FINELY HARMONIC FUNCTIONS WITH BOUNDED DIRICHLET INTEGRAL WITH RESPECT TO THE GREEN MEASURE

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Abstract.

We consider finely harmonic functions \( h \) on a fine, Greenian domain \( V \subset \mathbb{R}^d \) with bounded Dirichlet integral wrt. \( G_m \), i.e.

\[
\int_V |\nabla h(y)|^2 G(x, y) \, dm(y) < \infty \quad \text{for} \quad x \in V,
\]

(*)

where \( m \) denotes the Lebesgue measure, \( G(x, y) \) the Green function. We use Brownian motion and stochastic calculus to prove that such functions \( h \) always have boundary values \( h^* \) along a.a. Brownian paths. This partially extends results by Doob, Brelot and Godefroid, who considered ordinary harmonic functions with bounded Dirichlet integral wrt. \( m \) and Green lines in stead of Brownian paths.

As a consequence of Theorem 1 we obtain several properties equivalent to (*), one of these being that \( h \) is the harmonic extension to \( V \) of a random "boundary" function \( h^* \) (of a certain type), i.e. \( h(x) = E^x[h^*] \) for all \( x \in V \). Another application is that the polar sets are removable singularity sets for finely harmonic functions satisfying (*). This is in contrast with the situation for finely harmonic functions with bounded Dirichlet integral wrt. \( m \).
§1. Introduction and statement of results

Properties of harmonic functions with bounded Dirichlet integral have been studied by several authors. In 1962 Doob [4], extending earlier works by Brelot and Godefroid, proved that a harmonic function \( h \) on a domain \( V \) in \( \mathbb{R}^d \) \( (d \geq 2) \) admitting a Green function and with a bounded Dirichlet integral, i.e.

\[
(1.1) \quad \int_V |\nabla h|^2 dm < \infty
\]

(where \( m \) denotes Lebesgue measure in \( \mathbb{R}^d \)) always has a fine boundary function \( h^* \) and \( h \to h^* \) along the Green lines of \( V \). Doob (and Brelot and Godefroid) used a measure on the space of all Green lines.

In this article we use Brownian motion and stochastic calculus to prove a result of this type and establish a corresponding \( L^2 \)-isometry (Theorem 1) in the more general situation when \( h \) is a finely harmonic function on a fine domain \( V \) in \( \mathbb{R}^d \) with a Green function \( G \). The assumption that \( h \) has a finite Dirichlet integral is replaced by the assumption that

\[
(1.2) \quad \int_V |\nabla h(y)|^2 G(x,y) dm(y) < \infty \quad \text{for all } x \in V,
\]

i.e. that \( h \) has a finite Dirichlet integral wrt. the Green measure. (It is known (Debiard and Gaveau [2]) that \( \nabla h \) exists a.e. wrt. \( m \) on \( V \).)
In the case when $h$ is harmonic in the ordinary sense on an ordinary Greenian domain $V$ then (1.1) is a stronger assumption than (1.2), because $G(x,y) \to 0$ as $y \to \partial V$ (the boundary of $V$) and the singularity of $G(x,y)$ at $y = x$ is $m$-integrable. In the general fine situation it turns out that

(1.1) implies that (1.2) holds quasi-everywhere,

i.e. everywhere outside some polar set.

To see this let $W$ be a bounded subset of $V$ and assume that (1.1) holds. Then by the Fubini theorem

$$
\int \left( \int |\nabla h(y)|^2 G(x,y) dm(y) \right) dm(x) = \int |\nabla h(y)|^2 \left( \int G(x,y) dm(x) \right) dm(y)
$$

$$
< \infty, \text{ since } \sup_{W} \left( \int G(x,y) dm(x) \right) < \infty.
$$

So (1.2) holds for a.a. $x \in W$ wrt. $m$.

In particular, the function $H(x) = \int |\nabla h(y)|^2 G(x,y) dm(y)$ is not infinite everywhere in $V$. But then it follows from Theorem 2.4 in Fuglede [9] that $H(x)$ is a fine potential in $V$ and therefore finite quasi-everywhere, as asserted.

