal{D}$. To compute ∇R, we use the formula

2) $\quad (\nabla_X R)(Y,W,Z) = \nabla_X\{R(Y,W)Z\} - R(\nabla_X Y,W)Z - R(Y,\nabla_X W)Z - R(Y,W)\nabla_X Z.$

This is to be understood as follows: There is a (simply connected) split solvable Lie group $\mathcal{G}$, with Lie algebra $\mathfrak{g}$, acting simply transitively on $\mathcal{D}$ ([2], [5]). Hence we can identify $\mathcal{D}$ with $\mathcal{G}$. The elements of $\mathfrak{g}$ are left invariant vector fields on $\mathcal{G}$, and transferring the Bergman metric to $\mathcal{G}$, it too becomes invariant (i.e. elements of $\mathcal{G}$ are isometries). The metric defines the Riemannian connection ∇, and ∇ will be left invariant too, i.e. $\nabla_X Y$ is a field in $\mathfrak{g}$ if X and Y are. Similarly $R(Y,W)Z \in \mathfrak{g}$ if $Y,W,Z \in \mathfrak{g}$. So when we compute $\nabla_X\{R(X,Y)Z\}$, we can use lemma 1 of [5]. We also recall the mappings $T_X \in \operatorname{End}\mathfrak{t}$, $R_X \in \operatorname{End}_{\mathbb{C}}\mathfrak{u}$ for $X \in \mathfrak{t}$:

$$T_X Y = -j\nabla_X Y, \quad R_X u = -j\nabla_X u.$$

(See Lemmas 3 and 10 of [5]). Here j is the complex structure on $T_o\mathcal{D}$ transferred to $\mathfrak{g} \cong T_o\mathcal{D}$.

Recalling that $\mathfrak{t}$ is a sum of certain $\mathfrak{k}_\alpha$'s and that $j\mathfrak{h} \subset \mathfrak{t}$, we have

<u>Lemma 1</u>. For $X \in \mathfrak{k}_\alpha \subset \mathfrak{t}$, $Y \in \mathfrak{k}_\beta \in \mathfrak{t}$ and $H \in \mathfrak{h}$, we have

$$(\nabla_X R)(Y,H,Z) = \begin{cases} \{\alpha(H)-\beta(H)\}[T_X,T_Y]Z + [T_{jH},T_{XY}]Z & \text{if } Z \in \mathfrak{t} \\ \{\alpha(H)-\beta(H)\}[R_X,R_Y]Z + [R_{jH},R_{XY}]Z & \text{if } Z \in \mathfrak{u}\,. \end{cases}$$

(Here $XY = YX := T_X Y$).

Proof. Using 1) and 2) and also lemma 1 of [5] we find

$(\nabla_X R)(Y,H,Z) = \nabla_X\{\beta(H)\nabla_Y Z\} - R(\nabla_X Y,H)Z - R(Y,-\alpha(H)X)Z - \beta(H)\nabla_Y\nabla_X Z =$

$= \beta(H)[\nabla_X,\nabla_Y]Z + R(-j\nabla_X Y, jH)Z - \alpha(H)R(X,Y)Z$, by standard properties of the curvature. (Recall that $R(jU,jV) = R(U,V)$ since $\mathcal{D}$ is Kählerian). Now $\mathcal{T}$ is an abelian ideal of $\mathfrak{g}$, hence $[X,Y] = 0$, and so $R(X,Y) = [\nabla_X,\nabla_Y]$. This gives

$$(\nabla_X R)(Y,H,Z) = \{\beta(H)-\alpha(H)\}R(X,Y)Z + R(-j\nabla_X Y, jH)Z.$$

Since ∇ commutes with j on a Kählerian manifold, we have further

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = (-j\nabla_Y)(-j\nabla_X Z) - (-j\nabla_X)(-j\nabla_Y Z)$$

$$= \begin{cases} Y(XZ) - X(YZ) = [T_Y,T_X]Z & \text{if } Z \in \mathcal{T} \\ [R_Y,R_X]Z & \text{if } Z \in \mathcal{U}. \end{cases}$$

This gives the first term on the right hand side in the lemma. It also gives the second, by applying it to $R(-j\,\nabla_X Y, jH)Z = R(XY,jH)Z$, since $-j\nabla_X Y = XY \in \mathcal{T}$, $jH \in \mathcal{T}$.

q.e.d.

