THE SINGULARITIES OF THE 3-SECANT CURVE ASSOCIATED TO A SPACE CURVE

by

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ABSTRACT

Let $C$ be a curve in $\mathbb{P}^3$ over an algebraically closed field of characteristic zero. We assume that $C$ is non-singular and contains no plane component except possibly an irreducible conic.

In [GP] one defines closed $r$-secant varieties to $C$, $r \in \mathbb{N}$. These varieties are embedded in $G$, the Grassmannian of lines in $\mathbb{P}^3$. We study the local geometry of $T$, the 3-secant variety (curve), at points not contained in the 4-secant variety. The multiplicity of $T$ at any such point is determined in terms of the local geometry of $C$ at the secant points. Furthermore we give a geometrical interpretation of the tangential directions of $T$ at a singular point, and we give a criterion for whether all tangential directions are distinct or not.

Assume that the set of 4-secants is finite, and let $\tilde{T}$ be the curve obtained by blowing up the ideal of 4-secants in $T$. Although $\tilde{T}$ is not contained in $G$, we show that essentially the same results hold for $\tilde{T}$ at any point whose fibre of the blowing-up map is reduced at the point.

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§1. INTRODUCTION.

In this paper we consider a non-singular curve $C$ in $\mathbb{P}^3_K$, where $K$ is an algebraically closed field of characteristic 0. We assume that $C$ contains no plane component except possibly an irreducible conic.

There are various ways of studying the scheme of 3-secants to $C$. In [La 1] one defines secant schemes from a functorial point of view. See also [La 2] where a generalized trisecant lemma is given. In this paper we follow the approach of [GP]. There one studies a curve $T$ in the Grassmannian of lines in $\mathbb{P}^3$ whose points correspond to lines intersecting $C$ at least 3 times, counted with multiplicity.

We will always assume that the set of 4-secants is finite. Let $\tilde{T}$ denote the curve obtained by blowing up $T$ in the scheme of 4-secants. The goal of this paper is to study the local nature of $\tilde{T}$. If a fibre of this blowing-up map is reduced at a point, the dimension of the tangent space of $\tilde{T}$ at this point is at most 2. We give conditions on $C$ that determine the multiplicity of $\tilde{T}$ at such a point (Theorem 2.3.1), and we give a geometrical interpretation of the tangential directions of $\tilde{T}$ at the point (Theorem 2.3.2).
§2. DEFINITIONS AND MAIN RESULTS.

2.1.

Let $G$ be the Grassmannian of lines in $\mathbb{P}^3$. Given a line $L \subset \mathbb{P}^3$, we denote by $\ell(L)$ (or $\ell$) the corresponding point in $G$. Let

$$F = \{(P, \ell(L)) \in \mathbb{P}^3 \times G | P \in L\}$$

Let $p : F \to \mathbb{P}^3$ and $q : F \to G$ be the natural projection maps, and set $\mathcal{E} = p^{-1}(C)$.

Let $j$-secant scheme associated to $C$ is defined by $\mathcal{F}_{j-1}(q_* \mathcal{O}_C)$, the $j$-th Fitting ideal of the $\mathcal{O}_G$-module $q_* \mathcal{O}_C$. The support of this ideal is clearly \{\ell(L) \mid \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_L) > 3\}. We denote the 3-secant scheme by $T$.

By the Trisecant lemma (see e.g. [M], [Ab], [An], [Sa]) $T$ is a curve in $G$. We always assume that the scheme of 4-secants is finite. Let $\phi : \tilde{T} \to T$ denote the blowing-up of the sheaf of ideals $\mathcal{F}^3(q_* \mathcal{O}_C) \mathcal{O}_T$.

