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ON SMOOTH DOMAINS

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1. In this short note we will discuss regularization of plurisubharmonic functions. More precisely, we will address the following problem:

Question. Assume Ω is a bounded domain in \mathbb{C}^n ($n \geq 2$) with smooth (C^∞) boundary and that $\rho: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is a (discontinuous) plurisubharmonic function. Does there exist a sequence

$$\{\rho_n\}_{n=1}^\infty, \quad \rho_n: \Omega \rightarrow \mathbb{R}, \quad \text{of } C^\infty$$

plurisubharmonic functions such that

$$\rho_n \searrow \rho \quad \text{pointwise?}$$

If ρ is continuous, the answer to the above question is yes (see Richberg [3]). On the other hand, when ρ is allowed to be discontinuous and Ω is not required to have a smooth boundary, the answer is in general no (see [1], [2] for this and related questions).

Our result in this paper is that the answer to the above question is no. We present a counterexample in the next section. The construction leaves open what happens if we make the further requirement that Ω has real analytic boundary. Another question, suggested to the author by Grauert, is obtained by replacing Ω by a compact complex manifold with smooth boundary, and assuming

continuity of ρ .

In the next section we need of course both to construct the domain Ω and the function ρ . These constructions are intertwined and therefore we need at first to define approximate solutions Ω_1 and ρ_1 and then use both to define Ω and ρ . The geometric properties we seek of Ω are the following. There exists an annulus $A \subset \bar{\Omega}$ such that $\partial A \subset \Omega$. Furthermore there exist concentric circles C_1, C_2, C_3 in the relative interior of A arranged by increasing radii such that $C_1, C_3 \subset \partial \Omega$ and $C_2 \subset \Omega$. Finally there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli such that $A_n \rightarrow A$ and $A_n \subset \Omega \forall n$. The properties we seek of ρ are as follows. The function ρ is strictly positive on C_2 and is strictly negative on ∂A . A simple application of the maximum principle now shows that smoothing is impossible.

The example we construct is in \mathbb{C}^2 . This is with no loss of generality as one obtains then an example in \mathbb{C}^n by crossing with a smooth domain in \mathbb{C}^{n-2} , rounding off the edges and pulling back ρ to the new domain.

2. All domains and functions which we will consider in $\mathbb{C}^2(z, w)$ will be invariant under rotations in the z -plane, i.e. will depend only on $|z|$. They will also be invariant under the map $(z, w) \rightarrow (1/z, w)$. Because of the latter we will describe only those points (z, w) in these domains or domains of definitions for which $|z| \leq 1$.

If U is a domain in $\mathbb{C}^2(z, w)$, we let U_z denote the part of U over z , i.e. $U_z := \{(\eta, w) \in \mathbb{C}^2; \eta = z \text{ and } (\eta, w) \in U\}$. Abusing notation we will also take U_z to mean the set $\{w \in \mathbb{C}; (z, w) \in U\}$.

Similarly, if $\sigma : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is a function, then σ_z denotes the restriction of σ to U_z .

Let A be the annulus in \mathbb{C}^2 given by $A = \{(z, w); w = 0 \text{ and } 1/2 \leq |z| \leq 2\}$. This is then the limit of a sequence of annuli $\{A_n\}_{n=1}$ where $A_n = \{(z, w); w = 1/n \text{ and } 1/2 \leq |z| \leq 2\}$. We will next describe a bounded domain Ω_1 in \mathbb{C}^2 with C^∞ boundary containing all A_n 's (and hence A) in its closure. It will suffice to describe $\Omega_{1,z}$ for various z 's. That these can be made to add up to a domain with C^∞ boundary will be clear throughout.

Choose a sequence of positive numbers $\{r_k\}_{k=1}^\infty$, $0 < r_1 < r_2 < \dots < 1$, with $r_3 = 1/2$. We let $\Omega_{1,z} = \emptyset$ if $|z| \leq r_1$ and $\Omega_{1,z}$ be a nonempty disc, concentric about the origin if $r_1 < |z| \leq r_4$. Recall that $\Omega_{1,z} = \Omega_{1,|z|}$ for all z . If $r_2 \leq |z| \leq r_4$ we make the extra assumption that $\Omega_{1,z}$ has radius 2. For $|z| > r_4$ we will break the symmetry in the w -direction at first by letting $\Omega_{1,z}$ gradually approach the shape of an upper-disc. (This is a rough description to be made more precise below.) Increasing $|z|$ further we will rotate this approximate upper half disc 180° clockwise until it becomes approximately a lower half disc. Then we proceed by reversing the process, first by rotating counterclockwise back to an approximate upper half disc and then expanding this back to a disc of radius 2 near $|z| = 1$. As mentioned earlier, if $|z| > 1$, then $\Omega_{1,z} := \Omega_{1,1/z}$.

