PLURISUBHARMONIC FUNCTIONS
ON SMOOTH DOMAINS
by
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1. In this short note we will discuss regularization of plurisubharmonic functions. More precisely, we will address the following problem:

**Question.** Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^n (n \geq 2) \) with smooth \( (C^\infty) \) boundary and that \( \rho : \Omega \to \mathbb{R} \cup \{ -\infty \} \) is a (discontinuous) plurisubharmonic function. Does there exist a sequence \( \{ \rho_n \}_{n=1}^{\infty} \), \( \rho_n : \Omega \to \mathbb{R} \), of \( C^\infty \) plurisubharmonic functions such that

\[ \rho_n \nearrow \rho \text{ pointwise?} \]

If \( \rho \) is continuous, the answer to the above question is yes (see Richberg [3]). On the other hand, when \( \rho \) is allowed to be discontinuous and \( \Omega \) is not required to have a smooth boundary, the answer is in general no (see [1], [2] for this and related questions).

Our result in this paper is that the answer to the above question is no. We present a counterexample in the next section. The construction leaves open what happens if we make the further requirement that \( \Omega \) has real analytic boundary. Another question, suggested to the author by Grauert, is obtained by replacing \( \Omega \) by a compact complex manifold with smooth boundary, and assuming
continuity of $\rho$.

In the next section we need of course both to construct the domain $\Omega$ and the function $\rho$. These constructions are intertwined and therefore we need at first to define approximate solutions $\Omega_1$ and $\rho_1$ and then use both to define $\Omega$ and $\rho$. The geometric properties we seek of $\Omega$ are the following. There exists an annulus $A \subset \Omega$ such that $\partial A \subset \Omega$. Furthermore there exist concentric circles $C_1, C_2, C_3$ in the relative interior of $A$ arranged by increasing radii such that $C_1, C_3 \subset \partial \Omega$ and $C_2 \subset \Omega$. Finally there exists a sequence $\{A_n\}_{n=1}^\infty$ of annuli such that $A_n \to A$ and $A_n \subset \Omega \forall n$. The properties we seek of $\rho$ are as follows. The function $\rho$ is strictly positive on $C_2$ and is strictly negative on $\partial A$. A simple application of the maximum principle now shows that smoothing is impossible.

The example we construct is in $\mathbb{C}^2$. This is with no loss of generality as one obtains then an example in $\mathbb{C}^n$ by crossing with a smooth domain in $\mathbb{C}^{n-2}$, rounding off the edges and pulling back $\rho$ to the new domain.

2. All domains and functions which we will consider in $\mathbb{C}^2(z,w)$ will be invariant under rotations in the $z$-plane, i.e. will depend only on $|z|$. They will also be invariant under the map $(z,w) \to (1/z,w)$. Because of the latter we will describe only those points $(z,w)$ in these domains or domains of definitions for which $|z| \leq 1$.

If $U$ is a domain in $\mathbb{C}^2(z,w)$, we let $U_z$ denote the part of $U$ over $z$, i.e. $U_z = \{(n,w) \in \mathbb{C}^2 ; n = z \text{ and } (n,w) \in U\}$. Abusing notation we will also take $U_z$ to mean the set $\{w \in \mathbb{C} ; (z,w) \in U\}$. 
Similarly, if \( \sigma : U \rightarrow \mathbb{R} \cup \{-\infty\} \) is a function, then \( \sigma_z \) denotes the restriction of \( \sigma \) to \( U_z \).

Let \( A \) be the annulus in \( \mathbb{C}^2 \) given by
\[
A = \{ (z,w); \ w = 0 \text{ and } 1/2 \leq |z| \leq 2 \}.
\]
This is then the limit of a sequence of annuli \( \{A_n\}_{n=1}^\infty \) where
\[
A_n = \{ (z,w); \ w = 1/n \text{ and } 1/2 \leq |z| \leq 2 \}.
\]
We will next describe a bounded domain \( \Omega^1 \) in \( \mathbb{C}^2 \) with \( C^\infty \) boundary containing all \( A_n \)'s (and hence \( A \)) in it's closure. It will suffice to describe \( \Omega^1_z \) for various \( z \)'s. That these can be made to add up to a domain with \( C^\infty \) boundary will be clear throughout.

