1. Introduction

In [5] we gave a $j$-algebraic characterization of quasi-symmetric domains among bounded, homogeneous domains. In this paper we specialize to symmetric Siegel domains. We use the notation and definitions of [5]. A quasi-symmetric domain $\mathcal{D}(\Omega, F) = \{(z, w) \in U \times V | \text{Im} z - F(u, u) \in \Omega\}$ (U is an $\mathbb{R}$-vector space, $V$ is a $C$-vector space, $F : V \times V \rightarrow \mathbb{C}$ is "$\Omega$-hermitian", $\Omega$ is a "nice" cone in $U$) is symmetric precisely when

$$R_aR_bF(b, d)^b = R_F(R_a b, d)^b \quad \forall a \in U, \forall b, d \in V,$$

where $U \ni a = R_a \in \text{End}(V)$ is Satake's linear map. See [3], [4].

Now in the $j$-algebraic description of $\mathcal{D}(\Omega, F)$ given in [2], we have $U = \mathcal{C}$ and $V = \mathcal{U}$, where $\mathcal{C} = \mathcal{C} + j\mathcal{C} + \mathcal{U}$ is the corresponding $j$-algebra. Here $\mathcal{C}$ is a Lie algebra, $j \in \text{End}(\mathcal{C})$ satisfies $j^2 = -\text{Id}$ and $[X, Y] + j[jX, Y] + j[X, jY] - [jX, jY] = 0 \forall X, Y \in \mathcal{C}$, $\mathcal{C}$ is an abelian ideal of $\mathcal{C}$, $j\mathcal{C}$ is a subalgebra, $[\mathcal{C}, \mathcal{U}] \subset \mathcal{C}$, $[j\mathcal{C}, \mathcal{U}] \subset \mathcal{C}$ and $[\mathcal{C}, \mathcal{U}] = 0$. Also there is a linear form $w$ on $\mathcal{C}$ such that $w[jX, X] > 0$ if $X \neq 0$ and $w[jX, jY] = w[X, Y]$. Then we have the $j$-invariant positive definite inner product $\langle X, Y \rangle = w[jX, Y]$ on $\mathcal{C}$. Also ([2]) $\mathcal{C} = \mathcal{H} + \sum \mathcal{H}_\alpha$, vector space direct sum, where $\mathcal{H} = [\mathcal{C}, \mathcal{C}]^\perp$ is the $\langle , \rangle$-orthogonal complement to $[\mathcal{C}, \mathcal{C}]$, and $[\mathcal{C}, \mathcal{C}] = \sum \mathcal{H}_\alpha$ with root spaces $\mathcal{H}_\alpha = \{X \in [\mathcal{C}, \mathcal{C}] | [H, X] = \alpha(h) X \forall H \in \mathcal{H}\}$ where the root $\alpha$ is a linear form on $\mathcal{H}$. Here $\mathcal{H}$ is an abelian subalgebra. It is shown in [2] that if $\alpha_1, \ldots, \alpha_p$ are all the roots $\alpha$ such that $j\mathcal{H}_\alpha \subset \mathcal{H}$, then $\mathcal{H} = j\mathcal{H}_{\alpha_1} + \ldots + j\mathcal{H}_{\alpha_p}$
and \( \dim \mathfrak{k} = p \), and further that all roots are of the form \( \alpha_k, i\alpha_k \) with \( 1 \leq k \leq p \), \( \frac{1}{2}(\alpha_k \pm \alpha_m) \) with \( 1 \leq k < m \leq p \). We have

\[
  j \mathfrak{k}_{\frac{1}{2}}(\alpha_k + \alpha_m) = \mathfrak{k}_{\frac{1}{2}}(\alpha_k - \alpha_m) \quad \text{and} \quad j \mathfrak{k}_{\frac{1}{2}} \alpha_k = \mathfrak{k}_{\frac{1}{2}} \alpha_k.
\]

We put [2]

\[
  \mathcal{C} = \sum_{k=1}^{p} \mathfrak{k}_{\alpha_k} + \sum_{1 \leq k < m \leq p} \mathfrak{k}_{\frac{1}{2}(\alpha_k + \alpha_m)} \quad \text{and} \quad \mathcal{U} = \sum_{k=1}^{p} \mathfrak{k}_{i\alpha_k}
\]

