DEGENERATIONS OF COMPLETE

TWISTED CUBICS

by

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1. Introduction

Let $C \subset \mathbb{P}^3$ be a twisted cubic curve. Denote by $\Gamma \subset \text{Grass}(1,3)$ its tangent curve (curve of tangent lines) and by $C^* \subset \check{\mathbb{P}}^3$ its dual curve (curve of osculating planes). The curve Γ is rational normal, of degree 4, while C^* is again a twisted cubic. The triple (C, Γ, C^*) is called a (non degenerate) complete twisted cubic. By a degeneration of it we mean a triple $(\bar{C}, \bar{\Gamma}, \bar{C}^*)$, where $\bar{C}$ (resp. $\bar{\Gamma}$, resp. $\bar{C}^*$) is a flat specialization of C (resp. Γ, resp. C^*). Thus we work with Hilbert schemes rather than Chow schemes: let H denote the irreducible component of $\text{Hilb}^{3n+1}(\mathbb{P}^3)$ containing the twisted cubics, $\check{H}$ the corresponding component of $\text{Hilb}^{3n+1}(\check{\mathbb{P}}^3)$, and G the component of $\text{Hilb}^{4n+1}(\text{Grass}(1,3))$ containing the tangent curves of twisted cubics. The space of complete twisted cubics is the closure $T \subset H \times G \times \check{H}$ of the set of non degenerate complete twisted cubics.

In this paper we show how to obtain Schubert's 11 first order degenerations ([S], pp.164-166) of complete twisted cubics, viewed as elements of $H \times G \times \check{H}$, "via projections", i.e., by constructing 1-dimensional families of curves on various kinds of cones. In particular, we describe the ideals of the degenerated curves. A similar study was done by Alguneid [A], who viewed the degenerations as cycles (rather than flat specializations), and who gave equations for the complexes of lines associated to the degenerated cycles by using the theory of complete collineations.

An ultimate goal in the study of degenerations of complete twisted cubics, is of course to verify Schubert's results in the enumerative theory of twisted cubics. As long as one, as Schubert does, restricts oneself to only impose conditions that involve points, tangents, and osculating planes (and not secants, chords, osculating lines, ...), the space T is a compactification of the space of twisted cubics that contains enough information. In other words, one would like to describe the Chow ring of T in terms of cycles corresponding to degenerate complete

twisted cubics, and in terms of cycles representing the various Schubert conditions. One approach would be to study the Chow ring of H and the blow-up map T → H. In a joint work with Michael Schlessinger we prove that the 12-dimensional scheme H is in fact smooth, and, moreover, that H intersects the other (15-dimensional) component H' of $Hilb^{3m+1}(\mathbb{P}^3)$ transversally along an 11-dimensional locus (H' contains plane cubic curves union a point in $\mathbb{P}^3$, and H ∩ H' consists of plane cubics with an embedded point) This result, together with further investigations of the map T → H, will be the subject of a forthcoming paper.

2. Degenerations via projections

Since all twisted cubics are projectively equivalent, we shall fix one, $C \subset \mathbb{P}^3 = \mathbb{P}^3_k$ (k algebraically closed field of characteristic 0), given by the ideal

$$I = (X_0X_2-X_1^2, X_1X_3-X_2^2, X_0X_3-X_1X_2).$$

Hence C has a parameter form

$$X_0 = u^3, \ X_1 = u^2v, \ X_2 = uv^2, \ X_3 = v^3.$$

The tangent curve Γ of C, viewed as a curve in $\mathbb{P}^5$ via the Plücker embedding of Grass(1,3), has a parameter form $(t = \frac{v}{u})$ given by the 2-minors of

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \end{pmatrix},$$

hence by

$$Y_0 = v^4, \ Y_1 = 2uv^3, \ Y_2 = u^2v^2,$$
$$Y_3 = 3u^2v^2, \ Y_4 = 2u^3v, \ Y_5 = u^4.$$