2.2:

Assume $\text{rk}(\mathcal{O}_C \otimes \mathcal{O}_L) = n \geq 3$, and $L \cap C = \{P_1, \ldots, P_k\}$. Let

$$n_i = \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_{L})_{P_i} \quad \text{for } i = 1, \ldots, k.$$

Then

$$n = \sum_{i=1}^{k} n_i.$$
blowing-up map, that \( \phi^{-1}(\ell(L)) \) is reduced at the point \( \ell_x \), and that \( L \cap C = \{ P_1, \ldots, P_k \} \). We also assume that \( T \) is singular at \( \ell_x \). Let \( m_i \) be the intersection number \( I(P_i, C \cap H) \), for \( i = 1, \ldots, k \), where \( H \) is the special plane described in Proposition 2.2.1. Then we have the following:

**Theorem 2.3.1:** The dimension of the tangent space of \( \tilde{T} \) at \( \ell_x \) is 2, and the multiplicity of \( \tilde{T} \) at \( \ell_x \) is \( m \), where

\[
m = \min_{i, n_i \neq 0} \left[ \frac{m_i}{n_i} \right]
\]

\([x]\) means the integral part of the real number \( x \)

**Remark.** Pick a fixed set of 2 generators of the maximal ideal \( \mathcal{M} \) of \( \hat{\mathcal{O}}_{T, \ell_x} \). We show in §3 and §4 that (up to a multiplicative constant) modulo \( \mathcal{M}^{m+1} \) there is a unique homogeneous relation of order \( m \) between these generators. The \( m \) (not necessarily distinct) factors of this relation correspond to the tangential directions of \( \tilde{T} \) at the point \( \ell_x \). We say that a multiple factor corresponds to a multiple tangential direction.
a.) deg M = m+1 , and M has a singularity of order at least m at Q.
b.) I(P_i, C \cap M) \geq (m+1)n_i x, for i = 1, ..., k.
c.) L \notin \text{Sing.}(M), and mL \subset M \cap H
d.) Modulo the square of the ideal defining L, the equation defining M is equal to the equation of a cone of degree m+1 with vertex at Q.

Theorem 2.3.3.: The singularity of T at the point \( \zeta_x \) is non-ordinary with \( Q \in L \) corresponding H-dually to a multiple tangential direction iff there exists a surface N such that

a.) N is a cone of degree m with vertex at Q.
b.) I(P_i, C \cap N) \geq (m+1)n_i x, for i = 1, ..., k.
c.) L \notin \text{Sing.}(N).

Remark: Theorem 2.3.3. says that a tangential direction is multiple iff some surface M satisfying the properties of Theorem 2.3.2. breaks up into the union of a cone of degree m and a plane not containing any of the points of \( L \cap C \).
loss of generality we assume that $\alpha_i, n_i \neq 0$, for $i = 1, \ldots, k$. We assume that $T$ is singular at the point $\mathcal{E}_x$, and that the fibre $\phi^{-1}(\mathcal{E}(L))$ is reduced at $\mathcal{E}_x$. We choose $Y = 0$ as the equation of the unique plane $H$ described in Proposition 2.2.1.

Let $R$ be $\hat{O}_{G,L}$. There exists a regular system of parameters $(a,b,c,d)$ of $R$ such that the inverse image of $F$ in $A^3 \times \text{Spec } R = \text{Spec } R[X,Y,Z]$ is defined by

$$X = a+bZ, \ Y = c+dZ.$$ 

The completion of the local ring of $\mathfrak{e} = p^{-1}(C)$ at $(\mathcal{E}, P_i)$ is $A_i = R[[Z-z_i]]/\alpha_i$, where $\alpha_i$ is generated by

$$f_i = a+bZ - \sum_{j \geq n_i} \alpha_i, (Z-z_i)^j, \ g_i = c+dZ - \sum_{j \geq n_i} \beta_i, (Z-z_i)^j,$$

for $i = 1, \ldots, k$.

When $n = 3$, we have:

$$F^2(q_\bullet \mathcal{O}_G, \mathcal{E}) = F^2(\bigoplus_{i=1}^k A_i) \mathcal{O}_G, \mathcal{E} = \sum_{i=1}^k F^{n_i-1} (A_i) \mathcal{O}_G, \mathcal{E}.$$ 

By Weierstrass' Preparation Theorem ([ZS], p. 140-141, 145) there is a distinguished polynomial $S_i$ of degree $n_i$ in
3.2.

