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DEFINABILITY AND FORCING IN E-RECURSION *

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§ 0 Introduction

This paper will give a short review of forcing techniques in the setting of E-Recursion without the use of indices (for the approach via indices and detailed proofs of previous results mentioned here the reader is directed to Sacks [1980] or Sacks - Slaman [1980]). We follow an approach which is index-free for the sake of clarity. The fundamentally new tool in this setting, the Moschovakis Phenomenon (MP), was first isolated by Sacks in showing that many generic extensions preserve E-closure. Further applications of forcing in E-Recursion may be found in Slaman [1981] and Griffor - Normann [1981].

E-Recursion was introduced by D. Normann [1978] as a natural generalization of normal Kleene recursion in objects of finite type in order to facilitate the study of degrees of functionals. Normann's index-free approach emphasized the role of computations as opposed to hierarchies and indices which obscured that role.

In sections 1 - 3 we review the forcing technology briefly without indices as well as the results of Sacks concerning the preservation of E-closure in extensions via posets with chain conditions or closure conditions. Section 4 discusses the role of selection and definability in Cohen extensions and in section 5 the independence of the well-foundedness of the Σ^1_3 E-degrees of reals. Here we use the absoluteness results of Lévy for extensions via semi-homogeneous posets.

Section 6 - 8 address the problem of extending 1-sections, while sections 9 - 11 develope the methods required for extending k-sections (for $k \geq 2$) of $^{k+2}\mathbb{E}$ in a non-trivial way.

In section 12 we show that the RE-degrees of a ground model are unaffected by set forcing with effective notions of forcing.

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§ 1 The Forcing Technology

We say that a set $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if for all $r \in \mathbb{P}$ there exists a $d \in D$ such that r and d are competible (i.e. have a common extension in $\mathbb{P}$). A set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over A ($\mathbb{P}$-generic/A) if

(i) G is a directed set;

(ii) $g \in G$ and $p \leq_{\mathbb{P}} g$, then $p \in G$; and

(iii) every dense $D \subseteq \mathbb{P}$ which is first order definable over $\langle A, \in \rangle$ with parameters from A satisfies $G \cap D \neq \emptyset$.

$A[G]$ is then the least E-closed set containing A with G as an element (set forcing) restricted to sets of rank less than $\kappa = OR \cap A$.

The ramified language will be given with an eye to questions of effectiveness: $\mathcal{L}^*$ is defined effectively in A. The terms of $\mathcal{L}^*$ are built using parameters from A such that those involving only $b \in A$ are present in $E(b)$.

Symbols: $\in$, $=$; unranked variables $x,y,\cdots$; ranked variables $x^\lambda, y^\lambda \cdots$ for $\lambda < \kappa$; logical connectives $\wedge$, $\neg$; and the quantifier $\exists$.

Formulae are built up using these symbols and a class of constants C, defined by induction, i.e. we will name all elements of $A[G]$ in A. For $x \in A$ we define C^x by an induction of length $\kappa = OR \cap A$.

Definition

$$C^x_o = \{\underline{b} \mid b \in TC(x) \vee b = X\} \cup \{G\};$$

$C^x_{\alpha+1}$ satisfies: $C^x_\alpha \subseteq C^x_{\alpha+1}$ and if

$\varphi(v_0,\cdots,v_n)$ is in $\mathscr{L}^*$ with free variables in $v_0,\cdots,v_n$ and quantifiers variables of the form x^β, $\beta \leq \alpha$, then

$\hat{x}^\alpha \varphi(x^\alpha, c_1,\cdots,c_n) \in C^x_{\alpha+1}$, if

$c_1,\cdots,c_n \in C^x_\alpha$;

$C^x_\lambda = \bigcup_{\alpha<\lambda} C^x_\alpha$, if $\lim(\lambda)$ and $\lambda < \kappa$

$C^x = \bigcup_{\alpha<\kappa} C^x_\alpha$ and $C = \bigcup_{x\in A} C^x$ and each

$c \in C$ is a symbol in $\mathscr{L}^*$.

We say that a formula $\varphi \in \mathscr{L}^*$ is ranked, if all bound variables in φ are ranked and assign an ordinal ($\underline{rank(\varphi)}$) to each $\varphi \in \mathscr{L}^*$ as follows (in decreasing order of importance) :

(i) the number of unranked quantifiers;

(ii) ordinals associated with ranked quantifiers and constant terms;

(iii) logical complexity.

The forcing relation $p \Vdash \varphi$ is defined by induction on rank(φ). Apart from the clauses given by the schemator of E-Recursion, all clauses are standard (see Sacks-Slaman [1980]). The symbol $\underline{x}$ denotes a term. We consider the bounding scheme and composition :

First suppose

$$\{e\}^G(x,\vec{y}) = \bigcup_{z\in x} \{\{e_0\}^G(z,\vec{y})\}, \quad \text{then}$$

$p \Vdash |\{e\}^G(\underline{x},\vec{\underline{y}})| = \lambda$ iff

(a) $p \Vdash \forall z\in\underline{x}\ \exists\gamma<\lambda$

$[|\{e_0\}^G(z,\vec{\underline{y}})| = \gamma]$; and

(b) $p \Vdash \forall\sigma<\lambda\ \exists z\in\underline{x}$

$[|\{e_0\}^G(z,\vec{\underline{y}})| \geq \sigma]$. If we have

$$\{e\}^G(x,\vec{y}) = \{e_0\}^G(\{e_1\}^G(x,\vec{y}),x,\vec{y}), \quad \text{then}$$

$p \Vdash |\{e\}^G(\underline{x},\vec{\underline{y}})| = \sigma$ iff there exists $\sigma_1,\sigma_2 \leq \sigma$ such that

$$p \Vdash |\{e_1\}^G(\underline{x},\vec{\underline{y}})| = \sigma_1 \wedge \{e_1\}^G(\underline{x},\vec{\underline{y}} = \underline{z}$$

and $p \Vdash |\{e_0\}^G(\underline{z},\underline{x},\vec{\underline{y}})| = \sigma_2$, where $\sigma = \max(\sigma_1,\sigma_2)$.

Remark We have not explicitly defined what it means to say $p \Vdash \{e_1\}^G(\underline{x},\vec{\underline{y}}) = \underline{z}$, however for such a computation which converges there is an index which gives the characteristic function of the set

which is its value. Proceeding inductively this is the same as forcing that these functions values are the same as those of the term $\underline{z}$ on all appropriate arguments. (i.e. terms of lower rank).

Applications are often simplified by considering the 'weak' forcing relation $\Vdash^*$ defined by,

$$P \Vdash^* \varphi \quad \text{iff} \quad p \Vdash \neg\neg\varphi.$$

We shall assume the standard result that if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic/A, then

$$A[G] \models \varphi \quad \text{iff} \quad \exists p \in G[p \Vdash^* \varphi].$$

§ 2 Preserving E-closure: Closure Conditions

Now assume that A is E-closed and $\mathbb{P} \in A$. To show that E-closure is preserved by a generic extension of A (A[G] is E-closed), Sacks shows that for $x \in A$, $\vec{y} \subset A^n$ for some $n \in \omega$, the relation

$$p \Vdash^* \{e\}^G(\underline{x}, \underline{\vec{y}})\downarrow \quad \text{is } \underline{RE}.$$

<u>Lemma 2.0</u> (Sacks) Suppose $\gamma \in OR \cap A$, then the relation

$p \Vdash^* \varphi$ restricted to φ's of ordinal rank $\leq \gamma$ and quantifiers restricted to $E(z)$ for $z \in A$ is recursive in $\gamma, z, \mathbb{P}$.

<u>Proof</u> Sacks' proof proceeds by induction on the definition of the forcing relation. Consider only the cases $\neg$ and $\exists x^\beta$. Let $\psi \equiv \neg\varphi$ and suppose $p \Vdash \psi$, then by definition (iii) :

$$\forall q \leq_{\mathbb{P}} p \neg (q \Vdash \varphi).$$

By induction hypothesis and the bounding principle we have the desired conclusion.

Now let $\varphi \equiv \exists x^{\beta} \psi$ and suppose $p \Vdash \varphi$, then by definition (v)

$p \Vdash \psi(c)$ for some $c \in C^{x}_{\beta}$ where x is the parameter from A in ψ. By induction hypothesis $p \Vdash \psi(c)$ is recursive in $\gamma, z, \mathbb{P}$. C^{x}_{β} is recursive in x, β and by the bounding principle applied to that procedure $p \Vdash \varphi$ is recursive in $\gamma, z, \mathbb{P}$. The remaining cases are routine.