Choose a sequence of positive numbers \( \{r_k\}_{k=1}^\infty \),
\[ 0 < r_1 < r_2 < \ldots < 1, \quad \text{with } r_3 = 1/2. \]
We let \( \Omega_{1,z} = \emptyset \) if \( |z| \leq r_1 \) and \( \Omega_{1,z} \) be a nonempty disc, concentric about the origin if \( r_1 < |z| \leq r_4 \). Recall that \( \Omega_{1,z} = \Omega_1,|z| \) for all \( z \).
If \( r_2 \leq |z| \leq r_4 \) we make the extra assumption that \( \Omega_{1,z} \) has radius 2. For \( |z| > r_4 \) we will break the symmetry in the \( w \)-direction at first by letting \( \Omega_{1,z} \) gradually approach the shape of an upper-disc. (This is a rough description to be made more precise below.) Increasing \( |z| \) further we will rotate this approximate upper half disc \( 180^0 \) clockwise until it becomes approximately a lower half disc. Then we proceed by reversing the process, first by rotating counterclockwise back to an approximate upper half disc and then expanding this back to a disc of radius 2 near \( |z| = 1 \). As mentioned earlier, if \( |z| > 1 \), then \( \Omega_{1,z} := \Omega_1,1/z \).

We now return to the more precise description of \( \Omega_{1,z} \) for \( |z| > r_4 \).
Writing \( w = u + iv \) in real coordinates \( u,v \), let \( v = f(u) \) be
a $C^\infty$ function defined for $u \in \mathbb{R}$ with $f(u) = 0$ if $u \leq 0$ or $u \geq 2$, $f \geq 0$ and $f(u) = 0$ on $(0,2)$ if and only if $u = 1/n$ for some positive integer $n$. We may assume that $|f|, \ |f'|, \ |f''|$ are very small and therefore in particular that the graph of $f$ only intersects the boundary of any disc $\Delta(0;R) = \{|w| < R\}$ in exactly two points. If $r_4 < |z| < r_5$, we let $\Omega_{1,z}$ be a subdomain of $\Delta(0;2)$ containing those $u + iv \in \Delta(0;3/2)$ for which $v \geq f(u)$. When $r_5 \leq |z| \leq r_6$ we choose $\Omega_{1,z}$ independent of $z$ with the properties that $\Omega_{1,z} \subset \Delta(0;7/4) \cap \{v > f(u)\}$ and $\Delta(0;3/2) \cap \{v > f(u)\} \subset \Omega_{1,z}$.

Let $\epsilon(x)$ be a real $C^\infty$ function on $\mathbb{R}$ with $\epsilon(x) = 0$ if $x \leq r_6$, $\epsilon(x) = \pi$ if $x \geq r_7$ and $\epsilon'(x) > 0$ if $r_6 < x < r_7$.

Then we can rotate $\Omega_{1,z}$ $180^\circ$ clockwise for $r_6 \leq |z| \leq r_7$ by defining $\Omega_{1,z} = e^{-i\epsilon(|z|)} \Omega_{1,r_6}$ for such $z$. Further, we let $\Omega_{1,z} = \Omega_{1,r_7}$ when $r_7 \leq |z| \leq r_8$. Reversing the procedure, we rotate $\Omega_{1,z}$ back $180^\circ$ when $r_8 \leq |z| \leq r_9$ so that $\Omega_{1,r_9}$ again equals $\Omega_{1,r_6}$. Continuing, we let $\Omega_{1,z} = \Omega_{1,r_9}$ whenever $r_9 \leq |z| \leq r_{10}$. Reversing the procedure between $r_4$ and $r_5$ we obtain $\Omega_{1,z}$'s, $r_{10} \leq |z| \leq r_{11}$ so that in particular $\Omega_{1,r_{11}}$ is the disc $\Delta(0,2)$. When $r_{11} < |z| \leq 1$, we let $\Omega_{1,z}$ always be this same disc. This completes the construction of $\Omega_{1}$.

