S U B T R A N S V E R S A L I T Y
A N D B L O W I N G U P

by

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The notion of subtransversality is due to A. Andreotti; it was introduced in [1] and further studied in [2]. The definition is algebraic rather than geometric and goes well with certain standard operations in analytic geometry. In the present paper we show that in the smooth case subtransversality, or rather subtransversality after blowing up have a simple geometric meaning. (theorems 1.1, 1.2 and 3.1). In particular it generalizes and elucidates the results of section 19 in [1].

This paper extends and includes the results of [2].

1. <u>Preliminaries and statements</u>. We recall a few concepts from [1]. Let X and Y be smooth (i.e. C^∞-) manifolds, $\dim X > 0$, and let A and B be closed submanifolds of X and Y. We denote by $C^\infty(X,A;Y,B)$ the set of smooth maps $g:X \to Y$ such that $g(A) \subseteq B$. This is a closed subset of $C^\infty(X,Y)$ in the Whitney topology (= the fine C^∞-topology).

Furthermore, denote by $C^\infty_x(X)$ the local ring of germs of smooth functions at $x \in X$. An ideal $I \subseteq C^\infty_a(X)$ is <u>regular</u> of codimension k if I has k generators $h_1,h_2,\ldots,h_k$ such that $dh_1\wedge\ldots\wedge dh_k \neq 0$. This requires I to be a proper ideal of $C^\infty_a(X)$. In addition we consider $I = C^\infty_a(X)$ to be a regular ideal of codimension k for any integer k. Then $V(I) = \{x \in (X,a) | h(x) = 0 \; \forall h \in I\}$ is the germ of a smooth submanifold of X at a of codimension k (empty if $I = C^\infty_a(X)$). Clearly a mapping $g:X \to Y$ is transverse to B at $a \in X$ if and only if $C^\infty_a(X)\cdot g^* I(B)_{g(a)}$ is a regular ideal of codimension k, where k is the codimension of B at $g(a)$ and $I(B)_{g(a)} \subseteq C^\infty_{g(a)}(Y)$ is the ideal of smooth germs at $g(a)$ vanishing on B.

Next, let $g \in C^\infty(X,A;Y,B)$ and let $a \in A$; then $C^\infty_a(X)\cdot g^* I(B)_{g(a)} \subseteq I(A)_a$. Consider the conductor ideal $c_g(I(A)_a, I(B)_{g(a)}) \subseteq C^\infty_a(X)$. By definition $h \in c_g(I(A)_a, I(B)_{g(a)})$ if and only if $h\cdot I(A)_a \subseteq C^\infty_a(X)\cdot g^* I(B)_{g(a)}$. We say that g is <u>subtransverse</u> to B at a if $c_g(I(A)_a, I(B)_{g(a)})$ is regular of codimension equal the codimension of B at $g(a)$, and <u>strongly subtransverse</u> to B at a if $c_g(I(A)_a, I(B)_{g(a)}) + I(A)_a$ is regular of codimension equal the sum of the codimensions of A and B at a and b.

Finally, let $\tilde{X}$ be the blow-up of X along A and $\sigma:\tilde{X} \to X$ the collapse mapping. Then $\tilde{X}$ is canonically a smooth manifold with $\tilde{A} = \sigma^{-1}(A)$ a codimension one submanifold, [3], §3. A mapping $g \in C^\infty(X,A;\, Y,B)$ is (strongly) σ-subtransverse to B at a if $g \circ \sigma$ is (strongly) subtransverse to B at any point of $\sigma^{-1}\{a\}$.

The geometric content of these definitions is given by the following

Theorem 1.1. *Let* $g \in C^\infty(X,A;\, Y,B)$. *Then the statements*

(i) g *is strongly* σ*-subtransverse to* B *at all points of* A
(ii) Ng *is transverse to* O_B *outside* O_A

are equivalent.

Here $Ng\colon NA \to NB$ is the normal bundle mapping, and O_A and O_B are the zero-sections of NA and NB. The theorem follows from Proposition 2.2 and Theorem 3.1 of section 2 and 3.

We will consider in more detail the case where g is a product mapping $f\times f\colon N\times N \to P\times P$ and A and B are the diagonals Δ_N and Δ_P respectively. The normal bundles NA and NB can then be identified with the tangent bundles TN and TP. In this case we have the following sharper result.

Theorem 1.2. *Let* $f\colon N \to P$ *be a smooth mapping*. *Then the statements*

(i) $f\times f$ *is* σ*-subtransverse to* Δ_P *at all points of* Δ_N.
(ii) $f\times f$ *is strongly* σ*-subtransverse to* Δ_P *at all points of* Δ_N.
(iii) Tf *is transverse to* O_P *outside* O_N

are equivalent.

Here $Tf\colon TN \to TP$ is the tangent bundle mapping and O_N and O_P are the zero-sections of TN and TP.

The theorem is a corollary of Lemma 2.4, Proposition 2.5 and Theorem 3.1 of section 2 and 3. Theorem 3.1 gives yet another characterization of σ-subtransversality.

2. Double points and residual singularities.

Let $\tilde{W} = W(X,A)$ be the blow-up of X along A. Thus W is obtained from W by suitably replacing A with PNA, the projectivized normal bundle of A, see for instance [3], §4. Set $N-A = W_1$ and $PNA = W_2$, so that $W = W_1 \cup W_2$.