As a consequence of Theorem 1 we obtain several properties equivalent to (1.2), one of these being that $h$ is the harmonic extension to $V$ of a random function $h^*$ (of a certain type), i.e. $h(x) = E^x[h^*]$ for all $x \in V$ (Theorem 2). Another application is that the polar sets are removable singularity sets for a finely harmonic function $h$ satisfying (1.2) (Theorem 3). This result is in contrast with the situation for finely harmonic functions $h$ satisfying (1.1). In this case it is known that polar sets need not be removable singularity sets (see Fuglede [8], Théorème 12 and p. 153). Thus the condition (1.1) does not imply (1.2) in general.
§2. Boundary behaviour and removable singularity sets

In the following \( B_t(\omega) \), \( \omega \in \Omega \), \( t \geq 0 \) will denote Brownian motion in \( \mathbb{R}^d \) \( (d \geq 2) \). The probability law of \( B_t \) starting at \( x \in \mathbb{R}^d \) is denoted by \( P^x \) and \( E^x \) is the expectation operator wrt. \( P^x \).

For a finely open set \( V \subseteq \mathbb{R}^d \) we will let \( \tau_V = \inf\{t > 0 : B_t \not\in V\} \) be the first exit time from \( V \) \( (\tau_V = \infty \text{ if } B_t \in V \text{ for all } t > 0) \). If \( \tau_V < \infty \text{ a.s. the harmonic measure } \lambda_x^V \text{ at } x \text{ wrt. } V \text{ is defined by} \)

\[
\lambda_x^V = \frac{\int f \, d\lambda_x^V}{\int f \, d\lambda_x^V},
\]

if \( f \) is a bounded, continuous real function on \( \partial V \), the boundary of \( V \).

The Green function \( G(x,y) \) of a fine domain \( V \subseteq \mathbb{R}^d \) is defined by

\[
G(x,y) \, dm(y) = \int_0^\infty p^x[B_s \in dy, s < \tau_V] \, ds,
\]

provided the integral converges.

Intuitively, \( G(x,y) \, dm(y) \) is the expected length of time Brownian motion starting at \( x \) stays in \( dm(y) \) before it exits from \( V \).

See Chung [1] for more information.

**Lemma 1.** Let \( h \) be a finely harmonic function in a finely open set \( V \subseteq \mathbb{R}^d \) with a Green function \( G \). Let \( \tau_V \) be the first exit time from \( V \). Then

\[
E^x[\int_0^{\tau_V} |\nabla h(B_s)|^2 \, ds] = \int_V |\nabla h(y)|^2 G(x,y) \, dm(y)
\]

for all \( x \in V \).
Proof. By the Fubini theorem we have ($\chi$ denotes the indicator function)

$$E^x[\int_0^{\tau_V} |\varphi h(B_s)|^2 ds] = E^x[\int_0^{\infty} |\varphi h(B_s)|^2 \chi[0,\tau_V](s)ds]$$

$$= \int (\int |\varphi h(y)|^2 \mathbb{P}[B_s \in dy, s < \tau_V])ds$$

$$= \int |\varphi h(y)|^2 (\int \mathbb{P}[B_s \in dy, s < \tau_V]ds) = \int |\varphi h(y)|^2 G(x,y)dm(y),$$

which proves Lemma 1.

**Lemma 2.** Let $f$ be a real, finely continuous function on $\mathbb{R}^d$. Then

$$t \rightarrow f(B_t(\omega))$$

is continuous on $[0,\infty)$, for a.a. $\omega \in \Omega$.

Proof. By Theorem 3.5.1 in Chung [1] the function $t \rightarrow f(B_t(\omega))$ is right continuous on $[0,\infty)$, a.s. Left continuity follows by the same argument as in the proof of Theorem 4.5.9 in the same book: Choose $c > 0$ and define the reverse process

$$\tilde{B}_t = \begin{cases} B_{c-t} & \text{for } 0 \leq t \leq c \\ B_0 + B_t - B_c & \text{for } c < t \end{cases}$$

Then $\tilde{B}_t$ is again a Brownian motion, so $t \rightarrow f(\tilde{B}_t)$ is right continuous, a.s. Since this holds for all $c > 0$ the function $t \rightarrow f(B_t)$ is left continuous, a.s.

**Lemma 3.** Let $U \subset \mathbb{R}^d$ be finely open and let $\tau$ be a stopping time. Then for a.a. $\omega$ we have:

If $B_t(\omega) \in U$ then there exists $\varepsilon > 0$ such that $B_t(\omega) \in U$ for all $t \in (\tau(\omega) - \varepsilon, \tau(\omega))$. 

Proof. Since the fine topology is completely regular we can for each \( x \in U \) find a finely continuous function \( y \mapsto f_x(y) \) on \( \mathbb{R}^d \) such that \( 0 \leq f_x \leq 1 \), \( f_x \equiv 1 \) on \( \mathbb{R}^d \setminus U \) and \( f_x(x) = 0 \).

