§ 0. Introduction

Let $X = \mathbb{P}^3_k$ denote the projective 3-space over an algebraically closed field k of characteristic zero. Given an integer n, denote by $M(n)$ the moduli space for stable rank-2 vector bundles on X with Chern classes $c_1 = 0$ and $c_2 = n$, see [9]. In his survey article [10], M. Schneider asks the following question: Are $M(3)$ and $M(4)$ nonsingular, and do they have only two components?

In this paper we answer this question affirmatively for $M(3)$, and we also prove that both components are rational. Our main tool in the proof will be a careful study of the restriction of a bundle to all lines through a fixed point P in X. By a theorem of Grothendieck [4] any vector bundle on a projective line is a direct sum of linebundles. In particular, if E is a rank-2 bundle on X with $c_1(E) = 0$, and $L \subseteq X$ is a line, then $E_L \cong \mathcal{O}_L(\gamma) \oplus \mathcal{O}_L(-\gamma)$ for some integer $\gamma = \gamma(L) \geq 0$. Following Barth [1] we say that L is a jumping line for E if $\gamma(L) \neq 0$. A jumping line L is said to be multiple if $\gamma(L) > 1$. The well-known theorem of Grauert-Mülich [1] states that if E is stable (in this case this is equivalent to $H^0(X,E) = 0$), then the general line is not a jumping line.

If $P \in X(k)$ is a closed point, denote by $M^P(n)$ the open subscheme of $M(n)$ parametrizing bundles E satisfying the following two conditions:

(i) There exists a non-jumping line for E through P.

(ii) There are no multiple jumping lines for E through P.

Recall the α-invariant of Atiyah-Rees:

$\alpha(E) := \dim_k H^1(X, E(-2)) \mod 2$. It is known [6,cor.2.4] that $\alpha \in \mathbb{Z}/2\mathbb{Z}$ is constant in connected families. In particular, if $M_\alpha(n) \subseteq M(n)$ is the subscheme parametrizing bundles E with $\alpha(E) = \alpha$, then it follows that $M(n)$ is a disjoint union of $M_0(n)$ and $M_1(n)$.

For each $\alpha \in \mathbb{Z}/2\mathbb{Z}$ and each $P \in X(k)$, put $M_\alpha^P(n) = M^P(n) \cap M_\alpha(n)$. We can now state our main results:

Theorem 1. For each α and P, $M_\alpha^P(3)$ is a nonsingular, irreducible and rational variety of dimension 21.

Theorem 2. For each α, the $M_\alpha^P(3)$ form an open covering of $M_\alpha(3)$.

As an immediate corollary follows

Theorem Both $M_0(3)$ and $M_1(3)$ are nonsingular, irreducible and rational varieties of dimension 21.

Remark With only slight modifications (due to the fact that $M_{Y^\vee}(2)$ is not a fine moduli space, [11]) the same method shows that $M(2) = M_0(2)$ is a nonsingular, irreducible and unirational variety of dimension 13. This was first proved by R. Hartshorne in [5].

The material is divided as follows:

§ 1 describes certain data characterizing a bundle corresponding to a point of $M^P(n)$.

§ 2 uses these data to prove theorem 1.

§ 3 contains a proof of theorem 2.

§ 4 contains a short discussion on the general case, i.e. $n = c_2(E) \geq 4$.

§ 1

(1.1) Fix throughout this § a closed point P of $X = \mathbb{P}^3_k$. Denote by $f : \tilde{X} \to X$ the blowing up of P, and let $g : \tilde{X} \to Y = \mathbb{P}^2_k$ be the morphism induced by projecting X from P. If $G(1,3)$ denotes the Grassmannian of lines in X, we may identify Y with the special Schubert variety in $G(1,3)$ corresponding to lines containing P. Under this identification, $\tilde{X} \subseteq X \times Y$ is the restriction of the incidence correspondence in $X \times G(1,3)$. The Picard group of $\tilde{X}$ is freely generated by two elements $\mathcal{O}_{\tilde{X}}(\tau) := f^*\mathcal{O}_X(1)$ and $\mathcal{O}_{\tilde{X}}(\sigma) := g^*\mathcal{O}_Y(1)$. Let $B = f^{-1}(P) \subseteq \tilde{X}$ be the exceptional divisor; then the divisor class of B is $(\tau - \sigma)$. Furthermore, $g : \tilde{X} \to Y$ can be identified with the projective bundle $\mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(1))$. In particular, there is a surjection $\mathcal{O}_{\tilde{X}} \oplus \mathcal{O}_{\tilde{X}}(\sigma) \to \mathcal{O}_{\tilde{X}}(\tau)$ inducing an isomorphism $\mathcal{O}_Y \oplus \mathcal{O}_Y(1) \to g_*\mathcal{O}_{\tilde{X}}(\tau)$. Finally, the relative dualizing sheaf of $\tilde{X}$ over Y is $\omega_g = \mathcal{O}_{\tilde{X}}(\sigma - 2\tau)$.

(1.2) Let E be a stable rank-2 vector bundle on X with Chern classes $c_1(E) = 0$ and $c_2(E) = n$, satisfying the following two conditions:

(i) There exists a non-jumping line for E thorugh P.

(ii) There are no multiple jumping line for E through P.

Put $\tilde{E} = f^*E$, and $F = g_*\tilde{E}$, and let $\psi : g^*F \to \tilde{E}$ be the natural map.