give \( \mathcal{U} \) the complex structure \( j \). It is easy to see that \( [\mathfrak{k}_{\alpha}, \mathfrak{k}_{\beta}] = \mathfrak{k}_{\alpha + \beta} \) and that \( \mathfrak{k}_{\alpha} \perp \mathfrak{k}_{\beta} \) if \( \alpha \neq \beta \). (\( \mathfrak{k}_{\alpha + \beta} = (0) \) if \( \alpha + \beta \) is no root). Also \( \dim \mathfrak{k}_{\alpha_k} = 1 \), and there is a unique element \( E_k \in \mathfrak{k}_{\alpha_k} \setminus \{0\} \)

such that \( [jE_k, E_k] = E_k \). Put \( E := E_1 + \ldots + E_p \). The adjoint representation of the subalgebra \( j\mathcal{C} \) on the ideal \( \mathcal{C} \) gives a corresponding representation of the simply connected group \( G_0 \) whose Lie algebra is \( j\mathcal{C} \). Then [2] \( \Omega := G_0 \cdot E \) is an open, convex cone in \( \mathcal{C} \) with vertex at the origin, and not containing an entire straight line. By construction \( \Omega \) is homogeneous, i.e. \( \text{Gl}(\Omega) := \{ g \in \text{Gl}(\mathcal{C}) \mid g \Omega = \Omega \} \) is transitive on \( \Omega \). Finally,

\[
  F(u,v) := \frac{1}{4}[ju,v] + \frac{1}{4}i[u,v]
\]

is an \( \Omega \)-hermitian form \( F: \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{C} \). See [2]. (Of course the \( \frac{1}{4} \) is inessential).

We can now state

**Theorem.** If a quasi-symmetric, irreducible, bounded, homogeneous domain \( \mathcal{D} \) is described by the \( j \)-algebra \( \mathcal{G} = \mathcal{C} + j\mathcal{C} + \mathcal{U}, \mathcal{w} \), then \( \mathcal{D} \) is symmetric if and only if

\[
  R_F(b,d)b = 0 \quad \text{whenever} \quad b \in \mathcal{U}_m, \quad d \in \mathcal{U}_n, \quad m \neq n.
\]

**Remark.** A similar theorem was proved by Dorfmeister [1] in his big set up.

Since any bounded, homogeneous domain can be described by a \( j \)-algebra [2] the theorem gives a simple algebraic characterization
of the symmetric, bounded domains, given the \( j \)-algebraic reali-
tation.

The rest of this paper is devoted to proving the theorem \( j \)-alge-
braically, using notation and results from \([5]\). The proof is
rather computational.

2. Proof of the theorem.

I. Assume the condition is satisfied. We show that (1) is satis-
fied.

(a) If \( b \in \mathcal{V}_n \), \( d \in \mathcal{V}_m \), \( m < n \), then \( R_F(R_a b, d) b = 0 \) \( \forall a \in \mathcal{F} \)

Indeed, considering cases, we use \([5]\), \( \S \, 2 \):

\[
R_E(\Sigma) = \frac{1}{2} u_k, \quad R_{L_{kn}}(\Sigma u_1) = \frac{1}{2} [jL_{kn}, u_n] +
\]

(2)

\[
\frac{1}{2} (adj L_{kn})' u_k \in \mathcal{V}_k, \quad \forall L_{kn} \in \mathcal{F}(k, n) = \mathcal{F}(a_k + a_n),
\]

where \((\cdot)'\) means transpose w.r.t. \( \langle, \rangle \). Then:

i) If \( a = E_k \), \( k \neq n \), then \( R_a b = 0 \).

ii) If \( a = E_n \), then \( R_F(R_a b, d) b = R_F(\frac{1}{2} b, d) b = \frac{1}{2} R_F(b, d) b = 0 \).

iii) If \( a = L_{kl} \in \mathcal{F}(k, l) \), \( n \neq k, l \), then \( R_a b = 0 \).

iv) If \( a = L_{kn} \), then \( R_a b = \frac{1}{2} [jL_{kn}, b] \in \mathcal{V}_k \), so

\( F(R_a b, d) \in F(\mathcal{V}_k, \mathcal{V}_m) \in \mathcal{F}(k, m) \).

Here \( k, m < n \). Now \( R_a b = 0 \) for \( \forall a \in \mathcal{F}(k, m) \). Thus, \( a \to R_a \in \text{End}_c(\mathcal{V}) \) is extended
linearly to \( \mathcal{V} \to \text{End}_c(\mathcal{V}) \).

v) If \( a \in L_{hl} \), then \( R_a b = \frac{1}{2} (adj L_{hl})' b \in \mathcal{V}_l \), \( F(R_a b, d) \in \mathcal{F}(m, l) \),

and \( R_F(R_a b, d) b = 0 \) again, as in iv).
(b) Suppose \( m = n \). We have \( F(b,d) \in \mathcal{K}_n \), where \( \mathcal{K}_n = \mathcal{K}_n^* \), and \( RF(b,d) b \in \mathcal{U}_n \). Then

i) If \( a = E_k \), \( k \neq n \), then \( Ra(\mathcal{U}_n) = 0 \) implies that both sides of 1) vanish.