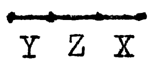
Now recall condition (A) of [4], § 3:

(A) $Y(ZX) - (YZ)X = 0$ if X,Y and Z are elements of $\mathcal{T}$ connected as follows: $\underset{Y}{\bullet}\overset{\beta}{\text{——}}\underset{Z}{\bullet}\overset{\alpha}{\text{——}}\underset{X}{\bullet}$. (See [4]. So $X,Y,Z \in \sum_{1\le k<m\le p} \mathcal{k}_{(k,m)} \subset \mathcal{T}$, where $\mathcal{k}_{(k,m)} := \mathcal{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$ and $\alpha_1,\dots,\alpha_p$ are the <u>basic</u> roots).

In this situation $XY = 0$ by [4], § 3, so $(\nabla_X R)(Y,H,Z) = \{\alpha(H)-\beta(H)\}[T_X,T_Y]Z = \{\alpha(H)-\beta(H)\}\{X(YZ) - Y(XZ)\}$.

We can choose H such that $\alpha(H) \neq 0$, $\beta(H) = 0$. Then, by the commutativity of the product on $\mathcal{T}$, we see that:

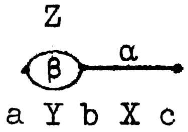
<u>Lemma 2:</u> (A) is equivalent to

(A_∇) $(\nabla_X R)(Y,H,Z) = 0$ for $X,Y,Z \in \mathfrak{t}$ connected as 

and $H \in \mathfrak{h}$.

We now check condition (B) of [4], § 3:

(B) $(XY)Z + (XZ)Y = (YZ)X$ for elements $X,Y,Z \in \mathfrak{t}$ connected as



, where $\alpha = \frac{1}{2}(\alpha_b + \alpha_c)$, $\beta = \frac{1}{2}(\alpha_a + \alpha_b)$.

Now $(\nabla_X R)(Y,H,Z) = \{\alpha(H)-\beta(H)\}\{X(YZ)-Y(XZ)\} + (jH)((XY)Z)-(XY)((jH)Z)$.

Here $(XY)Z \in \mathfrak{k}_{(b,c)} = \mathfrak{k}_\alpha$ and $XY \in \mathfrak{k}_{(a,c)}$, by [4], § 3.

Now let $H = jE_r$, where the $jE_1,\ldots,jE_p$ is a certain basis for $\mathfrak{h}$. (See [4], [5]).

If $r \neq a,b,c$, then $(\nabla_X R)(Y,jE_r,Z) = 0$, by [4].

If $r = a$, then $(\nabla_X R)(Y,jE_a,Z) = -\frac{1}{2}\{X(YZ)-Y(XZ)\} + (XY)(E_a Z)$

$= \frac{1}{2}\{-(YZ)X + (XZ)Y + (XY)Z\}$, by [4].

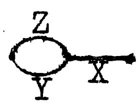
If $r = b$, then $(\nabla_X R)(Y,jE_b,Z) = 0\{X(YZ)-Y(XZ)\} - \frac{1}{2}(XY)Z + (XY)(\frac{1}{2}Z) = 0$, by [4].

If $r = c$, then $(\nabla_X R)(Y,jE_c,Z) = \frac{1}{2}\{X(YZ) - Y(XZ)\} - E_c((XY)Z)$

$= \frac{1}{2}\{(YZ)X - (XZ)Y - (XY)Z\}$, by [4].