The ideal of Γ in $\mathbb{P}^5$ is

$$J = (Y_3 - 3Y_2, 4Y_0Y_2 - Y_1^2, Y_0Y_4 - Y_1Y_2, Y_1Y_4 - 4Y_2^2, 4Y_0Y_5 - Y_1Y_4,$$
$$Y_1Y_5 - Y_2Y_4, 4Y_2Y_5 - Y_4^2).$$

The dual curve $C^* \subset \check{\mathbb{P}}^3$ has a parameter form given by the 3-minors of

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 1 & 3t \end{pmatrix}$$

hence by

$$\check{X}_0 = v^3, \quad \check{X}_1 = 3uv^2, \quad \check{X}_2 = 3u^2v, \quad \check{X}_3 = u^3.$$

Since Γ is also equal to the tangent curve of C^* (under the canonical isomorphism Grass(lines in $\mathbb{P}^3$) $\cong$ Grass(lines in $\check{\mathbb{P}}^3$)) and C is the dual curve of C^* (see e.g. [P], §5), any type of degeneration $(\bar{C}, \bar{\Gamma}, \bar{C}^*)$ gives another type - called the dual degeneration - by reading the triple backwards.

Let $A \subset \mathbb{P}^3$ be a linear space, and choose a complement $B \subset \mathbb{P}^3$ of A. By projecting C onto B from the vertex A we obtain a degeneration of C: we construct a family $\{C_a\}$ of twisted cubics, contained in the cone of the above projection, over Spec $k[a] - \{0\}$. This family has a unique extension to a flat family over Spec $k[a]$, and the "limit curve" C_0 is thus a flat specialization of $C = C_1$ (see also [H], p.259, for the case $A = $ a point). Note that interchanging the roles of A and B gives a limit curve equal to the curve C_∞ obtained in the similar way by letting $a \to \infty$, and C_∞ has the dual degeneration type of C_0. The type of degeneration obtained depends of course on the dimension and position of A and B w.r.t. C.

To find generators for the ideals of the degenerated curves, for chosen A and B, we start by writing down a parameter form of C_a, $a \neq 0$. (It is often convenient to introduce new coordinates at this point.) Then we determine enough generators for the ideal I_a of C_a, so that they specialize ($a = 0$) to generators for the ideal I_0 of C_0. (Whenever C_0 acquires an embedded point, it turns out that a cubic generator is needed in addition to

the (three standard) quadratic ones.)

The parameter form of C_a, $a \neq 0$, gives a parameter form of Γ_a, its tangent curve. As above we find generators for the ideal J_a of Γ_a that specialize to generators for J_0.

Similarly, one could work out the ideal of C_0^*. However, by a duality argument it is clear that C_0 will have the degeneration type obtained (from C) by interchanging the roles of A and B. That is, C_0^* will be of the same type as C_∞, or, the degeneration type of C^* is equal to the dual degeneration type of C. For example, consider the degeneration type λ: A is a (general) point, B a (general) plane. When C degenerates along the cone over it, with vertex A , onto the plane B , its osculating planes degenerate towards the plane B . In $\check{\mathbb{P}}^3$, this means that C_a^* degenerates on the cone with vertex the plane $\check{A} \subset \check{\mathbb{P}}^3$ towards the point $\check{B} \in \check{\mathbb{P}}^3$. This degeneration type we call λ'; in general, we shall denote the dual degeneration by a "prime" in this way.

3. Schubert's 11 degenerations

We now give a list of Schubert's 11 types of degenerations, in his order and using his names for them.