(1.3) Lemma

(i) ψ is injective and $R^1 g_* \tilde{E} = 0$.

(ii) $\chi(\tilde{E}(\mu\sigma)) = (\mu+2)(\mu+1-n)$ for all $\mu \in \mathbb{Z}$.

(iii) F is a rank-2 bundle on Y with Chern classes

$$c_1(F) = -n \quad \text{and} \quad c_2(F) = \tfrac{1}{2}n(n+1)$$

Proof: (i) is just a translation of the conditions (i) and (ii) of (1.2). (ii) follows from the Riemann-Roch theorem on $\tilde{X}$. Then (iii) follows from (i) and (ii) and the Riemann-Roch theorem on Y.

(1.4) It follows that $\overset{2}{\wedge}\psi$ is a non-zero section of $\mathcal{O}_{\tilde{X}}(n\sigma)$. Let $S \subseteq \tilde{X}$ be the zero-scheme of this section, and $C \subseteq Y$ the plane curve defined by the induced section of $\mathcal{O}_Y(n)$. Then $S = g^{-1}C$. Let $h : S \to C$ be the restriction of g. Denote by K the cokernel of $\psi : g^*F \to \tilde{E}$. Also, put $\theta = R^1 g_*(-\tau-\sigma)$.

(1.5) Proposition

(i) K is an invertible $\mathcal{O}_S$-sheaf.

(ii) θ is an invertible $\mathcal{O}_C$-sheaf.

(iii) $K \cong h^*\theta(2\sigma-\tau)$.

(iv) The restriction of F to C is $F_C = \theta^{-1}(-1) \oplus \theta^{-1}(-2)$.

(v) $\theta^2 = \mathcal{O}_C(n-3) = \omega_C$.

Proof: (i): Let $y \in C$ be a closed point. Since $R^1 g_* \tilde{E} = 0$, it follows that $K_{g^{-1}(y)} \cong \mathcal{O}_{g^{-1}(y)}(-1)$. Hence K is locally generated by one element. Let $x \in S$ be a closed point, and put $A = \mathcal{O}_{\tilde{X},x}$.

The exact sequence

(*1) $$0 \to g^*F \to \tilde{E} \to K \to 0$$

gives, when localized at x, an exact sequence $0 \to 2A \xrightarrow{\psi_x} 2A \to K_x \to 0$. So ψ_x is given by a 2×2 matrix. Since $\dim_k K \otimes k(x) = 1$, not all entries are in the maximal ideal. Therefore one of them is a unit, and from this it is clear that $K_x \cong A/\det(\psi_x) = \mathcal{O}_{S,x}$.

(ii) and (iii): Note that $K(\tau)$ induces the trivial linebundle on the fibers of h. Therefore, if we temporarily put $L = h_*K(\tau)$, it follows that the natural map $h^*L \to K(\tau)$ is an isomorphism and that L is a linebundle on C. If we show that $\theta = L(-2)$, both (ii) and (iii) will follow. Twist (*1) by $(-\tau)$ and apply R^1g_* to get $\theta(1) = R^1g_*\tilde{E}(-\tau) = R^1g_*K(-\tau) = R^1h_*K(-\tau)$.

Now relative duality gives

$$R^1h_*K(-\tau) = R^1h_*h^*L(-2\tau) = L(-1)\otimes R^1h_*\omega_h = L(-1),$$

since $\omega_h = \mathcal{O}_S(\sigma-2\tau)$. Combining these two strings of equalities, we obtain $\theta(1) = L(-1)$.

(iv): The restriction of (*1) to S induces an exact sequence $0 \to N \to \tilde{E}_S \to K \to 0$. Since $\overset{2}{\wedge}\tilde{E} = \mathcal{O}_{\tilde{X}}$, it follows that $N = K^{-1} = h^*\theta^{-1}(\tau-2\sigma)$. **Taking** h_*, we find the equality

$$h_*\tilde{E}_S = h_*N = \theta^{-1}(-2)\otimes(\mathcal{O}_C \oplus \mathcal{O}_C(1)) = \theta^{-1}(-1)\oplus\theta^{-1}(-2).$$

On the other hand, since $R^1g_*\tilde{E} = 0$, the natural base change map $F_C \to h_*\tilde{E}_S$ is an isomorphism.

(v): From (iv) we have $\overset{2}{\wedge}F_C = \theta^{-2}(-3)$. But by (1.3), $\overset{2}{\wedge}F_C = \mathcal{O}_C(-n)$. Thus $\theta^2 = \mathcal{O}_C(n-3)$.

(1.6) Lemma There is a short exact sequence on Y

$$(*2) \qquad 0 \to F \to 2\mathcal{O}_Y \to \theta(2) \to 0$$

Proof Restrict (*1) to the exceptional divisor B and then push it down to Y via the isomorphism $g|_B$.

(1.7) For technical reasons we also introduce the dual exact sequences to (*1) and (*2). First note that

$$\underline{\mathrm{Ext}}^1_{\mathcal{O}_Y}(\theta(2),\theta_Y) = \theta^{-1}(-2)\otimes\mathcal{O}_Y(n) = \theta(1), \text{ and}$$

$$\underline{\mathrm{Ext}}^1_{\mathcal{O}_{\tilde{X}}}(K,\mathcal{O}_{\tilde{X}}) = K^{-1}\otimes\mathcal{O}_{\tilde{X}}(n\sigma) = h^*\theta(\sigma+\tau).$$

Hence the sequences are

$$(*1)^\vee \qquad 0 \to \tilde{E}^\vee \to g^*F^\vee \to h^*\theta(\sigma+\tau) \to 0$$

$$(*2)^\vee \qquad 0 \to 2\mathcal{O}_Y \to F^\vee \to \theta(1) \to 0 .$$

Again, $(*2)^\vee$ can be obtained from $(*1)^\vee$ by restricting to B and pushing down to Y.