Proof of Theorem 2.3.1. in Case 1:

Let $m = \min m_i$, where $m_i = I(P_i, C \cap H)$, $i = 1, \ldots, k$, and $H$ is the special plane described in Proposition 2.2.1. Since the equation of $H$ is $Y = 0$, $m_i$ is $\min\{j|\beta_{ij} \neq 0\}$. We will show that the multiplicity of $T$ at $\ell$ is $m$.

Using Weierstrass' Preparation Theorem ([ZS], p. 140-141, 145) we obtain:

$$S_i = z_i + s_{i,0}(a,b), \text{ where } s_{i,0} \equiv \frac{a+z_ib}{\alpha_{i,1}} \mod (a,b)^2$$

$$T_i = t_{i,0} \equiv c+z_id - \left(\frac{a+z_ib}{\alpha_{i,1}}\right)^m \beta_{i,m} \mod ((a,b) \cdot (c,d), (a,b,c,d)^{m+1})$$

We have: $\mathcal{O}_{T,\ell} = R/(t_{1,0}, t_{2,0}, t_{3,0})$ where $R = k[[a,b,c,d]]$.

We denote by $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ the images of $a,b,c,d$ in $\mathcal{O}_{T,\ell}$.

Let $M = (\bar{a}, \bar{b}, \bar{c}, \bar{d})$. It is clear that $M = (\bar{a}, \bar{b})$.

A little calculation gives that the unique relation between $\bar{a}$ and $\bar{b} \mod M^{m+1}$ is:

$$(*): \frac{m}{j} \sum_{j=0}^{m} \frac{1}{\alpha_{i,1}} a_i^{m-j} b_i^j \equiv 0,$$

where

$$y_j = \frac{\beta_{1,m}}{\alpha_{1,1}} (z_2-z_3)z_1^j + \frac{\beta_{2,m}}{\alpha_{2,1}} (z_3-z_1)z_2^j + \frac{\beta_{3,m}}{\alpha_{3,1}} (z_1-z_2)z_3^j$$
For the proof of (ii) we may assume \( Q = (0,0,0,1) \) without loss of generality. Then a (1-fold) tangential direction to \( T \) at \( \ell \) corresponds H-dually to \( Q \) iff the coefficient \( \gamma_m \) of (*) is zero.

\[
\gamma_m = \frac{\beta_{1,m}}{\alpha_{1,1}} (z_2 - z_3) z_1^m + \frac{\beta_{2,m}}{\alpha_{2,1}} (z_3 - z_1) z_2^m + \frac{\beta_{3,m}}{\alpha_{3,1}} (z_1 - z_2) z_3^m = 0
\]

is equivalent to

\[
\begin{vmatrix}
\beta_{1,m} z_1^m & \alpha_{1,1}^m z_1 & \alpha_{1,1}^m \\
\beta_{2,m} z_2^m & \alpha_{2,1}^m z_2 & \alpha_{2,1}^m \\
\beta_{3,m} z_3^m & \alpha_{3,1}^m z_3 & \alpha_{3,1}^m \\
\end{vmatrix} = 0
\]

Using the local parametrizations of \( C \) at the \( P_i \) in \( L \cap C \) we see that this is equivalent to the existence of a triple \((r_1, r_2, r_3)\) with \( r_1 \neq 0 \), such that the surface with affine equation

\[
r_1 YZ^m + r_2 X^m Z + r_3 X^m = 0
\]

intersects \( C \) at least \( m+1 \) times at \( P_i \), for \( i = 1, 2, 3 \). On the other hand any surface satisfying properties a.) and c.) of Theorem 2.3.2. ii.) has affine equation

\[
r_1 YZ^m + r_2 X^m Z + r_3 X^m + r_4 YZ^{m-1} = 0 \quad \text{modulo } (XY, Y^2, X^{m+1})
\]
The equation of such a surface $N$ is obviously homogeneous of degree $m$. The image of the function $x_jz^{m-j}$ is of order exactly $j$, for $j = 0, \ldots, m$, in $O_{C,P_i}$, for each $P_i$ different from $Q$. Hence no term of the form $x_jz^{m-j}$ occurs in the equation of $N$, for $j = 0, 1, \ldots, m-1$. Hence the equation of $N$ is 
\[ r_1 Y Z^{m-1} + r_2 X^m = 0 \mod (XY, Y^2). \] Since $L \not\subset \text{Sing.}(N)$, $r_1 \neq 0$. Hence we have proved Theorem 2.3.3. in Case 1.