Hence we have

<u>Lemma 3.</u> (B) is equivalent to

(B_∇) $(\nabla_X R)(Y,H,Z) = 0$ for $X,Y,Z \in \mathfrak{t}$ connected as 

and $H \in \mathfrak{h}$.

Now in [4], § 4 quasi-symmetry was shown to be equivalent to

(A), (B), (C), (D) and (Q). The condition (C) was translated to a curvature condition in [5], and the condition (Q) is equivalent to

$$(\tilde{Q}) \qquad R_{XY} = R_X R_Y + R_Y R_X \quad \text{for} \quad X, Y \in \mathcal{T},$$

by [3]. We try to express $(\tilde{Q})$ by $(\nabla_X R)(Y, H, Z)$ with $Z \in \mathcal{U} = \sum_{t=1}^{p} \mathcal{U}_t$. (See [5] for the decomposition of $\mathcal{U}$). In [4], § 2 we calculated that

$$R_W Z_t \in \begin{cases} (0) & \text{if} \quad t \neq a, b \\ \mathcal{U}_b & \text{if} \quad t = a \\ \mathcal{U}_a & \text{if} \quad t = b \end{cases}, \text{ where } W \in \mathcal{K}_{(a,b)},\ Z_t \in \mathcal{U}_t, \text{ and that}$$

3)

$$R_{E_r} Z_t = \begin{cases} 0 & \text{if} \quad t \neq r \\ \tfrac{1}{2} Z_t & \text{if} \quad t = r \end{cases}$$

Using this and the multiplication on $\mathcal{T}$ ([4], § 3, [5]), we check by Lemma 1

$$(\nabla_X R)(Y, jE_r, Z_t) \quad \text{for} \quad X \in \mathcal{K}_\alpha = \mathcal{K}_{(a,b)},\ Y \in \mathcal{K}_\beta = \mathcal{K}_{(c,d)}.$$

If X and Y are disconnected ($\underset{X}{\bullet\!-\!\bullet}\ \ \underset{Y}{\bullet\!-\!\bullet}$), then we get zero, and in this case also $R_{XY} = R_0 = 0$, $R_X R_Y = R_Y R_X = 0$ (See 3)), so $(\tilde{Q})$ is satisfied.

If X and Y are connected as $\overset{a\ \ \alpha\ \ b\ \ \beta\ \ c}{\underset{X \qquad Y}{\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet}}$, then $(\nabla_X R)(Y, jE_r, Z_t) = 0$ unless $r, t = a, b$. Using 3) and Lemma 1, we find

$$(\nabla_X R)(Y, jE_a, Z_a) = \tfrac{1}{2}\{-R_Y R_X Z_a + R_{XY} Z_a\} = \tfrac{1}{2}\{-(R_X R_Y + R_Y R_X) Z_a + R_{XY} Z_a\},$$

since $R_Y Z_a = 0$ by 3). Similarly,

$(\nabla_X R)(Y, jE_c, Z_a) = \tfrac{1}{2}\{(R_X R_Y + R_Y R_X) Z_a - R_{XY} Z_a\}$, and also the cases with $t = b$ go like this. Hence we have

<u>Lemma 4</u>. i) $(\tilde{Q})$ is satisfied for $\overset{}{\underset{X}{\bullet\!-\!\bullet}}\ \underset{Y}{\bullet\!-\!\bullet}$ and $(\nabla_X R)(Y,H,Z) = 0$ for $H \in \mathfrak{h}$, $Z \in \mathfrak{U}$ in this case.

ii) $(\tilde{Q})$ is satisfied for $\underset{X\ \ Y}{\bullet\!-\!\bullet\!-\!\bullet}$ if and only if $(\nabla_X R)(Y,H,Z) = 0$ for $H \in \mathfrak{h}$, $Z \in \mathfrak{U}$.

Now we have to check the case $a \overset{Y}{\underset{X}{\bigcirc}} b$. Assume $a < b$.