Definition Let $\langle p, a \rangle$ and $\langle q, b \rangle \in \mathbb{P} \times C$ and let

$\langle p, a \rangle >_S \langle q, b \rangle$ iff $q \leq_{\mathbb{P}} p$ and

$q \Vdash^{*} {}'b$ is a subcomputation of a'.

Lemma 2.1 (Sacks) Suppose $\mathbb{P}$ is well-orderable in A and that $<_S$ is well-founded below $\langle p, a \rangle$, $\exists q \in \mathbb{P}\ \exists \gamma < \kappa$, q and γ uniformly recursive in $p, a, \mathbb{P}$ such that

$$q \Vdash^{*} |\{a_2\}^{G}(a_1)| = \gamma, \quad \text{where} \quad a = \langle a_0, a_1 \rangle .$$

Corollary 2.2 The relation $p \Vdash^{*} \{e\}^{G}(\underline{x}, \vec{\underline{y}})\downarrow$ is RE in $\mathbb{P}$.

The procedure defined in the lemma allows one to reduce the forcing of an apparently $\Sigma_1(A)$ formula (i.e. there exists a well-founded computation tree) effectively to a ranked formula.

As Sacks and Slaman [1980] remark, what has been shown here is that the $<_S$ height of $\langle q_0, a\rangle$ is recursive in p,a and bounds the value of $|\{e\}^G(x,\vec{y})|$, where G is $\mathbb{P}$-generic/A extending q_0. This result can be sharpened if $\mathbb{P}$ is homogeneous.

Definition A partially ordered $\langle X, <_X\rangle$ is homogeneous, if for all $p,q \in X$ there exists an automorphism π of $\langle X, <_X\rangle$ such that

$$\pi(p) = q \ .$$

Proposition 2.3 Under the assumption of the lemma and assuming $\mathbb{P}$ is homogeneous we get effectively q,γ from $p,a,\mathbb{P}$ such that

$$q \Vdash^* |\{a_0\}^G(a_1)| = \gamma \quad \text{and if} \quad G \subseteq \mathbb{P}$$

is any $\mathbb{P}$-generic/A extending p, then

$$|\{a_0\}^G(a_1)|^{A[G]} = \gamma$$

Proof Take p,γ as in the lemma and first minimize $\tau \leq \gamma$ such that for some $q \in \mathbb{P}$

$$q \Vdash^* |\{a_0\}^G(a_1)| = \tau$$

(using the effectiveness of the forcing relation for formulae of bounded rank). Now minimize the condition for this τ $(=\tau_0)$, i.e. take

$$q_0 = \mu q \in \mathbb{P}[q \Vdash^* |\{a_0\}^G(a_1)| = \tau_0]$$

(using the well-ordering of $\mathbb{P}$). Since $\mathbb{P}$ is homogeneous, any generic extending $p \in \mathbb{P}$ satisfies :

$$A[G] \models |\{a_0\}^G(a_1)| = \tau_0 \ .$$

Countable closure of $\mathbb{P}$ is one way of insuring the closure of A[G]. The virtue of countable closure is its ability to exploit the MP. Consider a procedure applied to a pair $\langle p,\tau\rangle$, where $p \in \mathbb{P}$ is a forcing condition and τ is a term in the associated forcing language :

(i) if $p \Vdash \tau \downarrow$, then we produce by induction a bound less than κ on $\|\tau\|$;

(ii) if $p \Vdash \tau \uparrow$, then we build a sequence

$\langle p_n, \tau_n \rangle_{n\in\omega}$ such that

$\forall_n [(p_{n+1} \leq p_n)$ and

$p_n \Vdash$ "τ_n is a subcomputation of τ_{n-1}"].

By countable closure we take p_∞ such that $\forall_n [p_\infty \leq_{\mathbb{P}} p_n]$, then

$p_\infty \Vdash \langle \tau_n \rangle_{n\in\omega}$ is a Moschovakis Witness for τ" .

Lemma 2.4 (Sacks) Suppose $\mathbb{P}$ is countable closed in A and $<_S$ is not well-founded below $\langle p,a\rangle$, then there exists a term t and a condition q such that $q \Vdash^{*}$' t is a MW for a'.

Sacks' theorem on countable closure is now immediate.

Theorem 2.5 (Sacks) Suppose A is E-closed, $A \models$ MP, $\mathbb{P} \in A$ is well-orderable in A such that

$A \models$'$\mathbb{P}$ is countable closed'

If G is $\mathbb{P}$-generic/A, then A[G] is E-closed and satisfies MP.

The existence of $\mathbb{P}$-generics over E-closed A is not provable in general for uncountable A. We say that $G \leq \mathbb{P}$ is $\mathbb{P}$-bounded generic/A, if G is generic with respect to all sentences of bounded rank in $\mathcal{L}^*$ (i.e. G meets the associated dense subsets of $\mathbb{P}$). Sacks [1980] first noticed that such a generic is often sufficient for applications.

Lemma 2.6 (Sacks) Suppose A and $\mathbb{P}$ satisfy the conditions of the above theorem and that for some transitive set X

$$A = E(X) \quad \text{(the E-closure of } X\text{)}$$

and that X is well-orderable in A. If $\gamma\ (< \kappa = OR \cap A)$ is the height of the shortest such well-ordering of X in A and

$A \models$ "γ is regular", then a $\mathbb{P}$-bounded generic over A exists, where

$$\mathbb{P} = \{f: \gamma \to \{0,1\} \mid \overline{\overline{f}}^A < \gamma\}.$$

Proof (sketch) Since $A = E(X)$, every set $z \in A$ is recursive in some $\tau < \gamma$ (modulo the parameter giving the well-ordering of X in type γ). Thus the senteces of bounded rank in $\mathcal{L}^*$ can be recursively enumerated by γ such that the enumeration restricted to an initial segment of γ is bounded below κ.

The forcing relation for these sentences (essentially those giving computation tuples) is RE in $\mathbb{P}$. Using the well-ordering of $\mathbb{P}$ define by transfinite recursion $p: \gamma \to \gamma$ by $\tau < \gamma$

$\tau = \alpha + 1$: $p(\tau)$ is the least $p \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p(\alpha)$

and p decides $\varphi_{\|\tau\|}$, if $\tau\downarrow$ and $p(\tau)$ is $p(\alpha)$ otherwise, where $\varphi_{\|\tau\|}$ is the $\|\tau\|^{th}$ sentence of $\mathcal{L}^*$ of bounded rank.

$\text{limit}(\tau)$: $p(\tau) = \bigcup_{\partial<\tau} p(\gamma)$

Claim: For all $\sigma < \gamma$, $p"\sigma$ is bounded below γ.

Proof (claim) Given $\sigma < \gamma$ we have that $G_\sigma = \{\tau < \sigma |$ τ codes a convergent computation$\}$ is an element of A (we have identified X with γ via the well-ordering). Using G_σ, $p"\sigma$ is an element of A and by the assumption that γ is regular in A, $p"\sigma$ is bounded below γ.

The first application of forcing in the setting of E-Recursion was due to Sacks [1980] where he made use of the above result concerning forcing with countably closed posets. Sacks showed that if there exists a recursively regular well-ordering of 2^ω recursive in ${}^3\mathbb{E}$ and a real, then the 2-sc(${}^3\mathbb{E}$) is not RE in any real.

§ 3 Antichain Conditions and E-closure

Antichain conditions on $\mathbb{P}$ are yet another way of preserving E-closure. For the sake of completeness we mention the results of Sacks in this direction.

Definition Let A be E-closed and $\mathbb{P} \in A$ be a poset, then

(i) $x \subseteq \mathbb{P}$ is an antichain if all elements of x are incomparable via $<_{\mathbb{P}}$;

(ii) an antichain x is maximal if every element of $\mathbb{P}$ is comparable via $<_{\mathbb{P}}$ with some element of x;

(iii) $\mathbb{P}$ satisfies the δ-chain condition (δ-cc) in A, if every $\mathbb{P}$-antichain in A has A-cardinality less than δ.

For example, if $\mathbb{P}$ has the β^+-cc in A then every $\mathbb{P}$-antichain in A has A-cardinality less than or equal to β. As a consequence any effective phenomenon in $A[G]$ can be restricted to at most β many possibilities in A.

Theorem 3.0 (Sacks) Let A be E-closed, $\mathbb{P} \in A$, $\gamma \in A$ such that

(i) $\mathbb{P}$ has the γ^+-cc in A;

(ii) there is an $a \in A$ cuch that $\langle a,x \rangle$ selects from γ for all $x \in A$;

(iii) each $x \in A$ is well-orderable in A.

Then if G is $\mathbb{P}$-generic/A we have that $A[G]$ is E-closed.