3.5.
In Case 2 we use Weierstrass' Preparation Theorem to obtain the unique relation between $a$ and $b$ modulo $M^{m+1}$:

\[
\begin{bmatrix}
1 & z_1 \left( \frac{a+z_1b}{\alpha_1,2} \right)^m \cdot b_{1,2m} \\
0 & 1 \left( \frac{a+z_1b}{\alpha_1,2} \right)^m \cdot b_{1,2m+1} \cdot \left( \frac{a+z_1b}{\alpha_1,2} \right)^{m-1} \cdot \left( \frac{b}{\alpha_1,2} - \frac{a+z_1b}{\alpha_1,2} \cdot \alpha_1,3 \right) \cdot b_{1,2m} \\
0 & z_2 \left( \frac{a+z_2b}{\alpha_2,1} \right)^m \cdot b_{2,m}
\end{bmatrix} = 0
\]

where $m$ is the number given in Theorem 2.3.1:

In Case 3 the corresponding relation is

\[
\begin{bmatrix}
1 & \left( \frac{a+z_3b}{\alpha_3,1} \right)^m \cdot \beta_{3,1} \\
0 & 1 \left( \frac{a+z_3b}{\alpha_3,1} \right)^m \cdot \beta_{3,1+1} \cdot \left( \frac{a+z_3b}{\alpha_3,1} \right)^{m-1} \cdot \left( \frac{b}{\alpha_3,1} - \frac{a+z_3b}{\alpha_3,1} \cdot \alpha_3,2 \right) \cdot \beta_{3,1} \\
0 & z_3 \left( \frac{a+z_3b}{\alpha_3,1} \right)^m \cdot \beta_{3,m}
\end{bmatrix} = 0
\]
§4. THE LAST PART OF THE PROOFS OF THE MAIN RESULTS.

We now consider an arbitrary point \( \ell_x \in T \), reduced in its fibre over \( \ell(L) \in T \). We keep the notation from §2.1, §2.2, §3.1 and assume that \( \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_L) = n \geq 3 \).

Hence we identify points \( \ell_x \in T \) over \( \ell(L) \) with k-tuples \( (n_{1x}, \ldots, n_{kx}) \), where \( 0 \leq n_{ix} \leq n_i \), for \( i = 1, \ldots, k \), \( \Sigma_{i=1}^k n_{ix} = 3 \) \( (L \cap C = \{P_1, \ldots, P_k\}) \),

\[
\text{rk}(\mathcal{O}_C \otimes \mathcal{O}_L)_{P_i} = n_i)
\]

To prove Theorems 2.3.1., 2.3.2., 2.3.3. it is enough to show:

a.) The fibre \( \tilde{T} \times \text{Spec} K(\ell) = \phi^{-1}(\ell) \) is reduced at \( \ell_x \) iff in the corresponding k-tuple \( (n_{1x}, \ldots, n_{kx}) \) we have: \( n_{ix} \) is either 0 or \( n_i \), for \( i = 1, \ldots, k \).

b.) \( n_{ix} \) is either 0 or \( n_i \), for \( i = 1, \ldots, k \Rightarrow \mathcal{O}_x \mathcal{T} = R/\mathfrak{a}_x \), where \( R = \hat{\mathcal{O}}_G, \ell \), and \( \mathfrak{a}_x \) is the ideal in \( \mathcal{T}, \ell_x \) generated by the coefficients of those \( T_i \) (see §3.1) such that \( n_{ix} = n_i \).
Now a.) follows directly from Lemma 3.5, [GP], p. 22. Under the assumption of b.) this lemma also gives:

\[ R \cong R_x^{\text{def}} \left[ R[x_0, \ldots, x_{n-4}] / (r_0, r_1, \ldots, r_{n-4}) \right] (a, b, c, d, x_0-x_0, \ldots, x_{n-4}-x_{n-4}) \]