We have $(\nabla_X R)(Y,H,Z) = [R_{jH}, R_{XY}]Z$ in this case, by Lemma 1. Since $XY = YX$, $(\nabla_X R)(Y,H,Z)$ is symmetric in X and Y in this case, and hence $(\nabla_X R)(Y,H,Z) = 0$ if we can prove that $(\nabla_X R)(X,H,Z) = 0$.

We have $X^2 = \frac{|X|^2}{2\kappa}(E_a+E_b)$. (Here $\kappa = |E_1|^2 = \dots = |E_p|^2$, assuming condition (C). See [4], § 3). Let $H = jE_r$, $Z = Z_t \in \mathfrak{U}_t$. Then $(\nabla_X R)(X, jE_r, Z_t) = 0$ unless $r,t = a,b$, by 3). We have

$$(\nabla_X R)(X, jE_a, Z_a+Z_b) = -\frac{|X|^2}{2\kappa}\{R_{E_a}((R_{E_a}+R_{E_b})(Z_a+Z_b)) - (R_{E_a}+R_{E_b})(R_{E_a}(Z_a+Z_b))\} = -\frac{|X|^2}{2\kappa}\{\tfrac{1}{4}Z_a - \tfrac{1}{4}Z_a\} = 0,$$ and similarly $(\nabla_X R)(X, jE_b, Z_a+Z_b) = 0$. So $(\nabla_X R)(Y,H,Z) = 0$.

The condition $(\tilde{Q})$ in this case is equivalent to $R_{X^2} = 2R_X^2$, by symmetry in X and Y. Let $Z = \sum_{t=1}^{p} Z_t$. Then

4) $$R_{X^2}Z = \frac{|X|^2}{2\kappa}(R_{E_a}+R_{E_b})Z = \frac{|X|^2}{4\kappa}(Z_a+Z_b).$$

Also $R_X|_{\mathfrak{U}_r} = 0$ for $r \neq a,b$, by 3), and 4) shows that

$$R_{X^2} = 2R_X^2 \text{ on } \mathfrak{U}_r \text{ for } r \neq a,b.$$

In general $R_X Z = \frac{1}{2}\{[jX,Z] + (\operatorname{ad} jX)'Z\}$ by [4], § 2. Also

$R_X Z_a = \frac{1}{2}(\operatorname{ad} jX)' Z_a \in \mathfrak{U}_b$ and $2R_X^2 Z_a = \frac{1}{2}[jX, (\operatorname{ad} jX)' Z_a] \in \mathfrak{U}_a$, by [4], § 2.

Let $U \in \mathcal{U}_a$. If $R_{X^2} = 2R_X^2$ on $\mathcal{U}_a$, then by 4)

$$\frac{|Z|^2}{4\kappa}\langle Z_a, U\rangle = \tfrac{1}{2}\langle [jX, (\mathrm{ad}\, jX)'\, Z_a], U\rangle = \tfrac{1}{2}\langle (\mathrm{ad}\, jX)'\, Z_a, (\mathrm{ad}\, jX)' U\rangle,$$

So $\frac{\sqrt{2\kappa}}{|X|}(\mathrm{ad}\, jX)' : \mathcal{U}_a \to \mathcal{U}_b$ is an isometry. Conversely, if this map is an isometry, then $R_{X^2} = 2R_X^2$ on $\mathcal{U}_a$. By [4], § 4, we see that the isometric nature of the above map is equivalent to condition $(\tilde{D})$.

Finally, $R_X Z_b = \tfrac{1}{2}[jX, Z_b] \in \mathcal{U}_a$ and $2R_X^2 Z_b = \tfrac{1}{2}(\mathrm{ad}\, jX)'\,[jX, Z_b] \in \mathcal{U}_b$, by [4], § 2. Let $U \in \mathcal{U}_b$. If $R_{X^2} = 2R_X^2$ on $\mathcal{U}_b$, then by 4)

$$\frac{|X|^2}{4\kappa}\langle Z_b, U\rangle = \tfrac{1}{2}\langle (\mathrm{ad}\, jX)'\,[jX, Z_b], U\rangle = \tfrac{1}{2}\langle [jX, Z_b], [jX, U]\rangle,$$

so $\frac{\sqrt{2\kappa}}{|X|}\mathrm{ad}\, jX : \mathcal{U}_b \to \mathcal{U}_a$ is an isometry. Conversely, if this map is an isometry, then $R_{X^2} = 2R_X^2$ on $\mathcal{U}_b$. By [2], p. 61, this map is always an isometry. Hence we have

<u>Lemma 5</u>. $(\tilde{Q})$ is satisfied for $\underset{X}{\overset{Y}{\bigcirc}}$ if and only if $(\tilde{D})$ holds.

Also $(\nabla_X R)(Y, H, Z) = 0$ for $\underset{X}{\overset{Y}{\bigcirc}}$, $H \in \mathfrak{h}$, $Z \in \mathcal{U}$.

It remains to check $(\tilde{Q})$ if one or both of X and Y are of the form E_r. (Recall that $\mathfrak{t} = \sum_{t=1}^{p} \mathbb{R}E_t + \sum_{1\le k<m\le p} \mathfrak{k}_{(k,m)}$, by [2], [4] or [5].

i) If $X = E_r$, $Y \in \mathfrak{k}_{(c,d)}$, then

$$XY = \begin{cases} 0 & \text{if } r \neq c,d \\ \tfrac{1}{2}Y & \text{if } r = c,d \end{cases}, \text{ by [4], § 3.}$$

Let $Z_t \in \mathcal{U}_t$. Then $R_{XY}Z_t = \begin{cases} 0 & \text{if } t \neq a,b \\ \tfrac{1}{2}R_Y Z_t & \text{if } t = a,b. \end{cases}$

$$\text{So}\quad R_{XY}Z_t = \begin{cases} 0 & \text{if} \quad r,t \neq a,b \\ \tfrac{1}{2}R_Y Z_t & \text{if} \quad r,t = a,b. \end{cases}$$

Further, by 3), $R_{E_r}R_Y Z_t + R_Y R_{E_r} Z_t = \begin{cases} 0 & \text{if} \quad r,t \neq a,b \\ \tfrac{1}{2}R_Y Z_t & \text{if} \quad r,t = a,b. \end{cases}$

(One of the terms on the left will always vanish). So $(\tilde{Q})$ holds in this case.

ii) If $X = E_r$, $Y = E_s$, then $E_r E_s = \begin{cases} 0 & \text{if} \quad r \neq s \\ E_r & \text{if} \quad r = s \end{cases}$, by [4], § 3.

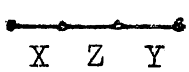
Let $Z_t \in \mathfrak{U}_t$. Then $R_{E_r E_s} Z_t = \begin{cases} 0 & \text{if} \quad r \neq t \text{ or } s \neq t \\ \tfrac{1}{2}Z_t & \text{if} \quad r = s = t \end{cases}$, by 3).

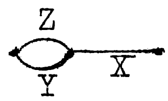
Further, by 3), $R_{E_r}R_{E_s}Z_t + R_{E_s}R_{E_t}Z_t = \begin{cases} 0 & \text{if} \quad r \neq t \quad \text{or} \quad s \neq t \\ \frac{1}{4}Z_t + \frac{1}{4}Z_t = \tfrac{1}{2}Z_t & \text{if} \quad r = s = t. \end{cases}$

So $(\tilde{Q})$ holds in this case too.

Putting together all the above, we get

Proposition 1. Quasi-symmetry is equivalent to the conditions (C), (D), $(\tilde{D})$ together with the vanishing of $(\nabla_X R)(Y,H,Z)$ for $H \in \mathfrak{h}$ and

i) $X,Y,Z \in \mathfrak{t}$ connected as 

and 

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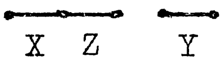
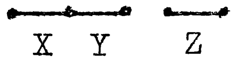
ii) $X,Y \quad \sum_{1 \leq k < m \leq p} \mathfrak{k}_{(k,m)} \subset \mathfrak{t}$, $Z \in \mathfrak{u}$.