We construct a smooth manifold $E = E(X,A;\ Y,B)$ over W depending functorially on (X,A) and (Y,B). First, set $E = E_1 \cup E_2$ where

$$E_1 = \{(x,y) \mid x \in X - A\ ,\ y \in Y\}$$

$$E_2 = \{(x,l,y,\phi) \mid x \in A,\ y \in B,\ \ell \in PN_xA,\ \phi \in \mathrm{Hom}\ (\ell, N_yB)\}.$$

Then there is a natural projection π of E onto W defined by

$$\pi(x,y) = x \qquad (\text{on } E_1)$$

$$\pi(x,\ell,y,\phi) = (x,\ell) \qquad (\text{on } E_2)$$

Secondly, for every $g \in C^\infty(X,A;\ Y,B)$ there is an induced mapping $\hat{g}\colon W \to E$, which is a section of π, defined by

$$\hat{g}(x) = (x,g(x)) \qquad (\text{on } W_1)$$

$$\hat{g}(x,\ell) = (x,\ell,g(x),\ Ng|\ell) \qquad (\text{on } W_2)$$

When Y is a point and $B=Y$, then $E(X,A;\ Y,B) = W(X,A)$ (as a set), and π is the identity mapping.

We need a smooth structure on E. Set $\dim X = m$, $\dim A = r$ and $\dim Y = q$, $\dim B = s$. First notice that E_1 and E_2 are naturally smooth manifolds of dimensions $m+q$ and $(m-1)+q$ over the smooth manifolds W_1 and W_2. In fact $E_1 = (X-A) \times Y$. As for E_2 let LNA be the tautological line bundle over PNA, and $\mathrm{Hom}(LNA,\ NB)$ the corresponding vector bundle over $PNA \times B$; then $E_2 = \mathrm{Hom}\ (LNA,\ NB)$.

<u>Lemma</u> 2.1. $E = E(X,A;\ Y,B)$ <u>has a canonical smooth structure</u>

<u>compatible with that of</u> E_1 <u>and</u> E_2, <u>such that</u> $\pi \in C^\infty(E,W)$ <u>and</u> $\hat{g} \in C^\infty(W,E)$ <u>for any</u> $g \in C^\infty(X,A;\ Y,B)$.

<u>In particular</u> $E(X,A;\ Y,B) = W(X,A)$ (<u>as a manifold</u>) <u>when</u> Y <u>is a point and</u> $B=Y$.

<u>Proof</u>: Consider the case $X = \underline{\underline{R}}^m$, $Y = \underline{\underline{R}}^q$, $A = \underline{\underline{R}}^r \times \{0\} \subset X$ and $B = \underline{\underline{R}}^s \times \{0\} \subset Y$. Define $A_k \subset E$, $1 \leq k \leq m-r$, by $A_k = A_{k1} \cup A_{k2}$ where

$$A_{k1} = \{(x,y) \in E_1 \mid x_{r+k} \neq 0\}$$
$$A_{k2} = \{(x,\ell,y,\phi) \in E_2 \mid \ell_k \neq 0\}$$

and $(\ell_1,\ldots,\ell_{m-r})$ are homogeneous coordinates for ℓ. Evidently $E = A_1 \cup \ldots \cup A_{m-r}$.

Next, define mappings $\alpha_k\colon A_k \to \underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q$ $(1 \leq k \leq m-r)$ by

$$\alpha_k(x,y) = (x,\underline{\underline{R}}x'',y',y''/x_{r+k}) \qquad (\text{on } A_{k1})$$
$$\alpha_k(x,\ell,y,\phi) = (x,\ell,y',\phi(\ell_{1k},\ldots,\ell_{m-r,k})) \qquad (\text{on } A_{k2})$$

where $x = (x',x'') \in \underline{\underline{R}}^r \times R^{m-r}$, $y = (y',y'') \in \underline{\underline{R}}^s \times \underline{\underline{R}}^{q-s}$ and $\ell_{ik} = \ell_i/\ell_k$ for $1 \leq i \leq m-r$.

Clearly α_k is injective for all k. We topologize A_k so that α_k is a homeomorphism onto its image. Then $A_k \cap A_\ell$ is an open subset of A_k and A_ℓ for each k and ℓ, as is quickly checked, and the topology induced by A_k on $A_k \cap A_\ell$ coincides with the topology induced by A_ℓ since the mappings $\alpha_\ell \circ \alpha_k^{-1}$ are continuous and therefore homeomorphisms. Consequently there is a unique topology E such that each space on A_k occurs as an open subspace of E. It is easy to see that E is a Hausdorff space.

We show that $\alpha_k(A_k)$ is a $(m+q)$ - dimensional smooth submanifold of $\underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q$. Set $U_k = \underline{\underline{R}}^m \times \underline{\underline{P}}_k^{m-r-1} \times \underline{\underline{R}}^q$ where $\underline{\underline{P}}_k^{m-r-1}$ is the affine open coordinate set $\{L \in \underline{\underline{P}}^{m-r-1} \mid L_k \neq 0\}$ in $\underline{\underline{P}}^{m-r-1}$. Then $\alpha_k(A_k) \subset U_k$ for $k = 1,\ldots,m-r$; in fact (ξ, L, η) is in $\alpha_k(A_k)$ if and only if $L_k \neq 0$ and $\xi_{r+i} L_k = \xi_{r+k} L_i$ for $1 \leqslant i \leqslant m-r$.

Define $\theta_k\colon U_k \to \underline{\underline{R}}^{m-r-1}$ by $\theta_k(\xi, L, \eta) = (\xi_{r+1} - L_{1k}\xi_k, \ldots, \xi_m - L_{m-r,k}\xi_k)$ where the k-th component $(=0)$ is omitted. Then θ_k is a submersion onto $\underline{\underline{R}}^{m-r-1}$. Since $\alpha_k(A_k) = \theta_k^{-1}\{0\}$, it follows that $\alpha_k(A_k)$ is a smooth submanifold of U_k, hence of $\underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q$, of codimension $m-r-1$.