We now return to the more precise description of $\Omega_{1,z}$ for $|z| > r_4$. Writing $w = u + iv$ in real coordinates u, v , let $v = f(u)$ be

a C^∞ function defined for $u \in \mathbb{R}$ with $f(u) = 0$ if $u \leq 0$ or $u \geq 2$, $f \geq 0$ and $f(u) = 0$ on $(0,2)$ if and only if $u = 1/n$ for some positive integer n . We may assume that $|f|$, $|f'|$, $|f''|$ are very small and therefore in particular that the graph of f only intersects the boundary of any disc $\Delta(0;R) = \{|w| < R\}$ in exactly two points. If $r_4 < |z| < r_5$, we let $\Omega_{1,z}$ be a subdomain of $\Delta(0;2)$ containing those $u + iv \in \Delta(0;3/2)$ for which $v \geq f(u)$. When $r_5 \leq |z| \leq r_6$ we choose $\Omega_{1,z}$ independent of z with the properties that $\Omega_{1,z} \subset \Delta(0;7/4) \cap \{v > f(u)\}$ and $\Delta(0;3/2) \cap \{v > f(u)\} \subset \Omega_{1,z}$. Let $\theta(x)$ be a real C^∞ function on \mathbb{R} with $\theta(x) = 0$ if $x \leq r_6$, $\theta(x) = \pi$ if $x \geq r_7$ and $\theta'(x) > 0$ if $r_6 < x < r_7$. Then we can rotate $\Omega_{1,z}$ 180° clockwise for $r_6 \leq |z| \leq r_7$ by defining $\Omega_{1,z} = e^{-i\theta(|z|)} \Omega_{1,r_6}$ for such z . Further, we let $\Omega_{1,z} = \Omega_{1,r_7}$ when $r_7 \leq |z| \leq r_8$. Reversing the procedure, we rotate $\Omega_{1,z}$ back 180° when $r_8 \leq |z| \leq r_9$ so that $\Omega_{1,z} = \Omega_{1,r_9}$ again equals Ω_{1,r_6} . Continuing, we let $\Omega_{1,z} = \Omega_{1,r_9}$ whenever $r_9 \leq |z| \leq r_{10}$. Reversing the procedure between r_4 and r_5 we obtain $\Omega_{1,z}$'s, $r_{10} \leq |z| \leq r_{11}$ so that in particular $\Omega_{1,r_{11}}$ is the disc $\Delta(0,2)$. When $r_{11} < |z| \leq 1$, we let $\Omega_{1,z}$ always be this same disc. This completes the construction of Ω_1 .

The next step is to define an (almost) plurisubharmonic function ρ_1 . Let $\{\epsilon_n\}_{n=1}^\infty$ be a sufficiently rapidly decreasing sequence of positive numbers, $\epsilon_n \searrow 0$. Then $\sigma_1(w) := \sum_{n=1}^\infty \epsilon_n \log |w - \frac{1}{n}|$

is a subharmonic function on the complex plane and $\sigma_1(0) \in (-\infty, 0)$.

Letting $\sigma(w) = \sigma_1(w) + 1 - \sigma_1(0)$ we obtain a subharmonic function on $\mathbb{C}(w)$ with $\sigma(0) = 1$ and $\sigma(1/n) = -\infty \forall n \in \mathbb{Z}^+$. If the constant $K > 0$ is chosen large enough, the plurisubharmonic function $\sigma(w) + K \log(|z|/r_5)$ will be strictly less than -1 at all points $(z, w) \in \Omega_1$ for which $|z| \leq r_4$. The function $\rho_1 : \Omega_1 \rightarrow \mathbb{R}$ is defined by the equations $\rho_1(z, w) = \rho_1(1/z, w)$ and $\rho_1(z, w) = \max \{ \sigma(w) + K \log(|z|/r_5), -1 \}$ when $|z| \leq 1$. Then ρ_1 is the restriction to Ω_1 of the similarly defined function on \mathbb{C}^2 and ρ_1 is plurisubharmonic at all points (z, w) with $|z| \neq 1$. This completes the construction of ρ_1 .

We have two main problems left. The annuli A_n all lie partly in the boundary of Ω_1 , so Ω_1 has to be bumped slightly so that they all lie in the interior. However, this bumping should not change the extent to which A lies in the boundary. The other main problem is the failure of plurisubharmonicity of ρ_1 at $|z| = 1$. We will change ρ_1 near $|z| = 1$ so that it will equal $\max\{\sigma(w), -1\}$ in a neighbourhood of this set. In order to deal with both these problems, we will at first construct a subharmonic function $\tau(w)$ which can be used for patching purposes.

