Compact ergodic groups of automorphisms

by

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Abstract. It is shown that if G is a compact ergodic group of $*$-automorphisms on a unital C^*-algebra A then the unique G-invariant state is a trace. Hence if A is a von Neumann algebra then it is finite.

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1. Introduction. Let A be a unital C^*-algebra, G a compact group and α a strongly continuous representation of G as an ergodic group of $*$-automorphisms of A, i.e. $\alpha_g(x) = x$ for all $g \in G$ implies x is a scalar operator. It was shown in [9] that if G is abelian and A a von Neumann algebra then A is automatically finite and the (necessarily unique) G-invariant state is a trace. Since then it has been an open problem whether the same is true without the assumption that G be abelian, see the introduction to [6]. In the present paper we solve this problem to the affirmative by showing that if G acts ergodically on the unital C^*-algebra A, then the G-invariant state is a trace. In the course of the proof of the theorem it will be shown that if D is an irreducible representation of G and $A(D)$ the corresponding spectral subspace in A, see below, then the multiplicity of D in $A(D)$ is not greater than the dimension of D. A consequence of this is that if G is second countable acting on a C^*-algebra then the action is cyclic if and only if it is ergodic.

The problem solved in this paper immediately raises the problem of classification of compact ergodic actions on C^*- or von Neumann algebras. If G is abelian this has been done completely in [1] and [6], and we can from those examples find nonabelian finite extensions of abelian ergodic actions on the hyperfinite II_1-factor. Another construction is to let for each positive integer i, G_i be an ergodic compact group of automorphisms on the complex $n_i \times n_i$ matrices, and then let the product group $G = \prod_{i=1}^{\infty} G_i$ act on the infinite tensor product of the matrix algebras in the obvious way. Then the GNS-representation due to the trace gives rise to an ergodic action of G on the hyperfinite factor. This is as far as we can go at present and we leave two basic problems open: (1) If a compact group acts ergodically on a II_1-factor M, is M hyperfinite ? (2) Find an example of a simple compact group acting ergodically on a II_1-factor.

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2. Compact ergodic groups.

Let A be a unital C^*-algebra, G a compact group, and suppose α is a strongly continuous representation of G as $*$-automorphisms of A, so $g \to \alpha_g(x)$ is norm continuous for all $x \in A$. We assume the action is ergodic on A, i.e. $\alpha_g(x) = x$ for all $g \in G$ only if x is a scalar operator. Then for each $x \in A$, $\int \alpha_g(x)dg$ is a scalar operator $\omega(x)1$, where dg is the normalized Haar measure on G. ω so defined is the unique G-invariant state on A.

If $f \in L^1(G)$ we denote by $\alpha(f)$ the operator on A defined by

$$\alpha(f)(x) = \int f(g)\alpha_g(x)dg .$$

Let D be an irreducible unitary representation of G and χ_D its normalized character $\chi_D(g) = \dim D \, \mathrm{Tr}(D_g^{-1})$, where Tr is the usual trace on the Hilbert space of dimension $\dim D$. Then $\alpha(\chi_D)$ is a projection of A onto a norm closed subspace $A(D)$ of A called the spectral subspace of D in A, see [3]. By [11, § 4.4.2] $A(D)$ is the set of $x \in A$ such that the linear span of $\alpha_g(x)$, $g \in G$, is finite dimensional and splits into a direct sum of irreducible components all unitarily equivalent to D.

Proposition 2.1. Let A be a unital C^*-algebra, G a compact group and α a strongly continuous representation of G as an ergodic group of $*$-automorphisms of A. Let D be an irreducible unitary representation of G, $A(D)$ the spectral subspace of D in A and $m(D)$ the multiplicity of D in $A(D)$. Then we have

(i) $m(D) \leq d$.

(ii) $\dim A(D) \leq d^2$.