λ A = general point (not on C , not on any tangent)

   B = general plane (not osculating, not containing any tangent)

Take $A = (0,1,0,1)$, B: $X_3 + X_1 = 0$,

and new coordinates:

$$X_0' = X_2 + X_0, \quad X_1' = X_3 + X_1, \quad X_2' = X_2 - X_0, \quad X_3' = X_3 - X_1.$$

Then C_a, $a \neq 0$, is given by

$$X_0' = uv^2 + u^3, \quad X_1' = av^3 + au^2 v, \quad X_2' = uv^2 - u^3, \quad X_3' = v^3 - u^2 v.$$

$$I_a = (a^2(X_0' - X_2')(X_0' + X_2') - (X_1' - aX_3')^2,$$

$$X_1'^2 - a^2 X_3'^2 - a^2(X_0' + X_2')^2, -X_1' X_2' + aX_0' X_3',$$

$$(X_0' - X_2')(X_1' + aX_3')^2 - a^2(X_0' + X_2')^3)$$

$$I_0 = (X_1'^2, X_1'X_3', X_1'X_2', X_3'^2(X_0'-X_2') - X_2'^2(X_0'+X_2'))$$

$$= ((X_3+X_1)^2, (X_3+X_1)(X_3-X_1), (X_3+X_1)(X_2-X_0),$$

$$X_0(X_3-X_1)^2 - X_2(X_2-X_0)^2)$$

Hence: C_0 is a plane nodal cubic with a nonplanar embedded point at the node.

λ' A = general plane

B = general point

Take A: $X_3+X_1 = 0$, B = (0,1,0,1) and coordinates as for λ.

$$I_a = ((X_0'-X_2')(X_0'+X_2')-(aX_1'-X_3')^2, a^2X_1'^2-X_3'^2-(X_0'+X_2')^2,$$

$$X_0'X_3' - aX_1'X_2')$$

$$I_0 = (X_0'^2-X_2'^2-X_3'^2, X_3'^2+(X_0'+X_2')^2, X_0'X_3')$$

$$= (4X_0X_2-(X_3-X_1)^2, X_2(X_2+X_0), (X_3-X_1)(X_2+X_0))$$

Hence: C_0 is the union of three skew lines through the point (0,1,0,1).

To find the degenerated tangent curve Γ_0 of λ (or of λ'):

Γ_a is given (in coordinates $Y_0', \ldots, Y_5'$ on $\mathbb{P}^5$ corresponding to $X_0', \ldots, X_3'$ on $\mathbb{P}^3$) on parameter form

$$Y_0' = v^4-2u^2v^2+u^4, \quad Y_1' = 4auv^3, \quad Y_2' = -av^4+4au^2v^2+au^4,$$

$$Y_3' = v^4+4u^2v^2-u^4, \quad Y_4' = 4u^3v, \quad Y_5' = av^4+2au^2v^2+au^4.$$

$$J_0 = (Y_2'-2Y_5', Y_1'^2, Y_1'Y_5', Y_5'^2, 12Y_3'Y_5'-7Y_1'Y_4', 7Y_0'Y_5'+11Y_3'Y_5',$$

$$Y_0'Y_1'-2Y_4'Y_5'-Y_1'Y_3', (4(Y_0'-Y_3')^2+12Y_3'(Y_0'-Y_3')-3Y_4'^2)^2$$

$$-(5Y_0'+4Y_3')^2((Y_0'-Y_3')^2+3Y_4'^2)).$$

Hence: Γ_0 is a plane tricuspidal quartic, with embedded points at the cusps.

κ A = point on a tangent, not on C

B = plane containing a tangent, not osculating.

Take A = (0,1,0,0), B: $X_1 = 0$. Then C_a, $a \neq 0$, is given by

$$X_0 = u^3, \quad X_1 = au^2v, \quad X_2 = uv^2, \quad X_3 = v^3,$$

$$I_a = (a^2X_0X_2-X_1^2, X_1X_3-aX_2^2, aX_0X_3-X_1X_2, X_0X_3^2-X_2^3),$$

$$I_0 = (X_1^2, X_1X_2, X_1X_3, X_0X_3^2-X_2^3).$$

<u>Hence</u>: C_0 is a cuspidal cubic, in the plane $X_1 = 0$, with a nonplanar embedded point at the cusp.

κ' A = plane containing a tangent, not osculating.