Our first approximation to τ will be τ_1 .

The domain of τ_1 will be

$D := \{w; |w| < 2, w \notin (-2, 0], w \notin \{1/n\}\}$. The properties we will require of τ_1 are that $\tau_1(u+iv) = 0$ when $v \geq f(u)$, $\tau_1(u+iv) \geq 1$ when $v \leq 0$, τ_1 is C^∞ and τ_1 is strongly subharmonic at all

points $u + iv$ with $v < f(u)$.

Let K_0 denote the compact set $\{w = u + iv; |w| \leq 2 \text{ and } v \geq f(u)\}$. Since K_0 is polynomially convex, there exists a C^∞ subharmonic function $\lambda_0 : \mathbb{C} \rightarrow [0, \infty)$ which vanishes precisely on K_0 and which is strongly subharmonic on $\mathbb{C} - K_0$. Choose an increasing sequence of compact sets

$F_1 \subset \text{int } F_2 \subset F_2 \subset \text{int } F_3 \subset \dots \subset D, \quad D = \bigcup F_\ell$. Letting $K_\ell = K_0 \cup F_\ell$

we may even assume that each bounded component of $\mathbb{C} - K_\ell$ clusters

at some $1/n$ and in particular therefore that there are only

finitely many of these components. With these choices it is

possible for each $\ell \geq 1$ to find a non-negative C^∞ function λ_ℓ

such that $\lambda_\ell|_{K_\ell} \equiv 0$, $\lambda_\ell \geq 1$ and strongly subharmonic on

$\{u+iv \in K_{\ell+2} - \text{int } K_{\ell+1}; v \leq 0\}$ and λ_ℓ fails to be subharmonic

only on a relatively compact subset of $(\text{int } K_{\ell+3} - K_{\ell+2}) \cap \{v < 0\}$.

But then, if $\{C_\ell\}_{\ell=0}^\infty$ is a sufficiently rapidly increasing sequence,

$\tau_1 := \sum_{\ell=0}^\infty C_\ell \lambda_\ell$ has all the desired properties.

We next want to push the singularities of τ_1 at the points $1/n$ over to the origin. First, let us choose discs $\Delta_n = \Delta(1/n, \rho_n)$

small enough so that $\sigma(w) + K \log 1/r_5 < -1$ on each Δ_n . We will first perturb τ_1 inside each Δ_n .

We can make a small perturbation of the situation by making a small translation parallel to the v -axis in the negative direction in a smaller disc about $1/n$ patched with the identity outside a slightly larger disc in Δ_n to obtain a new C^∞ function $\tau_2 \geq 0$ and a

new C^∞ function $v = f_1(u)$ with the properties that $f_1 \leq f$, $f_1 < f$

near $1/n$, $f_1 = f$ away from $1/n$ and $\tau_2 = 0$ when $v \geq f_1(u)$,

$\tau_2 \geq 1$ when $v \leq 0$ except in very small discs about $1/n$ and

$$\tau_3 = \begin{cases} 0 & \text{when } v \geq f_1(u) \\ \tau_2 + (v - f_1(u))^2 & \text{otherwise} \end{cases} \quad \text{is strongly subharmonic when } v < f_1(u).$$

The singularities of τ_1 at the points $1/n$ have thus been moved down to the points $\rho_n = 1/n + if_1(i/n)$. Let $\Delta'_n = \Delta(1/n, \rho'_n)$, $0 < \rho'_n \ll \rho_n$ be discs on which $\tau_3 \equiv 0$. We may assume that $p_n \notin \bar{\Delta}'_n$. Let γ be a curve from p_1 to 0 passing in the lower half plane through all the p'_n 's and avoiding all the $\bar{\Delta}'_n$'s. We can assume say that γ is linear between p_n and p_{n+1} . Let V be a narrow tubular neighbourhood of $\gamma - \{0\}$ also lying in the lower half-plane and avoiding all the $\bar{\Delta}'_n$'s. The restriction $\tau_3|_V$ is C^∞ , subharmonic and ≥ 1 except for singularities at each p_n . Let $\tau_4 \geq 1$ be a C^∞ function on V which agrees with $\tau_3|_V$ on $V \cap V'$, V' some open set containing $\partial V - \{0\}$. A construction similar to the one for τ_1 yields a C^∞ subharmonic function $\tau_5 \geq 0$ on $\mathbb{C} - (0)$ which vanishes outside V and is such that $\tau_4 + \tau_5$ is subharmonic on V . Finally, let $\tau : \{(w) < 2, w \notin [-2, 0]\} \rightarrow \mathbb{R}^+$ be the C^∞ subharmonic function given by $\tau = \tau_3$ outside V and $\tau = \tau_4 + \tau_5$ on V . Then $\tau = 0$ on each Δ'_n and $\tau(w) = 0$ when $v \geq f_1(u)$ except possibly on a concentric disc Δ'_n , $\Delta'_n \subset \subset \Delta''_n \subset \subset \Delta_n$. Also, $\tau(w) \geq 1$ when $v \leq 0$, $w \notin \cup \Delta''_n$. This completes the construction of the patching function τ .