Remark (a) Sacks' argument proceeds by approximating computations in $A[G]$ by building antichains in A. The reader is directed to Slaman [1981] for the proof.

(b) Slaman notices that Sacks' proof actually yields that if $a \in A$ and $a \subseteq OR$, then $\kappa_r^{a,G} = \kappa_r^a$.

Corollary 3.1 (Sacks) C.c.c. ($\aleph_1$-cc) set forcing (with (iii) of the theorem) preserves E-closure.

Proof Use Gandy Selection.

§ 4 Cohen Reals

In this section we consider the result of adding Cohen reals to $E(x)$. First we address the question posed in the previous section concerning the preservation of E-closure.

Let $X \in V$ be infinite and transitive and consider $E(x)$. Let the poset

$$\mathbb{P} = \{f: \omega \to \{0,1\} \mid f \text{ is a partial function and } \mathrm{dom}(f) \text{ is finite}\}$$

and for $p,q \in \mathbb{P}$, let $p \leq_{\mathbb{P}} q$ iff p extends q set-theoretically. $\mathbb{P}$ is just the Cohen poset for adding a new real and $\mathbb{P} \in E(x)$ satisfying

$$\mathbb{P} \leq_E \emptyset \ .$$

Lemma 4.C (Sacks) With $\mathbb{P}$ as above let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic/$E(x)$ then

(i) $\cup G = f: \omega \to \{0,1\}$;

(ii) $E(x)[f]$ is E-closed; and

(iii) $(\kappa_0^f)^{E(x)[f]} = (\kappa_0^{\emptyset})^{E(x)}$

(ii) follows immediately from (iii), while (i) is a standard density argument. Using the fact that the forcing relation is RE: assume

$$\{e\}(f)\downarrow \text{ in } E(x)[f],$$

then letting G be the term for f in $\mathscr{L}^*$ we have that there exists a $p \in G$ such that

$$\{e\}(\ )\downarrow \ .$$

The set of integers (under some standard coding of $\mathbb{P}$ as integers) :

$$\{p \in \mathbb{P} \mid p \Vdash \{e\}(f)\downarrow\}$$ is RE and, by Gandy Selection, we can effectively select such a p.

The proof that E-closure is preserved by generic extension in Lemma 4.0 only required selection over subsets of $\mathbb{P}$: we have selection over subsets of $\mathbb{P}$ if there exists a function φ E-recursive in some $p \in E(x)$ such that for all $A \subseteq \mathbb{P}$, if A is RE in some $p' \in E(x)$

and non-empty then $\varphi(p') \subseteq A$. In particular $\varphi(p')$ is defined and gives a non-empty recursive subset of A.

Proposition 4.1 Suppose $\mathbb{P} \in E(x)$, x is transitive and we have selection over subsets of $\mathbb{P}$. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic/E(x), then

$$(\kappa_0^G)^{E(x)[G]} = (\kappa_0^{\mathbb{P}})^{E(x)}$$

(in particular E(x)[G] is E-closed.

Proof As before consider $\{e\}(G)\downarrow$ in E(x)[G]; then there exists a $p \in G$ such that $p \Vdash \{e\}(G)\downarrow$. Consider

$\mathbb{P}(e) = \{p \in \mathbb{P} \mid p \Vdash \{e\}(G)\downarrow\}$, which is RE and a subset of $\mathbb{P}$. If φ is the selection function over subsets of $\mathbb{P}$ and $a \in E(x)$ is an index for $\mathbb{P}(e)$, then $\varphi(a)$ is a non-empty REC subset of $\mathbb{P}(e)$. The bounding principle applied to the computation giving the height forced by some element of $\varphi(a)$ on $\{e\}(G)$ yields a bound on the height of $\{e\}(G)$ (for any G) recursive in $\mathbb{P}$.

Now consider the case of Kleene recursion in ${}^3\mathbb{E}$. Harrington [1973] showed that $E(2^\omega) = L_{\kappa_1}{}^3\mathbb{E}(2^\omega)$. Let the 1-section of ${}^3\mathbb{E}$ be defined by: $1.sc({}^3\mathbb{E}) = \{a \subseteq \omega \mid a \leq_{{}^3\mathbb{E}} 2^\omega\}$. If every real is constructable, then $L_{\kappa_1}{}^3\mathbb{E}(2^\omega) = L_{\kappa_1}{}^3\mathbb{E}$ and a naturalquestion is whether a real b Cohen-generic/$L_{\kappa_1}{}^3\mathbb{E}$ satisfies : $b \in 1\text{-}sc({}^3\mathbb{E})$ in $L_{\kappa_1}{}^3\mathbb{E}[a]$.

Sacks showed that such a real computes no more ordinals than $\emptyset$ in the ground model. A result of Levy [1970] will allow us to answer this question negatively in a strong sense.

Definition If $\mathbb{P}$ is a poset we say that $\mathbb{P}$ is semi-homogeneous iff $\forall p,p' \in \mathbb{P}$ there exists an automorphism of $\mathbb{P}$ $\pi: \mathbb{P} \to \mathbb{P}$ such that $\pi(p)$ and p' are compatible (i.e. $\exists q \in \mathbb{P}$ such that $q \leq \pi(p)$ and $q \leq p'$).

Using this condition on $\mathbb{P}$, Lévy shows the following remarkable result about generic extensions via $\mathbb{P}$.

Theorem 4.2 (Lévy) Assume $\mathbb{P}$ is semi-homogeneous and let M be a countable model of ZF with $\mathbb{P} \in M$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic/M and $N = M[G]$. Then we have that for every $x \in N$ and $y \in M$:

$$x \in [HOD(y)]^N \to x \in M.$$

Remark HOD(y) are those sets hereditarily ordinal definable from y. A closer look at Lévy's proof reveals that the same ordinal parameters suffice to define x in M as did in N. The proofs of Lévy's result is a transfinite induction on rank (see Lévy [1970]).

Lemma 4.3 Let $\mathbb{P}$ be the Cohen poset for adding a real, then $\mathbb{P}$ is semi-homogeneous.

Proof $\mathbb{P} = \{f: \omega \to \{0,1\} \mid f$ partial with finite domain

so given $p,p' \in \mathbb{P}$: if p and p' are compatible, the identity automorphism will suffice. Otherwise let

$$B = \{n \in \omega \mid n \in dom(p) \cap dom(p') \wedge p(n) \neq p'(n)\}$$

and consider the case where $B = \{n_0\}$ (the general case is similar). Let

$$m = \max(dom(p), dom(p'))$$

and define a permutation $\rho: \omega \Leftrightarrow \omega$ by $z \in \omega$

$$\rho(z) = \begin{cases} m+1, & \text{if } z = n_0 \\ n_0, & \text{if } z = m+1 \\ z, & \text{otherwise.} \end{cases}$$

Then ρ induces an automorphism $\pi: \mathbb{P} \to \mathbb{P}$ given by : $q \in \mathbb{P}$

$$\mathrm{dom}(\pi(q)) = \{\rho(n) \mid n \in \mathrm{dom}(q)\}$$

and for $z \in \mathrm{dom}(\pi(q))$ we let

$$\pi(q)(z) = q(\rho^{-1}(z)).$$

Then if we consider n_0 above, we have

$n_0 \notin \mathrm{dom}(\pi(p))$ and

$\pi(p)(\rho(n_0)) = p(\rho^{-1}(\rho(n_0))) = p(n_0)$ and so $\pi(p)$ and p' are compatible with extension $q = \pi(p) \cup p'$.

Thus if we force with tnis $\mathbb{P}$ over L, the following fact shows that there is no hope of extending $1\text{-sc}(^3\mathbb{E})$.

Fact 4.4 Let M be a transitive model of ZF and let

$X \in (k\text{-DC}(^{k+1}\mathbb{E}))$, then

$X \in \mathrm{HOD}^M$. To see this notice that for any n, type (n) is definable and the ordinal of the computation X itself is ordinal definable in M.

Combining these results we can now show,

Theorem 4.5 Let $\mathbb{P}$ be the Cohen poset for adding a real to L and let $a \subseteq \omega$ be $\mathbb{P}$-generic/L, then

$$(1 - \mathrm{sc}(^3\mathbb{E}))^{L[a]} = (1 - \mathrm{sc}(^3\mathbb{E}))^{L}$$

Proof Assume that $b \in (2^\omega)^{L[a]} \setminus (2^\omega)^L$ and suppose that $b \leq_{{}^3\mathbb{E}} 2^\omega$ in $L[a]$, then $b \in OD^{L[a]}$ and since $b \subseteq \omega$, we have that $b \in HOD^{L[a]}$. By theorem 4.2 $b \in L$, contradicting the choice of b. If $b \in L$ such that

$$b \in (1 - sc({}^3\mathbb{E}))^{L[a]},$$

then by Lemma 4.0 $b \leq_{{}^3\mathbb{E}} 2^\omega$, for some $\gamma < (\kappa_0^{2^\omega})^L$ and by the remark following Theorem 4.2 we have $b \leq_{{}^3\mathbb{E}} 2^{\omega,\gamma}$ in L, as desired.