B = point on tangent, not on C.

$$I_a = (X_0X_2-a^2X_1^2, aX_1X_3-X_2^2, X_0X_3-aX_1X_2).$$

$$I_0 = (X_0X_2, X_2^2, X_0X_3).$$

<u>Hence</u>: C_0 is the union of the line $X_2 = X_3 = 0$ with the double line $X_0 = X_2 = 0$ (doubled on a quadratic cone with vertex $(0,1,0,0)$).

The tangent curve Γ_a of κ (or κ') is given by

$$Y_0 = v^4, \quad Y_1 = 2auv^3, \quad Y_2 = au^2v^2,$$

$$Y_3 = 3u^2v^2, \quad Y_4 = 2u^3v, \quad Y_5 = av^4.$$

$$J_0 = (Y_2, Y_1^2, Y_1Y_3, Y_1Y_4, Y_1Y_5, Y_0Y_5, Y_2Y_5, 27Y_0Y_4^2-4Y_3^3).$$

<u>Hence</u>: Γ_0 is a cuspidal cubic, in the plane $Y_1 = Y_2 = Y_5 = 0$, with a nonplanar embedded point at the cusp $(1,0,0,0,0,0)$ (this point corresponds to the flex tangent of C_0), union the line $Y_0 = Y_1 = Y_2 = Y_3 = 0$, intersecting the cubic in its flex $(0,0,0,0,1,0)$ (corresponding to the cusp tangent of C_0).

ω A = point on C

B = osculating plane

Take $A = (0,0,0,1)$, B: $X_3 = 0$.

C_a, $a \ne 0$, is given by

$$X_0 = u^3, \quad X_1 = u^2v, \quad X_2 = uv^2, \quad X_3 = av^3,$$

$$I_a = (X_0X_2-X_1^2, X_1X_3-aX_2^2, X_0X_3-aX_1X_2).$$

$$I_0 = (X_0X_2-X_1^2, X_0X_3, X_1X_3).$$

<u>Hence</u>: C_0 is the union of a conic, in the plane $X_3 = 0$, with the line $X_0 = X_1 = 0$.

ω'   A = osculating plane

B = point on C

$$I_a = (X_0X_2 - X_1^2, aX_1X_3 - X_2^2, aX_0X_3 - X_1X_2).$$

$$I_0 = (X_0X_2 - X_1^2, X_1X_2, X_2^2).$$

<u>Hence</u>: C_0 is the triple line $X_1 = X_2 = 0$ (tripled on a quadratic cone with vertex $(0,0,0,1)$).

The tangent curve Γ_a of ω (or ω') is given by

$$Y_0 = av^4, \quad Y_1 = 2auv^3, \quad Y_2 = u^2v^2, \quad Y_3 = 3au^2v^2,$$

$$Y_4 = 2u^3v, \quad Y_5 = u^4.$$

$$J_0 = (Y_3, Y_1^2, Y_1Y_4, Y_1Y_5, Y_0Y_5, Y_0Y_4 - Y_1Y_2, 4Y_2Y_5 - Y_4^2).$$

<u>Hence</u>: Γ_0 is the union of a conic, in the plane $Y_0 = Y_1 = Y_3 = 0$, with the double line $Y_1 = Y_3 = Y_4 = Y_5 = 0$.

θ   To obtain this degeneration, we choose A to be a "line-plane" (L,U), s.t. for some $x \in C$, $x \in L \subset U$, $\text{tg}_x \subset U$, $L \ne \text{tg}_x$, U not osculating - and B a "point-line" (P,L'), s.t. for some $x \in C$, $L' \subset \text{osc}_x$, $\{P\} = L' \cap \text{tg}_x$, $x \ne P$, $P \notin C$. Then we form a 2-dimensional family $\{C_{a,b}\}$, where the parameter a corresponds to projecting C from U to P, and b to projecting from L to L'. Taking a = b we obtain a 1-dimensional family $\{C_{a,a}\}$.