By means of α_k we pull back the smooth structure on $\alpha_k(A_k)$ to A_k. We now need to show that A_k and A_ℓ induce the same smooth structure on the open set $A_k \cap A_\ell$ for any two k and ℓ. But this holds since the mappings $\alpha_\ell \circ \alpha_k^{-1}$ are smooth and therefore diffeomorphisms. Thus $E = A_1 \cup \ldots \cup A_{m-r}$ receives a smooth structure in which $A_1,\ldots,A_{m-r}$ are open submanifolds.

For $q = 0$, i.e. $B = Y = \{0\}$, we clearly get $E = W$. (Alternatively define the smooth structure on $W(\underline{\underline{R}}^m, \underline{\underline{R}}^r)$ as that of $E(\underline{\underline{R}}^m, \underline{\underline{R}}^r; 0,0)$.) Throughout the paper we shall use primed letters $A_k', \alpha_k', \ldots$ in the particular case $E = W$, i.e. primed letters refer to W. Then we have a commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\alpha_k} & \underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q \\ \pi \big\downarrow & & \big\downarrow \mathrm{pr} \\ A_k' & \xrightarrow{\alpha_k'} & \underline{\underline{R}}^n \times \underline{\underline{P}}^{m-r-1} \end{array}$$

showing that π is smooth on A_k, $1 \leqslant 1 \leqslant m-r$. Thus π is smooth (on E).

Finally we need to check that $\hat{g}:W \to E$ is smooth for smooth g. Obviously it suffices to check this at a point $(x,\ell) \in W_2$. Let k be such that $(x,\ell) \in A'_k$. We have $\hat{g}(A'_k) \subset A_k$ and therefore a map $\tau_k:\alpha'_k(A'_k) \to \alpha_k(A_k)$ defined by the commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\alpha_k} & \alpha_k(A_k) \\ {\scriptstyle \hat{g}}\uparrow & & \uparrow{\scriptstyle \tau_k} \\ A'_k & \xrightarrow{\alpha'_k} & \alpha'_k(A'_k) \end{array}$$

Extend τ_k to a mapping $T_k:U'_k \to U_k$ in the following way: Write

$$g_{s+i}(\xi) = \sum_{j=1}^{m-r} \xi_{r+j} G_{ij}(\xi) \quad , \quad 1 \leqslant i \leqslant q-s,$$

with the $G_{ij}(\xi) = \int_0^1 \frac{\partial g_{s+i}}{\partial x_{r+j}} (\xi', t\xi'')\, dt$ for $\xi = (\xi', \xi'') \in \underline{\underline{R}}^r \times \underline{\underline{R}}^{m-r}$, such that $G_{ij}(\xi) = \frac{\partial g_{s+i}}{\partial x_{r+j}}(\xi)$ when $\xi'' = 0$. Now set

$$T_k(\xi,L) = (\xi, L, g_1(\xi), \ldots, g_s(\xi), \sum_{j=1}^{m-r} L_{jk} G_{1j}(\xi), \ldots, \sum_{j=1}^{m-r} L_{jk} G_{q-s,j}(\xi))$$

Then T_k extends τ_k as claimed. Since T_k is smooth, so is τ_k. Consequently $\hat{g}$ is smooth.

This concludes the proof in the affine case $X = \underline{\underline{R}}^m$, $Y = \underline{\underline{R}}^q$. The extension to the flat case, where X and Y are diffeomorphic to $\underline{\underline{R}}^m$ and $\underline{\underline{R}}^q$, is by transport of structure; the result is easily seen to be independent of the choice of diffeomorphisms. The extension to the general case is then by patching over coordinate neighbourhoods in X and Y, thereby constructing the germ of E along E_2 compatible with E_1, and joining the result to E_1. The procedure is straightforward. We omit further details.

Remark 1. By construction E_1 and E_2 are built in as submanifolds of E. Since E_1 is an open submanifold, E_2 is a closed submanifold of E.

2. There is also a smooth projection $\pi_2 : E \to Y$ defined by

$$\pi_2(x,y) = y \qquad \text{(on } E_1\text{)}$$

$$\pi_2(x,\ell,y,\phi) = y \qquad \text{(on } E_2\text{)}$$

More symmetrically we have the smooth projections

$$X \xleftarrow{\pi_1} E \xrightarrow{\pi_2} Y$$

where $\pi_1 = \sigma \circ \pi$. Thus the extension $\hat{g}$ of g fits into the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{g}} & E \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ X & \xrightarrow{g} & Y \end{array}$$

We next define a special submanifold Z of E. Let $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x,y) \in E_1 \mid y \in B\}$$

$$Z_2 = \{(x,\ell,y,\phi) \in E_2 \mid \phi = 0\}$$

Then $Z \subset E$; we claim that Z is a closed submanifold of E. First notice that $Z \cap E_1 = Z_1$ is certainly a closed submanifold of E_1. If $a \in E_2$ is in the closure of Z, then $a \in E(U, U \cap A;\ V, V \cap B)$ for suitable coordinate systems (U,ϕ) and (V,ψ) in X and Y such that $\phi(U \cap A) = \underline{\underline{R}}^r \times \{0\}$ and $\psi(V \cap B) = \underline{\underline{R}}^s \times \{0\}$. Thus $a \in Z$ if $Z \cap E(U, U \cap A;\ V, V \cap B)$ is closed in $E(U, U \cap A; V, V \cap B)$. Moreover, Z is a submanifold of E locally around a if $Z \cap E(U, U \cap A;\ V, V \cap B)$ is a submanifold of $E(U, U \cap A;\ V, V \cap B)$.