Hence

\[ B_{\bar{\mathfrak{n}}_x} = R_x / (V_0, V_1, V_2) . \]

Since \( R_x \) is a regular local ring, \( R_x[Z] \) is a UFD. Identifying all polynomials with their images in \( R_x[Z] \), we have in \( R_x[Z] \):

\[ S = S'S'' = \Omega \Delta , \]

where \( S' = \prod_{i, n_{ix}=0} S_i \), \( S'' = \prod_{i, n_{ix} \neq 0} S_i \), and \( \Delta \) is some power series. Since \( \Omega_x \) and \( \Delta_x = \prod_{i} (Z-z_i)^{n_{ix}} \) have no common factors, we obtain \( S' = \Omega \) in \( R_x[Z] \). Hence

\[ V_i = \tau(\Delta S'Z^i) , i = 0, 1, 2 , \]

and one easily obtains that \((V_0, V_1, V_2)\) generates the
§5. **EXAMPLES AND REMARKS.**

Let $C$ be a smooth complete intersection of two cubic surfaces. Then $C$ contains no line and possesses no 4-secants. Assume $L$ is a 3-secant corresponding to a point $\ell \in T$.

As a consequence of Proposition 2.2.1, the following 3 statements are equivalent:

a.) $T$ is singular at $\ell$.

b.) There exists a plane $H$ and a cubic surface $F_3$ containing $C$ such that $H \cap F_3 \supset L_2$, where $L_2$ is the double line in $H$ with support $L$.

c.) There exists a cubic surface $F_3$ containing $C$ and having 2 (possibly coinciding) singularities on $L$.

If $T$ is singular at $\ell$, the surface $F_3$ of b.) is the same as surface $F_3$ of c.).

Furthermore, as a consequence of the results in §2, we have:

The multiplicity of $T$ at the point $\ell$ is 3 iff there exists a plane $H$ and a cubic surface $F_3$ containing $C$ such that $H \cap F_3 = L_3$ where $L$ is the triple line in $H$ with support $L$. The multiplicity of $T$ at any point $\ell$ is at most 3.
\[ \sum_{i, n_i \geq n_i} \left[ (m+1)n_{ix} - \min(m_i, (m+1)n_{ix}) \right] = 1 , \]

where \( m_i = I(P_i, C \cap H) \) for the special plane \( H \).
Then the point corresponding \( H \)-dually to the \( m \)-fold tangential direction is the unique \( P_i \in L \cap C \)
such that \( m_i = (m+1)n_{ix} - 1 \).

(2) Using essentially the same methods as in the proofs of the results in §2.3 we may examine the local geometry of \( T \) at non-singular points of \( T \).
Let \( L \) be the tangent line to \( T \) at a non-singular point \( \ell(L) \). Then we have for example:

\( \ell \) is a flex on \( T \), i.e. \( \text{rk}(\mathcal{O}_T \otimes \mathcal{O}_L) \geq 3 \), iff

\( L \cap C = \{P_1, P_2, P_3\} \), \( n_1 = n_2 = n_3 = 1 \), and
\( I(P_1, C \cap H) = 1 \), \( I(P_j, C \cap H) \geq 3 \), \( j = 2, 3 \), for some plane \( H \), or:

\( L \cap C = \{P_1, P_2\} \), \( n_1 = 1 \), \( n_2 = 2 \), and
\( I(P_1, C \cap H) = 1 \), \( I(P_2, C \cap H) \geq 6 \), for some plane \( H \).

(3) When the fibre \( \phi^{-1}(\ell) \) is not reduced at a point \( \ell_x \in \tilde{T} \), the multiplicity formula given in Theorem 2.3.1.
is no longer valid, although Proposition 2.2.1. holds.
In addition the dimension of the tangent space of \( \tilde{T} \) at \( \ell_x \) may be 3.
References:


