"A symmetry-condition for quasi-symmetric domains.

1. Introduction

In [5] we gave a j-algebraic characterization of quasi-symmetric domains among bounded, homogeneous domains. In this paper we specialize to symmetric Siegel domains. We use the notation and definitions of [5]. A quasi-symmetric domain $\mathfrak{D}(\Omega,F) = \{(z,u) \in U_{\mathbb{C}} \times V \mid \operatorname{Im} z - F(u,u) \in \Omega\}$ (U is an $\mathbb{R}$-vector space, V is a $\mathbb{C}$-vector space, $F : V \times V \to U_{\mathbb{C}}$ is "Ω-hermitian", Ω is a "nice" cone in U) is symmetric precisely when

$$(1) \qquad R_a R_{F(b,d)} b = R_{F(R_a b,d)} b \ \forall a \in U,\ \forall b,d \in V,$$

where $U \ni a \to R_a \in \operatorname{End}(V)$ is Satake's linear map. See [3], [4]. Now in the j-algebraic description of $\mathfrak{D}(\Omega,F)$ given in [2], we have $U = \mathfrak{t}$ and $V = \mathfrak{u}$, where $\mathfrak{g} = \mathfrak{t} + j\mathfrak{t} + \mathfrak{u}$ is the corresponding j-algebra. Here $\mathfrak{g}$ is a Lie algebra, $j \in \operatorname{End}(\mathfrak{g})$ satisfies $j^2 = -\mathrm{Id}$ and $[X,Y] + j[jX,Y] + j[X,jY] - [jX,jY] = 0 \ \forall X,Y \in \mathfrak{g}$, $\mathfrak{t}$ is an abelian ideal of $\mathfrak{g}$, $j\mathfrak{t}$ is a subalgebra, $[\mathfrak{u},\mathfrak{u}] \subset \mathfrak{t}$, $[j\mathfrak{t},\mathfrak{u}] \subset \mathfrak{u}$ and $[\mathfrak{t},\mathfrak{u}] = 0$. Also there is a linear form ω on $\mathfrak{g}$ such that $\omega[jX,X] > 0$ if $X \neq 0$ and $\omega[jX,jY] = \omega[X,Y]$. Then we have the j-invariant positive definite inner product $\langle X,Y \rangle := \omega[jX,Y]$ on $\mathfrak{g}$. Also ([2]) $\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{k}_\alpha$, vector space direct sum, where $\mathfrak{h} = [\mathfrak{g},\mathfrak{g}]^\perp$ is the $\langle\,,\rangle$-orthogonal complement to $[\mathfrak{g},\mathfrak{g}]$, and $[\mathfrak{g},\mathfrak{g}] = \sum_\alpha \mathfrak{k}_\alpha$ with <u>root spaces</u> $\mathfrak{k}_\alpha = \{X \in [\mathfrak{g},\mathfrak{g}] \mid [H,X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$ where the <u>root</u> α is a linear form on $\mathfrak{h}$. Here $\mathfrak{h}$ is an abelian subalgebra. It is shown in [2] that if $\alpha_1,\ldots,\alpha_p$ are all the roots α such that $j\mathfrak{k}_\alpha \subset \mathfrak{h}$, then $\mathfrak{h} = j\mathfrak{k}_{\alpha_1} + \ldots + j\mathfrak{k}_{\alpha_p}$

and $\dim \mathfrak{h} = p$, and further that all roots are of the form $\alpha_k, \frac{1}{2}\alpha_k$ with $1 \leq k \leq p$, $\frac{1}{2}(\alpha_k \pm \alpha_m)$ with $1 \leq k < m \leq p$. We have $j\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)} = \mathfrak{k}_{\frac{1}{2}(\alpha_k-\alpha_m)}$ and $j\mathfrak{k}_{\frac{1}{2}\alpha_k} = \mathfrak{k}_{\frac{1}{2}\alpha_k}$. We put [2] $\mathcal{C} := \sum_{k=1}^{p} \mathfrak{k}_{\alpha_k} + \sum_{1\leq k<m\leq p} \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$ and $\mathcal{U} := \sum_{k=1}^{p} \mathfrak{k}_{\frac{1}{2}\alpha_k}$ and give $\mathcal{U}$ the complex structure j. It is easy to see that $[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha+\beta}$ and that $\mathfrak{k}_\alpha \perp \mathfrak{k}_\beta$ if $\alpha \neq \beta$. ($\mathfrak{k}_{\alpha+\beta} := (0)$ if $\alpha + \beta$ is no root). Also $\dim \mathfrak{k}_{\alpha_k} = 1$, and there is a unique element $E_k \in \mathfrak{k}_{\alpha_k} - \{0\}$ such that $[jE_k, E_k] = E_k$. Put $E := E_1 + \dots + E_p$. The adjoint representation of the subalgebra $j\mathcal{C}$ on the ideal $\mathcal{C}$ gives a corresponding representation of the simply connected group $\mathcal{G}_0$ whose Lie algebra is $j\mathcal{C}$. Then [2] $\Omega := \mathcal{G}_0 \circ E$ is an open, convex cone in $\mathcal{C}$ with vertex at the origin, and not containing an entire straight line. By construction Ω is homogeneous, i.e. $Gl(\Omega) := \{g \in Gl(\mathcal{C}) \mid g\Omega = \Omega\}$ is transitive on Ω. Finally,

$F(u,v) := \frac{1}{4}[ju,v] + \frac{1}{4}i\,[u,v]$ is an Ω-hermitian form $F: \mathcal{U} \times \mathcal{U} \to \mathcal{C}_{\mathbb{C}}$. See [2]. (Of course the $\frac{1}{4}$ is inessential). We can now state

<u>Theorem</u>. If a quasi-symmetric, irreducible, bounded, homogeneous domain $\mathcal{D}$ is described by the j-algebra $(\mathfrak{g} = \mathcal{C} + j\mathcal{C} + \mathcal{U}, \omega)$, then $\mathcal{D}$ is symmetric if and only if

$$R_{F(b,d)}b = 0 \quad \text{whenever} \quad b \in \mathcal{U}_m,\ d \in \mathcal{U}_n,\quad m \neq n.$$

<u>Remark</u>. A similar theorem was proved by Dorfmeister [1] in his big set up.

Since any bounded, homogeneous domain can be described by a j-algebra [2] the theorem gives a simple algebraic characterization of the symmetric, bounded domains, given the j-algebraic reali a-tion.

The rest of this paper is devoted to proving the theorem j-algebraically, using notation and results from [5]. The proof is rather computational.

2. Proof of the theorem.

I. Assume the condition is satisfied. We show that (1) is satisfied.

(a) If $b \in \mathcal{U}_n$, $d \in \mathcal{U}_m$, $m<n$, then $R_{F(R_a b,d)} b = 0 \ \forall a \in \mathcal{L}$

Indeed, considering cases, we use [5], § 2:

$$R_{E_k}(\Sigma u_l) = \tfrac{1}{2} u_k, \quad R_{L_{kn}}(\Sigma u_l) = \tfrac{1}{2}[jL_{kn}, u_n] + \tag{2}$$

$$\tfrac{1}{2}(\operatorname{ad} j\, L_{kn})' u_k \in \mathcal{U}_k + \mathcal{U}_n \quad \forall L_{kn} \in \mathfrak{k}_{(k,n)} := \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_n)},$$

where $(\ )'$ means transpose w.r.t. $\langle , \rangle$. Then:

i) If $a = E_k$, $k \neq n$, then $R_a b = 0$.

ii) If $a = E_n$, then $R_{F(R_a b,d)} b = R_{F(\frac{1}{2}b,d)} b = \tfrac{1}{2} R_{F(b,d)} b = 0$.

iii) If $a = L_{kl} \in \mathfrak{k}_{(k,l)}$, $n \neq k,l$, then $R_a b = 0$.

iv) If $a = L_{kn}$, then $R_a b = \tfrac{1}{2}[jL_{kn}, b] \in \mathcal{U}_k$, so $F(R_a b,d) \in F(\mathcal{U}_k, \mathcal{U}_m) \in \mathfrak{k}_{(k,m)\mathbb{C}}$. Here $k,m<n$. Now $R_{\tilde{a}} b = 0$ for $\tilde{a} \in \mathfrak{k}_{(k,m)\mathbb{C}}$. ($\mathcal{L} \ni a \to R_a \in \operatorname{End}_{\mathbb{C}}(\mathcal{U})$ is extended linearly to $\mathcal{L}_{\mathbb{C}} \to \operatorname{End}_{\mathbb{C}}(\mathcal{U})$).

v) If $a \in L_{nl}$, then $R_a b = \tfrac{1}{2}(\operatorname{ad} j\, L_{nl})' b \in \mathcal{U}_l$, $F(R_a b,d) \in \mathfrak{k}_{(m,l)\mathbb{C}}$, and $R_F(R_a b,d)\, b = 0$ again, as in iv).

(b) Suppose $m = n$. We have $F(b,d) \in \mathfrak{k}_{n\mathbb{C}}$, where $\mathfrak{k}_n := \mathfrak{k}_{\alpha_n}$, and $R_{F(b,d)}\, b \in \mathcal{U}_n$. Then

i) If $a = E_k$, $k \neq n$, then $R_a(\mathcal{U}_n) = 0$ implies that both sides of 1) vanish.

ii) If $a = E_n$, then $R_a|_{\mathcal{U}_n} = \frac{1}{2}\mathrm{Id}$, hence

$$R_a R_{F(b,d)}\, b = \tfrac{1}{2} R_{F(b,d)}\, b = R_{F(\frac{1}{2}b,d)}\, b = R_{F(R_a b,d)}\, b$$

iii) If $a = L_{kl} \in \mathfrak{k}_{(k,l)}$, $n \neq k,l$, then $R_a(\mathcal{U}_n) = 0$, so both sides of 1) vanish.

iv) If $a = L_{kn}$, then by 2) we have $R_a R_{F(b,d)} b = \frac{1}{2}[jL_{kn}, R_{F(b,d)} b] \in \mathcal{U}_k$, and $R_a b = \frac{1}{2}[jL_{kn}, b] \in \mathcal{U}_k$, and we want to show that

$$[jL_{kn}, R_{F(b,d)} b] = R_{F([jL_{kn},b],d)} b. \tag{3}$$

Now $[jb,d] = \lambda E_n$, some λ. Applying ω, we get $[jb,d] = \frac{\langle b,d\rangle}{\varkappa} E_n$, where $\varkappa = \omega(E_n)$ (independent of n for an irreducible, quasi-symmetric domain, by [5]). Using the form of F, we see $4F(b,d) = \frac{1}{\varkappa}\{\langle b,d\rangle - i\langle jb,d\rangle\}E_n$. By 2) we see $R_{4F(b,d)} b = \frac{1}{2\varkappa}\{\langle b,d\rangle - i\langle jb,d\rangle\} b$, where $ib := jb$. We get

$$8[jL_{kn}, R_{F(b,d)} b] = \frac{1}{\varkappa}\{\langle b,d\rangle [jL_{kn}, b] - \langle jb,d\rangle [jL_{kn}, jb]\}. \tag{4}$$

Further, using 2) and the form of F, we get

$$8R_{F([jL_{kn},b],d)} b = [j[j[jL_{kn},b],d],b] + j[j[[jL_{kn},b],d],b]. \tag{5}$$

We have

(6) $[ju,v] = [jv,u]$ for $u \in \mathcal{U}_a$, $v \in \mathcal{U}_b$, $a \neq b$, and

(7) $j[jL_{kn},b] = [jL_{kn},jb]$ for $L_{kn} \in \mathcal{k}_{(k,n)}$, $b \in \mathcal{U}_n$.

Both of these identities are proved by the four-term defining relation for a j-algebra, by considering the root-spaces (some of which may be zero) in which the terms lie.

A particular case of 6) is

(8) $[j[jL_{kn},b],d] = -[[jL_{kn},b],jd]$.

Further, by Leibniz identity, one proves, using above results and the fact ([5], § 2) that $[jL_{kn},E_n] = L_{kn}$:

(9) $[[jL_{kn},b],d] = [[jL_{kn},d],b] - \frac{\langle jb,d\rangle}{\varkappa} L_{kn}$.

Using 8), 9) and 7), and the j-invariance of $\langle , \rangle$, we find

(10) $[j[j[jL_{kn},b],d],b] = -[j[j[jL_{kn},d],b],b] + \frac{\langle b,d\rangle}{\varkappa}[jL_{kn},b]$.

Again, by 7) and 9), we have

(11) $j[j[[jL_{kn},b],d],b] = j[j[[jL_{kn},d],b],b] - \frac{\langle jb,d\rangle}{\varkappa}[jL_{kn},jb]$.

By 4) and 5) we get now, using 10 and 11):

(12) $8\{-R_{F}([jL_{kn},b],d)^{b} + [jL_{kn},R_{F}(b,d)^{b}]\}$

$= [j[j[jL_{kn},d],b],b] - j[j[[jL_{kn},d],b],b]$.

Now let $v := [jL_{kn},d] \in \mathcal{U}_k$, $(k<n)$. Then by assumption, and using 2) and 6):

$0 = 8R_{F}(b,v)^{b} = 2R_{[jb,v]}^{b} + 2jR_{[b,v]}^{b}$

$= [j[jb,v],b] + j[j[b,v],b] = [j[jv,b],b] - j[j[v,b],b]$

= right hand side of 12). This proves 3).

v) If $a = L_{nl} \in \mathcal{k}_{(n,1)}$, $(n<1)$, then the calculation is similar. Instead of adja, we must use, according to 2), $(\mathrm{adj}a)'$. So we want to prove

$$(13)\qquad (\mathrm{adj}L_{nl})' R_{F(b,d)} b = R_{F((\mathrm{adj}L_{nl})'b,d)} b .$$

In place of 4) we have

$$(14)\qquad 8(\mathrm{adj}L_{nl})' R_{F(b,d)} b = \frac{1}{\varkappa}\{\langle b,d\rangle(\mathrm{adj}L_{nl})'b - \langle jb,d\rangle(\mathrm{adj}L_{nl})'(jb)\},$$

and in place of 5) we have

$$(15)\qquad 8R_{F((\mathrm{adj}L_{nl})'b,d)} b = (\mathrm{adj}[j(\mathrm{adj}L_{nl})'b,d])'b + j(\mathrm{adj}[(\mathrm{adj}L_{nl})'b,d])'b.$$

In place of 7) we have, by [5], § 2,

$$(16)\qquad j(\mathrm{adj}L_{nl})'u = (\mathrm{adj}L_{nl})'(ju) \quad \text{for} \quad u \in \mathcal{U}_n .$$

In place of the Leibnitz identity we have, in the quasi-symmetric case, the condition (Q'') of [5], § 4:

$$(\mathrm{adj}L)'[b,d] = [(\mathrm{adj}L)'b,d] + [b,(\mathrm{adj}L)'d] \quad \text{for} \quad b,d \in \mathcal{U},\ L \in \mathcal{T} .$$

Using this and the fact ([5], § 4) that $(\mathrm{adj}L_{nl})'E_n = E_n$, we obtain in place of 9):

$$(17)\qquad [(\mathrm{adj}L_{nl})'b,d] = [(\mathrm{adj}L_{nl})'d,b] - \frac{\langle jb,d\rangle}{\varkappa} L_{nl} .$$

We then get, in place of 10) and 11):

$$(18)\qquad (\mathrm{adj}[j(\mathrm{adj}L_{nl})'b,d])'b = -(\mathrm{adj}[j(\mathrm{adj}L_{nl})'d,b])'b + \frac{\langle b,d\rangle}{\varkappa}(\mathrm{adj}L_{nl})'b,$$

$$(19)\qquad j(\mathrm{adj}[(\mathrm{adj}L_{nl})'b,d])'b = j(\mathrm{adj}[(\mathrm{adj}L_{nl})'d,b])'b - \frac{\langle jb,d\rangle}{\varkappa}(\mathrm{adj}L_{nl})'(jb).$$

In place of 12) we get

(20) $8\{-R_{F((\mathrm{adj}L_{nl})'b,d)}b + (\mathrm{adj}L_{nl})'R_{F(b,d)}b\}$

$= (\mathrm{adj}[j(\mathrm{adj}L_{nl})'d,b])'b - j(\mathrm{adj}[(\mathrm{adj}L_{nl})'d,b])'b$.

Letting $v := (\mathrm{adj}L_{nl})'d \in \mathcal{U}_l$, $(l > n)$, we get as before (using assumptions)

$0 = 8R_{F(b,v)}b = (\mathrm{adj}[jb,v])'b + j(\mathrm{adj}[b,v])'b$

$= (\mathrm{adj}[jv,b])'b - j(\mathrm{adj}[v,b])'b$ = right hand side of 20).

This proves 13).

(c) Suppose $m > n$. One proves easily, practically the same way as in (a), that $R_{F(R_a b,d)}b = 0 \ \forall a$ in this case.

(d) Now since condition 1) is not linear in b, we still have something to do. Suppose $b = \Sigma b_k$ with $b_k \in \mathcal{U}_k$. Expanding $R_a R_{F(\Sigma b_k,d)}\Sigma b_l$ and $R_{F(R_a \Sigma b_k,d)}\Sigma b_l$, and noting that we have proved that our assumption implies $R_a R_{F(b_k,d)}b_k = R_{F(R_a b_k,d)}b_k$, we need to establish the equality

(21) $R_a R_{F(b_k,d)}b_l + R_a R_{F(b_l,d)}b_k = R_{F(R_a b_k,d)}b_l + R_{F(R_a b_l,d)}b_k \ \forall \ k < l.$

(By symmetry, and by what we have proved, 21) will then hold for $\forall\, k,l$).

We can assume $d = d_n \in \mathcal{U}_n$.

(α) Suppose $n \neq k,l$. Then $F(b_k,d_n) \in \mathcal{k}_{(k,n)\mathbb{C}}$ implies $R_{F(b_k,d_n)}b_l = 0$, and similarly $R_{F(b_l,d_n)}b_k = 0$. Hence left hand side of 21) equals zero.

i) If $a = E_r$, then $R_a b_k = \frac{1}{2} b_k$ if $r = k$, and $R_a b_k = 0$ if $r \neq k$. Similarly for l, and since $R_{F(b_k,d_n)} b_l = 0 = R_{F(b_l,d_n)} b_k$, the right hand side of 21) vanishes.

ii) If $a = L_{st} \in \mathcal{K}_{(s,t)}$, and if $n < k < l$, then a possibly non-zero right hand side can occur only when s or t equals k or l by 2). In fact 2) shows that we need only check the case $s = k$, $t = l$, since otherwise both terms on the right hand side vanish. Now in terms of the Jordan product $\circ$ on U (see [3], [4]) we have $L \circ F(u,v) = F(R_L u, v) + F(u, R_L v)$, because of quasi-symmetry. Furthermore (see [3]) $R_{L \circ M} = R_L R_M + R_M R_L$. So the right hand side of 21) equals

$$R_{L_{kl} \circ F(b_k,d_n) - F(b_k, R_{L_{kl}} d_n)} b_l + R_{F(R_{L_{kl}} b_l, d_n)} b_k$$

$$= R_{L_{kl}} R_{F(b_k,d_n)} b_l - R_{F(b_k, R_{L_{kl}} d_n)} b_l$$

$$+ \{R_{F(b_k,d_n)} R_{L_{kl}} b_l + R_{F(R_{L_{kl}} b_l, d_n)} b_k\}.$$

The first two terms vanish since $F(b_k,d_n) \in \mathcal{K}_{(n,k)\mathbb{C}}$ and $R_{L_{kl}} d_n = 0$ (see 2)). Putting $\tilde{b}_k := R_{L_{kl}} b_l \in \mathcal{U}_k$, the expression equals

$$R_{F(b_k,d_n)} \tilde{b}_k + R_{F(\tilde{b}_k,d_n)} b_k = R_{F(b_k+\tilde{b}_k,d_n)} (b_k+\tilde{b}_k) - R_{F(b_k,d_n)} b_k - R_{F(\tilde{b}_k,d_n)} \tilde{b}_k = 0,$$

by assumptions.

The cases $k < n < l$ and $k < l < n$ are similar.

(β) Suppose $k < n < l$. Then $F(b_k,d_k) \in \mathcal{K}_{k\mathbb{C}}$, so $R_{F(b_k,d_k)} b_l = 0$.

Further $R_{F(b_1,d_k)}b_k \in \mathcal{U}_1$ since $F(b_1,d_k) \in \mathfrak{k}_{(k,1)\mathbb{C}}$, and the left hand side of 21) is $R_a R_{F(b_1,d_k)} b_k$.

i) If $a = E_r$, then the left hand side of 21) is contained in $R_{\mathfrak{k}_r}(\mathcal{U}_1)$, which is non-zero only if $r = 1$, and in that case the left hand side equals $\frac{1}{2}R_{F(b_1,d_k)}b_k = R_{F(\frac{1}{2}b_1,d_k)}b_k$

$= R_{F(R_a b_1,d_k)}b_k$ = right hand side of 21), since $R_a b_k = 0$. For $r \neq k,1$ the right hand side equals zero, as we want. For $r = k$ we have $R_a b_1 = 0$, $R_a b_k = \frac{1}{2}b_k$, so the right hand side equals $\frac{1}{2}R_{F(b_k,d_k)}b_1 = 0$, since $F(b_k,d_k) \in \mathfrak{k}_{k\mathbb{C}}$, again as we want.

ii) If $a = L_{st} \in \mathfrak{k}_{(s,t)}$, we have to check that

$$R_a R_{F(b_1,d_k)} b_k = R_{F(R_a b_k,d_k)} b_1 + R_{F(R_a b_1,d_k)} b_k .$$

Here the right hand side equals

$$R_{F(R_a b_k,d_k)} b_1 + R_{a \circ F(b_1,d_k) - F(b_1,R_a d_k)} b_k$$

$$= R_{F(R_a b_k,d_k)} b_1 + R_a R_{F(b_1,d_k)} b_k + R_{F(b_1,d_k)} R_a b_k - R_{F(b_1,R_a d_k)} b_k .$$

So we have to check that

$$R_{F(R_a b_k,d_k)} b_1 + R_{F(b_1,d_k)} R_a b_k = R_{F(b_1,R_a d_k)} b_k .$$

Here the left hand side is non-zero only if $(s,t) = (k,1)$ (see 2)), in which case it equals zero, since, letting $\tilde{b}_1 := R_a b_k \in \mathcal{U}_1$, we have $R_{F(\tilde{b}_1,d_k)} b_1 + R_{F(b_1,d_k)} \tilde{b}_1 = 0$, just as in case α) ii) above. The right hand side is non-zero only if $R_a d_k \neq 0$ and $F(b_1,R_a d_k) \in \mathfrak{k}_{(k,1)\mathbb{C}}$.

But this is impossible (see 2)).

(γ) If $k < l = n$, the argument is as in case (β). This proves one way of the theorem.

II. Now suppose we have symmetry. Then in 1) let $b \in \mathcal{U}_k$, $d \in \mathcal{U}_l$, $k \neq l$, and thus $R_{F(b,d)}b \in \mathcal{U}_l$ (see 2)).
Let $a = E_l$. Then left hand side of 1) equals $\frac{1}{2}R_{F(b,d)}b$, and the right hand side vanishes since $R_a b = 0$.
Hence $R_{F(b,d)}b = 0$. q.e.d.

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