(1.8) Lemma: We have the following equality of cohomology groups:

$$H^1(X,E(-2)) = H^0(Y,F^\vee(-1)) = H^0(Y,\theta)$$

Proof: Clearly, $H^1(X,E(-2)) = H^0(Y,R^1g_*\tilde{E}(-2\tau))$ by the Leray spectral sequence for g. On the other hand, since $R^1g_*\tilde{E} = 0$ the relative duality map

$$R^1g_*(\underline{\mathrm{Hom}}_{\mathcal{O}_{\tilde{X}}}(\tilde{E},\omega_g)) \to \underline{\mathrm{Hom}}_Y(g_*E,\mathcal{O}_Y) = F^\vee$$

is an isomorphism [7, thm.(21)]. Noting that $\omega_g = \mathcal{O}_{\tilde{X}}(\sigma-2\tau)$ and that $\tilde{E} \cong \tilde{E}^\vee$, we obtain $\tilde{E}(-2\tau) \cong \underline{\mathrm{Hom}}_{\tilde{X}}(\tilde{E},\omega_g)\otimes\mathcal{O}_{\tilde{X}}(-\sigma)$. Putting all this together, we get the first equality. The second is an immediate consequence of $(*2)^{\vee}$.

(1.9) <u>Proposition</u> Assume that $H^1(E(-2)) = 0$. Then there is an exact sequence on Y

$$0 \to n\mathcal{O}_Y(-2) \xrightarrow{m} n\mathcal{O}_Y(-1) \to \theta \to 0$$

where the matrix of m can be taken to be symmetric.

<u>Proof</u>: It is well known that $\tilde{X} \subseteq Y \times X$ has a resolution of the form

$$0 \to \mathcal{O}_Y(-1) \boxtimes \mathcal{O}_X(-2) \to \Omega^1_Y(1) \boxtimes \mathcal{O}_X(-1) \to \mathcal{O}_{Y\times X} \to \mathcal{O}_{\tilde{X}} \to 0.$$

Tensor this by $\mathcal{O}_Y(-1) \boxtimes E(-1)$ to get two exact sequences

$$0 \to \mathcal{O}_Y(-2) \boxtimes E(-3) \to \Omega^1_Y \boxtimes E(-2) \to A \to 0$$

$$0 \to A \to \mathcal{O}_Y(-1) \boxtimes E(-1) \to \tilde{E}(-\tau-\sigma) \to 0$$

Taking the $R^i pr_{Y*}$ sequences of these and using that $H^i(X,E(-2)) = 0$ for all i, we get an exact sequence

$$0 \to \mathcal{O}_Y(-2) \otimes_k H^2(E(-3)) \xrightarrow{m} \mathcal{O}_Y(-1) \otimes_k H^1(E(-1)) \to \theta \to 0.$$

By Riemann-Roch, $\dim_k H^1(E(-1)) = \dim_k H^2(E(-3)) = n$. Finally, since $\theta^2 = \omega_C$, or equivalently $\underline{Ext}^1_{\mathcal{O}_Y}(\theta,\mathcal{O}_Y) = \theta$, it is easily verified that the map m is selfadjoint, hence its matrix can be taken to be symmetric.

(1.10) In his paper [2], Barth introduces a certain condition called $(\alpha 2)$ on selfadjoint maps $m: H \otimes_k \mathcal{O}_Y(-2) \to H^* \otimes_k \mathcal{O}_Y(-1)$, where H is an n-dimensional k-vector space. Twist by $\mathcal{O}_Y(2)$

and take global sections to get a map $H \to H^* \otimes_k \Gamma(\mathcal{O}_Y(1))$. It induces a map $\alpha : \Gamma(\mathcal{O}_Y(1))^* \otimes_k H \to H^*$, and the condition ($\alpha 2$) is that for each nonzero $h \in H$, the image of $\Gamma(\mathcal{O}_Y(1))^* \otimes_k (h)$ in H^* should have dimension at least 2. We claim that if m is injective and of rank $\geq n-1$ everywhere, then ($\alpha 2$) holds. In particular it holds for the map m in (1.9).

Indeed, choose coordinates Y_o, Y_1, Y_2 in Y such that m has rank n at the point $(1,0,0)$. We may then assume that the matrix of m can be written $Y_oI_n + Y_1A + Y_2B$, where I_n is the identity $n \times n$ matrix. If ($\alpha 2$) does not hold, let h be a column vector such that $\dim_k \mathrm{Span}\{h, Ah, Bh\} = 1$. Then h is a common eigenvector for A and B. By a suitable orthogonal change of basis we may assume that $h = (1,0,\ldots,0)^t$. It follows that the matrix of m can be written in the form

$$Y_oI_n + Y_1 \begin{bmatrix} a & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix} + Y_2 \begin{bmatrix} b & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{bmatrix}$$

From this it is clear that $\mathrm{rank}\, m \leq n-2$ at each point of intersection of the two curves $Y_o + aY_1 + bY_2$ and $\det(Y_oI_{n-1} + Y_1A' + Y_2B')$.