ii) If \( a = E_n \), then \( Ra|_{\mathcal{U}_n} = \frac{1}{n}Id \), hence

\[
RaF(b,d) b = \frac{1}{n}RF(b,d) b = RF(\frac{1}{n}b,d) b = RF(Ra b,d) b
\]

iii) If \( a = L_{kl} \in \mathcal{K}_n \), \( n \neq k,l \), then \( Ra(\mathcal{U}_n) = 0 \), so both sides of 1) vanish.

iv) If \( a = L_{kn} \), then by 2) we have \( RaRF(b,d) b = \frac{1}{n}[jL_{kn},RF(b,d)b] \in \mathcal{U}_n \), and \( Ra b = \frac{1}{n}[jL_{kn},b] \in \mathcal{U}_k \), and we want to show that

\[
[jL_{kn},RF(b,d)b] = RF([jL_{kn},b],d)b.
\]

Now \([jb,d] = \lambda E_n \), some \( \lambda \). Applying \( w \), we get

\[
[jb,d] = \frac{\langle b,d \rangle}{\kappa} E_n \text{, where } \kappa = w(E_n) \text{ (independent of } n \text{ for an irreducible, quasi-symmetric domain, by [5]).}
\]

Using the form of \( F \), we see

\[
4F(b,d) = \frac{1}{\kappa} [\langle b,d \rangle - i\langle jb,d \rangle] E_n.
\]

By 2) we see

\[
RaF(b,d)b = \frac{1}{2\kappa} [\langle b,d \rangle - i\langle jb,d \rangle] b, \text{ where } ib := jb. \text{ We get}
\]

\[
8[jL_{kn},RF(b,d)b] = \frac{1}{\kappa} [\langle b,d \rangle [jL_{kn},b] - \langle jb,d \rangle [jL_{kn},jb]].
\]

Further, using 2) and the form of \( F \), we get

\[
8RF([jL_{kn},b],d)b = [j[j[jL_{kn},b],d],b] + j[j[[jL_{kn},b],d],b].
\]
We have

(6) \([ju,v] = [jv,u]\) for \(u \in \mathcal{L}_a, v \in \mathcal{L}_b, a \neq b\), and

(7) \(j[jL_{kn}, b] = [jL_{kn}, jb]\) for \(L_{kn} \in \mathcal{K}(k,n), b \in \mathcal{L}_n\).

Both of these identities are proved by the four-term defining relation for a \(j\)-algebra, by considering the root-spaces (some of which may be zero) in which the terms lie. A particular case of 6) is

(8) \([jL_{kn}, b], d] = -[[jL_{kn}, b], jd].\)

Further, by Leibniz identity, one proves, using above results and the fact ([5], § 2) that \([jL_{kn}, E_n] = L_{kn}\):

(9) \([[jL_{kn}, b], d] = [[jL_{kn}, d], b] - \frac{\langle jb, d \rangle}{\kappa} L_{kn}.\)

Using 8), 9) and 7), and the \(j\)-invariance of \(\langle , \rangle\), we find

(10) \([j[jL_{kn}, b], d], b] = -[j[jL_{kn}, d], b], b] + \frac{\langle b, d \rangle}{\kappa} [jL_{kn}, b].\)

Again, by 7) and 9), we have

(11) \([j][jL_{kn}, b], d], b] = j[j[[jL_{kn}, d], b], b] - \frac{\langle jb, d \rangle}{\kappa} [jL_{kn}, jb].\)

By 4) and 5) we get now, using 10 and 11):

(12) \(8[RF([jL_{kn}, b], d) b + [jL_{kn}, R_F(b, d)] b]\)

\(= [j][jL_{kn}, d], b], b] - [j][jL_{kn}, d], b].\)

Now let \(v = [jL_{kn}, d] \in \mathcal{L}_k, (k < n)\). Then by assumption, and using 2) and 6):

\(0 = 8R_F(b, v) b = 2R jb, v] b + 2jR[b, v] b\)

\(= [j][jb, v], b] + [j][j[b, v], b] = [j][jv, b], b] - [j][v, b], b] \)
= right hand side of 12). This proves 3).

v) If \( a = L_{nl} \in \mathcal{K}(n,l) \), \( n < 1 \), then the calculation is similar. Instead of \( \text{adj} a \), we must use, according to 2), \( (\text{adj} a)' \). So we want to prove

\[
(13) \quad (\text{adj} L_{nl})' R_F(b,d)b = R_F((\text{adj} L_{nl})'b,d)b.
\]