The next step is to define an (almost) plurisubharmonic function $\rho_1$. Let $\{\epsilon_n\}_{n=1}^\infty$ be a sufficiently rapidly decreasing sequence of positive numbers, $\epsilon_n \downarrow 0$. Then $\sigma_1(w) = \sum_{n=1}^\infty \epsilon_n \log|w - \frac{1}{n}|$
is a subharmonic function on the complex plane and \( \sigma_1(0) \in (-\infty, 0) \).

Letting \( \sigma(w) = \sigma_1(w) + 1 - \sigma_1(0) \) we obtain a subharmonic function on \( \mathbb{C}(w) \) with \( \sigma(0) = 1 \) and \( \sigma(1/n) = -\infty \) \( \forall n \in \mathbb{Z}^+ \). If the constant \( K > 0 \) is chosen large enough, the plurisubharmonic function \( \sigma(w) + K \log(|z|/r_5) \) will be strictly less than \(-1\) at all points \((z,w) \in \Omega_1\) for which \( |z| \leq r_4 \). The function \( \rho_1 : \Omega_1 \to \mathbb{R} \) is defined by the equations \( \rho_1(z,w) = \rho_1(1/z,w) \) and

\[
\rho_1(z,w) = \max \{ \sigma(w) + K \log(|z|/r_5), -1 \} \quad \text{when} \quad |z| \leq 1.
\]

Then \( \rho_1 \) is the restriction to \( \Omega_1 \) of the similarly defined function on \( \mathbb{C}^2 \) and \( \rho_1 \) is plurisubharmonic at all points \((z,w)\) with \(|z| \neq 1\).

This completes the construction of \( \rho_1 \).

We have two main problems left. The annuli \( A_n \) all lie partly in the boundary of \( \Omega_1 \), so \( \Omega_1 \) has to be bumped slightly so that they all lie in the interior. However, this bumping should not change the extent to which \( A \) lies in the boundary. The other main problem is the failure of plurisubharmonicity of \( \rho_1 \) at \(|z| = 1\). We will change \( \rho_1 \) near \(|z| = 1\) so that it will equal \( \max\{\sigma(w), -1\} \) in a neighbourhood of this set. In order to deal with both these problems, we will at first construct a subharmonic function \( \tau(w) \) which can be used for patching purposes.

Our first approximation to \( \tau \) will be \( \tau_1 \).

The domain of \( \tau_1 \) will be

\[
D = \{ w; |w| < 2, w \notin (-2, 0], w \notin \{1/n\} \}. \quad \text{The properties we will require of} \quad \tau_1 \quad \text{are that} \quad \tau_1(u+iv) = 0 \quad \text{when} \quad v \geq f(u), \tau_1(u+iv) \geq 1 \quad \text{when} \quad v \leq 0, \quad \tau_1 \quad \text{is} \quad C^\infty \quad \text{and} \quad \tau_1 \quad \text{is strongly subharmonic at all}....
points \( u + iv \) with \( v < f(u) \).

Let \( K_0 \) denote the compact set \( \{ w = u + iv; \ |w| \leq 2 \ \text{and} \ v \geq f(u) \} \). Since \( K_0 \) is polynomially convex, there exists a \( C^\infty \) subharmonic function \( \lambda_0 : \mathbb{C} \rightarrow [0,\infty) \) which vanishes precisely on \( K_0 \) and which is strongly subharmonic on \( \mathbb{C} - K_0 \). Choose an increasing sequence of compact sets
\[ F_1 \subset \text{int } F_2 \subset F_2 \subset \text{int } F_3 \subset \ldots \subset D, \ D = U F_\ell. \]
Letting \( K_\ell = K_0 \cup \text{int } F_\ell \), we may even assume that each bounded component of \( \mathbb{C} - K_\ell \) clusters at some \( 1/n \) and in particular therefore that there are only finitely many of these components. With these choices it is possible for each \( \ell \geq 1 \) to find a non-negative \( C^\infty \) function \( \lambda_\ell \) such that \( \lambda_\ell |_{K_\ell} = 0, \ \lambda_\ell \geq 1 \) and strongly subharmonic on
\[ \{ u + iv \in K_{\ell+2} - \text{int } K_{\ell+1}; \ v \leq 0 \} \] and \( \lambda_\ell \) fails to be subharmonic only on a relatively compact subset of \( \{ \text{int } K_{\ell+3} - K_{\ell+2}\} \cap \{ v < 0 \} \).