Take $L: X_1 - X_3 = X_2 = 0$, $U: X_2 = 0$ and $L': X_0 = X_1 + X_3 = 0$, $P = (0,0,1,0)$. In new coordinates $X_0, X_1' = X_1 - X_3$, $X_2, X_3' = X_1 + X_3$, $C_{a,b}$ is given by $X_0 = abu^3$, $X_1' = au^2v - av^3, X_2 = uv^2, X_3' = abu^2v + abv^3$

$$I_{a,b} = (4abX_0X_2 - (X_3' + bX_1')^2, (X_3' + bX_1')(X_3' - bX_1') - 4a^2b^2X_2^2,$$
$$X_0(X_3' - bX_1') - ab(X_3' + bX_1')X_2, X_0(X_3' - bX_1')^2 - 4a^3b^3X_2^3).$$

By letting a = b, rewriting the generators, and letting a = 0, we obtain

$$I_0 = (X_0(X_1 + X_3), (X_1 - X_3)(X_1 + X_3), (X_1 + X_3)^2, X_0((X_1 - X_3)^2 - X_0X_2)).$$

<u>Hence</u>: C_0 is the union of a conic, in the plane $X_1 + X_3 = 0$, with its tangent line at $(0,0,1,0)$, and with that point as a nonplanar embedded point.

θ' A = "point-line"

  B = "line-plane".

$$I_{a,b} = (4bX_0X_2 - a(bX_3' + X_1')^2, a^2(bX_3' + X_1')(bX_3' - X_1')$$
$$-4X_2^2, abX_0(bX_3' - X_1') - (bX_3' + X_1')X_2).$$

Taking $a = b$ and $a = 0$ gives

$$I_0 = (4X_0X_2 - (X_1 - X_3)^2, (X_1 - X_3)X_2, X_2^2)$$

<u>Hence</u>: C_0 is the line $X_1 - X_3 = X_2 = 0$ tripled on a quadratic cone with vertex $(1,0,1,0)$.

The tangent curve Γ_a of θ (or θ') (for $a = b$) is given by

$$Y_0' = av^4 - au^2v^2, \quad Y_1' = 4a^2uv^3, \quad Y_2' = v^4 + u^2v^2,$$

$$Y_3' = 3a^3u^2v^2 + a^3u^4, \quad Y_4' = 2au^3v, \quad Y_5' = -3a^2u^2v^2 + a^2u^4.$$

$$J_0 = (Y_3', Y_0'Y_1', Y_1'^2, Y_1'Y_4', Y_1'Y_5', Y_0'Y_5', 4Y_2'Y_5' - Y_0'^2 - Y_4'^2).$$

<u>Hence</u>: Γ_0 is the union of a conic, in the plane $Y_0' = Y_1' = Y_3' = 0$, with the two lines $Y_1' = Y_3' = Y_5' = Y_0'^2 + Y_4'^2 = 0$, with the common point of intersection, $(0,0,1,0,0,0)$, as an embedded point (this point corresponds to the line - tangent to the conic - of C_0).

δ A = a line, not contained in any osculating plane, and intersecting C in exactly one point.

  B = a line, not intersecting C, contained in exactly one osculating plane.

Take A: $X_0 - X_2 = X_1 = 0$, B: $X_0 + X_2 = X_3 = 0$ and change coordinates: $X_0' = X_0 - X_2, X_1, X_2' = X_0 + X_2, X_3$. Then C_a is given by

$$X_0' = u^3 - uv^2, \quad X_1 = u^2v, \quad X_2' = au^3 + auv^2, \quad X_3 = av^3.$$

$$I_a = (X_2'^2 - a^2X_0'^2 - 4a^2X_1^2, 4aX_1, X_3 - (X_2' - aX_0')^2,$$
$$(X_2' + aX_0')X_3 - aX_1(X_2' - aX_0')).$$

By changing the generators, we see

$$I_0 = ((X_0+X_2)^2, (X_0+X_2)X_3, X_0^2-X_2^2-2X_1X_3).$$

Hence: C_0 is the union of the line $X_0+X_2 = X_1 = 0$ with the double line $X_0+X_2 = X_3 = 0$, and is contained in a smooth quadric.