In place of 4) we have

\[
(14) \quad 8(\text{adj} L_{nl})' R_F(b,d)b = \frac{1}{\mathcal{K}} \{ \langle b,d \rangle (\text{adj} L_{nl})'b - \langle jb,d \rangle (\text{adj} L_{nl})'(jb) \},
\]

and in place of 5) we have

\[
(15) \quad 8R_F((\text{adj} L_{nl})'b,d)b = (\text{adj}[j(adj) L_{nl}']b,d)b + j(adj[L_{nl}']b,d)'.
\]

In place of 7) we have, by [5], § 2,

\[
(16) \quad j(adj L_{nl})'u = (adj L_{nl})'(ju) \text{ for } u \in \mathcal{L}_n.
\]

In place of the Leibnitz identity we have, in the quasi-symmetric case, the condition \( Q'' \) of [5], § 4:

\[
(adj L)'[b,d] = [(adj L)'b,d] + [b,(adj L)'d] \text{ for } b,d \in \mathcal{L}, L \in \mathcal{G}.
\]

Using this and the fact ([5], § 4) that \( (adj L_{nl})'E_n = E_n \), we obtain in place of 9):

\[
(17) \quad [(adj L_{nl})'b,d] = [(adj L_{nl})'d,b] - \frac{\langle jb,d \rangle}{\mathcal{K}} L_{nl}.
\]

We then get, in place of 10) and 11):

\[
(18) \quad (adj[j(adj) L_{nl}']b,d)']b = -(adj[j(adj) L_{nl}']d,b)']b + \frac{\langle b,d \rangle}{\mathcal{K}} (adj L_{nl})'b,
\]

\[
(19) \quad j(adj[(adj) L_{nl}']b,d)']b = j(adj[(adj) L_{nl}']d,b)']b - \frac{\langle jb,d \rangle}{\mathcal{K}} (adj L_{nl})'(jb).
\]
In place of 12) we get

\[(20) \quad 8\{ -RF((adj_{Ln})'b,d)b + (adj_{Ln})'RF(b,d)b \} \]

\[= (adj[j(adj_{Ln})'d,b])'b - j(adj[(adj_{Ln})'d,b])'b. \]

Letting \( v := (adj_{Ln})'d \in \Omega_1 \), \((1 > n)\), we get as before (using assumptions)

\[0 = 8RF(b,v)b = (adj[jb,v])'b + j(adj[b,v])'b \]

\[= (adj[jv,b])'b - j(adj[v,b])'b = \text{right hand side of 20}. \]

This proves 13).

(c) Suppose \( m > n \). One proves easily, practically the same way as in (a), that \( RF(R_{ab}b,d)b = 0 \ \forall a \) in this case.

(d) Now since condition 1) is not linear in \( b \), we still have something to do. Suppose \( b = \Sigma b_k \) with \( b_k \in \Omega_k \). Expanding \( RaRF(\Sigma b_k,d)\Sigma b_1 \) and \( RF(Ra \Sigma b_k,d)\Sigma b_1 \), and noting that we have proved that our assumption implies \( RaRF(b_k,d)b_k = RF(R_{ab}b_k,d)b_k' \), we need to establish the equality

\[(21) \quad RaRF(b_k,d)b_1 + RaRF(b_1,d)b_k = RF(R_{ab}b_k,d)b_1 + RF(R_{ab}b_1,d)b_k \forall k < 1. \]

(By symmetry, and by what we have proved, 21) will then hold for \( \forall k,l \).

We can assume \( d = d_n \in \Omega_n \).

(a) Suppose \( n \neq k, l \). Then \( F(b_k,d_n) \in \hat{K}_{(k,n)} \) implies \( RF(b_k,d_n)b_1 = 0 \), and similarly \( RF(b_1,d_n)b_k = 0 \). Hence left hand side of 21) equals zero.
Similarly for 1, and since \( R^F(b_k, d_n)b_l = 0 = R^F(b_1, d_n)b_k \), the right hand side of (21) vanishes.

ii) If \( a = L_{st} e \in \mathcal{F}(s, t) \), and if \( n < k < 1 \), then a possibly non-zero right hand side can occur only when \( s \) or \( t \) equals \( k \) or 1 by 2). In fact 2) shows that we need only check the case \( s = k, t = 1 \), since otherwise both terms on the right hand side vanish. Now in terms of the Jordan product on \( U \) (see [3], [4]) we have \( L^*F(u, v) = F(R_L u, v) + F(u, R_L v) \), because of quasi-symmetry. Furthermore (see [3])

\[
R_{L^*M} = R_L R_M + R_M R_L.
\]