(1.11) In the particular case $n = 3$, it is trivial that also condition ($\alpha 3$) holds for m (see [2]). So by Barth's work, θ is the θ-characteristic of a uniquely determined vector bundle on $Y^\vee$, the projective plane dual to Y. We will use this later.

(1.12) To sum up some of the results so far, the given bundle E determines the following data:

(i) A plane curve $C \subseteq Y$ of degree n.

(ii) A θ-characteristic θ on C, i.e. a linebundle θ with $\theta^2 = \omega_C$.

(iii) A two-dimensional subspace V of $H^0(Y, \theta(2))$ generating $\theta(2)$, inducing an exact sequence

$$(*2) \qquad 0 \to F \to 2\mathcal{O}_Y \to \theta(2) \to 0$$

(iv) A surjection $\lambda : g^*F^\vee \to g^*\theta(\sigma+\tau)$, determined up to multiplication with a non-zero scalar.

Conversely, it is clear that the bundle E is uniquely determined by these data, as $\tilde{E}^\vee \cong \ker\lambda$. In fact, given any data (i)-(iv), we claim that they arise from a unique bundle on X.

Indeed, define $\tilde{E} = (\ker\lambda)^\vee$. To compute the restriction of $\tilde{E}$ to B, we restrict λ to B to get a surjection $F^\vee \xrightarrow{\lambda_B} \theta(1)$, where we identify B with Y via g. Restricted to C, this gives an exact sequence

$$0 \to \theta(2) \to F_C^\vee \to \theta(1) \to 0$$

from which we easily compute

$$\mathrm{Hom}(F^\vee, \theta(1)) = \mathrm{Hom}(F_C^\vee, \theta(1)) = \mathrm{Hom}(\theta(1), \theta(1)) = k.$$

On the other hand, the dual $(*2)^\vee$ of $(*2)$ gives

$$0 \to 2\mathcal{O}_Y \to F^\vee \xrightarrow{\lambda'} \theta(1) \to 0$$

It follows that λ_B is a scalar multiple of λ'. In particular, $\tilde{E}_B \cong (\ker\lambda_B)^\vee \cong 2\mathcal{O}_B$ is the trivial bundle. The proof of the claim is now completed by the following proposition:

(1.13) Proposition (See also [11, for the case of a surface) Let $\tilde{E}$ be a rank-2 bundle on $\tilde{X}$ such that $\tilde{E}_B \cong 2\mathcal{O}_B$. Then $E := f_*\tilde{E}$ is locally free and the natural map $f^*E \to \tilde{E}$ is an isomorphism.

Proof: The question being local on X, we may replace X by any open affine $\mathcal{U}$ containing P, and $\tilde{X}$ by $\tilde{\mathcal{U}} = f^{-1}\mathcal{U}$. Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{\mathcal{U}}}(-B) \to \mathcal{O}_{\tilde{\mathcal{U}}} \to \mathcal{O}_B \to 0$$

Tensor by $\tilde{E}$ and take global sections to get

$$H^0(\tilde{\mathcal{U}},\tilde{E}) \to H^0(B,\tilde{E}_B) \to H^1(\tilde{\mathcal{U}},\tilde{E}(-B)).$$

Since $\mathcal{U}$ is affine, $H^1(\tilde{\mathcal{U}},\tilde{E}(-B)) = H^0(\mathcal{U},R^1f_*\tilde{E}(-B))$.

Now Grothendieck's "theorem on formal functions" [5,III,11.1] implies that $R^1f_*\tilde{E}(-B) = 0$, since $H^1(E_{\mu B}(-B)) = 0$ for all μ, where μB denotes the μ-tuple scheme structure on the divisor $B \subseteq F$. This follows by induction on μ, the exact sequence $0 \to \mathcal{O}_B(-\mu B) \to \mathcal{O}_{(\mu+1)B} \to \mathcal{O}_{\mu B} \to 0$ and the fact that $\mathcal{O}_B(-B)$ is the positive generator of $\operatorname{Pic} B$. Since $H^1(\tilde{\mathcal{U}},\tilde{E}(-B)) = 0$, the isomorphism $2\mathcal{O}_B \to \tilde{E}_B$ can be extended to a map $2\mathcal{O}_{\tilde{\mathcal{U}}} \to \tilde{E}$ on $\tilde{\mathcal{U}}$. This map must be an isomorphism on some open set of $\tilde{\mathcal{U}}$ containing B. Now the proposition follows easily.

(1.14) Proposition Assume given data (i), (ii) and (iii) as in (1.12). Then there exist surjections $\lambda : g^*F^\vee \to g^*\theta(\sigma+\tau)$ if and only if $F_C \cong \theta^{-1}(-1) \oplus \theta^{-1}(-2)$. **In this** case,

$$\dim_k \operatorname{Hom}(g^*F^\vee, g^*\theta(\sigma+\tau)) = \begin{cases} 4 & \text{if } n = 1 \\ 5 & \text{if } n \geq 2. \end{cases}$$

Furthermore, if $n \leq 3$, then F_C always splits as above.

Proof If there exists a surjection λ, then F_C splits by (1.12) and (1.5,iv). Conversely, suppose $F_C \cong \theta^{-1}(-1) \oplus \theta^{-1}(-2)$. Then $\mathrm{Hom}_{\tilde{X}}(g^*F^\vee, g^*\theta(\sigma+\tau)) = \mathrm{Hom}_S(g^*F_C^\vee, g^*\theta(\sigma+\tau)) \cong$ $\mathrm{Hom}_S(g^*\theta(\sigma) \oplus g^*\theta(2\sigma), g^*\theta(\sigma+\tau)) = \mathrm{Hom}_S(\mathcal{O}_S \oplus \mathcal{O}_S(\sigma), \mathcal{O}_S(\tau))$, surjections corresponding to surjections. By (1.1), surjections $\mathcal{O}_S \oplus \mathcal{O}_S(\sigma) \to \mathcal{O}_S(\tau)$ exist. To compute the dimension, we have $\mathrm{Hom}_S(\mathcal{O}_S \oplus \mathcal{O}_S(\sigma), \mathcal{O}_S(\tau)) = H^0(\mathcal{O}_S(\tau) \oplus \mathcal{O}_S(\tau-\sigma)) = H^0(\mathcal{O}_C \oplus \mathcal{O}_C(1) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C)$ from which the assertion follows. To prove the last claim, note that there is always an exact sequence

$$0 \to \theta^{-1}(-1) \to F_C \to \theta^{-1}(-2) \to 0$$

The obstruction for splitting this sequence lies in $H^1(\mathcal{O}_C(1))$, which is zero for $n \leq 3$.

§ 2. The universal family.

(2.1) In this § we study the spaces $M_0^P(3)$ and $M_1^P(3)$. Keep all the notation from (1.1). The construction is based on the results of § 1. We describe $M_0^P(3)$ in detail first, and afterwards we point out the changes needed to get a similar description of $M_1^P(3)$.

(2.2) Let $M_{Y^\vee}$ denote the fine moduli space of stable rank-2 vector bundles on the projective plane $Y^\vee$ dual to Y [9,thm.7.17]. Since $M_{Y^\vee}$ carries a universal family, there is a corresponding universal θ-characteristic θ_1 which is a sheaf on $Y \times M_{Y^\vee}$, flat over $M_{Y^\vee}$. Let $C_1 \subseteq Y \times M_{Y^\vee}$ be defined by the zero-th Fitting ideal of θ_1. Let $N \subseteq M_{Y^\vee}$ be the maximal open subset such that θ is a line-bundle on C, where θ (resp. C) denotes the restriction of θ_1 (resp. C_1) to $Y \times N \subseteq Y \times M_{Y^\vee}$. Now $pr_{N^*}(\theta(2))$ is locally free of rank 6 on N and its formation commutes with base change on N. Let $G_1 = \mathrm{Grass}(2,(pr_{N^*}(\theta(2)))^\vee)$ denote the Grassmanian of 2-subbundles of $pr_{N^*}(\theta(2))$, and let K_1 denote the universal subbundle. Now there is a natural map $\mu : pr_{G_1}^* K_1 \to \theta(2)_{Y \times G_1}$ on $Y \times G_1$. Let $G \subseteq G_1$ be the maximal open subset of G_1 such that μ is surjective over G, and let K be the restriction of K_1 to G, and put $F = \ker \mu$, a sheaf on $Y \times G$. Abusing notation, also denote by θ the pullback of θ to $Y \times G$.

On G, define a sheaf R as follows:

$$R = pr_{G^*}\underline{Hom}_{\tilde{X} \times G}((g \times 1)^* F^\vee, (g \times 1)^* \theta(\sigma + \tau)),$$

and put $Q_1 = \mathbb{P}_G(R^\vee)$. It is easily checked that R commutes with base change and is locally free of rank 5, by (1.14). Let $Q \subseteq Q_1$ denote the open subvariety corresponding to surjections. It is clear that we get a rank-2 bundle $\tilde{E}$ on $\tilde{X} \times Q$ by taking the kernel of the universal homomorphism coming from the universal 1-quotient on $\mathbb{P}_G(R^\vee)$.

(2.3) <u>Proposition</u> Q is a nonsingular, irreducible and rational variety of dimension 21.

<u>Proof</u>: By [2], N is a nonsingular, irreducible and rational variety of dimension 9. The fibers of $G \to N$ have dimension 8 and the fibers of $Q \to G$ have dimension 4. Also, both $G \to N$ and $Q \to G$ are constructed as open subvarieties of Grassmanians

on locally free sheaves, which clearly implies the proposition.

(2.4) Proposition Let Q be any nonsingular variety and $\widetilde{E}$ a rank-2 vector bundle on $\widetilde{X} \times Q$ such that for each closed point q of Q, the restriction of $\widetilde{E}$ to $B \times \{q\} \subseteq \widetilde{X} \times Q$ is trivial. Then $E := (f \times 1)_* \widetilde{E}$ is locally free on $X \times Q$ and the natural map $(f \times 1)^* E \to \widetilde{E}$ is an isomorphism.

Proof: The question being local on Q, we may replace Q by $\operatorname{Spec} A$, A a regular local ring. We proceed by induction on $\dim A$, the case $\dim A = 0$ being (1.13). If $\dim A > 0$, let t be a regular parameter.

Applying $(f \times 1)_*$ to the exact sequence

$$0 \to \widetilde{E} \xrightarrow{t} \widetilde{E} \to \widetilde{E}/t\widetilde{E} \to 0 \quad \text{we get}$$

$$0 \to (f \times 1)_* \widetilde{E} \xrightarrow{t} (f \times 1)_* \widetilde{E} \to (f \times 1)_* (\widetilde{E}/t\widetilde{E}) \to 0 .$$

By induction, $(f \times 1)_* (\widetilde{E}/t\widetilde{E})$ is locally free of rank 2 on $X \times \operatorname{Spec}(A/t)$. Nakayamas lemma concludes the proof.