δ' $\quad I_a = (a^2X_2'^2-X_0'^2-4X_1^2, 4aX_1X_3-(aX_2'-X_0')^2,$

$$a(aX_2'+X_0')X_3-X_1(aX_2'-X_0'))$$

$$I_0 = ((X_0-X_2)^2, (X_0-X_2)X_1, X_1^2) = (X_0-X_2, X_1)^2$$

Hence: C_0 is the triple line $X_0-X_2 = X_1 = 0$ (tripled by taking its 2nd order neighbourhood in $\mathbb{P}^3$).

The tangent curve Γ_a for δ (or δ') is given by

$$Y_0' = a^2v^4+3a^2u^2v^2, \quad Y_1' = 2auv^3, \quad Y_2' = au^2v^2-au^4,$$

$$Y_3' = -av^4+3au^2v^2, \quad Y_4' = 4au^3v, \quad Y_5' = u^2v^2+u^4.$$

$$J_0 = (Y_0', Y_1'(2Y_1'+Y_4'), Y_2'(2Y_1'+Y_4'), Y_4'(2Y_1'+Y_4'),$$

$$Y_2'^2+Y_1'^2, Y_4'(Y_3'-2Y_2'), Y_2'(Y_3'-2Y_2'))$$

Hence: Γ_0 is the union of the two lines

$Y_0' = 2Y_1'+Y_4' = Y_3'-2Y_2' = Y_1'^2+Y_2'^2 = 0$, with the double line $Y_0' = Y_1' = Y_2' = Y_4' = 0$.

η $\quad A$ = general line, i.e. $A \cap C = \emptyset$,

$\qquad$ A not contained in an osculating plane

$\quad B$ = general line (same conditions as for A, since these are self-dual!)

Take $A: X_0-X_3 = X_1+X_2 = 0$, $B: X_0+X_3 = X_1-X_2 = 0$, and change coordinates:

$X_0' = X_0-X_3, \quad X_1' = X_1-X_2, \quad X_2' = X_1+X_2, \quad X_3' = X_0+X_3$. Then C_a is given by

$$X_0' = u^3-v^3, \quad X_1' = au^2v-auv^2, \quad X_2' = u^2v+uv^2,$$

$$X_3' = au^3+av^3.$$

$$I_a = ((aX_0' + X_3')(aX_2' - X_1') - (X_1' + aX_2')^2, (X_1' + aX_2')(X_3' - aX_0')$$

$$- (aX_2' - X_1')^2, X_3'^2 - a^2 X_0'^2 - a^2 X_2'^2 + X_1'^2)$$

$$I_0 = (X_1'^2, X_1' X_3', X_3'^2) = (X_1 - X_2, X_0 + X_3)^2$$

<u>Hence</u>: C_0 is the tripled line $X_1 - X_2 = X_0 + X_3 = 0$
(tripled as in ε').

η' is of the same type as η, since the conditions on
A, B are self dual.

The tangent curve Γ_a of η is given by

$$Y_0' = av^4 + 2auv^3 - 2au^3 v - au^4$$

$$Y_1' = -a^2 v^4 + 2a^2 uv^3 + 2a^2 u^3 v - a^2 u^4$$

$$Y_2' = 2au^2 v^2$$

$$Y_3' = 6au^2 v^2$$

$$Y_4' = v^4 + 2uv^3 + 2u^3 v + u^4$$

$$Y_5' = -av^4 + 2auv^3 - 2au^3 v + au^4 .$$

$$J_0 = (Y_3' - 3Y_2', Y_0' Y_1', Y_1'^2, Y_1' Y_2', Y_1' Y_5', Y_0' Y_5' - Y_1' Y_4' + 3Y_2'^2,$$

$$4Y_1' Y_4' - 6Y_0' Y_5' + 3Y_0'^2 - Y_5'^2)$$

<u>Hence</u>: Γ_0 is the union of four lines in the three-space
$Y_3' - 3Y_2' = Y_1' = 0$, with an embedded point (sticking out of
that space) at their common point of intersection.

