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THE SINGULARITIES OF THE 3-SECANTCURVE ASSOCIATED TO A SPACE CURVE

by

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ABSTRACT

Let C be a curve in $\mathbb{P}^3$ over an algebraically closed field of characteristic zero. We assume that C is non-singular and contains no plane component except possibly an irreducible conic.

In [GP] one defines closed r-secant varieties to C, $r \in \mathbb{N}$. These varieties are embedded in G, the Grassmannian of lines in $\mathbb{P}^3$. We study the local geometry of T, the 3-secant variety (curve), at points not contained in the 4-secant variety. The multiplicity of T at any such point is determined in terms of the local geometry of C at the secant points. Furthermore we give a geometrical interpretation of the tangential directions of T at a singular point, and we give a criterion for whether all tangential directions are distinct or not.

Assume that the set of 4-secants is finite, and let $\tilde{T}$ be the curve obtained by blowing up the ideal of 4-secants in T. Although $\tilde{T}$ is not contained in G, we show that essentially the same results hold for $\tilde{T}$ at any point whose fibre of the blowing-up map is reduced at the point.

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KEY WORDS AND PHRASES: Space curve, 3-secant curve, local geometry.

§1. INTRODUCTION.

In this paper we consider a non-singular curve C in $\mathbb{P}^3_K$, where K is an algebraically closed field of characteristic 0. We assume that C contains no plane component except possibly an irreducible conic.

There are various ways of studying the scheme of 3-secants to C. In [La 1] one defines secant schemes from a functorial point of view. See also [La 2] where a generalized trisecantlemma is given. In this paper we follow the approach of [GP]. There one studies a curve T in the Grassmannian of lines in $\mathbb{P}^3$ whose points correspond to lines intersecting C at least 3 times, counted with multiplicity.

We will always assume that the set of 4-secants is finite. Let $\tilde{T}$ denote the curve obtained by blowing up T in the scheme of 4-secants. The goal of this paper is to study the local nature of $\tilde{T}$. If a fibre of this blowing-up map is reduced at a point, the dimension of the tangent space of $\tilde{T}$ at this point is at most 2. We give conditions on C that determine the multiplicity of $\tilde{T}$ at such a point (Theorem 2.3.1), and we give a geometrical interpretation of the tangential directions of $\tilde{T}$ at the point (Theorem 2.3.2).

§2. DEFINITIONS AND MAIN RESULTS.

2.1.

Let G be the Grassmannian of lines in $\mathbb{P}^3$. Given a line $L \subset \mathbb{P}^3$, we denote by $\ell(L)$ (or ℓ) the corresponding point in G. Let

$$F = \{(P,\ell(L)) \in \mathbb{P}^3 \times G \,|\, P \in L\}$$

Let $p : F \to \mathbb{P}^3$ and $q : F \to G$ be the natural projection maps, and set $\mathcal{C} = p^{-1}(C)$.

Let j-secant scheme associated to C is defined by $F^{j-1}(q_*\mathcal{O}_{\mathcal{C}})$, the j-1'th Fitting ideal of the $\mathcal{O}_G$-module $q_*\mathcal{O}_{\mathcal{C}}$. The support of this ideal is clearly $\{\ell(L)\,|\,\mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_L) \geq 3\}$. We denote the 3-secant scheme by T. By the Trisecant lemma (see e.g. [M], [Ab], [An], [Sa]) T is a curve in G. We always assume that the scheme of 4-secants is finite. Let $\phi : \tilde{T} \to T$ denote the blowing-up of the sheaf of ideals $F^3(q_*\mathcal{O}_{\mathcal{C}})\mathcal{O}_T$.

2.2.

Assume $\mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_L) = n \geq 3$, and $L \cap C = \{P_1,\ldots,P_k\}$.