§ 5 ${}^3\mathbb{E}$ - Degrees of Reals

We will use Lévy's result to show that the well-foundedness of the set of degrees of reals modulo ${}^3\mathbb{E}$ under the induced ordering is independent of ZF. This answers a question of Normann and also one of Sacks concerning the relative computability of mutually Cohen generic reals.

Definition If $a \subseteq \omega$, then the degree of a mod ${}^3\mathbb{E}$ is

$$[a]_{{}^3\mathbb{E}} = \{b \subseteq \omega \mid a \leq_{{}^3\mathbb{E}} b \wedge b \leq_{{}^3\mathbb{E}} a\}$$

and $\mathcal{D}({}^3\mathbb{E}) = \{[a]_{{}^3\mathbb{E}} \mid a \subseteq \omega\}$. Therefore $[a]_{{}^3\mathbb{E}}\ [b]_{{}^3\mathbb{E}} \in \mathcal{D}({}^3\mathbb{E})$:

$$[a]_{{}^3\mathbb{E}} \leq [b]_{{}^3\mathbb{E}} \quad \text{iff} \quad \exists a_0 \in [a]_{{}^3\mathbb{E}}\ \exists b_0 \in [b]_{{}^3\mathbb{E}}$$

such that $a_0 \leq_{{}^3\mathbb{E}} b_0$.

Proposition 5.0 (V = L) $\langle \mathcal{D}(^3\mathbb{E}), \leq \rangle$ is well-founded.

Proof Let $\leq_L$ denote the well-ordering of L, then $\leq_L \cap (2^\omega)^L$ is recursive in $^3\mathbb{E}, (2\omega)^L$. Given $a \in (2^\omega)^L$ we can effectively compute $|a|_{\leq_L}$, the height of a in the well-ordering, and a counting of $|a|_{\leq_L}$. Thus for every $b \in (2^\omega)^L$ with $b \leq_L a$, b is recursive in $^3\mathbb{E}, (2^\omega)^L$ and some integer (b's place in the counting of $|a|_{\leq_L}$). This shows that in L the degree ordering follows $\leq_L$ and is therefore well-founded.

Corollary 5.1 $Con(ZF) \rightarrow Con(ZF + \langle \mathcal{D}(^3\mathbb{E}), \leq \rangle$ is well-founded).

We will now show that the mildest possible extension of L adding reals, namely adding a single Cohen real, yields an infinite descending path through this ordering.

Theorem 5.2 Let M be a countable, transitive model of ZF + V = L and let $a \subseteq \omega$ be Cohen-generic/M, then

$$M[a] \models \text{"} \langle \mathcal{D}(^3\mathbb{E}), \leq \rangle \text{ is not well-founded"}.$$

Proof M fulfils the condition of Lèvy's theorem and the Cohen poset for adding a rent is semi-homogeneous as we've shown. Define the following splitting of the Cohen real a:

$a_{0,0}$ = even part of a

$a_{0,1}$ = odd part of a and

in general at stage n:

$a_{n+1,0}$ = even part of $a_{n,0}$

$a_{n+1,1}$ = odd part of $a_{n,0}$

A standard argument shows that $\forall_n$ $a_{n,0}$ and $a_{n,1}$ are mutually Cohen generic. By Lèvy's result we have in $L[a]$: $\forall n \in \omega$

$$a_{n,0} \not\leq_{^3\mathbb{E}} a_{n,1} \quad \text{and}$$

$$a_{n,1} \not\leq_{^3\mathbb{E}} a_{n,0} \,. \quad \text{As a result}$$

$$\forall_{i\in\omega} [a_{0,i+1} \not\leq_{^3\mathbb{E}} a_{0,i}] \quad \text{and} \quad a_{0,0} \not\leq_{^3\mathbb{E}} a.$$

The sequence $\{a_{0,i} \mid i\in\omega\} \in N$ and hence

$$N \models \text{"}<\mathcal{D}(^3\mathbb{E}),\subseteq> \text{ is not well-founded"}.$$

§ 6 Extending the 1-sc($^3\mathbb{E}$)

Recall that the extension via a Cohen real a in the previous section satisfies :

$$(\kappa_1^{^3\mathbb{E}})^L = (\kappa_1^{^3\mathbb{E}})^{L[a]} \,.$$

If we are willing to give up this constraint we can extend the 1-sc($^3\mathbb{E}$) by forcing over a well-known partially ordered set.

Theorem 6.0 Let M be a countable, transitive model of $ZF + V = L$ and let $a \subseteq \omega$ be $\mathrm{Col}(\omega, \aleph_1)$-generic/$M$. ($\mathrm{Col}(\omega, \aleph_1)$ is the Lèvy poset for collapsing $\aleph_1$ to ω). Then

$$(1\text{-sc}(^3\mathbb{E}))^M \not\subseteq (1\text{-sc}(^3\mathbb{E}))^{M[a]}.$$

Proof Define the complete set of integers relative to $^3\mathbb{E}$ by

$$C = \{<e,m> \mid \{e\}(^3\mathbb{E}, m)\downarrow\}, \text{ then}$$

$C \in L_{\kappa_1^{3\mathbb{E}}}$ but $C \notin 1\text{-sc}(^3\mathbb{E})$ in L. In $M[a]$, Σ_1^L is recursive in $^3\mathbb{E}, 2^\omega$ and therefore

$$(\kappa_1^{3\mathbb{E}})^L \leq_{^3\mathbb{E}} 2^\omega \quad \text{in } M[a].$$

Thus if we denote by C^M the interpretation of C in M, then using $(\kappa_1^{3\mathbb{E}})^L$, C^M is recursive in $^3\mathbb{E}, 2^\omega$ in $M[a]$, i.e.

$$C^M \in (1\text{-sc}(^3\mathbb{E}))^{M[a]},$$

as desired.

A reasonable question is whether we can extend the $1\text{-sc}(^3\mathbb{E})$ as above without violating $\kappa_1^{3\mathbb{E}}$ of the ground model. In the next section we provide such an example.

§ 7 Jensen - Johnsbråten Reals and $1\text{-sc}(^3\mathbb{E})$

Here we consider a forcing extension preserving $\kappa_1^{3\mathbb{E}}$, but extending the $1\text{-sc}(^3\mathbb{E})$.

The relevant theorem is an improvement of Solovay's result [1967] (that it is consistent with ZF to assume that there is a non-constructable Δ_3^1 subset of ω by Jensen - Johnsbråten [1974].

Theorem 7.0 (Jensen - Johnsbråten) : There exists a π_2^1 formula φ such that the following are provable in ZF :

(a) $\varphi(x) \to x \subseteq \omega$;

(b) $V = L \to \neg\exists x \varphi(x)$

(c) $\omega f = \omega_1 \to \exists^{\leq 1} x \varphi(x)$

(d) $Con(ZF) \to Con(ZF + GCH + \omega_1^L = \omega_1 + \exists a(\varphi(a) \wedge V = L[a]))$;

(e) If $M \models ZFC + \omega_1^L = \omega_1 + \varphi(a)$ and N is a cardinal preserving extension of M, then $N \models \varphi(a)$.

If $\{a\} \in \pi^1_2$ (i.e. a is implicitly π^1_2-definable), then $a \in \Delta^1_3$. It is this definability ($a \in \Delta^1_3$ clearly implies that $a \leq_{3\mathbb{E}} \emptyset$) and the chain condition on the necessary iterated forcing that gives the desired result. For the proof of Theorem 7.0 consult Jensen-Johnsbråten [1974] or Devlin-Johnsbråten [1975].

Theorem 7.1 There is a countable chain condition (c.c.c.) iterated forcing (set forcing) $\mathbb{P}_\omega$ such that if G is $\mathbb{P}_\omega$-generic/$L_{K_1{}^3\mathbb{E}}$, then

(i) $1\text{-}sc^{L}({}^3\mathbb{E}) \subsetneq 1\text{-}sc^{L[G]}({}^3\mathbb{E})$; and

(ii) $L_{K_1{}^3\mathbb{E}}[G]$ is E-closed.

Proof Jensen-Johnsbråten show that the necessary trees are $\Sigma_1(L_{\omega_1} L)$ and are hence recursuve in ${}^3\mathbb{E}$ in L. The real coding $\langle b_n \mid n \in \omega \rangle$ the sequence of branches through these ω-many trees is Δ^1_3 and also recursive in ${}^3\mathbb{E}$, which gives (i).