Let \( D_x \subset U \) be a fine neighbourhood of \( x \) such that \( f_x < \frac{1}{2} \) on \( D_x \). The family \( \{D_x\}_{x \in U} \) covers \( U \), so by Doob's quasi-Lindelöf principle ([3]) we can find a countable subfamily \( \{D_{x_k}\}_{k=1}^{\infty} \) such that

\[
K = U \setminus \bigcup_{k=1}^{\infty} D_{x_k}
\]

is polar. Put

\[
f = \sum_{k=1}^{\infty} 2^{-k} f_{x_k}.
\]

Then \( f \) is finely continuous, \( f \equiv 1 \) on \( \mathbb{R}^d \setminus U \) and \( f < 1 \) on \( U \setminus K \). Assume \( B_t \in U \). Since \( K \) is polar \( B_t \notin K \) and therefore \( f(B_t) < 1 \), a.s. By Lemma 2 \( t \mapsto f(B_t) \) is continuous a.s. So for a.a. \( \omega \) there exists \( \varepsilon > 0 \) such that \( f(B_t) < 1 \) for \( \tau - \varepsilon < t < \tau \). This implies that \( B_t \in U \) for \( \tau - \varepsilon < t < \tau \) and Lemma 3 is proved.

**Lemma 4.** Let \( h \) be a finely harmonic function in a fine domain \( V \subset \mathbb{R}^d \). Then there exists an increasing sequence of fine bounded domains \( V_n \subset V \) such that with

\[
\tau_n = \tau_{V_n}
\]

we have

\[
(2.3) \quad \tau_n + \tau_V \quad \text{a.s. as } n \to \infty
\]

and

\[
(2.4) \quad E^x[\int_{\tau_n}^{\tau} h^2(B_s) \, ds] < \infty
\]

for all \( n \) and all \( x \in V \).
Proof. Choose $x \in V$. Then there exists a fine bounded neighbourhood $U_x \ni x$ with compact closure $\overline{U}_x \subset V$ and a sequence of functions $h_n$ harmonic (in the ordinary sense) in a neighbourhood of $\overline{U}_x$ such that $h_n \to h$ uniformly on $\overline{U}_x$.

(Fuglede [7], Theorem 4.1.)

Put $\tau = \tau_{U_x}$. Then by Ito's formula

$$h_n(B_\tau) - h_n(x) = \int_0^\tau \nabla h_n(B_s) dB_s$$

for all $n$.

So by the basic isometry for Ito integrals

$$E^X[(h_n(B_\tau) - h_n(x))^2] = E^X[\int_0^\tau |\nabla h_n(B_s)|^2 ds]$$

i.e.

$$E^X[h_n^2(B_\tau)] = h_n^2(x) + E^X[\int_0^\tau |\nabla h_n(B_s)|^2 ds]$$

since $E^X[h_n(B_\tau)] = h_n(x)$, for all $n$.

Letting $n \to \infty$ we obtain, using Lemma 1,

$$E^X[h^2(B_\tau)] = h^2(x) + E^X[\int_0^\tau |\nabla h(B_s)|^2 ds] < \infty.$$ 

The family $\{U_x\}_{x \in V}$ covers $V$, so by Doob's quasi-Lindelöf principle [3] we can find a countable subfamily denoted by $\{W_n\}$ such that

$$\bigcup_{n=1}^\infty W_n = V \setminus K,$$

where $K$ is a polar set. Now define

$$V_n = \bigcup_{k=1}^n W_k ; \quad n = 1, 2, \ldots .$$

Since $K$ is polar (2.3) holds.

We prove (2.4) by induction: The argument above proves that (2.4) holds for $n = 1$. To prove the induction step assume that it holds for $n = k$. Put $S_0 = \tau_k$, $T = \tau_{k+1} (= \tau_{V_k \cup W_{k+1}})$. Define
\[ S_1 = \inf \{ t > S_0 ; B_t \notin W_{k+1} \} \]
\[ S_2 = \inf \{ t > S_1 ; B_t \notin V_k \} \]

and inductively
\[ S_{2j+1} = \inf \{ t > S_{2j} ; B_t \notin W_{k+1} \} \]
\[ S_{2j+2} = \inf \{ t > S_{2j+1} ; B_t \notin V_k \} ; \quad j = 0, 1, 2, \ldots \]

Then \{S_j\} is an increasing sequence of stopping times. Since \( S_j \leq T < \infty \) a.s. the limit
\[ S = \lim_{j \to \infty} S_j \]
exists a.s. and \( S \leq T \).