<u>Remark</u>: By choosing other A's and B's we can obtain
further types of degenerations. For example, consider the
degeneration obtained by taking A = a chord of C , B = an
axis of C (i.e., the intersection of two osculating
planes). Then C_0 is the union of three skew lines,
meeting in 2 points, whereas its dual is a triple line (2nd
order nbhd. of a line in $\check{\mathbb{P}}^3$). The tangent curve Γ_0 is
the union of two double lines.

On the next page, we give a figure showing Schubert's
11 degenerate complete twisted cubics. Each triple should
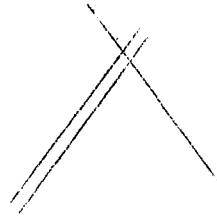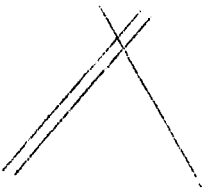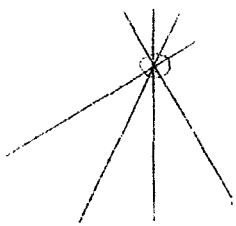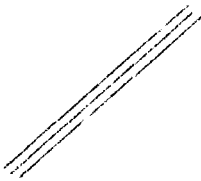also be read backwards!
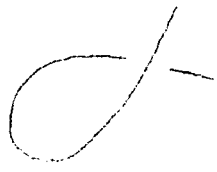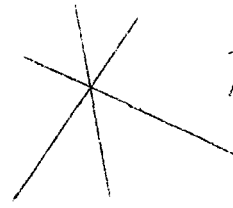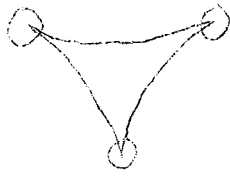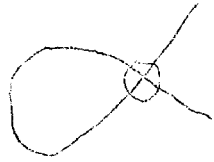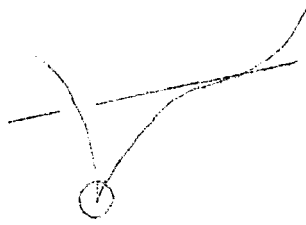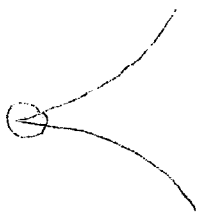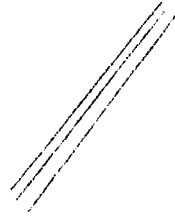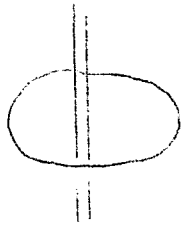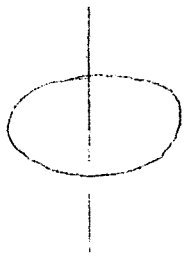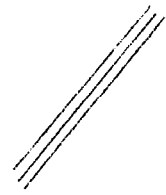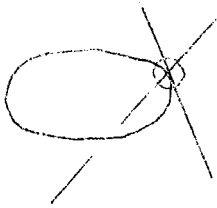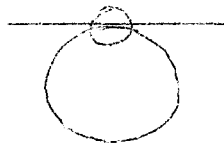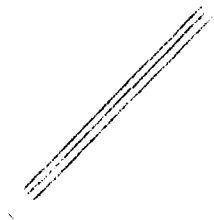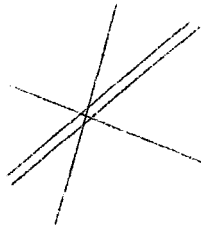
4. Some remarks on H and T