(ii) follows from theorem 3.0 and each stage in the iteration is c.c.c.. The iteration is given by:

$$\mathbb{P}_0 = T_0 \quad \text{(under the reverse ordering)}$$

$$\mathbb{P}_{n+1} = T^*_{n+1} \quad \text{over} \quad M_{n+1} = L[\langle b_0, \cdots, b_{n+1} \rangle],$$

then

$$\mathbb{P}_\omega = \lim_{\rightarrow} \langle \mathbb{P}_n \mid n \in \omega \rangle$$

Each $\mathbb{P}_n$ is c.c.c. and hence the direct limit is also c.c.c.. The desired model is thus the direct limit of $M_n = L_{\kappa_1}{}^3\mathbb{E}[\langle b_0, \cdots, b_n \rangle]$ and is given by a true iteration.

§ 8 Almost Disjoint Codes and 1-sc($^{k+2}\mathbb{E}$)

We consider here the effect of adding reals which are almost disjoint codes for subsets of $\aleph_1$ upon the 1-sc($^3\mathbb{E}$) as a characteristic case. First we give a brief outline of this notion of forcing.

Let $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ be a family of almost disjoint subsets of ω and let $X \subseteq \omega_1$. Define $\mathbb{P}_{\mathcal{A},X}$ as follows :

A <u>condition</u> is a function from a subset of ω into $\{0,1\}$ such that

a (i) $\mathrm{dom}(p) \cap A_\alpha$ is finite for every $\alpha \in X$;

(ii) $\{n \mid p(n) = 1\}$ is finite.

The set $\mathbb{P}_{\mathcal{A},X}$ is partially ordered by inverse indusion : $P \leq q$ iff p extends q. If p and q are incompatible, then $\{n \mid p(n) = 1\} \neq \{n \mid q(n) = 1\}$ and so $\mathbb{P}_{\mathcal{A},X}$ satisfies the c.c.c. Thus if $\mathbb{P}_{\mathcal{A},X} \in L_{\kappa_1}{}^3\mathbb{E}$ and $f: \omega \to \{0,1\}$ is $\mathbb{P}_{\mathcal{A},X}$-generic/L, then $L_{\kappa_1}{}^3\mathbb{E}[f]$ is E-closed by Sacks (see Slaman [1981]).

This example of a generic cannot extend 1-sc($^{k+2}\mathbb{E}$), $k \geq 1$.

Theorem 8.0 Suppose $\mathbb{P}_{\mathcal{O}1,X} \in L_{\kappa_1^{k+2}\mathbb{E}}$ and if is $\mathbb{P}_{\mathcal{O}1,X}$-generic/$L_{\kappa_1^{k+2}\mathbb{E}}$,

then

$$f \notin (1\text{-sc}(^{k+2}\mathbb{E}))^{L[f]}$$

Proof We consider the case $k = 1$ and $X \subseteq \aleph_1^L$ for simplicity. As before we use the result of Lévy and Fact 4.4.

Suppose that $f \leq_{3_{\mathbb{E}}} \emptyset$, then $f \in OD^N$ by lemma . Since $f \subseteq \omega$, f is an element of HOD^N. All that remains is to show that $\mathbb{P}_{\mathcal{O}1,X}$ satisfies the hypothesis of Lévy's theorem.

Lemma 8.1 The poset $\mathbb{P}_{\mathcal{O}1,X}$ for almost disjoint coding is semi-homogeneous.

Proof We can view two conditions as

$$p = \langle k, (A_1, \cdots, A_n)\rangle, \qquad p' = \langle h, (B_1, \cdots, B_m)\rangle,$$

where k and h are finite subsets of ω and the A_i and B_j ($i \leq n$, $j \leq m$) are finite subsets of $\{A_\alpha \mid \alpha \in X\}$.

We find a permutation $p: \mathbb{N} \to \mathbb{N}$ as follows : let

$$A = \bigcup_{i \leq n} A_i \quad \text{and} \quad B = \bigcup_{j \leq m} B_j, \quad \text{then}$$

$$x \in k \to p(x) \in h \vee p(x) \notin B$$

$$x \in h \to p^{-1}(x) \in k \vee p^{-1}(x) \notin A.$$

Let $s_0 < s_1 < s_2$ be integers such that

(i) $x \in k \cup h \Rightarrow x < s_0$;

(ii) $\overline{\overline{[s_0,s_1) \setminus B}} \geq \overline{\overline{k}}$

(iii) $\overline{\overline{[s_1,s_2) \setminus A}} \geq \overline{\overline{h}}$

Define as follows :

$x \geq s_2$, let $p(x) = x$ thus ρ will be a permutation on $[0,s_2)$:

$$\left.\begin{array}{l} x \in k \cap h, \text{ let } p(x) = x \\ x \in k \setminus h, \text{ let } p(x) \in [s_0,s_1) \setminus B \\ x \in h \setminus k, \text{ let } p^{-1}(x) \in [s_1,s_2) \setminus A \end{array}\right\} pH$$

By taking $=$ in (ii) and (iii) above p gives a permutation.

To define the automorphism $\pi : \mathbb{P} \to \mathbb{P}$ take $\pi(p)$ for $p = \langle k,A \rangle$ to be

$\langle \rho(k), \rho(A) \rangle$ where

$\rho(k) = \{\rho(n) \mid n \in k\}$ and

$\rho(A) = \{\{\rho(n) \mid n \in b\} \mid b \in A\}$.

thus by Lemma 2

$f \in M$ which is absurd since f was taken $\mathbb{P}_{\mathcal{J},X}$-generic/M.

Corollary 8.2 If we take $\mathbb{P}_{\mathcal{J},X}$ to be the generalization of almost disjoint codes to regular κ over L by taking the appropriate family $\mathcal{J}$ where $\kappa = {}_n, \exists n \in \omega$ where

$X \subseteq \text{type}(n)$ and

G is $\mathbb{P}_{\mathcal{J},X}$-generic/$L$ with $\mathbb{P}_{\mathcal{J},X} \in L_{\kappa_1^{n+2}}\mathbb{E}$ then

$G \notin (n\text{-sc}({}^{n+2}\mathbb{E}))^{L[G]}$

Proof Using the fact that L is the ground model and every element of L is $HOD^{L[G]}$, so if

$$G \in OD^{L[G]}, \quad \text{then} \quad G \in HOD^{L[G]}.$$

The argument that this $\mathbb{P}_{\mathcal{O}\mathcal{T},X}$ satisfies semi-homogeneity is suitably altered to handle the limit ordinals involved. The argument G preserves E-closure uses Theorem 3.0 and selection over type(n-1).

Until now we have been primarily concerned with 1-sections. In the next section we study n-sections for $n > 1$ for the Kleene functionals $^{k+2}\mathbb{E}$ for $k \geq 2$. The 2-sc($^3\mathbb{E}$) is determined completely by the reals and thus cannot be extended without adding new reals.

§ 9 Extending the 2-sc($^4\mathbb{E}$)

We shall argue here that we can by forcing add an element of the 2-sc($^4\mathbb{E}$) = $\{X \subseteq 2^\omega \mid X$ is recursive in $^4\mathbb{E}\}$ over L without violating $\kappa_1^{^4\mathbb{E}}$. The techniques involved had to confront the obstacle posed by Lévy's result concerning posets satisfying semi-homogeneity which states that forcing with such a poset cannot add new elements of HOD(x) for any ground model set x.

The natural solution here is to resort to a poset $\mathbb{P}$ which has the identity as its only automorphism. We force over the rigid Souslin tree constructed by Jensen [1968] in L and using his methods for showing that the resulting ω_1-tree is Souslin we show that the only ω_1-path in the extension is the generic path. This yields the

definability required for arguing that this path (viewed as a subset of $(2^{\omega})^{L}$ via $<_{L}$) is recursive in ${}^{4}\mathbb{E}$.

If we work over L, then if we force with a semi-homogeneous poset $\mathbb{P}$ Lévy's result and lemma show that there is no hope of extending the 2-sc(${}^{4}\mathbb{E}$) without adding new reals (and hence having done so trivially). To see this suppose N is such a generic extension of L and

$$X \in (2\text{-sc}({}^{4}\mathbb{E}))^{N}.$$

Then $X \in OD^{N}$ and if no new reals were added in forcing over L, we would have that $X \in HOD^{N}$. By Lévy and semi-homogeneity $X \in L$ and definable in the same ordinal parameters, hence

$$X \in (2\text{-sc}({}^{4}\mathbb{E}))^{L}.$$

Fact: If $\mathbb{P}$ is a notion of forcing such that the only automorphism is the identity, then $\mathbb{P}$ does not satisfy semi-homogeneity (just take p and q in $\mathbb{P}$ incompatible).