Since \( B_{S_{2j+1}} \in \partial_f W_{k+1} \) for all \( j \) (\( \partial_f \) denotes fine boundary) we must have \( B_S \notin W_{k+1} \) a.s., by Lemma 3.
Similarly \( B_S \notin V_k \) a.s. Thus \( S > T \) and therefore \( S = T \).

Therefore it suffices to prove that
\[ (2.5) \quad E^X[h^2(B_{S_j})] = h^2(x) + E^X[\int_0^{S_j} |\nabla h(B_s)|^2 ds] \quad \text{for all } j. \]

For if (2.5) is established then the induction step of (2.4) follows by bounded convergence if we let \( j \to \infty \). (Recall that \( h \) is bounded on \( V_{k+1} \).)

We establish (2.5) by induction on \( j \). The strong Markov property states that if \( \tau \) is a stopping time and \( \eta \) is measurable wrt. \( \{B_s ; s \geq 0\} \), then
\[ (2.6) \quad E^X[\theta_{\tau} \eta | B_{\tau}] = E^\tau[\eta], \]
where \( \theta_t \) is the shift operator:
\[ \theta_t(g_1(B_{t_1}) \ldots g_i(B_{t_i})) = g_1(B_{t_1+t}) \ldots g_i(B_{t_i+t}). \]

(See Dynkin [5], Theorem 3.11, p. 100 or Øksendal [10], (7.15).)
Assume (2.5) holds for a given \( j \). For simplicity put 
\( a = S_j \), \( b = S_{j+1} \). Then, using (7.16) in [8]

\[
E^x[h^2(B_b)] = E^x[E^x[h^2(B_b)|B_a]] = E^x[B_a[h^2(B_b)]]
\]

\[
= E^x[h^2(B_a)] + E^x[B_a[\int_0^b |\nabla h(B_s)|^2 ds]]
\]

\[
= h^2(x) + E^x[a^{\int_0^b |\nabla h(B_s)|^2 ds} + E^x[B_a[\psi]],
\]

(2.7)

where \( \psi = \int_0^b |\nabla h(B_s)|^2 ds = \int_0^\infty |\nabla h(B_s)|^2 \chi_{[s,\infty)}(b) ds \).

Since \( E^x[E^x[\psi]] = E^x[E^x[\theta_a|\psi|B_a]] = E^x[\theta_a|\psi] \)
and

\[
\theta_a|\psi = \int_0^\infty |\nabla h(B_{a+s})|^2 \cdot \chi_{[a+s,\infty)}(b) ds
\]

\[
= \int_a^b |\nabla h(B_u)|^2 \chi_{[u,\infty)}(b) du = \int_a^b |\nabla h(B_s)|^2 ds \]

we obtain from (2.7) that

\[
E^x[h^2(B_b)] = h^2(x) + E^x[B_a[\int_0^b |\nabla h(B_s)|^2 ds]]
\]

which establishes the induction step of (2.5) and thus completes the proof of Lemma 4.

Let \( V_n, \tau_n \) be as in Lemma 4. Then we let \( B_n \) denote the \( \sigma \)-algebra of subsets of \( \Omega \) generated by the random variables

\( \{B_k, \tau_n \} \) and we define

\[
B = \cap_{n=1}^\infty B_n,
\]

i.e., \( B \) is the tail field of the sequence \( \{B_{\tau_n} \} \).
THEOREM 1. Let \( h \) be a finely harmonic function in a fine domain \( V \subset \mathbb{R}^d \) with a Green function \( G \), and assume that

\[
\int_V |\nabla h(y)|^2 G(x,y) \, dm(y) < \infty \quad \text{for all } x \in V.
\]

Then there exists a function \( h^* \in L^2(\Omega, P^x) \) for all \( x \) such that

\[
\lim_{t \uparrow \tau_V} h(B_t) = h^* \quad \text{a.s. \( \tau^x \)}
\]

and

\[
E^x[(h(B_t^{\uparrow \tau_V}) - h^*)^2] \to 0 \quad \text{as } t \to \infty, \quad \text{for all } x \in V.
\]

We may regard \( h^* \) as a generalized (random) boundary value function of \( h \), in the sense that \( h^* \) is measurable wrt. the tail field \( \mathbb{B} \) and \( h \) is the "harmonic extension" of \( h^* \) to \( V \), i.e.