Proof. If E is an irreducible unitary representation of G either α has no subrepresentation equivalent to E or there is an irreducible subspace V_E of A such that $\alpha | V_E$ is equivalent to E. Then $V_E \subset A(E)$, as follows from the characterization of $A(E)$ given above. Let D be as in the proposition. We may assume $V_D \neq 0$.

Consider A as imbedded in the Hilbert space obtained in the GNS-representation due to the invariant state ω. Thus $(a,b) = \omega(b^*a)$ is the inner product on A. Let $d = \dim D$. Then we can choose $a_1, \dots, a_d$ in V_D so they form an orthonormal basis for V_D. Then the map P_D defined by

$$P_D(a) = \sum_{i=1}^{d} (a, a_i) a_i$$

is a projection of A onto V_D, and since ω is G-invariant $\alpha_g(P_D(a)) = P_D(\alpha_g(a))$ for all $a \in A$. Thus the subspace

$(\iota - P_D)(A(D))$ of A, ι denoting the identity map, is a closed G-invariant subspace of A orthogonal to V_D. If $(\iota - P_D)(A(D)) \neq 0$ it contains an irreducible subspace V_E [7], and E is unitarily equivalent to D. Considering $P_D + P_E$ we have found a norm continuous projection onto $V_D + V_E$, and we can do this for any finite set of irreducible representations D_i equivalent to D, such that the spaces V_{D_i} are pairwise mutually orthogonal.

We fix now a finite set J of unitarily equivalent irreducible representations $D_1, \ldots, D_N$ such that their irreducible subspaces V_{D_k} of $A(D)$ are nonzero and pairwise mutually orthogonal. We shall show $N \leq d$, which will prove the proposition.

Choose $a_{ik} \in V_{D_k}$, $i = 1, \ldots, d$, so that they form an orthonormal basis for V_{D_k}, and such that they have the same action under G, i.e. there is an irreducible unitary representation $g \to (u_{rs}(g))$ of G into the complex $d \times d$ matrices M_d satisfying

$$\alpha_g(a_{ik}) = \sum_{j=1}^{d} u_{ij}(g) a_{jk}, \qquad k \in J. \tag{2.1}$$

For each pair $j,k \in J$ we have

$$\begin{aligned}\alpha_g\Big(\sum_{i=1}^{d} a_{ij}^* a_{ik}\Big) &= \sum_i \alpha_g(a_{ij})^* \alpha_g(a_{ik}) \\ &= \sum_{i,r,s} \overline{u_{ir}(g)}\, a_{rj}^* u_{is}(g) a_{sk} \\ &= \sum_r a_{rj}^* a_{rk}.\end{aligned}$$

Since G is ergodic $\sum_i a_{ij}^* a_{ik}$ is a scalar operator, the scalar being found by the computation

$$\omega\Big(\sum_i a_{ij}^* a_{ik}\Big) = \sum_i (a_{ik}, a_{ij}) = \sum_i \delta_{jk} = \delta_{jk} d.$$

Thus we have shown

$$\sum_{i=1}^{d} a_{ij}^* a_{ik} = \delta_{jk} d 1, \qquad j,k \in J. \tag{2.2}$$

Similarly we can find complex numbers c_{jk} such that

$$\sum_{i=1}^{d} a_{ij} a_{ik}^* = c_{jk}\, d1 , \qquad j,k \in J . \tag{2.3}$$

The $N \times N$ matrix (c_{jk}) is clearly self-adjoint, so we can find a unitary $N \times N$ matrix (α_{rs}) such that

$$\sum_{l,m=1}^{N} \alpha_{kl}\, c_{lm}\, \overline{\alpha_{jm}} = \delta_{jk} \lambda_j , \qquad j,k \in J$$

with $\lambda_j \in \mathbb{R}$. Let $a'_{ij} = \sum_{k=1}^{N} \alpha_{jk} a_{ik}$. Then $a'_{ij} \in \sum_{k=1}^{N} V_{D_k}$, and they form an orthonormal basis for $\sum_{k=1}^{N} V_{D_k}$. Note that