The following theorem of Jensen (see Devlin-Johnsbråten [1975] gives us the required notion of forcing for extending the 2-sc(${}^{4}\mathbb{E}$).

Definition A partially ordered set $X = \langle X, \leq \rangle$ is rigid, if $id \restriction X$ is the only automorphism on X.

Theorem 9.0 (Jensen) Assume $\diamondsuit$. Then there exists a rigid Souslin tree.

For our purposes work in L, then $\diamondsuit$ holds and there exists a rigid Souslin tree T, which is in fact $\Sigma_1(L_{\omega_1})$ and hence recursive in ${}^{4}\mathbb{E}$, $2^{2\omega}$ in L. Viewing T as its coding

$$T \in (2\text{-sc}({}^{4}\mathbb{E}))^{L},$$

so let us consider the result of forcing with the poset corresponding

to $\mathbb{P}$ over L (we also use T to refer to the Souslin algebra derived from T). T satisfies the c.c.c., so if G is T-generic/L, then L[G] is a cardinal and cofinality preserving extension of L. By the following lemma we have a bit more.

Lemma 9.1 If G is T-generic/L, then

$$(2^{\omega})^{L[G]} = (2^{\omega})^{L}$$

Proof Suppose not and let $\dot{f}: \omega \to \omega$ be a term for a real $f \in (2^{\omega})^{L[G]} \setminus (2^{\omega})^{L}$. In L[G] consider the following map defined by induction on ω:

$n \to p_n$ given by

p_0 = least $p \in G$ such that $\exists m \in \omega$ with

$$p \Vdash \dot{f}(o) = m \ ;$$

given $p_0, \cdots, p_n$ let

$$p_{n+1} = \text{least } y \in G \text{ such that } q \leq p_n \text{ and } \exists m$$
$$q \Vdash \dot{f})n+1) = m$$

Claim $F: \omega \to \omega_1$ defined by

$F(n) = \bigcup \mathrm{dom}(p_n)$ is unbounded in ω_1

Proof Otherwise $\exists \delta < \omega_1$ such that

$$\bigcup_{n \in \omega} F(n) \leq \delta.$$

But then $f: \omega \to \omega$ is definable from $G \restriction \delta + 1 \in L$ contradicting the choice of f.

Clean F up by taking $F': \omega \longrightarrow \omega_1$ and let $F'(n) = \alpha_n$. Each α_n is countable via some $a_n \in WO$ and letting a code the family

$\{a_n\}_{n\in\omega}$ in a standard way we get in $L[G]$ $g: \omega \leftrightarrow \omega_1$ contradicting the fact that $L[G]$ was a cardinal preserving extension of L.

Remark Thus no new reals are added and if we can show that $\cup G$ is definable from T in $L[G]$, then the following theorem, giving the uniqueness of $\cup G$ as a path, will yield the desired non-trivial extension of $2\text{-sc}(\Delta^1_4 E)$ in $L[G]$.

Theorem 9.2 Let G be T-generic/L, then $\cup G$ is the only branch through T in $L[G]$.

Proof Suppose not and let $b \in L[G]$ be a branch through T such that $b \neq \cup G$. Then there exists an $\alpha < \omega_1$ such that $b(\alpha) \neq \cup G(\alpha)$, take the least such α_0. Let τ be a term in LST such that for $\vec{\alpha}$ a finite vector of ordinals:

$$\tau^{L[G]}(\vec{\alpha}, G) = b \quad \text{(take the least such in the sense of } <_L).$$

By the same argument showing that no new reals are added we have that $(\forall \alpha < \omega_1)(b \restriction \alpha \in L)$ and $\exists \beta < \omega_1$ such that $b \restriction \alpha_0 + 1 \in L_\beta$.

The term $\tau \in L\omega_2$ so proceeds now as in the proof of rigidity including τ and $\alpha_0 + 1$ in the chain of elementary substructures used in Devlin-Johnsbråten [1975].

Corollary 9.3 If we denote by $\{a_\gamma \mid \gamma < \omega_1\}$ the well-ordering of $(2^\omega)^L$ and

$$G^* = \{a_\gamma \mid \gamma \in G\}$$

and G is T-generic/L, then

$$G^* \in 2\text{-sc}(\Delta^1_4 E)$$

Proof The predicate

$\varphi(T,x) \equiv x$ is a path through T

is recursive in ${}^{4}\mathbb{E}$ (using $\omega_1 \leq_{4\mathbb{E}} \emptyset$) and hence, so is the set

$\{x|\varphi(T,x)\} = \{G\}$

by the above theorem. Again using the well-ordering of $(2^{\omega})^{L}$ recursive in ${}^{4}\mathbb{E}$ we compute G^* from G.

§ 10 Extending the k-sc($^{k+2}\mathbb{E}$)

In this section we generalize the methods used to extend the 2-sc($^{4}\mathbb{E}$) to all finite types. We modify the proof of Jensen [1972] that there exists a rigid Souslin tree in L to prove the existence of a rigid κ-tree which is κ-Souslin in L. We then force over that tree preserving $\kappa_1^{k+2\mathbb{E}}$ for the appropriate k. Using the definability of the resulting κ-branch (actually its uniqueness in the extension) we conclude that it is recursive in ${}^{k+2}\mathbb{E},\emptyset$ and hence clearly extends the k-sc($^{k+2}\mathbb{E}$). Throughout we consider the case of the 3-sc($^{5}\mathbb{E}$). The generalization to all finite types is straightforward. We show that the extension of the section is non-trivial by showing that we add no new sets of lower type.

ω_2-Trees which are ω_2-Souslin.

In Jensen [1972] one constructs ω_2-trees which are ω_2Souslin, but the resulting tree is not obviously rigid. We modify that construction here using the main idea of the proof as presented in Devlin-Johnsbråten [1974] to produce an ω_2-Souslin tree which is rigid and later use the

strategy for showing that the tree is rigid to argue that forcing over that tree yields a model in which there is only one branch. We include a proof for those uninterested in Souslin trees, but curious about the coding.

Theorem 10.0 ($V = L$) There exists an ω_2-tree which is ω_2-Souslin and rigid.

Proof Let $\langle S_\alpha | \alpha < \omega_2 \rangle$ be the sequence given by $\diamondsuit$ in L. We wish to construct a Souslin tree T. The points of T will be ordinals less than ω_2. We shall construct T in stages T_α $(1 \leq \alpha < \omega_2)$ where T_α is to be the restriction of T to points of rank $< \alpha$. Hence T_α will be a normal tree of length α and T_β will be an end extension of T_α for $\beta > \alpha$. We define T by induction on α as follows.

Case 1. $\alpha = 1$, $T_1 = \{0\}$.

Case 2. $T_{\alpha+1}$ is defined. Define $T_{\alpha+2}$ by appointing to immediate successors for each maximal point of $T_{\alpha+1}$.

Case 3. lim(α) and T_ν is defined for $\nu < \alpha$. Set

$$T_\alpha = \bigcup_{\nu>\alpha} T_\nu .$$

Case 4. lim(α) and T_α is defined. We must define $T_{\alpha+1}$.

If $cf(\alpha) = \omega$ then define $T_{\alpha+1}$ by appointing a sucessor for each maximal point of T_α. Our work is to be done at α such that $cf(\alpha) = \omega_1$. By induction on $\alpha < \omega_2$ let $\delta(\alpha)$ be the least ordinal $\delta > \alpha$ such that,

(i) $L_\delta \prec L_{\omega_2}$ and

(ii) $\langle \delta(\nu) \mid \nu < \alpha \rangle \in L_\delta$ and

set $M_\alpha = L_\delta(\alpha)$. Then M_α has size $\leq \aleph_1$ for $\alpha < \omega_2$. If $\alpha < \omega_2$ and $\lim(\alpha)$ and $cf(\alpha) = \omega_1$, assume that $T_\alpha \in M_\alpha$.

To define $T_{\alpha+1}$ we force over M_α with $\mathbb{P} = \langle \mathbb{P} \leq_{\mathbb{P}} \rangle \in M_\alpha$ given by:

$$\mathbb{P} = \{p \mid \exists a \leq \omega_1 (|a| < \omega_1 \wedge p: a \to T_\alpha)\}$$

with $p \leq_{\mathbb{P}} q \leftrightarrow \mathrm{dom}(p) \supseteq \mathrm{dom}(q) \wedge \forall \alpha \in \mathrm{dom}(q)$

$$[p_\alpha \supseteq q_\alpha]$$

Notice that $M_{\alpha+1} \models \overline{\overline{M}}_\alpha = \aleph_1$ and also $M_\alpha \models$ '$\mathbb{P}$ is countably closed'.