Let

$$n_i = \mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_L)_{P_i}, \quad i = 1,\ldots,k .$$

Then

$$n = \sum_{i=1}^{k} n_i .$$

blowing-up map, that $\phi^{-1}(\ell(L))$ is reduced at the point ℓ_x, and that $L \cap C = \{P_1,\ldots,P_k\}$. We also assume that $\tilde{T}$ is singular at ℓ_x. Let m_i be the intersection number $I(P_i, C \cap H)$, for $i = 1,\ldots,k$, where H is the special plane described in Proposition 2.2.1. Then we have the following:

Theorem 2.3.1: The dimension of the tangent space of $\tilde{T}$ at ℓ_x is 2, and the multiplicity of $\tilde{T}$ at ℓ_x is m, where

$$m = \min_{i, n_{ix} \neq 0} \left[\frac{m_i}{n_i}\right]$$

([x] means the integral part of the real number x)

Remark. Pick a fixed set of 2 generators of the maximal ideal $\underline{\underline{M}}$ of $\hat{\mathcal{O}}_{\tilde{T},\ell_x}$. We show in §3 and §4 that (up to a multiplicative constant) modulo $\underline{\underline{M}}^{m+1}$ there is a unique homogeneous relation of order m between these generators. The m (not necessarily distinct) factors of this relation correspond to the tangential directions of $\tilde{T}$ at the point ℓ_x. We say that a multiple factor corresponds to a multiple tangential direction.

a.) deg $M = m+1$, and M has a singularity of order at least m at Q .

b.) $I(P_i, C \cap M) \geq (m+1)n_{ix}$, for $i = 1,\dots,k$.

c.) $L \not\subset$ Sing.(M) , and $m \cdot L \subset M \cap H$

d.) Modulo the square of the ideal defining L , the equation defining M is equal to the equation of a cone of degree $m+1$ with vertex at Q .

Theorem 2.3.3.: The singularity of $\tilde{T}$ at the point ℓ_x is non-ordinary with $Q \in L$ corresponding H-dually to a multiple tangential direction iff there exists a surface N such that

a.) N is a cone of degree m with vertex at Q .

b.) $I(P_i, C \cap N) \geq (m+1)n_{ix}$, for $i = 1,\dots,k$.

c.) $L \not\subset$ Sing.(N) .

Remark: Theorem 2.3.3. says that a tangential direction is multiple iff some surface M satisfying the properties of Theorem 2.3.2. breaks up into the union of a cone of degree m and a plane not containing any of the points of $L \cap C$.

loss of generality we assume that $\alpha_{i,n_i} \neq 0$, for $i = 1,\dots,k$. We assume that $\tilde{T}$ is singular at the point ℓ_x , and that the fibre $\phi^{-1}(\ell(L))$ is reduced at ℓ_x . We choose $Y = 0$ as the equation of the unique plane H described in Proposition 2.2.1.

Let R be $\hat{\mathcal{O}}_{G,\ell}$. There exists a regular system of parameters (a,b,c,d) of R such that the inverse image of F in $A^3 \times \operatorname{Spec} R = \operatorname{Spec} R[X,Y,Z]$ is defined by

$$X = a+bZ \ , \ Y = c+dZ \ .$$

The completion of the local ring of $\mathcal{C} = p^{-1}(C)$ at (ℓ, P_i) is $A_i = R[[Z-z_i]]/\mathfrak{a}_i$, where $\mathfrak{a}_i$ is generated by

$$f_i = a+bZ - \sum_{j \geq n_i} \alpha_{i,j}(Z-z_i)^j \ , \ g_i = c+dZ - \sum_{j \geq n_i} \beta_{i,j}(Z-z_i)^j \ ,$$

$$\text{for } i = 1,\dots,k \ .$$

When $n = 3$, we have:

$$F^2(q_*\mathcal{O}_{\mathcal{C}})\hat{\mathcal{O}}_{G,\ell} = F^2\left(\bigoplus_{i=1}^{k} A_i\right)\hat{\mathcal{O}}_{G,\ell} = \sum_{i=1}^{k} F^{n_i-1}(A_i)\hat{\mathcal{O}}_{G,\ell} \ .$$

By Weierstrass' Preparation Theorem ([ZS], p. 140-141, 145) there is a distinguished polynomial S_i of degree n_i in

3.2.