$$\alpha_g(a'_{ij}) = \sum_{r=1}^{d} u_{ir}(g) a'_{rj} ,$$

as is easily computed, hence we may replace a_{ij} by a'_{ij}, $i = 1,\dots,d$, $j \in J$, and still have that (2.1) is satisfied. We shall therefore do this and thus assume (2.1), (2.2), and the diagonal form of (2.3)

$$\sum_{i=1}^{d} a_{ij}\, a_{ik}^* = \delta_{jk} \lambda_j\, d1 , \qquad j,k \in J , \tag{2.4}$$

where $\lambda_j \in \mathbb{R}$. From (2.4) it is clear that $\lambda_j > 0$.

Denote by e the $d \times d$ matrix operator

$$e = \{ \sum_{k=1}^{N} a_{ik}\, a_{jk}^* \} \in A \otimes M_d , \qquad i,j \in \{1,\dots,d\} .$$

Clearly e is self-adjoint, and by (2.2) it satisfies

$$e^2 = \{ \sum_{k,l=1}^{N} \sum_{s=1}^{d} a_{ik}\, a_{sk}^*\, a_{sl}\, a_{jl}^* \} = \{ \sum_{k=1}^{N} a_{ik}\, a_{jk}^* d \} = de .$$

Hence $e = dp$ with p a projection, in particular $0 \leq e \leq d1$. Let τ denote the normalized trace on M_d. Then $\omega \otimes \tau$ is a state on $A \otimes M_d$, so by (2.4) we have

$$d \geq \omega \otimes \tau(e) = d^{-1} \sum_{k=1}^{N} \omega(\sum_{i=1}^{d} a_{ik}\, a_{ik}^*) = \sum_{k=1}^{N} \lambda_k . \tag{2.5}$$

We next assert that

$$(2.6)\qquad \omega(a_{ik}\, a_{jl}^*) = \delta_{ij}\,\delta_{kl}\,\lambda_k\ , \qquad i,j \in \{1,\ldots,d\},\ k,l \in J\ .$$

Indeed, fix $k,l \in J$, and let $\beta_{ij} = \omega(a_{ik}\, a_{jl}^*)$. Then (β_{ij}) is a $d \times d$ matrix which by (2.1) satisfies

$$\begin{aligned}\sum_{s=1}^{d} u_{is}(g)\beta_{sj} &= \omega(\sum_s u_{is}(g)a_{sk}\, a_{jl}^*)\\ &= \omega(\alpha_g(a_{ik})a_{jl}^*)\\ &= \omega(a_{ik}\,\alpha_{g^{-1}}(a_{jl})^*)\\ &= \sum_s \beta_{is}\, u_{sj}(g)\ .\end{aligned}$$

Therefore the matrix (β_{ij}) commutes with $(u_{is}(g))$ for all $g \in G$. Since the representation $g \to (u_{is}(g))$ is irreducible (β_{ij}) is a scalar operator, so (2.6) follows from (2.4).

Now consider the conjugate representation $\overline{D}$ to D. Since $a \in A(E)$ if and only if $\alpha(\chi_E)(a) = a$ for E an irreducible representation, it is immediate from the definition of $\alpha(\chi_E)$ that $a \in A(D)$ if and only if $a^* \in A(\overline{D})$. Thus by (2.6) if $b_{ij} = \lambda_j^{-\frac{1}{2}} a_{ij}^*$ then $\{b_{ij} : i = 1,\ldots,d\ ,\ j \in J\}$ form an orthonormal set in $A(\overline{D})$ for which (2.1) is replaced by

$$\alpha_g(b_{ik}) = \sum_{j=1}^{d} \overline{u_{ij}(g)}\, b_{jk}\ .$$

Since $g \to (\overline{u_{ij}(g)})$ is irreducible the space spanned by $\{b_{ik} : i = 1,\ldots,d\}$ is irreducible in $A(\overline{D})$ for each $k \in J$. Thus our previous discussion for D and the a_{ij} is valid for $\overline{D}$ and the b_{ij}. We have in particular by the equations (2.2) - (2.5)

$$(2.7)\qquad \sum_{i=1}^{d} b_{ij}\, b_{ik}^* = \delta_{jk}\,\mu_j\, d\, 1\ , \qquad j,k \in J\ ,$$

where $\mu_j > 0$ and $\sum_{j=1}^{N} \mu_j \leq d$. Computing we find by (2.2)

$$\omega(\sum_{i=1}^{d} b_{ij} b_{ij}^*) = \lambda_j^{-1} \omega(\sum_i a_{ij}^* a_{ij}) = \lambda_j^{-1} d ,$$

so that $\mu_j = \lambda_j^{-1}$ and therefore

$$\sum_{j=1}^{N} \lambda_j^{-1} \leq d . \tag{2.8}$$

Since $x+x^{-1} \geq 2$ whenever $x > 0$ we have by (2.5) and (2.8) that $2N \leq \sum_{j=1}^{N} (\lambda_j + \lambda_j^{-1}) \leq 2d$, so that $N \leq d$, as we wanted to show. Q.E.D.

Let A, G, and α be as in Proposition 2.1. Representing A in the GNS-representation defined by the G-invariant state ω we may assume $\omega(a) = (a\xi_0, \xi_0)$ for some cyclic vector ξ_0 for A in the Hilbert space. Furthermore there is a continuous unitary representation $g \to u_g$ of G on H such that $\alpha_g(a) = u_g a u_g^{-1}$ and $u_g \xi_0 = \xi_0$ for all $g \in G$, $a \in A$. Since ω is the unique G-invariant state on A, ω is the unique normal G-invariant state on the weak closure A^- of A, hence by [5], G is ergodic on A^- as well as A. Since the support projection for ω is a G-invariant projection in A^-, it is 1, hence ω is faithful on A^-, and ξ_0 is a separating vector for A^-. Let Δ denote the modular operator for ξ_0 with respect to A^-, and J the corresponding conjugation, so $a^*\xi_0 = J\Delta^{\frac{1}{2}} a\xi_0$ for all $a \in A^-$, see [10]. By [8] $u_g \Delta = \Delta u_g$ and $J u_g = u_g J$, $g \in G$, hence in particular the finite dimensional subspace $A(D)\xi_0$ is invariant under the action of $\Delta^{\frac{1}{2}}$, so under Δ, recall $A(D)\xi_0 = \{\int \chi_D(g) u_g a \, dg \, \xi_0 : a \in A\}$. By equation (2.6) we have with $N = m(D)$, so $\sum_{k=1}^{N} V_{D_k} = A(D)$,

$$(\Delta a_{ij}\xi_0, a_{kl}\xi_0) = (a_{kl}^*\xi_0, a_{ij}^*\xi_0) = \delta_{ik}\delta_{jl}\lambda_j = \lambda_j (a_{ij}\xi_0, a_{kl}\xi_0) .$$

Hence $a_{ij}\xi_0$ is an eigenvector for Δ with eigenvalue λ_j. Hence we have from (2.5) and (2.8)

Corollary 2.2. Let A, G, α, D be as in Proposition 2.1. Let ξ_0 be the cyclic vector in the GNS-representation defined by the G-invariant state ω. Then ξ_0 is separating for A^-, and if Δ is its modular operator then Δ leaves the finite dimensional vector space $A(D)\xi_0$ invariant. If λ is an eigenvalue for $\Delta|A(D)\xi_0$ then both $\lambda \leq \dim D$ and $\lambda^{-1} \leq \dim D$.

We shall also need the probably well known observation

Lemma 2.3. Let M be a von Neumann algebra and G an ergodic group of *-automorphisms of M. Suppose V is a nonzero globally G-invariant linear subspace of M. If $x \in M$, denote by $r(x)$ and $s(x)$ respectively the range and support projections of x. Then we have

$$\bigvee_{x\in V} r(x) = \bigvee_{x\in V} s(x) = 1 .$$

Proof. If α is a *-automorphism of M then α is ultraweakly continuous, so by the construction of $r(x)$ by spectral theory on the positive operator xx^*, we see that $\alpha(r(x)) = r(\alpha(x))$ for $x \in M$. Thus $\bigvee_{x\in V} r(x)$ and $\bigvee_{x\in V} s(x)$ are nonzero G-invariant projections in M, hence are equal to 1 by ergodcity.

Q.E.D.

3. Tensor representations. In this section we shall apply Herman Weyl's classical theory for representations of groups, to obtain estimates for the dimensions of irreducible subspaces of powers of G-invariant subspaces of an ergodic group G.

If V is a finite dimensional complex vector space we denote by $V^{(m)}$ the tensor product $V\otimes\cdots\otimes V$ (m times). If π is a representation of a group G on V, π has a corresponding representation π^m of G on $V^{(m)}$ defined by $\pi^m(g) = \pi(g)\otimes\cdots\otimes\pi(g)$.

Lemma 3.1. Let V be a finite dimensional complex vector space with $\dim V = n$. Consider $Gl(n,\mathbb{C})$ as acting on V and consider the corresponding representation of $Gl(n,\mathbb{C})$ on $V^{(m)} = V\otimes\cdots\otimes V$. Then any irreducible subspace U of $V^{(m)}$ satisfies

$$\dim U \leq (1+m)^{\frac{n(n-1)}{2}} .$$

Proof. Let π denote the representation of $Gl(n,\mathbb{C})$ on V. By [2, p. 192] we can decompose the representation π^m of $Gl(n,\mathbb{C})$ on $V^{(m)}$ into irreducible components as follows:

$$V^{(m)} = \sum_{\lambda_1+\cdots+\lambda_n=m} l_\lambda D_\lambda$$

where $\lambda = (\lambda_1,\ldots,\lambda_n)$, λ_i is a nonnegative integer for each $i \in \{1,\ldots,n\}$,

$$l_\lambda = \frac{m!}{\prod_{i,j} h_{ij}} , \qquad h_{ij} = 1+\lambda_i+\bar{\lambda}_j-(i+j) ,$$

and $\bar{\lambda}_j$ is the number of boxes in the j^{th} column in the Young tableau corresponding to λ [2, p. 192, eq. (23)]. $l_\lambda D_\lambda$ means that the irreducible representation D_λ is repeated l_λ times, and D_λ is the irreducible representation of $Gl(n,\mathbb{C})$ with highest weight $\lambda = (\lambda_1,\ldots,\lambda_n)$.

Set now $l_j = \lambda_j + n-j$ and $l_j^o = n-j$. Then the Weyl formula, see [2, p. 283, eq. (32)] gives that

$$\dim D_\lambda = \frac{\prod_{i<j}(l_i - l_j)}{\prod_{i<j}(l_i^o - l_j^o)} .$$

Hence

$$\dim D_\lambda = \prod_{1\le i<j\le n} \left(1+\frac{\lambda_i-\lambda_j}{i-j}\right) \le (1+m)^{\frac{n(n-1)}{2}}$$

Q.E.D.

Proposition 3.2. Let G be a group of $*$-automorphisms on a C^*-algebra A, and suppose V is a finite dimensional linear subspace of A which is globally invariant under G. Let $\dim V = n$, and let for $m \in \mathbb{N}$, V^m denote the linear subspace of A generated by products of m elements in V. Then V^m is again globally invariand under G, and for each subspace $U \subset V^m$ globally invariant and irreducible under the action of G we have

$$\dim U \le (1+m)^{\frac{n(n-1)}{2}}$$

Proof. Let π be the representation of G on V and π^m the corresponding representation on $V^{(m)}$. Let j_m be the m-linear map of $V^{(m)}$ onto V^m given by

$$j_m(x_1 \otimes \cdots \otimes x_m) = x_1 \cdots x_m .$$

Then j_m intertwines the representation π^m and the action of G on V^m, i.e.

$$j_m \circ \pi^m(g) = \pi(g) \circ j_m , \qquad g \in G .$$

Therefore j_m takes invariant subspaces of $V^{(m)}$ onto invariant subspaces of V^m. Since the dimension of the image of a subspace is not greater than the dimension of the subspace, it suffices to show that for any invariant subspace U of $V^{(m)}$ irreducible under the action of $\pi^m(G)$ we have $\dim U \leq (1+m)^{\frac{n(n-1)}{2}}$.

Denote by ι the representation of $Gl(n,\mathbb{C})$ on V, and ι^m the corresponding representation on $V^{(m)}$. Then $\pi^m(G) \subset \iota^m(Gl(n,\mathbb{C}))$. By Lemma 3.1 any irreducible invariant subspace for $\iota^m(Gl(n,\mathbb{C}))$ has dimension at most $(1+m)^{\frac{n(n-1)}{2}}$. Hence any subgroup and especially $\pi^m(G)$ also has the property that any irreducible invariant subspace has dimension at most $(1+m)^{\frac{n(n-1)}{2}}$. Thus $\dim U \leq (1+m)^{\frac{n(n-1)}{2}}$.

Q.E.D.

4. The main results.

Theorem 4.1. Let A be a unital C^*-algebra, G a compact group, and α a strongly continuous representation of G as an ergodic group of *-automorphisms of A. Then the unique G-invariant state on A is a trace.

Proof. Since G is compact A is generated by the spectral subspaces $A(D)$, as D runs through the irreducible unitary representations of G [7]. Thus it suffices to show that each $A(D)$ is contained in the centralizer of the invariant state, or equivalently by Corollary 2.2 and [10], to show that all the eigenvalues of Δ restricted to $A(D)\xi_0$ are equal to 1, ξ_0 being the G-invariant separating and cyclic vector in the GNS-representation due to the invariant state. Suppose λ is one of them. By Corollary 2.2 we may assume $\lambda \geq 1$. Let V be a G-invariant subspace of $A(D)$ such that $\Delta a\xi_0 = \lambda a\xi_0$ for all $a \in V$ and such that V is irreducible under the action of G. This is possible since $\Delta u_g = u_g\Delta$ for all $g \in G$. For each $m \in \mathbb{N}$, if V^m is the space generated by products of m elements in V, for each $a \in V^m$, $a\xi_0$ is an eigenvector for Δ with eigenvalue λ^m, as is easily seen since $y \to \Delta^{it} y \Delta^{-it}$ is an automorphism of the weak closure of A. Since G is ergodic an easy induction argument based on Lemma 2.3 shows that $V^m \neq 0$, and by Proposition 3.2 each subspace U of V^m which is globally invariant and irreducible under the action of G has dimension $\dim U \leq (1+m)^{\frac{n(n-1)}{2}}$, where $n = \dim V$. By Corollary 2.2 $\lambda^m \leq \dim U$, hence $\lambda^m \leq (1+m)^{\frac{n(n-1)}{2}}$. Thus

$$0 \leq \log\lambda \leq \frac{n(n-1)}{2m}\log(1+m),$$

which is arbitrarily small for large m, so that $\log\lambda = 0$, and $\lambda = 1$. Since λ was an arbitrary eigenvalue for Δ restricted to an arbitrary subspace $A(D)\xi_0$ with D an irreducible representation of G, $\Delta = 1$, and ξ_0 is a trace vector for A.

Q.E.D.

If M is a von Neumann algebra, G a topological group and α a representation if G as $*$-automorphisms of M, we say α is continuous if $g \to \rho(\alpha_g(x))$ is continuous on G for each $\rho \in M_*$, $x \in M$.

Corollary 4.2. Let M be a von Neumann algebra and G a compact group. If there is a continuous representation of G as an ergodic group of *-automorphisms on M then M is finite.

Proof. It is well known that the set A of $x \in M$ such that the function $g \to \alpha_g(x)$ is norm continuous on G is a C^*-algebra globally invariant under G and weakly dense in M. Let ω be a normal G-invariant state on M. Then $\omega|A$ is G-invariant, hence is a trace by Theorem 4.1. By density of A in M, ω is a trace on M. Since by ergodicity ω is faithful, M is finite.
Q.E.D.

The next result is a generalization of Corollary 4.2 and shows that compact automorphism groups in general have very large fixed point algebras.

Corollary 4.3. Let M be a von Neumann algebra of type III, G a compact group, and α a continuous representation of G as *-automorphisms of M. Then the fixed point algebra M^G of G in M contains no minimal projections.

Proof. $M^G = \{x \in M : \alpha_g(x) = x,\ g \in G\}$. Suppose to the contrary that e is a nonzero minimal projection in M^G. Then G acts ergodically on the reduced algebra M_e by $\alpha_g(exe) = e\alpha_g(x)e$. By Corollary 4.2 M_e is finite contradicting the fact that it is of type III since M is.
Q.E.D.

Let A be a C^*-algebra, G a group, and α a representation of G as *-automorphisms of A. Suppose ω is a G-invariant state. We say α is <u>cyclic with respect to</u> ω if there is $x \in A$

such that $\omega(y\alpha_g(x)) = 0$ for all $g \in G$ implies $y = 0$. We shall see below that if G is compact and α is a continuous representation of G as an ergodic group, then cyclicity of G means that the orbit of $x\xi_0$ under G in the GNS-representation due to the unique G-invariant trace, is dense in the Hilbert space.

Lemma 4.4. Let A be a unital C^*-algebra, G a compact group and α a strongly continuous representation of G as $*$-automorphisms of A. Suppose ω is a G-invariant state such that α is cyclic with respect to ω. Then α is an ergodic representation, and ω is the unique G-invariant state.

Proof. Let A^G denote the fixed point algebra of G in A. Since G is compact the adjoint of the map

$$y \to \int_G \alpha_g(y)dg$$

of A onto A^G defines an affine isomorphism between the G-invariant states of A and the state space of A^G. Suppose there is $x \in A$ such that $\omega(y\alpha_g(x)) = 0$ for all $g \in G$ implies $y = 0$. Then if $y \in A^G$ we have $\omega(y\alpha_g(x)) = \omega(\alpha_g^{-1}(y)x) = \omega(yx)$, so the functional $y \to \omega(yx)$ is injective on A^G. But this is only possible if A^G is the scalars.

Q.E.D.

The next theorem is a direct analogue for representations of compact groups as $*$-automorphisms on C^*-algebras, of a result of Greenleaf and Moskowitz on unitary representations on Hilbert space [4].

Theorem 4.5. Let A be a unital C^*-algebra and G a second countable compact group. Suppose α is a strongly continuous representation of G as $*$-automorphisms of A. Then α is an ergodic representation if and only if α is cyclic with respect to some G-invariant state.

Proof. By Lemma 4.4 we only have to show that if α is ergodic and ω is the unique G-invariant state, then α is cyclic with respect to ω. By Proposition 2.1 if D is an irreducible representation of G then its multiplicity in the spectral subspace $A(D)$ of A is not greater than $\dim D$. Thus there is $x_D \in A(D)$ of norm one such that the linear span of $\alpha_g(x_D)$, $g \in G$, equals $A(D)$. Indeed, in the notation of the proof of Proposition 2.1 we may choose $x_D = c\sum_{i=1}^{m(D)} a_{ii}$ for a suitable scalar $c > 0$. Since G is second countable and compact its dual $\hat{G}$ is countable, hence there is a countable number of spectral subspaces $A(D)$. Number them by $A(D_k)$, $k \in \mathbb{N}$. For each k choose $x_{D_k} \in A(D_k)$ of norm one as above, and let $x = \sum_{k=1}^{\infty} 2^{-k} x_{D_k}$ (if $\hat{G}$ is finite let the sum be finite). Then $\|x\| \leq 1$ and $x \in A$. We show that the linear span of the orbit of $x\xi_0$, ξ_0 being the G-invariant separating and cyclic vector in the GNS-representation due to ω, is dense in the underlying Hilbert space H, hence in particular that α is cyclic with respect to ω.

Let $\xi \in H$ satisfy $(\xi, \alpha_g(x)\xi_0) = 0$ for all $g \in G$. Let u denote the unitary representation of G on H such that $u_g a u_g^{-1} = \alpha_g(a)$ and $u_g \xi_0 = \xi_0$ for all $g \in G$, $a \in A$. Let D be an irreducible representation of G and χ_D the corresponding normalized character. Then $u(\chi_D) = \int \chi_D(g) u_g \, dg$ is the orthogonal projection of H onto the subspace $A(D)\xi_0$. Let $D = D_k$ be one of the irreducible representations described above. Then $\alpha(\chi_D)(x) = 2^{-k}x_D$. Let $h \in G$, then $\alpha(\chi_D)(\alpha_h(x))\xi_0 \in A(D)\xi_0$, hence we have

$$\begin{aligned}(u(\chi_D)\xi, u_h x_D \xi_0) &= (\xi, u(\chi_D)u_h x_D \xi_0) \\ &= 2^k(\xi, \alpha(\chi_D)\alpha_h(x)\xi_0) \\ &= 2^k \int \chi_D(g)(\xi, \alpha_{gh}(x)\xi_0)dg \\ &= 0\end{aligned}$$

by assumption on ξ. Since $\mathrm{span}\{u_h x_D \xi_0 : h \in G\} = A(D)\xi_0$, $u(\chi_D)\xi = 0$ for each $D = D_k$. Since the subspaces $A(D_k)\xi_0$ are mutually orthogonal and span H, $\xi = \sum_{k=1}^{\infty} u(\chi_{D_k})\xi = 0$.

Q.E.D.

References

1. S. Albeverio and R. Høegh-Krohn, Ergodic actions by compact groups on C*-algebras, Math. Zeitschrift, to appear.
2. A.O. Barut and R. Raczka, Theory of group representations and applications, PWN-Polish Scientific Publishers, Warszawa 1977.
3. D.E. Evans and T. Sund, Spectral subspaces for compact actions, Reports in Math. Phys., to appear.
4. F. Greenleaf and M. Moskowitz, Cyclic vectors for representations of locally compact groups, Math. Annalen 190 (1971), 265-288.
5. I. Kovács and J. Szücs, Ergodic type theorems in von Neumann algebras, Acta Sci. Math. 27 (1966), 233-246.
6. D. Olesen, G.K. Pedersen, and M. Takesaki, Ergodic actions of compact abelian groups, to appear.
7. K. Shiga, Representations of a compact group on a Banach space, J. Math. Soc. Japan, 7 (1955), 224-248.
8. E. Størmer, Automorphisms and invariant states of operator algebras, Acta Math. 127 (1971), 1-9.
9. E. Størmer, Spectra of ergodic transformations, J. Funct. Anal. 15 (1974), 202-215.
10. M. Takesaki, Tomita's theory of modular Hilbert algebras and its applications, Springer-Verlag, Lecture notes in Math, 128 (1970).
11. G. Warner, Harmonic analysis on semi-simple Lie groups, I, Springer-Verlag 1972.