Let $G \subseteq \mathbb{P}$ be the $<_L$-least $\mathbb{P}$-generic/M_α set. Since

$$M_{\alpha+1} \dashv L_{\omega_2} \quad \text{and} \quad M_\alpha \in M_{\alpha+1},$$

and $\exists f \in M_{\alpha+1}$ $f: \omega_1 \leftrightarrow M_\alpha$ and since $\mathbb{P}$ is countably closed generics exists in L_{ω_2} and by elementarity also in $M_{\alpha+1}$. Hence $T_\beta \in M_\beta$ for $\lim(\beta)$ will be trivial.

For $\gamma < \omega_1$, let

$$b_\gamma = \{p_\gamma \mid p \in G\}$$

Claim (i) each b_γ is an α-branch of T_α;

(ii) each b_γ is T_α-generic/M_α;

(iii) $b_\gamma \neq b_\delta$ for $\gamma \neq \delta$ less than ω_1;

(iv) if $\alpha_1, \cdots, \alpha_n$ are distinct, then

$b_{\alpha_1} \times \cdots \times b_{\alpha_n}$ is $(T_\alpha)^n$-generic/M_α;

(v) $T_\alpha = \bigcup_{\alpha < \omega_1} b_\alpha$

Proof (i), (ii) and (iii) follow easily from (iv) :

let $\alpha_1, \cdots \alpha_n$ be distinct ordinals less than ω_1, and let

$D \subseteq (T_\alpha)^n$ be dense and closed under extensions. Let

$$D^* = \{p \in \mathbb{P} | \langle p_{\alpha_1}, \cdots, p_{\alpha_n} \rangle \in D\} ,$$

then D^* is dense in $\mathbb{P}$ so let $p \in G \cap D^*$. By the choice of p

$$\langle p_{\alpha_1}, \cdots, p_{\alpha_n} \rangle \in b_{\alpha_1} \times \cdots \times b_{\alpha_n} \cap D$$

as desired. To see (v), let $\sigma \in T_\alpha$ and define

$$D' = \{p \in \mathbb{P} | \exists \gamma \in \mathrm{dom}(p)(p_\gamma \supseteq \sigma)\} ,$$

then D' is dense in $\mathbb{P}$ so let $p \in G \cap D'$.

Then $\exists_\gamma \in \mathrm{dom}(p)$ such that

$$\sigma \subseteq p_\gamma \in b_\gamma \quad \text{and}$$

so $\sigma \in b_\gamma$.

Now set $T_{\alpha+1} = \{\cup b_\alpha | \alpha < \omega_1\}$, then by (v) $T|(\alpha+1)$ is still normal and so $T = \bigcup_{\alpha<\omega_2} T_\alpha$ is a normal tree of length ω_2.

Claim T is ω_2-Souslin.

Proof It suffices to show that T has no ω_2-antichains so let $X \subseteq T$ be a maximal antichain. We show $\overline{\overline{X}} \leq \aleph_1$. Let A be the set of limit $\alpha < \omega_2$ such that $X \cap \alpha$ is a maximal antichain in T_α. A is club in ω_2.

Now let

$$\kappa_\gamma = OR \cap M_\gamma, \quad \text{for} \quad \gamma < \omega_2 .$$

$E = \{\alpha_\gamma \mid \gamma < \omega_2\}$ is also club in ω_2, hence there exists $\alpha \in A \cap E$ such that

$$S_\alpha = X \cap \alpha \quad \text{by} \quad .$$

By the construction of $T_{\alpha+1}$, then we have :

Every X of level α lies above an element of $X \cap \alpha$. Hence $X \cap \alpha$ is a maximal antichain in T and $X = X \cap \alpha$ has cardinality $< \aleph_2$.

The proof that T is rigid proceeds as in Jensen's proof for the rigid ω_1-Souslin tree.

Remark (i) Obvious modifications show that with $\diamondsuit_{\aleph_n}$ we construct a rigid $\aleph_n$ - Souslin tree.

(ii) T has the $\aleph_2$ -cc. by the above. By the construction at $cf(\alpha) = \omega$ stages and the fact that $\mathbb{P}$ at $cf(\alpha) = \omega_1$ stages was countably closed, T itself is countably closed. For κ as in (i) equal to $\aleph_n$ for $n \geq 2$ T will have the κ-c.c. and be $\aleph_{n-2}$ -closed. This fact will prove indispensable.

(iii) It is an interesting question whether $\diamondsuit$ is enough to produce a κ-Souslin tree for all κ not Mahlo. Jensen does so using $\square$.

§ 11 Forcing with Rigid ω_2 Souslin Trees

We will work over $L_{\kappa_1}{}^5\mathbb{E}$ and force with the ω_2-Souslin tree constructed in the previous section to extend non-trivially the 3-sc($^5\mathbb{E}$). The tree T is recursive in $^5\mathbb{E}, \emptyset$ since $T \in \Sigma_1(L_{\omega_2})$. Let G be T-generic, then the theorem guarantees that $\cup G$ preserves the E-closure. of $L_{\kappa_1}{}^5\mathbb{E}$ and more.

We shall argue that $G \leq_{5_{\mathbb{E}}} \emptyset$ on $L_{\kappa_1}{}^{5_{\mathbb{E}}}[G]$ by showing that G is the only path through T in $L_{\kappa_1}{}^{5_{\mathbb{E}}}[G]$.

Theorem 11.0 If G is T-generic/L, then $\cup G$ is the only branch through T in $L[G]$.

Proof Suppose not and let $b \in [T]$ in $L[G]$ such that $b \neq \cup G$. Then as before there exists a term $\tau \in L_{\omega_3}$ such that

$\tau^{L[G]} = b$, where τ depends on G and finitely many ordinal parameters. There also exists a $p \in G$ such that

$p \Vdash$ 'τ is a branch through T different from G'.

Now argue as in Jensen's proof of rigidity that, at some stage $\alpha < \omega_2$ in the construction, τ gives a branch through T_α different from $G \restriction \alpha$ and that

$$\tau \in M_\alpha[G \restriction \alpha]$$

but as branches we extended through the αth stage

$\tau \times G \restriction \alpha$ is $(T_\alpha)^2$-generic/M_α and hence by the product lemma

$$\tau \notin M_\alpha[G \restriction \alpha] \text{ , a contradiction.}$$

Corollary 11.1 $\cup G \leq_{5_{\mathbb{E}}} \emptyset$ in $L_{L_1}{}^{5_{\mathbb{E}}}[G]$

Proof $\cup G$ is the unique branch through T $T \leq_{5_{\mathbb{E}}} \emptyset$ and we test all such candidates.

Corollary 11.2 $(3\text{-sc}(^5\mathbb{E}))^{L_{\kappa_1}{}^5\mathbb{E}[G]} \not\subseteq L_{\kappa_1}{}^5\mathbb{E}$ and hence the extension of $3\text{-sc}(^5\mathbb{E})$ is achieved.

Proof Interpret UG as a subset of

$$(2^{2^\omega})^L .$$

In order to argue that the extension of $3\text{-sc}(^5\mathbb{E})$ is non-trivial, the following lemmae suffices.

Lemma 11.3 In $L_{\kappa_1}{}^5\mathbb{E}[G]$

(i) $\aleph_1^L$ is preserved

(ii) $\aleph_2^L$ is preserved

Proof (i) follows from the construction of T at $\lim(\alpha)$ with $cf(\alpha) = \omega$ where we extended all branches and the fact that $\mathbb{P}$ at $\lim(\alpha)$ with $cf(\alpha) = \omega$, was countably closed. Hence $\aleph_1^L$ is preserved.

(ii) follows from $\aleph_2$-c.c. which T satisfies.

Countable closure of T insures that, in addition, no new reals are added. Thus a new subset of the reals would be a new subset of $\aleph_1^L$. The following argument shows that no new subsets of the reals are added and hence that we have extended $3\text{-sc}(^5\mathbb{E})$ non-trivially.

Lemma 11.4 $(2^{\aleph_1})^L = (2^{\aleph_1})^{L[G]}$.