The construction of Ω can now be completed. A point $(z, 1/n) \in A_n$ lies in the boundary of Ω_1 only when $|z|$ or $1/|z|$ is in $[r_5, r_6] \cup [r_7, r_8] \cup [r_9, r_{10}]$. This set is contained in the open set $\{(z, w); |z| \text{ or } 1/|z| \in (r_4, r_{11}) \text{ and } w \in \Delta'_n\} =: U_n$. We let

Ω be a domain with C^∞ boundary which agrees with Ω_1 outside $\bigcup U_n$ and which contains all A_n in its interior.

Next we define the plurisubharmonic function $\rho : \Omega \rightarrow \mathbb{R}$.

Let $\sigma' = \max\{\sigma, -1\}$ and choose a constant $L \gg 1$ such that

$\rho_1 \leq L - 1$ on $\bar{\Omega}$. If $|z| \leq r_6$, let $\rho_z := \rho_{1,z}$. For

$r_5 \leq |z| \leq r_6$, this definition agrees with $\rho_z = \max\{\rho_{1,z}, \sigma' + L\tau\}$

since τ is then 0 and $\rho_1 = \sigma' + K \log(|z|/r_5)$. If $r_6 < |z| \leq r_8$,

let $\rho_z := \max\{\rho_{1,z}, \sigma' + L\tau\}$. For $r_7 \leq |z| \leq r_8$, this

definition agrees with $\rho_z = \sigma' + L\tau$. To see this, observe that if

$w \in \Delta_n''$, then $\rho_{1,z} = -1$ and $\sigma' = -1$ while $\tau \geq 0$. If on the

other hand $w \notin \bigcup \Delta_n''$, then $v < 0$ and $\sigma' + L\tau \geq -1 + L \geq \rho_1$.

If $r_8 < |z| \leq r_{10}$, let $\rho_z := \sigma' + L\tau$. For $r_9 \leq |z| \leq r_{10}$

this definition agrees with $\rho_z = \sigma'$ since $\tau = 0$. Also, if

$r_{10} \leq |z| \leq 1$, let $\rho_z := \sigma'$, and if $|z| > 1$, let $\rho_z := \rho_{1/z}$.

Then ρ is plurisubharmonic on Ω , $\rho(e^{i\theta}, 0) = 1 \quad \forall \theta \in \mathbb{R}$ and

$\rho(e^{i\theta}/2, 0) = \rho(2e^{i\theta}, 0) = -1 \quad \forall \theta \in \mathbb{R}$. If there exists a sequence

of C^∞ plurisubharmonic functions $\rho_m : \Omega \rightarrow \mathbb{R}$, $\rho_m \searrow \rho$, then

there exists an m for which $\rho_m(e^{i\theta}/2, 0), \rho_m(2e^{i\theta}, 0) < 0 \quad \forall \theta \in \mathbb{R}$.

Hence, for all large enough n , $\rho_m(e^{i\theta}/2, 1/n), \rho_m(2e^{i\theta}, 1/n) < 0$

$\forall \theta \in \mathbb{R}$. By the maximum principle applied to the annuli $A_n \subset \Omega$,

it follows that $\rho_m(e^{i\theta}, 1/n) < 0 \quad \forall \theta \in \mathbb{R}$ and all large enough n .

Hence, by continuity of ρ_m , $\rho_m(e^{i\theta}, 0) \leq 0 \quad \forall \theta \in \mathbb{R}$. This contradicts

the assumption that $\rho_m \geq \rho$ and therefore completes the counter-

example.

R E F E R E N C E S

1. Bedford, E.: The operator $(dd^c)^n$ on complex spaces (preprint)
2. Fornæss, J.E.: Regularizations of plurisubharmonic functions. Math. Ann. (to appear).
3. Richberg, R.: Stetige streng pseudokonvexe Funktionen. Math. Ann. 175, 251 - 286 (1968).