Proof of Theorem 2.3.1. in Case 1:

Let $m = \min_i m_i$, where $m_i = I(P_i, C \cap H)$, $i = 1,\dots,k$, and H is the special plane described in Proposition 2.2.1. Since the equation of H is $Y = 0$, m_i is $\min\{j | \beta_{ij} \neq 0\}$. We will show that the multiplicity of T at ℓ is m . Using Weierstrass' Preparation Theorem ([ZS], p. 140-141, 145) we obtain:

$$S_i = Z - z_i + s_{i,0}(a,b) \text{ , where } s_{i,0} \equiv -\frac{a+z_i b}{\alpha_{i,1}} \text{ modulo } (a,b)^2$$

$$T_i = t_{i,0} \equiv c + z_i d - \left(\frac{a+z_i b}{\alpha_{i,1}}\right)^m \beta_{i,m}$$

$$\text{modulo}((a,b)\cdot(c,d),(a,b,c,d)^{m+1})$$

We have: $\hat{O}_{T,\ell} = R/(t_{1,0},t_{2,0},t_{3,0})$ where $R = k[[a,b,c,d]]$. We denote by $\bar{a},\bar{b},\bar{c},\bar{d}$ the images of a,b,c,d in $\hat{O}_{T,\ell}$. Let $\underline{\underline{M}} = (\bar{a},\bar{b},\bar{c},\bar{d})$. It is clear that $\underline{\underline{M}} = (\bar{a},\bar{b})$. A little calculation gives that the unique relation between $\bar{a}$ and $\bar{b}$ mod $\underline{\underline{M}}^{m+1}$ is :

$$(*) \quad \sum_{j=0}^{m} \binom{m}{j} \bar{a}^{m-j}\, \bar{b}^{j} \equiv 0 \text{ , where}$$

$$\gamma_j = \frac{\beta_{1,m}}{\alpha_{1,1}^{m}}(z_2-z_3)z_1^j + \frac{\beta_{2,m}}{\alpha_{2,1}^{m}}(z_3-z_1)z_2^j + \frac{\beta_{3,m}}{\alpha_{3,1}^{m}}(z_1-z_2)z_3^j$$

For the proof of (ii) we may assume $Q = (0,0,0,1)$ without loss of generality. Then a (1-fold) tangential direction to T at ℓ corresponds H-dually to Q iff the coefficient γ_m of (*) is zero.

$$\gamma_m = \frac{\beta_{1,m}}{\alpha_{1,1}^m}(z_2-z_3)z_1^m + \frac{\beta_{2,m}}{\alpha_{2,1}^m}(z_3-z_1)z_2^m + \frac{\beta_{3,m}}{\alpha_{3,1}^m}(z_1-z_2)z_3^m = 0$$

is equivalent to

$$\det\begin{bmatrix} \beta_{1,m}z_1^m & \alpha_{1,1}^m z_1 & \alpha_{1,1}^m \\ \beta_{2,m}z_2^m & \alpha_{2,1}^m z_2 & \alpha_{2,1}^m \\ \beta_{3,m}z_3^m & \alpha_{3,1}^m z_3 & \alpha_{3,1}^m \end{bmatrix} = 0$$

Using the local parametrizations of C at the P_i in $L \cap C$ we see that this is equivalent to the existence of a triple (r_1,r_2,r_3) with $r_1 \neq 0$, such that the surface with affine equation

$$r_1YZ^m + r_2X^mZ + r_3X^m = 0$$

intersects C at least $m+1$ times at P_i, for $i = 1,2,3$.. On the other hand any surface satisfying properties a.) and c.) of Theorem 2.3.2. ii.) has affine equation

$$r_1YZ^m + r_2X^mZ + r_3X^m + r_4YZ^{m-1} = 0 \quad \text{modulo } (XY, Y^2, X^{m+1})$$

The equation of such a suface N is obviously homogeneous of degree m. The image of the function $X^j Z^{m-j}$ is of order exactly j, for $j = 0,\dots,m$, in $\mathcal{O}_{C,P_i}$, for each P_i different from Q. Hence no term of the form $X^j Z^{m-j}$ occurs in the equation of N, for $j = 0,1,\dots,m-1$. Hence the equation of N is $r_1 Y Z^{m-1} + r_2 X^m = 0 \mod (XY, Y^2)$. Since $L \not\subset \mathrm{Sing.}(N)$, $r_1 \neq 0$. Hence we have proved Theorem 2.3.3. in Case 1.

3.5.

In Case 2 we use Weierstrass' Preparation Theorem to obtain the unique relation between $\bar{a}$ and $\bar{b}$ modulo $\underline{\underline{M}}^{m+1}$:

$$(**)\quad \det \begin{bmatrix} 1 & z_1 & \left(\frac{\bar{a}+z_1\bar{b}}{\alpha_{1,2}}\right)^m \cdot \beta_{1,2m} \\ 0 & 1 & \left(\frac{\bar{a}+z_1\bar{b}}{\alpha_{1,2}}\right)^m \cdot \beta_{1,2m+1} + m\cdot\left(\frac{\bar{a}+z_1\bar{b}}{\alpha_{1,2}}\right)^{m-1} \cdot \left(\frac{\bar{b}}{\alpha_{1,2}} - \frac{\bar{a}+z_1\bar{b}}{\alpha_{1,2}^2}\cdot\alpha_{1,3}\right)\cdot \beta_{1,2m} \\ 0 & z_2 & \left(\frac{\bar{a}+z_2\bar{b}}{\alpha_{2,1}}\right)^m \cdot \beta_{2,m} \end{bmatrix} = 0$$

where m is the number given in Theorem 2.3.1:

In Case 3 the corresponding relation is

§4. THE LAST PART OF THE PROOFS OF THE MAIN RESULTS.

We now consider an arbitrary point $\ell_x \in \tilde{T}$, reduced in its fibre over $\ell(L) \in T$. We keep the notation from §2.1, §2.2, §3.1 and assume that $\mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_L) = n \geq 3$.

Hence we identify points $\ell_x \in \tilde{T}$ over $\ell(L)$ with k-tuples $(n_{1x},\ldots,n_{kx})$, where $0 \leq n_{ix} \leq n_i$, for $i = 1,\ldots,k$, $\sum_{i=1}^{k} n_{ix} = 3$ $(L \cap C = \{P_1,\ldots,P_k\}$, $\mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{\mathbb{P}^3}} \mathcal{O}_L)_{P_i} = n_i)$

To prove Theorems 2.3.1., 2.3.2., 2.3.3. it is enough to show:

a.) The fibre $\tilde{T} \times_T \mathrm{Spec}\, K(\ell) = \phi^{-1}(\ell)$ is reduced at ℓ_x iff in the corresponding k-tuple $(n_{1x},\ldots,n_{kx})$ we have: n_{ix} is either 0 or n_i, for $i = 1,\ldots,k$.

b.) n_{ix} is either 0 or n_i, for $i = 1,\ldots,k \Rightarrow$ $\hat{\mathcal{O}}_{\tilde{T},\ell_x} \simeq R/\mathfrak{a}_x$, where $R = \hat{\mathcal{O}}_{G,\ell}$, and $\mathfrak{a}_x$ is the ideal in R generated by the coefficients of those T_i (see §3.1) such that $n_{ix} = n_i$.

$(x_0,\ldots,x_{n-4})$ the maximal ideal $\underline{n}_x$ is the image of $(a,b,c,d,X_0-x_0,\ldots,X_{n-4}-x_{n-4})$ in B. When $\Omega_x = \prod_{i=1}^{k} (Z-z_i)^{n_i-n_{ix}}$, the k-tuple corresponding to ℓ_x is $(n_{1x},\ldots,n_{kx})$.