Proof Suppose not and let $X \subseteq \aleph_1$ satisfy

$$X \in (2^{\aleph_1})^{L[G]} \setminus (2^{\aleph_1})^L .$$

We will show that $\aleph_2^L$ is collapsed in L[G], giving a contradiction. By recursion on $\aleph_1$ define $f: \aleph_1 \to \aleph_2$ from G in L[G]:

$f(\gamma) = \mu p_0 \in G$ such that

$p_0 \Vdash \dot{X} \subseteq \check{\alpha}_1$ and $p_0 | 0 \in X$

$f(\tau+1) = \mu p_{\tau+1} \leq p_\tau$ such that

$$p_{\tau+1} \in G \begin{cases} p_{\tau+1} \Vdash \tau+1 \in \dot{X} \\ \text{or} \\ p_{\tau+1} \Vdash \tau+1 \notin \dot{X} \, . \end{cases}$$

If τ is limit ordered and $f(\gamma)$ has been defined $\forall \gamma < \tau$ let

$$f(\tau) = \mu p_\tau \leq \bigcup_{\gamma<\tau} p_\gamma \, .$$

(Since τ is countable and T is countable closed $\bigcup_{\gamma<\tau} p_\gamma \in T$) such that $P_\tau \in G$ and

$$p_\tau | \tau \in \dot{X}$$

Now define $F: \check{\alpha}_1 \to \check{\alpha}_2$ by taking

$F(\gamma) = \bigcup \mathrm{dom}(p\gamma)$

$\gamma F^{11} \leq \lambda < \check{\alpha}_2$, then $X \in L$ were done. Otherwise define $F': \check{\alpha}_1 \overset{1-1}{\to} \check{\alpha}_2$ by recursion from F. Placing together the collapses of ordinals less than $\check{\alpha}_2$ to $\check{\alpha}_1$ in the range of F' yields a collapse of $\check{\alpha}_2^L$ in $L[G]$, a contradiction.

As remarked above a straightforward generalization gives a way of non-trivially extending the k-sc($^{k+2}\mathbb{E}$) and as a result the n-sc($^{k+2}\mathbb{E}$), for $1 < n \leq k$. This is best possible since the k+1-sc($^{k+2}\mathbb{E}$) cannot be altered without changing the set of objects of type (k).

§ 12 Forcing and Reduction Procedures in E-Recursion

Since the question of Post's Problem was first posed for recursion in higher types some progress has been made, both positive and negative. In the presence of well-orderings Sacks has given a positive solution without a priority argument. Later Griffor gave a positive solution using a natural combinatorial principle which is consistent with the absence of well-orderings. On the negative side Normann [1979] showed that AD implies a negative answer and later Griffor [1981] strengthened Normann's result to show that under AD any regular RE set is REC.

Sacks asked whether it was possible to show that it is consistent with ZF that Post's Problem for $^3\mathbb{E}$ has a negative answer. In particular he asked whether it was possible to use forcing to produce a model of ZF where Post's Problem fails for $^3\mathbb{E}$. In this section we offer some evidence to the effect that new techniques will be required.

Definition Let $\mathbb{P}$ be a notion of forcing such that $\mathbb{P} \in L_{\kappa_1}{}^3\mathbb{E}$, then $\mathbb{P}$ is an effective notion of forcing iff the relation,

$$p \Vdash \varphi \equiv R(p,\varphi)$$

for $p \in \mathbb{P}$ and φ a formula is recursive in $^3\mathbb{E}, \mathbb{P}, \gamma$ when restricted to

$$\mathscr{L}_\gamma = \{\varphi \mid \text{rank}(\varphi) \leq \gamma\} .$$

Effectiveness is often used to prove that the generic over the poset preserves the closure one had in the ground model. An example is the Cohen poset for adding a real to $L_{\kappa_1}{}^3\mathbb{E}$ we saw in section 4, as are most set forcings.

Our main result is,

Theorem 12.0 Let $\mathbb{P} \in L_{\kappa_1}{}^3\mathbb{E}$ be an effective notion of forcing and $A, B \subseteq {}_{\kappa_1}{}^3\mathbb{E}$ such that A and B are both regular and hyperregular. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic/L, then :

$$\text{if } (A \leq_{{}^3\mathbb{E}} B)^{L_{\kappa_1}{}^3\mathbb{E}[G]} \quad \text{(with parameter)},$$

then

$$(A \leq_{{}^3\mathbb{E}} B)^{L} .$$

Remark In this case regularity corresponds to Jensen's [1972] amenability and hyperregularity to $L_{\kappa_1}{}^3\mathbb{E}[A]$ and $L_{\kappa_1}{}^3\mathbb{E}[B]$ being $\mathbb{E}$-closed. Note that if B is RE and hyperregular, then B is also regular.

Proof We shall show that if $B \subseteq L_{\kappa_1}{}^3\mathbb{E}$ is REC in $L_{\kappa_1}{}^3\mathbb{E}[G]$ where G is $\mathbb{P}$-generic/L, then B is REC on $L_{\kappa_1}{}^3\mathbb{E}$. The theorem will then follow from this fact by realativizing the argument to B and using the fact that B is regular and hyperregular.

Suppose that $B \subseteq L_{\kappa_1}{}^3\mathbb{E}$ and $\exists e \in \omega \quad \exists p \in L_{\kappa_1}{}^3\mathbb{E}[G]$ such that in $L_{\kappa_1}{}^3\mathbb{E}[G]$:

(i) $\{e\}(p, {}^3\mathbb{E},)$ is total; and

(ii) $\forall z \in L_{\kappa_1}{}^3\mathbb{E}[G]$,

$\{e\}(p, {}^3\mathbb{E}, z) = B(z)$.

Now p is given by some term in the forcing language $\tau(a_1, \cdots, a_n, \mathcal{A})$, where $a_1, \cdots, a_n$ can be taken to be reals in L and $\mathcal{A}$ is an unary predicate symbol denoting the set to be added via $\mathbb{P}$.

Remark Here we assume that the language for recursion on $L_{\kappa_1}{}^3\mathbb{E}$, $\mathscr{L}$, has been expanded to $\mathscr{L}'$ by introducing the new predicate symbol B (using regularity and hyperregularity of B) denoting B.

By the genericity of G, $\exists q \in G$ such that,

$$q \Vdash (\forall z)[\{e\}(\tau(\vec{\underline{a}},y),{}^3\mathbb{E},z)\downarrow \wedge \{e\}(\tau(\vec{\underline{a}},y),{}^3\mathbb{E},z) = B(z)]$$

Using $\mathbb{P},q,e$ as parameters we can now compute B on $L_{\kappa_1}{}^3\mathbb{E}$: if $\gamma \in L_{\kappa_1}{}^3\mathbb{E}$ then

$$\text{rank}(\{e\}(\tau(\vec{\underline{a}},y),{}^3\mathbb{E},\gamma) \underline{\sim} n) = \sigma$$

depending on the parameters a_n occurring in τ. Since $\mathbb{P}$ is an effective notion of forcing, the relation

$$q' \leq q \wedge q' \Vdash \{e\}(\tau(\vec{\underline{a}},y),{}^3\mathbb{E},\gamma) = n \equiv R(q')$$

is a relation recursive in $q,\mathbb{P},\sigma$, where we imagine σ as encoding in addition the finite sequence $\vec{\underline{a}}$.

We know $\exists q' \leq_{\mathbb{P}} q$ such that for some $n \in \{0,1\}$,

$$q' \Vdash \{e\}(\tau(\vec{\underline{a}},y),{}^3\mathbb{E},\gamma) = n \ .$$

Furthermore any $q' \leq_{\mathbb{P}} q$ which forces convergence, must force the correct value (i.e. $B(\gamma)$). Since we consider only extensions of q, which forces that B is given by e,p. Thus to compute B on L : $\gamma \in L_{\kappa_1}{}^3\mathbb{E}$ then

$$B(\gamma) = i \leftrightarrow \exists q' \in \mathbb{P}[q' \leq_{\mathbb{P}} q \wedge q' \Vdash \{e\}(\tau(\vec{\underline{a}},y){}^3\mathbb{E},\gamma) = i] \ .$$

The matrix on the right hand side is recursive in ${}^3\mathbb{E},\mathbb{P},q,\sigma$ since $\mathbb{P}$ is an effective notion of forcing and so by the bounding principle is closed under the quantifier $\exists q' \in \mathbb{P}$.

The proof here is formulated in terms of L, but the only necessary condition was that $\mathbb{P}$ be an effective notion of forcing. We made no use of the strong selection present in the setting of L and its definable well-ordering. Thus a forcing argument designed to establish the relative consistency of a failure of Post's Problem for ${}^3\mathbb{E}$ to ZF will be forced to resort to non-effective posets and, hence, have difficulty in preserving $\kappa_1 {}^3\mathbb{E}$. The proof can be altered in such a way that the result also holds for a class notion of forcing which can be 'localized', i.e. such that we require only a set of conditions to decide a given set of sentences. The Steel-collapse of a countable admissible ordinal is such a notion of forcing.

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