We want to simplify the statement of the proposition by having the vanishing of $(\nabla_X R)(Y,H,Z)$ for $X,Y \in \mathfrak{t}$, $H \in \mathfrak{h}$, $Z \in \mathfrak{g}$. In the proposition $X,Y \in \sum_{1 \leq k < m \leq p} \mathfrak{k}_{(k,m)}$, while $\mathfrak{t}$ also contains $\sum_{t=1}^{p} \mathfrak{k}_t$

where $\mathfrak{k}_t = \mathbb{R} E_t$ (the basic root spaces). So we have to check $(\nabla_X R)(Y,H,Z)$ in the cases

(1) 

, (2) 

, (3) 

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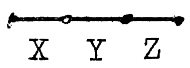
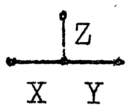
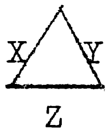
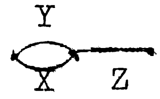
(4) 

, (5) 

, (6) 

, (7) 

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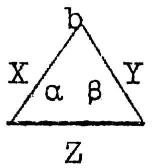
and in the cases

(8) $X = E_a$, $Y = E_b$, $Z = E_c$, (9) $X = E_a$, $Y = E_b$, $Z \in \mathfrak{k}_{(k,m)}$,
(10) $X = E_a$, $Y = E_b$, $Z \in \mathfrak{u}_t$, (11) $X = E_a$, $Y \in \mathfrak{k}_{(k,m)}$, $Z = E_c$,
(12) $X = E_a$, $Y \in \mathfrak{k}_{(k,m)}$, $Z \in \mathfrak{k}_{(c,d)}$, (13) $X = E_a$, $Y \in \mathfrak{k}_{(k,m)}$, $Z \in \mathfrak{u}_t$,
(14) $X \in \mathfrak{k}_{(a,b)}$, $Y \in \mathfrak{k}_{(k,m)}$, $Z = E_c$.

The remaining cases are covered by observing that according to Lemma 1 $(\nabla_X R)(Y,H,Z)$ is symmetric in X and Y in the cases we consider. We also let $H = jE_r$. Using 3) and the description of the multiplication on $\mathfrak{t}$ given in [4], § 3, it turns out that $(\nabla_X R)(Y,jE_r,Z)$ vanishes identically in most of the cases. As an example consider case (6):

a 

c , $X(YZ) = \frac{1}{2\varkappa}\langle X,YZ\rangle(E_a+E_b) = \frac{1}{2\varkappa}\langle XY,Z\rangle(E_a+E_b)$,

$Y(XZ) = \frac{1}{2\varkappa}\langle Y,XZ\rangle(E_b+E_c) = \frac{1}{2\varkappa}\langle XY,Z\rangle(E_b+E_c)$, (see [4], § 3),

so $[T_X,T_Y]Z = \frac{1}{2\varkappa}\langle XY,Z\rangle(E_a-E_c)$. Also $(XY)Z = \frac{1}{2\varkappa}\langle XY,Z\rangle(E_a+E_c)$,

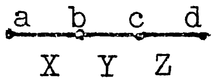
so by Lemma 1 $(\nabla_X R)(Y,jE_r,Z) =$

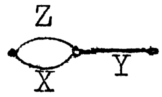
$$\tfrac{1}{2}\{\alpha_a(jE_r) - \alpha_c(jE_r)\}\cdot \frac{1}{2\varkappa}\langle XY,Z\rangle(E_a-E_c) - E_r((XY)Z) + (XY)(E_rZ).$$

This expression vanishes if $r \neq a, c$, and it equals

$$\frac{1}{4\varkappa}\langle XY,Z\rangle(E_a-E_c)-\frac{1}{2\varkappa}\langle XY,Z\rangle E_a+\frac{1}{2}\cdot\frac{1}{2\varkappa}\langle XY,Z\rangle(E_a+E_c)=0 \quad \text{if} \quad r=a$$

and similarly for $r = c$.