Now a.) follows directly from Lemma 3.5, [GP], p. 22. Under the assumption of b.) this lemma also gives:

$$R \cong R_x \overset{\text{def}}{\equiv} [R[X_0,\ldots,X_{n-4}]/(r_0,r_1,\ldots,r_{n-4})]_{(a,b,c,d,X_0-x_0,\ldots,X_{n-4}-x_{n-4})}$$

Hence

$$B_{\underline{n}_x} \simeq R_x/(V_0,V_1,V_2) .$$

Since R_x is a regular local ring, $R_x[Z]$ is a UFD. Identifying all polynomials with their images in $R_x[Z]$, we have in $R_x[Z]$:

$$S = S'S'' = \Omega\Delta ,$$

where $S' = \prod_{i,n_{ix}=0} S_i$, $S'' = \prod_{i,n_{ix}\neq 0} S_i$, and Δ is some power series. Since Ω_x and $\Delta_x = \prod_i (Z-z_i)^{n_{ix}}$ have no common factors, we obtain $S' = \Omega$ in $R_x[Z]$. Hence

$$V_i = \tau(\overline{DS'Z^i}) , \; i = 0,1,2 ,$$

and one easily obtains that (V_0,V_1,V_2) generates the

§5. EXAMPLES AND REMARKS.

Let C be a smooth complete intersection of two cubic surfaces. Then C contains no line and possesses no 4-secants. Assume L is a 3-secant corresponding to a point $\ell \in T$.

As a consequence of Proposition 2.2.1, the following 3 statements are equivalent:

a.) $\dot{T}$ is singular at ℓ.

b.) There exists a plane H and a cubic surface F_3 containing C such that $H \cap F_3 \supseteq L_2$, where L_2 is the double line in H with support L.

c.) There exists a cubic surface F_3 containing C and having 2 (possibly coinciding) singularities on L.

If T is singular at ℓ, the surface F_3 of b.) is the same as surface F_3 of c.).

Furthermore, as a consequence of the results in §2, we have:

The multiplicity of T at the point ℓ is 3 iff there exists a plane H and a cubic surface F_3 containing C such that $H \cap F_3 = L_3$ where L is the triple line in H with support L. The multiplicity of T at any point ℓ is at most 3.

$$\sum_{i, n_{ix} = n_i} [(m+1)n_{ix} - \min(m_i, (m+1)n_{ix})] = 1 ,$$

where $m_i = I(P_i, C \cap H)$ for the special plane H. Then the point corresponding H-dually to the m-fold tangential direction is the unique $P_i \in L \cap C$ such that $m_i = (m+1)n_{ix} - 1$.

(2) Using essentially the same methods as in the proofs of the results in §2.3 we may examine the local geometry of T at non-singular points of T. Let L be the tangent line to T at a non-singular point $\ell(L)$. Then we have for example:

ℓ is a flex on T, i.e. $\mathrm{rk}(\mathcal{O}_T \otimes_{\mathcal{O}_{\mathbb{P}^5}} \mathcal{O}_L)_\ell \geq 3$, iff

$L \cap C = \{P_1, P_2, P_3\}$, $n_1 = n_2 = n_3 = 1$, and $I(P_1, C \cap H) = 1$, $I(P_j, C \cap H) \geq 3$, $j = 2,3$, for some plane H, or:

$L \cap C = \{P_1, P_2\}$, $n_1 = 1$, $n_2 = 2$, and $I(P_1, C \cap H) = 1$, $I(P_2, C \cap H) \geq 6$, for some plane H.

(3) When the fibre $\phi^{-1}(\ell)$ is not reduced at a point $\ell_x \in \tilde{T}$, the multiplicity formula given in Theorem 2.3.1. is no longer valid, although Proposition 2.2.1. holds. In addition the dimension of the tangent space of $\tilde{T}$ at ℓ_x may be 3.

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