But then, if \( \{ \mathcal{C}_\ell \}_{\ell=0}^\infty \) is a sufficiently rapidly increasing sequence,
\[ \tau_1 : = \sum_{\ell=0}^{\infty} C_\ell \lambda_\ell \] has all the desired properties.

We next want to push the singularities of \( \tau_1 \) at the points \( 1/n \) over to the origin. First, let us choose discs \( \Delta_n = \Delta(1/n, \rho_n) \) small enough so that \( o(w) + K \log 1/r_n < -1 \) on each \( \Delta_n \).

We will first perturbe \( \tau_1 \) inside each \( \Delta_n \). We can make a small perturbation of the situation by making a small translation parallel to the \( v \)-axis in the negative direction in a smaller disc about \( 1/n \) patched with the identity outside a slightly larger disc in \( \Delta_n \) to obtain a new \( C^\infty \) function \( \tau_2 \geq 0 \) and a new \( C^\infty \) function \( v = f_1(u) \) with the properties that \( f_1 \neq f, f_1 < f \) near \( 1/n \), \( f_1 = f \) away from \( 1/n \) and \( \tau_2 = 0 \) when \( v \geq f_1(u) \), \( \tau_2 \geq 1 \) when \( v \leq 0 \) except in very small discs about \( 1/n \) and
\[ \tau_3 = \begin{cases} 0 & \text{when } v > f_1(u) \\ \tau_2 + (v - f_1(u))^2 & \text{otherwise} \end{cases} \]

is strongly subharmonic when \( v < f_1(u) \).

The singularities of \( \tau_1 \) at the points \( 1/n \) have thus been moved down to the points \( \rho_n = 1/n + i f_1(i/n) \). Let \( \Delta_n' = \Delta(1/n, \rho_n') \), \( 0 < \rho_n' << \rho_n \) be discs on which \( \tau_3 \equiv 0 \). We may assume that \( p_n \not\in \Delta_n' \).

Let \( \gamma \) be a curve from \( p_1 \) to \( 0 \) passing in the lower half plane through all the \( p_n' \)'s and avoiding all the \( \Delta_n' \)-s. We can assume say that \( \gamma \) is linear between \( p_n \) and \( p_{n+1} \). Let \( V \) be a narrow tubular neighbourhood of \( \gamma - \{0\} \) also lying in the lower half-plane and avoiding all the \( \Delta_n' \)-s. The restriction \( \tau_3|_V \) is \( C^\infty \), subharmonic and \( \geq 1 \) except for singularities at each \( p_n \). Let \( \tau_4 \geq 1 \) be a \( C^\infty \) function on \( V \) which agrees with \( \tau_3|_V \) on \( V \cap V', V' \) some open set containing \( \gamma \cup \{0\} \). A construction similar to the one for \( \tau_1 \) yields a \( C^\infty \) subharmonic function \( \tau_5 \geq 0 \) on \( C - \{0\} \) which vanishes outside \( V \) and is such that \( \tau_4 + \tau_5 \) is subharmonic on \( V \). Finally, let \( \tau : \{(w) < 2, w \not\in [-2,0] \} + \mathbb{R}^+ \) be the \( C^\infty \) subharmonic function given by \( \tau = \tau_3 \) outside \( V \) and \( \tau = \tau_4 + \tau_5 \) on \( V \). Then \( \tau = 0 \) on each \( \Delta_n' \) and \( \tau(w) = 0 \) when \( v \geq f_1(u) \) except possibly on a concentric disc \( \Delta_n', \Delta_n' \subset \Delta_n'' \subset \Delta_n \).