So the right hand side of (21) equals

\[
R^L_{kl}F(b_k, d_n) - F(b_k, R^L_{kl} d_n)b_l + R^F(R^L_{kl} b_l, d_n)b_k
\]

\[
= R^L_{kl} R^F(b_k, d_n)b_l - R^F(b_k, R^L_{kl} d_n)b_l
\]

\[
+ \{ R^F(b_k, d_n)R^L_{kl} b_l + R^F(R^L_{kl} b_l, d_n)b_k \}.
\]

The first two terms vanish since \( F(b_k, d_n) \in \mathcal{F}(n, k) \) and \( R^L_{kl} d_n = 0 \) (see 2)). Putting \( \bar{b}_k := R^L_{kl} b_l \in \mathcal{F}_k \), the expression equals

\[
R^F(b_k, d_n)\bar{b}_k + R^F(b_k, d_n)\bar{b}_k = R^F(b_k + \bar{b}_k, d_n)(b_k + \bar{b}_k) - R^F(b_k, d_n)b_k - R^F(b_k, d_n)\bar{b}_k = 0,
\]

by assumptions.

The cases \( k < n < 1 \) and \( k < 1 < n \) are similar.

(3) Suppose \( k < n < 1 \). Then \( F(b_k, d_k) \in \mathcal{F}_k \), so \( R^F(b_k, d_k)b_l = 0 \).
Further $R_F(b_1, d_k)^{b_k} \in \mathcal{L}_1$ since $F(b_1, d_k) \in \mathcal{P}(k, l) \not\subset$, and
the left hand side of (21) is $R_a F(b_1, d_k)^{b_k}$.

i) If $a = L_t$, then the left hand side of (21) is contained
in $R \mathcal{L}_1$, which is non-zero only if $r = 1$, and in that
case the left hand side equals $\frac{1}{R} F(b_1, d_k)^{b_k} = \frac{1}{R} F(b_1, d_k)^{b_k}
= R_F(R_a b_1, d_k)^{b_k} = \text{right hand side of (21)},$ since $R_a b_k = 0.$
For $r \neq k, l$ the right hand side equals zero, as we want.
For $r = k$ we have $R_a b_1 = 0$, $R_a b_k = 0$, so the right
hand side equals $\frac{1}{R} F(b_k, d_k)^{b_1} = 0$, since $F(b_k, d_k) \in \mathcal{P}(k, l),$ again as we want.

ii) If $a = L_s \in \mathcal{P}(s, t)$, we have to check that

$$R_a F(b_1, d_k)^{b_k} = R_F(R_a b_k, d_k)^{b_1} + R_F(R_a b_1, d_k)^{b_k}.$$  

Here the right hand side equals

$$R_F(R_a b_k, d_k)^{b_1} + R_a F(b_1, d_k)^{b_1} - F(b_1, R_a d_k)^{b_k}$$

$$= R_F(R_a b_k, d_k)^{b_1} + R_a F(b_1, d_k)^{b_1} + R_F(b_1, d_k)^{R_a b_k} - R_f(b_1, R_a d_k)^{b_k}.$$  

So we have to check that

$$R_F(R_a b_k, d_k)^{b_1} + R_F(b_1, d_k)^{R_a b_k} = R_F(b_1, R_a d_k)^{b_k}.$$  

Here the left hand side is non-zero only if $(s, t) = (k, l)$
(see 2)), in which case it equals zero, since, letting
$\mathcal{L}_1 : = R_a b_k \in \mathcal{L}_1$, we have $R_F(b_1, d_k)^{b_1} + R_F(b_1, d_k)^{b_1} = 0,$
just as in case a) ii) above. The right hand side is non-zero only if $R_a d_k \not= 0$ and $F(b_1, R_a d_k) \in \mathcal{P}(k, l) \not\subset$. 
But this is impossible (see 2)).

(γ) If \( k < l = n \), the argument is as in case (β). This proves one way of the theorem.

II. Now suppose we have symmetry. Then in 1) let \( b \in \mathcal{U}_k, \ d \in \mathcal{U}_l, \ k \neq l \), and thus \( R_F(b, d)b \in \mathcal{U}_l \) (see 2)).

Let \( a = E_l \). Then left hand side of 1) equals \( \frac{1}{2}R_F(b, d)b \), and the right hand side vanishes since \( R_a b = 0 \).

Hence \( R_F(b, d)b = 0 \). \( \text{q.e.d.} \)

Bibliography.