\[
h(x) = E^x[h^*] \quad \text{for all } x \in V.
\]

Moreover, we have the isometry

\[
E^x[(h^*)^2] = h^2(x) + \int_V |\nabla h(y)|^2 G(x,y) \, dm(y) \quad \text{for all } x \in V.
\]

Proof. Let \( V_n, \tau_n \) be as in Lemma 4. Choose \( n > m \) and \( x \in V \). Then

\[
E^x[h(B_{\tau_n} h(B_{\tau_m})] = E^x[E^x[h(B_{\tau_n})h(B_{\tau_m})|B_{\tau_m}] = E^x[h^2(B_{\tau_m})] \]

Therefore

\[
E^x[(h(B_{\tau_n}) - h(B_{\tau_m}))^2] = E^x[h^2(B_{\tau_n})] - 2E^x[h(B_{\tau_n})h(B_{\tau_m})] + E^x[h^2(B_{\tau_m})] \]

\[
= E^x[h^2(B_{\tau_n})] - E^x[h^2(B_{\tau_m})] = E^x[\int_{\tau_n}^{\tau_m} |\nabla h(B_s)|^2 ds] \]

\[
\leq \int_{V_n \setminus V_m} |\nabla h(y)|^2 G(x,y) \, dm(y) \to 0 \quad \text{as } m, n \to \infty.
\]
So the sequence of functions

\[ h_n = h(B_{\tau_n}) \]

converges in \( L^2(\Omega, P^X) \) to a function \( h^* \in L^2(\Omega, P^X) \).

In particular,

\[ h(x) = \lim_{n \to \infty} E^X[h(B_{\tau_n})] = E^X[h^*] \]

and

\[ E^X[(h^*)^2] = \lim_{n \to \infty} E^X[h^2(B_{\tau_n})] = h^2(x) + \int_V |\nabla h(y)|^2 G(x, y) \, dm(y) \]

by Lemma 1 and Lemma 4.

Moreover,

\[ (2.12) \quad h_n \to h^* \text{ a.s. wrt. } P^X. \]

Choose \( y \in V \). Then by the Harnack inequalities \( P^Y|B_n^\tau \) is boundedly (uniformly in \( n \)) absolutely continuous wrt. \( P^X|B_n^\tau \), if \( n \) is large enough.

So

\[ h_n \to h^* \text{ in } L^2(\Omega, P^Y) \text{ as well,} \]

and we have proved (2.10) and (2.11).

It remains to establish (2.8) and (2.9):

For all \( t \geq 0 \) and \( n \in \mathbb{N} \) we get, as before

\[ (2.13) \quad E^X[h^2(B_{t\tau_n}^\tau)] = h^2(x) + E^X[\int_0^{t\tau_n} |\nabla h(B_s)|^2 \, ds]. \]

The same procedure as above gives, for \( n > m \),

\[ E^X[h(B_{t\tau_n}^\tau) - h(B_{t\tau_m}^\tau)] = E^X[\int_{t\tau_m}^{t\tau_n} |\nabla h(B_s)|^2 \, ds] \to 0. \]

So letting \( n \to \infty \) in (2.13) we obtain, using (2.12)

\[ E^X[h^2(B_{t\tau}^\tau)] = h^2(x) + E^X[\int_0^{t\tau} |\nabla h(B_s)|^2 \, ds], \]

where \( h(B_{t\tau}^\tau) \) is interpreted as \( h^* \) if \( t = \tau \).
Again the same procedure as above gives that, for $t > s$,

$$
E^x[(h(B_{t\tau \Delta t}) - h(B_{s\tau \Delta t}))^2] = E^x[\int_{s\tau \Delta t}^{t\tau \Delta t} |\nabla h(B_s)|^2 ds] \to 0 \text{ as } s, t \to \tau.
$$

So $(h(B_{t\tau \Delta t}))_t$ converges in $L^2(\Omega, \mathcal{P}^x)$ as $t \to \tau$.

The limit is necessarily equal to $h^*$ and (2.8) and (2.9) follow.

Remark. Theorem 1 raises the following question: When is $h^*$ a genuine boundary function? In other words, when is $h^*$ $B_{\tau V}$-measurable, i.e. of the form $g(B_{\tau V})$ for some function $g \in L^2(\partial V, \lambda^x)$?