Remark This proposition is still valid if Q is singular.

(2.5) By (2.2), (2.3) and (2.4), there is a vectorbundle E on $X \times Q$ such that $\widetilde{E} = (f \times 1)^* E$. It is easily checked that E induces stable bundles with $c_1 = 0$, $c_2 = 3$ on each closed fibre of $X \times Q \to Q$. So by the universal property of a coarse moduli space, there is induced a morphism $Q \xrightarrow{i} M(3)$. It is easily checked that i is an open embedding, and that its image is precisely $M_0^P(3)$.

(2.6) The construction of $M_1^P(3)$ follows the same general lines; in fact (1.8) implies that $\alpha = 1$ if and only if $H^0(\theta) = 1$.

This follows from the fact that if $n = 3$, $\dim_k H^1(E(-2)) \leq 1$ [3,Prop.3.5]. But on any curve C of degree 3, the only θ-characteristic with a section is $\mathcal{O}_C$ itself. This shows that the only necessary changes are the following:

Let N be $\mathbb{P}^9$, parametrizing cubic curves in Y, let $C \subseteq Y \times N$ be the universal curve, and let $\theta = \mathcal{O}_C$. The rest of the construction goes through with no change, including the assertions about base change. In this way we get $M_1^P(3)$. The proof of Theorem 1 of the introduction is now complete.

§ 3. Proof of theorem 2.

(3.1) Let $X^\vee$ denote the projective 3-space dual to X, let $\Sigma \subseteq X \times X^\vee$ be the incidence correspondence, and let $p : \Sigma \to X$, $q : \Sigma \to X^\vee$ denote the natural maps. If P is a closed point of X, let $P^\vee = q(p^{-1}(P)) \subseteq X^\vee$ be the dual plane. Similarly, if $L \subseteq X$ is a line, let $L^\vee \subseteq X^\vee$ be the dual line, corresponding to the pencil of planes containing L. Let $\bar{\Sigma} = q^{-1}(L^\vee)$ and let $\bar{p} : \bar{\Sigma} \to X$ and $\bar{q} : \bar{\Sigma} \to L^\vee$ be the restrictions of p and q to $\bar{\Sigma}$. Then $\bar{p}$ is the blowing up of X with center L, and the divisor class of the exceptional divisor is $\bar{p}^*\mathcal{O}_X(1) \otimes \bar{q}^*\mathcal{O}_{L^\vee}(-1)$. In particular, there is an inclusion $\bar{p}^*\mathcal{O}_X(-1) \otimes \bar{q}^*\mathcal{O}_{L^\vee}(1) \to \mathcal{O}_{\bar{\Sigma}}$.

(3.2) Let E be a stable rank-2 vector bundle on X with $c_1(E) = 0$ and $c_2(E) = 3$. For each integer i, let $W_i \subseteq X^\vee$ be the closed subset corresponding to planes H such that $H^0(H, E_H(-i)) \neq 0$. Then $W_1 \subseteq W_0 \subseteq X^\vee$. By a theorem of Barth [1,thm.3] it follows that $W_0 \neq X^\vee$. Abusing language, we will say that a plane H is <u>stable</u> if it corresponds to a point of $X^\vee - W_0$, which really means that the restriction E_H of E to H is stable. Similarly, a point of $X^\vee - W_1$ is called <u>semistable</u>. Points of W_0 are called <u>not stable</u>, and points of W_1 are called <u>unstable</u>.

(3.3) <u>Lemma</u> If E_H is stable, then there is at most one multiple jumping line in H.

<u>Proof</u> By the Riemann-Roch theorem it follows that the first twist $E_H(1)$ has a section. Pick one, and let $Z \subseteq H$ be its zero-scheme. Z is a group of points of degree 4. It is easily seen that a line L with $E_L = \mathcal{O}_L(\gamma) \oplus \mathcal{O}_L(-\gamma)$ must intersect Z in $(\gamma+1)$ points if $\gamma \geq 2$. Since Z can have at most one trisecant, the lemma follows.

(3.4) <u>Lemma</u> If E_H is semistable, then there are only a finite number (in fact, at most 3) multiple jumping lines in H.

<u>Proof</u>: Similar to the proof of (3.3).

(3.5) <u>Lemma</u> Let $W \subseteq X^\vee$ be a proper closed subset with the property that $\bigcup_{L^\vee \subseteq W} L = X$. Then there is a point P_0 in X such that $P_0^\vee \subseteq W$.

<u>Proof</u> We may assume that W is irreducible. It is clear that W is a surface. Let P be a general point of X, then $W \cap P^\vee = \Gamma$ is an irreducible curve. On the other hand there exists a line L containing P such that $L^\vee \subseteq W$. It follows that $L^\vee \subseteq \Gamma$. Since Γ is irreducible, $\Gamma = L$. Therefore W must be a plane.

(3.6) <u>Lemma</u> There is at most one unstable plane.