Let $T_\lambda, T_\kappa, \ldots$ denote the closure of the set of points in T corresponding to degenerations of type $\lambda, \kappa, \ldots$, and let $H_\lambda, H_\kappa, \ldots$ denote the similarly defined sets in H. That the degenerations $\lambda, \kappa, \ldots$ are of first order, means that $T_\lambda, T_\kappa, \ldots$ are of codimension 1 in T; this is easily seen to be true by counting the parameters of each of the corresponding figures. Only H_λ and H_ω are of codimension 1 in H, so the (birational) projection map $\pi: T \to H$ blows up the other sets $H_\kappa, H_{\lambda'}, \ldots$. For example, H_κ has codimension 2 (there are ∞^{10} plane cuspidal cubics in P^3), and for a given $\bar{C} \in H_\kappa$, $\pi^{-1}(\bar{C}) =$

$$\left\{ (\bar{C}, \bar{\Gamma}, \bar{C}^*) : \begin{array}{l} \bar{\Gamma} = \text{a (uniquely determined) cuspidal} \\ \text{cubic union a line through the flex} \end{array} \right\}$$

Since "a line through the flex" corresponds to "a plane containing the cusp tangent of $\bar{C}$ ", we see that $\dim \pi^{-1}(\bar{C}) = 1$.

The set H_η $(=H_{\delta'})$ has the largest codimension, namely 8: all degenerations without an embedded point specialize to these. In this case, $\pi^{-1}(\bar{C})$ has dimension 7: the tangent curve is determined by choosing 4 point-planes through the line $\bar{C}_{red}$, which satisfy one relation between the cross-ratios (of the points and planes) (see e.g. [A], p.206, or recall that the four concurrent lines $\bar{\Gamma}_{red}$ span only a P^3).

Let N denote the normal sheaf of $\bar{C} \in H_\eta$ in P^3. One can prove, e.g. by taking a presentation of the ideal of $\bar{C}$, that $\dim H^0(N, \bar{C}) = 12$. It follows that H is smooth at $\bar{C}$, since $\dim H = 12$, and hence all points of $H - H_\lambda$ (i.e., those corresponding to Cohen-Macaulay curves, i.e., curves without an embedded point) are smooth on H.

Now consider H_λ. Any point in it can be specialized to one corresponding to a plane triple line with a nonplanar embedded point, e.g. given by the ideal $(X_1 X_3, X_2 X_3, X_3^2, X_1^3)$. In the work with M. Schlessinger, cited in the introduction, we prove that such a point is smooth on H, and hence that H is smooth.

<u>Remark</u>: The results $\dim H^0(N,\tilde{C}) = 12$ if $\tilde{C} \in H_\eta$, and $\dim H^0(N,\tilde{C}) = 16$ if $\tilde{C}$ is a plane triple line with embedded point, have also been obtained by Joe Harris; he also gives a list of possible degeneration types of a curve $C \in H$ (private communication).

As a final comment, let us mention an advantage of working with Hilbert schemes rather than Chow schemes: the existence of universal families of curves, which allows the following way of expressing Schubert's various conditions as cycles on T. Namely, let

$$\mathcal{C} \subset \mathbb{P}^3 \times T \qquad \mathcal{C}' \subset \mathrm{Grass}(1,3) \times T \qquad \mathcal{C}^* \subset \check{\mathbb{P}}^3 \times T$$
$$p \downarrow \qquad\qquad q \downarrow \qquad\qquad r \downarrow$$
$$T \qquad\qquad\quad T \qquad\qquad\qquad T$$

denote the universal families (pulled back to T from H, G, $\check{H}$ respectively). The condition, denoted ν by Schubert, for a curve C to intersect a given line L, is then represented by the cycle $T_\nu = p_*(\mathcal{C} \cap L \times T)$; the condition, Schubert's ρ, that the curve touches a given plane U, by $T_\rho = q_*(\mathcal{C}' \cap \sigma_{1,1} \times T)$, where $\sigma_{1,1}$ is the 2-plane in $\mathrm{Grass}(1,3)$ of lines in U, and so on. We plan to return to the question of determining the relations between these cycles and the cycles $T_\lambda, T_\kappa, \ldots$ – and to a study of the Chow ring of T.

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