Also, \( \tau(w) \geq 1 \) when \( v \leq 0, w \not\in \Delta_n'' \). This completes the construction of the patching function \( \tau \).

The construction of \( \Omega \) can now be completed. A point \((z,1/n) \in A_n \) lies in the boundary of \( \Omega \) only when \(|z| \) or \(1/|z|\) is in \([r_5,r_6] \cup [r_7,r_8] \cup [r_9,r_{10}] \). This set is contained in the open set \( \{(z,w); |z| \text{ or } 1/|z| \in (r_4,r_{11}) \text{ and } w \in \Delta_n' \} = : U_n \). We let
\(\Omega\) be a domain with \(C^\infty\) boundary which agrees with \(\Omega_1\) outside \(U U_n\) and which contains all \(A_n\)'s in its interior.

Next we define the plurisubharmonic function \(\rho : \Omega \rightarrow \mathbb{R}\). Let \(\sigma' = \max\{\sigma, -1\}\) and choose a constant \(L \gg 1\) such that \(\rho_1 \leq L - 1\) on \(\Omega\). If \(|z| \leq r_6\), let \(\rho_z := \rho_{1,z}\). For \(r_5 \leq |z| \leq r_6\), this definition agrees with \(\rho_z = \max\{\rho_{1,z}, \sigma' + L\tau\}\) since \(\tau\) is then 0 and \(\rho_1 = \sigma' + K \log(|z|/r_5)\). If \(r_6 < |z| \leq r_8\), let \(\rho_z := \max\{\rho_{1,z}, \sigma' + L\tau\}\). For \(r_7 \leq |z| \leq r_8\), this definition agrees with \(\rho_z = \sigma' + L\tau\). To see this, observe that if \(w \in \Delta_n\), then \(\rho_{1,z} = -1\) and \(\sigma' = -1\) while \(\tau \geq 0\). If on the other hand \(w \notin \Delta_n\), then \(v < 0\) and \(\sigma' + L\tau \leq 1 + L \geq \rho_1\). If \(r_8 < |z| \leq r_{10}\), let \(\rho_z := \sigma' + L\tau\). For \(r_9 \leq |z| \leq r_{10}\), this definition agrees with \(\rho_z = \sigma'\) since \(\tau = 0\). Also, if \(r_{10} \leq |z| \leq 1\), let \(\rho_z := \sigma'\), and if \(|z| > 1\), let \(\rho_z := \rho_{1/z}\).

Then \(\rho\) is plurisubharmonic on \(\Omega\), \(\rho(e^{i\theta}, 0) = 1\) \(\forall \theta \in \mathbb{R}\) and \(\rho(e^{i\theta}/2, 0) = \rho(2e^{i\theta}, 0) = -1\) \(\forall \theta \in \mathbb{R}\). If there exists a sequence of \(C^\infty\) plurisubharmonic functions \(\rho_m : \Omega \rightarrow \mathbb{R}\), \(\rho_m \not\equiv \rho\), then there exists an \(m\) for which \(\rho_m(e^{i\theta}/2, 0), \rho_m(2e^{i\theta}, 0) < 0\) \(\forall \theta \in \mathbb{R}\). Hence, for all large enough \(n\), \(\rho_m(e^{i\theta}/2, 1/n), \rho_m(2e^{i\theta}, 1/n) < 0\) \(\forall \theta \in \mathbb{R}\). By the maximum principle applied to the annuli \(A_n \subset \Omega\), it follows that \(\rho_m(e^{i\theta}, 1/n) < 0\) \(\forall \theta \in \mathbb{R}\) and all large enough \(n\). Hence, by continuity of \(\rho_m\), \(\rho_m(e^{i\theta}, 0) \leq 0\) \(\forall \theta \in \mathbb{R}\). This contradicts the assumption that \(\rho_m \geq \rho\) and therefore completes the counterexample.
REFERENCES