Consequently we are reduced to substanciating our claim in the affine case $X = \underline{\underline{R}}^m$, $Y = \underline{\underline{R}}^q$, $A = \underline{\underline{R}}^r \times \{0\} \subset X$ and $B = \underline{\underline{R}}^s \times \{0\} \subset Y$. Again, in the affine case it suffices to show that $Z \cap A_k$ is a closed submanifold of A_k for $k = 1,\ldots,m-r$. Let $\rho : \underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q \to \underline{\underline{R}}^{q-s}$ be the projection to the last $q-s$ cordinates. It is quickly checked that $\rho \mid \alpha_k(A_k)$ has constant rank $q-s$, i.e. that $\rho \circ \alpha_k$ has constant rank $q-s$. But $Z \cap A_k = (\rho \circ \alpha_k)^{-1}\{0\}$, and so $Z \cap A_k$ is indeed a closed submanifold of A_k.

Notice that Z_2 is a closed submanifold of Z. This follows by the same arguments as above if we use the projection $\lambda_k : \underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q \to \underline{\underline{R}}^{q-s+1}$ defined by

$$\lambda_k(\xi,\ell,\mu) = (\xi_{r+k}; \mu_{q-s+1},\ldots,\mu_q)$$

instead of ρ.

We have the following characterization of the maps $g \in C^\infty(X,A\colon Y,B)$ with $\hat{g}$ transverse to Z_2:

<u>Proposition</u> 2.2. <u>Let</u> $g \in C^\infty(X,A; Y,B)$ <u>and</u> $w = (a,\ell) \in W_2$. <u>Then</u> $\hat{g} \pitchfork Z_2$ <u>at</u> w <u>if and only if</u> $Ng \pitchfork O_B$ <u>at</u> $\ell - \{0\}$. <u>Moreover</u> $\hat{g} \pitchfork Z_2$ (on W) <u>if and only if</u> $Ng \pitchfork O_B$ <u>outside</u> O_A.

The first statement means that $Ng\colon NA \to NB$ is transverse to the zero-section $O_B \subset NB$ at $\nu \in NA$ for some (hence any) non-zero vector ν in $\ell \subset N_aA$.

<u>Proof</u>: Since $\hat{g}(W_1) \pitchfork Z_2 = \phi$, we obviously have $\hat{g} \pitchfork Z_2$ on W_1. The second statement in the proposition therefore follows from the first.

Assume $w = (a,\ell) \in W_2$ and set $t = \operatorname{rank}(Ng)_a$. By restricting to suitable coordinate patches around a and $g(a)$, it suffices to consider the case $X = \underline{\underline{R}}^m$, $Y = \underline{\underline{R}}^q$, $A = \underline{\underline{R}}^r \times \{0\} \subset X$, $B = \underline{\underline{R}}^s \times \{0\} \subset Y$, $a = 0$, $g(a) = 0$. In fact we may assume the coordinatisation at a and $g(a)$ performed such that

$g = (g_1, g_2): \underline{\underline{R}}^m \to \underline{\underline{R}}^s \times \underline{\underline{R}}^{q-s}$ with $g_1(0) = 0$ and

$$g_2(x) = (x_{r+1}, \dots, x_{r+t}, \psi(x)),$$

where $\psi: \underline{\underline{R}}^m \to \underline{\underline{R}}^{q-s-t}$ is a smooth mapping such that $\psi(A) = \{0\}$ and $D\psi(0) = 0$.

Now, let $v = (v', v'') \in \underline{\underline{R}}^t \times \underline{\underline{R}}^{m-r-t}$ be a non-zero vector and $\ell \in \underline{\underline{P}}^{m-r-1} = P\, N_0\, \underline{\underline{R}}^{m-r}$ the line spanned by v. We have $\hat{g}(0,\ell) = (0, \ell, 0, Ng(0) | \ell)$ with

$$Ng(0) = \left[\begin{array}{c|c} I_t & 0 \\ \hline 0 & 0 \end{array}\right]$$

Thus $Ng(0)v = v'$ and so

(i) $\hat{g}(0,\ell) \notin Z_2$ if and only if $v' \neq 0$.

Suppose $v' = 0$. With notations as before choose k such that $(0,\ell) \in A_k'$; then $\hat{g}(0,\ell) \in A_k$. Recall that $\lambda_k \circ \alpha_k : A_k \to \underline{\underline{R}}^{q-s+1}$ is a submersion and that $Z_2 \cap A_k = (\lambda_k \circ \alpha_k)^{-1}\{0\}$. Thus

$$\begin{array}{lll} & \hat{g} \pitchfork Z_2 \text{ at } (0,\ell) & \\ \iff & \lambda_k \circ \alpha_k \circ \hat{g} : A_k' \to \underline{\underline{R}}^{q-s+1} & \text{is submersive at } (0,\ell) \\ \iff & \lambda_k \circ \tau_k : \alpha_k'(A_k') \to \underline{\underline{R}}^{q-s+1} & \text{is submersive at } \alpha_k'(0,\ell) \\ \iff & \lambda_k \circ T_k \circ i_k' : \alpha_k'(A_k') \to \underline{\underline{R}}^{q-s+1} & \text{is submersive at } \alpha_k'(0,\ell) \end{array}$$

Here $i_k' : \alpha_k'(A_k') \to U_k'$ is the inclusion mapping,

$$\begin{tikzcd} \alpha_k'(A_k') \arrow[r, "i_k'"] & U_k' \arrow[r, "T_k"] \arrow[d, "\theta_k'"] & U_k \arrow[r, "\lambda_k"] & \underline{\underline{R}}^{q-s+1} \\ & \underline{\underline{R}}^{m-r-1} & & \end{tikzcd}$$