<u>Proof</u>: By Serre duality, $H^0(E_H(-1))$ is dual to $H^2(E_H(-2))$. Therefore $W_1 = \operatorname{Supp} R^2q_*(p^*E(-2))$.
The resolution $0 \to \mathcal{O}_X(-1) \boxtimes \mathcal{O}_{X^\vee}(-1) \to \mathcal{O}_{X\times X^\vee} \to \mathcal{O}_\Sigma \to 0$
induces an exact sequence on $X^\vee$:

$$H^2(E(-3)) \otimes \mathcal{O}_{X^\vee}(-1) \to H^2(E(-2)) \otimes \mathcal{O}_{X^\vee} \to R^2q_*(p^*E(-2)) \to H^3(E(-3)) \otimes \mathcal{O}_{X^\vee}(-1).$$

By Serre duality, $H^3(E(-3)) = 0$. The group $H^2(E(-3))$ has dimension 3, and $H^2(E(-2))$ has dimension at most 1 by [3,prop.3.5]. It follows that W_1 is empty or the intersection of three planes in $X^\vee$. It remains only to show that W_1 contains no line. Assume there is a line L in X such that $L^\vee \subseteq W_1$. In the notation of (3.1) this means that $\bar{q}_*(\bar{p}^*E(-1)) \neq 0$. On the other hand, the resolution.

$$0 \to \mathcal{O}_X(-1) \boxtimes \mathcal{O}_{L^\vee}(-1) \to \mathcal{O}_{X\times L^\vee} \to \mathcal{O}_\Sigma \to 0$$

gives an exact sequence

$$0 \to \bar{q}_*(\bar{p}^*E(-1)) \to H^1(E(-2)) \otimes \mathcal{O}_{L^\vee}(-1) \to H^1(E(-1)) \otimes \mathcal{O}_{L^\vee}.$$

Since $\dim_k H^1(E(-2)) = 1$ it follows that $\bar{q}_*(\bar{p}^*E(-1)) \cong \mathcal{O}_{L^\vee}(-1)$.
In particular, $\bar{p}^*E(-1) \otimes \bar{q}^*\mathcal{O}_{L^\vee}(1)$ has a global section.
But since $\bar{p}^*\mathcal{O}_X(-1) \otimes \bar{q}^*\mathcal{O}_{L^\vee}(1) \subseteq \mathcal{O}_\Sigma$ is the ideal of the exceptional divisor $\bar{p}^{-1}(L)$, we get a global section of E itself, which is impossible.

<u>Remark</u> The last part of this proof was pointed out to us by L. Ein and T. Sauer; they also showed us the similarity with Barth's proof of condition (α2) [2,p.67].

(3.7) <u>Proposition</u> There exists a point P of X such that no multiple jumping lines for E contain P.

<u>Proof</u> Consider the closed subset $\Lambda \subseteq G(1,3)$ corresponding to multiple jumping lines. If L is any line, let $\sigma_L \subseteq G(1,3)$ be the special linear complex of lines intersecting L. For each $L \in \Lambda$, we define a morphism $\varphi_L : \sigma_L \cap \Lambda - \{L\} \to L^\vee \subseteq X^\vee$ via $\varphi_L(L') =$ plane spanned by L and $L^\vee$. Define two closed subsets of Λ by $\Lambda_1 =$ closure of $\{L \in \Lambda$ such that φ_L is dominating$\}$, and $\Lambda_2 =$ closure of $\Lambda - \Lambda_1$.

The assertion of the proposition is $\bigcup_{L \in \Lambda} L \neq X$. Assume the contrary, then either $\bigcup_{L \in \Lambda_1} L = X$ or $\bigcup_{L \in \Lambda_2} L = X$.

<u>First case:</u> $\bigcup_{L \in \Lambda_1} L = X$. By (3.3) we have that $L^\vee \subseteq W_o$ for each $L \in \Lambda_1$. Now (3.5) implies that $W_o = P_1^\vee \cup \ldots \cup P_r^\vee \cup W^1$ for some points $P_1,\ldots,P_r$ of X and such that W^1 contains no plane. For each $j = 1,\ldots,r$ put $\Lambda_{1,j} = \{L \in \Lambda_1$ such that $P_j \in L\}$. It follows that there exists an index j such that $\bigcup_{L \in \Lambda_{1,j}} L = X$, i.e. all lines through P_j are multiple jumping lines. Since there is at most one unstable plane (3.6) we can find a semistable plane containing P_j. But this contradicts (3.4).

<u>Second case:</u> $\bigcup_{L \in \Lambda_2} L = X$. For a general $L \in \Lambda_2$, φ_L is not dominating. Since Λ_2 must have at least one component of dimension 2, it follows that $\sigma_L \cap \Lambda - \{L\}$ is infinite, so φ_L must have at least one infinite closed fibre. In other words, there exists a plane H containing L with infinitely many multiple jumping lines. By (3.4) and (3.6) there is only one such plane H_2, and hence $\bigcup_{L \in \Lambda_2} L \subseteq H_1 \neq X$, which gives the desired contradiction.

(3.8) Combining (3.7) with the Grauert-Mülich theorem, the proof of Theorem 2 is now complete.

§ 4.

(4.1) If we try to generalize the methods of § 2 and § 3 to higher values of $n = c_2(E)$, we immediately encounter difficulties of various kinds, to be pointed out presently. First of all, the proof of theorem 2 does no longer hold if $n \geq 4$, although we conjecture that the theorem still holds true, at least for bundles with $H^1(E(-2)) = 0$, the socalled <u>mathematical instantons</u>. More serious is the fact that we do not have a firm grip on the set of plane curves C occurring. Also, given C and θ, the problem of classifying surjections $2\mathcal{O}_Y \to \theta(2)$ such that the kernel F satisfies $F_C \cong \theta^{-1}(-1) \oplus \theta^{-1}(-2)$ seems difficult.

One is tempted to conjecture that, for mathematical instantons, θ always satisfies the condition $(\alpha 3)$ of Barth (compare (1.11)). This would imply that $M^P_{inst}(n)$ is irreducible, non-singular and unirational of dimension $(8n-3)$, where $M_{inst}(n) \subseteq M_o(n)$ corresponds to mathematical instantons. Unfortunately, this conjecture is false, as we will show presently.

(4.2) Let $C \subseteq Y$ be any nonsingular curve of degree 4, and let θ be a θ-characteristic on C, i.e. a linebundle with $\theta^2 = \mathcal{O}_C(1)$. Then there exists a surjection $2\mathcal{O}_Y \to \theta(2)$ such that the kernel F restricts to $\theta^{-1}(-1) \oplus \theta^{-1}(-2)$ on C.

<u>Proof</u>: Let G_1 be the Grassmannian of 2-planes in the 8-dimensional vector-space $H^o(\theta(2))$, and let $K \subseteq \mathcal{O}_{G_1} \otimes_k H^o(\theta(2))$ be the

universal subspace.

Write $\mathcal{L} = \theta(2)$. On $C \times G_1$ there is induced a natural map $\varphi : \mathcal{O}_C \boxtimes K \to \mathcal{L} \boxtimes \mathcal{O}_{G_1}$; let $W \subseteq C \times G_1$ be the zero-scheme of φ. Then $\operatorname{codim} W = 2$, and the class of W is $c_2(\mathcal{L} \times K^\vee) = pr_G^*(c_2K^\vee) + pr_Y^*(\mathcal{L}) \circ pr_{G_1}^*(c_1K^\vee)$ in the ring $A^\bullet(C \times G_1)$ of cycles modulo numerical equivalence. The projection formula then gives the following expression for the class of $pr_{G_1}(W)$ in $A^\bullet(G_1)$:

$$[pr_G(W)] = pr_{G*}[W] = \text{degree}(\mathcal{L}) \circ c_1(K^\vee) = 10\ c_1(K^\vee).$$

Furthermore, we claim that W is irreducible. Indeed, let $\Delta \subseteq C \times C$ be the diagonal, and put $\mathcal{E} = pr_{1*}(\mathcal{O}_{C\times C}(-\Delta) \otimes pr_2^*\mathcal{L})$. The formation of $\mathcal{E}$ commutes with base change. There is a natural map $\mathcal{E} \hookrightarrow \mathcal{O}_C \underset{k}{\otimes} H^0(\mathcal{L})$ inducing a map $\text{Grass}(2, \mathcal{E}^\vee) \to C \times G_1$. It is straightforward to check that the image is exactly W, and that it is birational onto W. Therefore W is irreducible (and reduced). In particular, $p(W)$ is irreducible. Put $G = G_1 - p(W)$. By the exact sequence $\mathbf{Z} \xrightarrow{\alpha} \operatorname{Pic} G_1 \to \operatorname{Pic} G \to 0$ where $\alpha(1) = [p(W)] = 10\ c_1(K^\vee)$ and the fact that $c_1(K^\vee)$ generates $\operatorname{Pic} G_1 \cong \mathbf{Z}$, it follows that $\operatorname{Pic} G \cong \mathbf{Z}/10\mathbf{Z}$ and is generated by the restriction of $c_1(K^\vee)$. Let K also denote the restriction of K to G, then there is an exact sequence on $Y \times G$:

$$0 \to F \to \mathcal{O}_Y \boxtimes K \to \mathcal{L} \boxtimes \mathcal{O}_G \to 0.$$

Restricting this sequence to $C \times G$ we get an induced sequence

$$0 \to \mathcal{L}^{-1}(1) \boxtimes \mathcal{O}_G \to F_{C\times G} \to \mathcal{L}^{-1} \boxtimes \overset{2}{\wedge} K \to 0$$

Twisting by $\mathcal{L} \boxtimes (\overset{2}{\wedge}K)^{-1}$ and applying pr_{G*}, we get a map $\delta : \mathcal{O}_G \to (\overset{2}{\wedge}K)^{-1}$. Since all this commutes with base change, it is clear that at any closed point of G where δ vanishes, the corresponding induced F will split on C as stated in (4.2). But by what we have computed above, $(\overset{2}{\wedge}K)^{-1} = \overset{2}{\wedge}(K^{\vee})$ is non-zero in Pic G. In other words, δ does have zeroes in G, and we are finished.

(4.1) <u>Remark</u> By [2,prop.5], a general quartic curve can <u>not</u> be the divisor of jumping lines for a stable rank-2 vector bundle on $\mathbb{P}^2 = Y^{\vee}$. Stated differently, (4.2) may be phrased as follows: G(1,3) is, at the same time, also the Grassmannian of lines in the dual projective 3-space $X^{\vee}$. If $\Delta \subseteq G(1,3)$ is a divisor of degree $n \geq 4$, it is <u>not</u> true that Δ is the jumping line divisor of a 2-bundle on X if and only if it is a jumping line divisor of a 2-bundle on $X^{\vee}$. In other words, the set of possible Δ is not compatible with the intrinsique symmetry of G(1,3).

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