In the remaining cases of the list $(\nabla_X R)(Y,H,Z)$ vanishes exactly when the quasi-symmetry conditions (A), (B) are satisfied. These are the cases (4) 

with $r = a, c$ (we have for instance $(\nabla_X R)(Y,jE_a,Z) = \frac{1}{2}\{X(YZ)-(XY)Z\}$). (The case 

is not in the list because the symmetry in X and Y reduces it to a case in Proposition 1).

So now the vanishing of $(\nabla_X R)(Y,H,Z)$ in Proposition 1 can be stated for $X,Y \in \mathfrak{t}$, $H \in \mathfrak{h}$, $Z \in \mathfrak{t}+\mathfrak{u}$. We now observe that $\mathfrak{g} = \mathfrak{t}+j\mathfrak{t}+\mathfrak{u}$ and that $(\nabla_X R)(Y,H,jZ) =$

$$= \nabla_X\{R(Y,H)jZ\} - R(\nabla_X Y,H)jZ - R(Y,\nabla_X H)jZ - R(Y,H)\nabla_X jZ$$

$$= j\{\nabla_X\{R(Y,H)Z\} - R(\nabla_X Y,H)Z - R(Y,\nabla_X H)Z - R(Y,H)\nabla_X Z\}$$

$= j(\nabla_X R)(Y,H,Z)$, where we use the fact that because $\mathfrak{D}$ is Kählerian, both ∇ and R commute with j (see[1]).

So then the vanishing in Proposition 1 can be stated for $X,Y \in \mathfrak{t}$, $H \in \mathfrak{h}$, $Z \in \mathfrak{g}$.

Letting $(\nabla R)(X,Y,H,Z) := (\nabla_X R)(Y,H,Z)$, and using [5] and the theorem there together with the definition of a <u>triangular</u> subgroup of Aut $\mathfrak{D}$ used in [5], we can now restate Proposition 1 as

<u>Theorem 1</u>. Let $\mathfrak{D}$ be an indecomposable bounded homogeneous domain, and let $\mathcal{G}$ be a triangular subgroup of Aut $\mathfrak{D}$

(= biholomorphic automorphisms) with Lie algebra $\mathfrak{g}$. Choose a base point o of $\mathcal{D}$ and give $\mathfrak{g}$ the structure of a j-algebra by $\mathfrak{g} \cong T_o\mathcal{D}$, the complex structure on $T_o\mathcal{D}$ and the Bergman metric on $T_o\mathcal{D}$. Then $\mathcal{D}$ is quasi-symmetric (i.e. biholomorphic to a quasi-symmetric Siegel domain in the sense of Satake) if and only if the following conditions hold:

(∇) $\quad \nabla R\big|_{\mathfrak{t} \times \mathfrak{t} \times \mathfrak{h} \times \mathfrak{g}} = 0$,

(C') $\quad \max\limits_{Y \in \mathfrak{h}, |Y|=1} |K(Y)| = \dim \mathfrak{h} \cdot \min\limits_{Y \in \mathfrak{h}, |Y|=1} |K(Y)|$, where

$K(Y) = \langle R(Y, jY)jY, Y \rangle$ (holomorphic sectional curvature),

(D) $\quad \sum\limits_{l=1}^{\dim \mathfrak{h}} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l + \alpha_k)}$ is independent of k.

($\tilde{D}$) $\quad \dim \mathfrak{u}_k$ is independent of k.

(Here $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]^{\perp}$, $\mathfrak{g} = \mathfrak{h} + \sum\limits_{\alpha} \mathfrak{k}_{\alpha} = \mathfrak{t} + j\mathfrak{t} + \mathfrak{u}$ as in [2], [4], [5], and the root spaces are found as described after condition (C') in the text of [5] for the case that (C') is satisfied).

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