Consequently we want to determine the range of $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0,\ell))$. Since $i_k'(\alpha_k'(A_k')) = \theta_k'^{-1}\{0\}$, we have range $Di_k'(\alpha_k'(0,\ell)) = \ker D\theta_k'(\alpha_k'(0,\ell))$, with $\alpha_k'(0,\ell) = (0,(v_1,\ldots,v_{m-r}))$. Now $D\theta_k'(\alpha_k'(0,\ell))$ has the matrix block form

$$\begin{bmatrix} 0 & I & -V_k' & 0 & 0 \\ 0 & 0 & -V_k'' & I & 0 \end{bmatrix}$$

where as usual I means an identity matrix and 0 a zero matrix. V_k' and V_k'' are the column matrices

$$\begin{bmatrix} v_{1,k} \\ \vdots \\ v_{k-1,k} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_{k+1,k} \\ \vdots \\ v_{m-r,k} \end{bmatrix}$$

where $v_{ik} = v_i/v_k$. (Recall that $v_k \neq 0$ since $(0,\ell) = (0,(v_1,\ldots,v_{m-r})) \in A_k'$.) In particular $v_{1,k} = \ldots = v_{t,k} = 0$ since $v' = 0$.

It now follows by straight forward computation that range $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0,\ell))$ is spanned by the t standard basis vectors $e_2,\ldots,e_{t+1}$ in $\underline{\underline{R}}^{q-s+1}$ together with the r vectors $(1 \leqslant i \leqslant r)$

$$(0,0,\ldots,0, \sum_{j=t+1}^{m-r} v_{jk} \frac{\partial^2\psi_1}{\partial x_i \partial x_{r+j}}(0),\ldots, \sum_{j=t+1}^{m-r} v_{jk} \frac{\partial^2\psi_{q-s-t}}{\partial x_i \partial x_{r+j}}(0))$$

and the vector

$$(2,0,\ldots,0, \sum_{i=t+1}^{m-r} \sum_{j=t+1}^{m-r} v_{ik}v_{jk} \frac{\partial^2\psi_1}{\partial x_{r+i} \partial x_{r+j}}(0),\ldots,$$

$$\sum_{i=t+1}^{m-r} \sum_{j=t+1}^{m-r} v_{ik}v_{jk} \frac{\partial^2\psi_{q-s-t}}{\partial x_{r+i} \partial x_{r+j}}(0)).$$

We therefore have

(ii) $\hat{g}(0,\ell) \in Z_2$ and $\hat{g} \pitchfork Z_2$ at $(0,\ell)$ if and only if the vectors

$$\left(0,0,\ldots,0,\sum_{j=t+1}^{m-r} v_j \frac{\partial^2\phi_1}{\partial x_i \partial x_{r+j}}(0),\ldots,\sum_{j=t+1}^{m-r} v_j \frac{\partial^2\phi_{q-s-t}}{\partial x_i \partial x_{r+j}}\right)$$

for $1 \leqslant i \leqslant r$ form a set of rank $q-s-t$ in $\underline{\underline{R}}^{q-s+1}$.

To complete the proof of proposition 2.2 we now appeal to the following elementary

Lemma 2.3. Let $g \in C^\infty(\underline{\underline{R}}^m, \underline{\underline{R}}^r \times \{0\};\ \underline{\underline{R}}^q, \underline{\underline{R}}^s \times \{0\})$ be a mapping of the form $g(x) = (g_1(x);\ x_{r+1},\ldots,x_{r+t},\phi(x))$ with $g_1\colon \underline{\underline{R}}^m \to \underline{\underline{R}}^s$, $\phi\colon \underline{\underline{R}}^m \to \underline{\underline{R}}^{q-s-t}$ such that $g_1(0) = 0$ and $\phi(\underline{\underline{R}}^r \times \{0\}) = \{0\}$, $D\phi(0) = 0$. Let $v = (v', v'')$ be a non-zero vector in the normal space $N_0(\underline{\underline{R}}^r \times \{0\}) = \underline{\underline{R}}^t \times \underline{\underline{R}}^{m-r-t}$.

Then $Ng \pitchfork O_{\underline{\underline{R}}^s \times \{0\}}$ at $(0,v) \in N(\underline{\underline{R}}^r \times \{0\})$ if and only if

either

(i) $v' \neq 0$ (then $Ng(0,v) \notin O_{\underline{\underline{R}}^s \times \{0\}}$)

or

(ii) $v' = 0$ (then $Ng(0,v) \in O_{\underline{\underline{R}}^s \times \{0\}}$) and the matrix

$$\begin{bmatrix} \sum_{j=t+1}^{m-r} v_j \frac{\partial^2\phi_1}{\partial x_1 \partial x_{r+j}}(0) & \ldots & \sum_{j=t+1}^{m-r} v_j \frac{\partial^2\phi_1}{\partial x_r \partial x_{r+j}}(0) \\ \\ \sum_{j=t+1}^{m-r} v_j \frac{\partial^2\phi_{q-s-t}}{\partial x_1 \partial x_{r+j}}(0) & \ldots & \sum_{j=t+1}^{m-r} v_j \frac{\partial^2\phi_{q-s-t}}{\partial x_r \partial x_{r+j}}(0) \end{bmatrix}$$

has rank $q-s-t$.

The proof of lemma 2.3 is left to the discretion of the reader.

We now consider the case where g is a product mapping $f\times f\colon N\times N \to P\times P$ and A and B are the diagonals Δ_N and Δ_P. For brevity we denote the mapping $\hat{g}\colon W \to E$ by f^Δ. We then have the following

Lemma 2.4. Let $f\colon N \to P$ be a smooth mapping. Then $f^\Delta \pitchfork Z$ on W_2 if and only if $f^\Delta \pitchfork Z_2$ on W_2. Moreover $f^\Delta \pitchfork Z_2$ on W_2 if and only if $f^\Delta \pitchfork Z_2$ (on W).

Proof: The last claim is obvious since $f^\Delta(W_1) \cap Z_2 = \phi$.

Let $(a,\ell) \in W_2$ and assume that $f^\Delta(a,\ell) \in Z_2$. By suitable coordinatisations we may assume $N = \mathbb{R}^n$, $a = 0$, $P = \mathbb{R}^p$, $f(a) = 0$. Using the diffeomorphism $\mu_n\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$ defined by $\mu_n(x,y) = (x,y-x)$, we may further identify the diagonal $\Delta_{\mathbb{R}^n}$ with $\mu_n(\Delta_{\mathbb{R}^n}) = \mathbb{R}^n\times\{0\}$ and similarly $\Delta_{\mathbb{R}^p}$ with $\mu_p(\Delta_{\mathbb{R}^p}) = \mathbb{R}^p\times\{0\}$. The product mapping $f\times f\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^p\times\mathbb{R}^p$ is then identified with $g = \mu_p \circ (f\times f) \circ \mu_n^{-1}$, which is given by $g(x,y) = (f(x), f(x+y)-f(x))$.

We know that $f^\Delta \pitchfork Z$ at $(0,\ell)$ if and only if $\rho \circ T_k \circ i_k'\colon \alpha_k'(A_k') \to \mathbb{R}^p$ is a submersion at $\alpha_k'(0,\ell)$. Since $\rho = pr_2 \circ \lambda_k$ where $pr_2\colon \mathbb{R}\times\mathbb{R}^p \to \mathbb{R}^p$ is the projection, this is equivalent to $\lambda_k \circ T_k \circ i_k'$ being transverse to $K = \mathbb{R}\times\{0\} \subset \mathbb{R}\times\mathbb{R}^p$ at $\alpha_k(0,\ell)$.

We show that $T_0K \subset$ range $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0,\ell))$. Thus if $f^\Delta \pitchfork Z$ at $(0,\ell)$, then $\lambda_k \circ T_k \circ i_k'$ is a submersion at $\alpha_k'(0,\ell)$ and so $f^\Delta \pitchfork Z_2$ at $(0,\ell)$.

As usual let $(\ell_1,\dots,\ell_n)$ be homogeneous coordinates for ℓ and set $\ell_{jk} = \ell_j/\ell_k$ when $\ell_k \neq 0$, $j = 1,\dots,n$. Define the smooth curve $c\colon \langle -\varepsilon,\varepsilon\rangle \to \alpha_k'(A_k')$ by $c(t) = (-t\ell_{1k},\dots,-t\ell_{nk}, 2t\ell_{1k},\dots,2t\ell_{nk},\ell)$; then $c(0) = \alpha_k'(0,\ell)$. Since

$$\lambda_k \circ T_k(\xi,L) = (\xi_k, \sum_{j=1}^{n} L_{jk} \int_0^1 \frac{\partial f}{\partial x_j} (\xi'+s\xi'')ds)$$

for $\xi = (\xi',\xi'') \in \underline{\underline{R}}^n \times \underline{\underline{R}}^n$, $L_k \neq 0$, we find

$$\lambda_k \circ T_k \circ i_k' \circ c(t) = (2t, \sum_{j=1}^{n} \ell_{jk} \int_0^1 \frac{\partial f}{\partial x_j} (t(2s-1)(\ell_{1k},\dots,\ell_{nk}))ds)$$

From this we get

$$\frac{d}{dt}(\lambda_k \circ T_k \circ i_k' \circ c)(0) = (2,0,\dots,0) \in T_0K$$

which confirms that T_0K sits in the range of $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0,\ell))$. Thus $f^\Delta \pitchfork Z_2$ on W_2 if $f^\Delta \pitchfork Z$ on W_2. The converse is of course trivial.

Using lemma 2.4 we can now give the following characterisation of the smooth maps $f\colon N \to P$ such that f^Δ is transverse to Z.

<u>Proposition</u> 2.5. <u>Let</u> $f\colon N \to P$ <u>be a smooth mapping and</u> w <u>a point of</u> W. <u>Then</u> $f^\Delta \pitchfork Z$ <u>at</u> w <u>if and only if</u>

(i) $f\times f \pitchfork \Delta_P$ <u>at</u> w, <u>in case</u> $w = (a,a') \in W_1$.

(ii) $Tf \pitchfork O_P$ <u>at</u> $\ell-\{0\}$, <u>in case</u> $w = (a,\ell) \in W_2$.

The second statement means that $Tf\colon TN \to TP$ is transverse to the zero-section $O_P \subset TP$ at $v \in TN$ for some (hence any) non-zero vector v in $\ell \subset T_aN$.

Proof: The case $w \in W_1$ is trivial, and the case $w \in W_2$ follows from proposition 2.2 and lemma 2.4 when we identify the normal bundles NA and NB with the tangent bundles TN and TP.

3. Subtransversality. The purpose of this section is to prove the following result.

Theorem 3.1. *Let $g \in C^\infty(X,A;\ Y,B)$. Then g is σ-subtransverse to B at all points of A if and only if $\hat{g} \pitchfork Z$ on W_2, and strongly σ-subtransverse if and only if $\hat{g} \pitchfork Z_2$ on W_2.*

Proof: Let $(a,\ell) \in W_2$ and $b = g(a)$. Again, by suitable coordinatisations we may assume that $X = \mathbb{R}^m$, $Y = \mathbb{R}^q$, $A = \mathbb{R}^r \times \{0\} \subset X$, $B = \mathbb{R}^s \times \{0\} \subset Y$ and that g is of the form $g(x) = (g_1(x);\ x_{r+1},\dots,x_{r+t}, \psi(x))$ with $g_1\colon \mathbb{R}^m \to \mathbb{R}^s, \psi\colon \mathbb{R}^m \to \mathbb{R}^{q-s-t}$ smooth mappings such that $g_1(0) = 0$, $\psi(A) = \{0\}$ and $D\psi(0) = 0$.

Let $(\ell_1,\dots,\ell_{m-r})$ be homogeneous coordinates for ℓ and assume $\ell_k \neq 0$, i.e. $\ell \in \mathbb{P}_k^{m-r-1}$. Define the projection $s_k\colon \mathbb{R}^m \times \mathbb{R}^{m-r-1} \times \mathbb{R}^q \to \mathbb{R}$ by $s_k(\xi,L,\mu) = \xi_{r+k}$ and let $s_k'\colon \mathbb{R}^m \times \mathbb{P}^{m-r-1} \to \mathbb{R}$ be equal s_k when $q = 0$.

Then $s_k' \circ i_k' \circ \alpha_k'\colon A_k' \to \mathbb{R}$ is a submersion, and $W_2 \cap A_k' = (s_k' \circ i_k' \circ \alpha_k')^{-1}\{0\}$. Therefore $I(W_2)_{(0,\ell)}$ is the principal ideal generated by the germ of $s_k' \circ i_k' \circ \alpha_k'$ at $(0,\ell)$.

Now let $\phi\colon \mathbb{R}^q \to \mathbb{R}^{q-s}$ and $\rho\colon \mathbb{R}^m \times \mathbb{P}^{m-r-1} \times \mathbb{R}^q \to \mathbb{R}^{q-s}$ be projections to the last $q-s$ coordinates. Recall the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{g}} & E \\ \sigma\downarrow & & \downarrow\pi_2 \\ X & \xrightarrow{g} & Y \end{array}$$

The ideal $I(B)_0$ is generated by the germs of $\phi_1,\dots,\phi_{q-s}$ at 0. The pullback by the mapping $g \circ \sigma$ is therefore generated by the germs of $\phi_j \circ \pi_2 \circ \hat{g}$ at $(0,\ell)$, $j = 1,\dots,q-s$.

Let $r_k\colon \underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times \underline{\underline{R}}^q \to \underline{\underline{R}}^q$ be the mapping $r_k(\xi,\ell,\mu) = (\mu',\xi_{r+k}\mu'')$ for $\mu = (\mu',\mu'') \in \underline{\underline{R}}^s \times \underline{\underline{R}}^{q-s}$, $1 \leqslant k \leqslant m-r$. Since $\pi_2 | A_k = r_k \circ i_k \circ \alpha_k$, we have

$\phi \circ \pi_2 \circ \hat{g} \mid A_k' = (s_k \rho) \circ i_k \circ \alpha_k \circ \hat{g} = (s_k \circ i_k' \circ \alpha_k')(\rho \circ T_k \circ i_k' \circ \alpha_k')$

with T_k as before. The conductor $c_g(I(W_2)_{(0,\ell)}, I(B)_0)$ is therefore the ideal generated by the germs of $\rho_j \circ T_k \circ i_k' \circ \alpha_k'$ at $(0,\ell)$, $j = 1,\dots,q-s$.

Finally, let $\lambda_k\colon \underline{\underline{R}}^m \times \underline{\underline{P}}^{m-r-1} \times R^q \to R^{q-s+1}$ be the projection $\lambda_k(\xi,\ell,\mu) = (\xi_{r+k},\mu'')$ for $\mu = (\mu',\mu'') \in \underline{\underline{R}}^s \times \underline{\underline{R}}^{q-s}$. Then $c_g(I(W_2)_{(0,\ell)}, I(B)_0) + I(W_2)_{(0,\ell)}$ is the ideal generated by the germs of $\lambda_{kj} \circ T_k \circ i_k' \circ \alpha_k'$ at $(0,\ell)$, $j = 1,\dots,q-s+1$.

For the first part of the theorem: Suppose $\ell_k \neq 0$ for some $k \leqslant t$. On U_k we have

$$\rho_k(T_k(\xi,L)) = \sum_{j=1}^{m-r} L_{jk} \int_0^1 \frac{\partial g_{s+k}}{\partial x_{r+j}} (\xi',t\xi'')dt = 1$$

Thus $c_g(I(W_2)_{(0,\ell)}, I(B)_0)$ contains the unit element in $C^\infty_{(0,\ell)}(W)$, and so by our convention is regular of codimension $q-s$ at $(0,\ell)$. But we have also $\hat{g}(0,\ell) \in E_2 - Z_2 = E_2 - Z$ (p.9 statement (i)).

Suppose on the other hand $\ell_1 = \dots = \ell_t = 0$. Then $c_g(I(W_2)_{(0,\ell)}, I(B)_0)$ is regular of codimension q-s if and only if $\rho \circ T_k \circ i_k'$ is a submersion at $\alpha_k'(0,\ell)$. But the last condition is equivalent to $\hat{g} \pitchfork Z$ at $(0,\ell)$; this follows by an argument analogous to that for the case $\hat{g} \pitchfork Z_2$ on page 9.

For the second part of the theorem: Suppose again $\ell_k \neq 0$ for some $k \leq t$. Then $c_g(I(W_2)_{(0,\ell)}, I(B)_0) + I(W_2)_{(0,\ell)} = C^{\infty}_{(0,\ell)}(W)$ and so is regular of codimension $q - s + 1$, and $\hat{g} \pitchfork Z_2$ at $(0,\ell)$ since $\hat{g}(0,\ell) \notin Z_2$.

Suppose on the other hand $\ell_1 = \ldots = \ell_t = 0$. Then $c_g(I(W_2)_{(0,\ell)}, I(B)_0) + I(W_2)_{(0,\ell)}$ is regular of codimension $q - s + 1$ if and only if $\lambda_k \circ T_k \circ i'_k$ is a submersion at $\alpha'_k(0,\ell)$. But this is equivalent to $\hat{g} \pitchfork Z_2$ at $(0,\ell)$ (p.9).

It follows that g is strongly σ-subtransverse to B at all points of A if and only if $\hat{g} \pitchfork Z_2$ on W_2. This completes the proof of theorem 3.1.

4. <u>Complements</u>. Let us again consider the case where g is a product mapping $f \times f\colon N \times N \to P \times P$ and A and B the diagonals Δ_N and Δ_P. The following is an easy consequence of theorems 2.4 and 2.5.

<u>Proposition</u> 4.1. <u>The smooth mappings</u> $f\colon N \to P$ <u>such that</u> $f^{\Delta} \pitchfork Z_2$ <u>form a dense open subset of</u> $C^{\infty}(N,P)$.

For the condition $f^{\Delta} \pitchfork Z_2$ is equivalent to $Tf \pitchfork O_P$ outside O_N, and the latter condition is satisfied for an open dense set of mappings f by a standard transversality argument.

The construction E is tailored to the study of the generic double points of f, as indicated by proposition 2.5. Let $D_f \subset N$ be the locus of genuine double points of f and $S_f \subset N$ the singular locus of f. Thus $x \in D_f$ if $f(x) = f(x')$ for some point $x' \neq x$, and $x \in S_f$ if $\ker Tf_x \neq \{0\}$.

<u>Proposition</u> 4.2. If $f\colon N \to P$ <u>is a proper smooth mapping such that</u> $f^{\Delta} \pitchfork Z_2$, <u>then</u> $\bar{D}_f = D_f \cup S_f$.

Proof. Let $\sigma_1 : W \to N$ be the smooth mapping $pr_1 \circ \sigma$, where $pr_1 : N \times N \to N$ is the projection to the first factor. Then $\sigma_1(x,\xi) = x$ for arbitrary $(x,\xi) \in W_1 \cup W_2$, and so $D_f = \sigma_1((f^\Delta)^{-1}(Z_1))$, $S_f = \sigma_1((f^\Delta)^{-1}(Z_2))$. Consequently

$$D_f \cup S_f = \sigma_1((f^\Delta)^{-1}(Z)).$$

Since f is proper, $\sigma_1 | (f^\Delta)^{-1}(Z)$ is also proper. Hence $D_f \cup S_f$ is a closed subset of N; in particular $\bar{D}_f \subseteq D_f \cup S_f$.

It remains to show that $S_f \subseteq \bar{D}_f$. Let $a \in S_f$, so that $(a,l) \in (f^\Delta)^{-1}(Z_2)$ for a suitable $l \subseteq T_a N$. Again, by means of coordinate systems at a and $f(a)$ we are reduced to the affine case $a = o \in \mathbb{R}^n$, $f(a) = o \in \mathbb{R}^p$. Choose $k \leq n$ such that $(o,l) \in A_k'$. Since $f^\Delta \pitchfork Z_2$, $\nu_k \circ \alpha_k \circ f^\Delta : A_k' \to \mathbb{R}^{p+1}$ is a submersion at (o,l), and we may choose a local coordinate system around (o,l) in W in which $\nu_k \circ \alpha_k \circ f^\Delta$ is presented as the standard projection $\nu_k \circ \alpha_k \circ f^\Delta (w_1,\ldots,w_{2n}) = (w_1,\ldots,w_{p+1})$. In this coordinate system, which flattens W into $\mathbb{R}^{2n}$ around (o,l), we have $(f^\Delta)^{-1}(Z_2) = \{w \in \mathbb{R}^{2n} | w_1 = \ldots = w_{p+1} = 0\}$ and $(f^\Delta)^{-1}(Z_1) = \{w \in \mathbb{R}^{2n} | w_2 = \ldots = w_{p+1} = 0 \text{ and } w_1 \neq 0\}$. Obviously then the origin $o \in (f^\Delta)^{-1}(Z_2)$ belongs to the closure of $(f^\Delta)^{-1}(Z_1)$. Backtracking this means that (a,l) belongs to the closure of $(f^\Delta)^{-1}(Z_1)$. By continuity this implies that $a = \sigma_1(a,l)$ belongs to the closure of $\sigma_1((f^\Delta)^{-1}(Z_1))$, i.e. to $\bar{D}_f$. Thus $S_f \subseteq \bar{D}_f$.

This gives at neat proof that $\bar{D}_f = D_f \cup S_f$ is a generic property for proper mappings, satisfied by those mappings $f \in C^\infty_{pr}(N,P)$ such that $Tf \pitchfork O_P$ outside O_N.

One can also prove a general transversality result.

Proposition 4.3. *Let* M *be a smooth submanifold of* E. *The smooth mappings* $f:N \to P$ *such that* $f^{\Delta} \pitchfork M$ *form a dense subset of* $C^{\infty}(N,P)$. *If* M *or* N *is compact, this subset is open*.

In general the openess property fails unless there is a compactness condition. E.g. proposition 4.1 holds because of the special character of the submanifold Z_2.

We omit the proof of proposition 4.3.

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