Any function of the form $g(B_{\tau V})$ is $\mathcal{B}$-measurable (since $B_{\tau V} = \lim_{n \to \infty} B_{\tau V_n}$ a.s.), but in general the family of $\mathcal{B}$-measurable functions may also contain functions which are not of this type.

For example, if

$$
V = \{(x_1, x_2); x_1^2 + x_2^2 < 1\} \setminus \{(x_1,0); x_1 \leq 0\} \subset \mathbb{R}^2
$$

and

$$
h(x_1, x_2) = \text{Arg}(x_1 + ix_2) = \text{Im} (\log(x_1 + ix_2)) ; \quad (x_1, x_2) \in V
$$

then $h$ has different boundary values as $B_{\tau V}$ approach a point $(x_1, x_2)$ on the negative real axis from above or below. So $h^*$ is not $B_{\tau V}$-measurable in this case.
**Theorem 2.** Let $h$ be a finely harmonic function on a fine domain $V \subset \mathbb{R}^d$ with a Green function $G$.

Then the following are equivalent:

(i) \[ \int_V |\nabla h(y)|^2 G(x,y) \, dm(y) < \infty \quad \text{for all } x \in V \]

(ii) There exists a $\mathcal{B}$-measurable function $h^* \in L^2(\Omega,\mathcal{P})$ for all $x$ such that

\[ h(x) = E^X[h^*] \quad \text{for all } x \in V \]

(iii) There exists a number $M < \infty$ such that

\[ E^X[h^2(B_\tau)] < M \]

for all stopping times $\tau < \tau_V$.

**Proof.**

(i) $\implies$ (ii) by Theorem 1

(ii) $\implies$ (iii): Suppose (ii) holds. Let $\tau < \tau_V$ be a stopping time. First assume that $\tau < \tau_n$ for some $n$. Then since $h^*$ is $\mathcal{B}$-measurable,

\[
E^X[h^2(B_\tau)] = E^X[(E^X[h^*])^2] \\
= E^X[(E^X[h^*|B_\tau])^2] \\
= E^X[(E^X[h^*|B_\tau])^2] \\
\leq E^X[(h^*)^2] = M
\]

In the general case we apply the above argument to $\tau \wedge \tau_n$ and obtain $E^X[h^2(B_{\tau \wedge \tau_n})] \leq M$. Letting $n \to \infty$ we get (iii).
(iii) => (i): If we choose $\tau = \tau_n$ as in Lemma 2 we get by Lemma 1

$$M \geq E^X[h^2(B_{\tau_n})] = h^2(x) + \int_{V_n} |\nabla h(y)|^2 G(x,y) \, dm(y),$$

and (i) follows.

This completes the proof of Theorem 2.

**Theorem 3.** Let $U \subset \mathbb{R}^d$ be a fine domain with a Green function $G$ and let $h$ be a finely harmonic function on $V = U \setminus F$, where $F$ is a polar set. Suppose

$$\int_U |\nabla h(y)|^2 G(x,y) \, dm(y) < \infty \quad \text{for all} \quad x \in V.$$  

Then $h$ extends to a finely harmonic function in $U$.

**Proof.** Choose finely open sets $V_n$ as in Lemma 2 such that

$$\bigcup_{n=1}^{\infty} V_n = U \setminus F \setminus K,$$

where $K$ is a polar set. Then by Theorem 1 there exists a $\mathcal{B}$-measurable function $h^* \in L^2(\Omega, \mathcal{F}^X)$ for all $x$ such that

$$h(x) = E^X[h^*] \quad \text{for all} \quad x \in V.$$  

Define

$$\tilde{h}(x) = E^X[h^*] ; x \in U.$$  

We claim that $\tilde{h}$ is finely harmonic in $U$.

To see this choose $x \in U$ and a fine neighbourhood $D$ of $x$ such that $\overline{D} \subset U$. Let $T$ be the first exit time from $D$. Since $K \cup F$ is polar we must have $T < \tau_n$ for some $n$. Hence
since $h^*$ is $\mathcal{B}$-measurable we get by the strong Markov property
\[
\tilde{h}(x) = E^x[h^*] = E^x[E^x[h^*|B_T]]
\]
\[
= E^x[E^T[h^*]] = \int_{\partial D} \tilde{h}(z) d\lambda^D(x) ,
\]
so that $\tilde{h}$ satisfies the required mean value property.

As pointed out to me by B. Fuglede it is possible to give a stronger, pointwise version of Theorem 3 by combining Theorem 3 with Theorem 2.4 in [9], mentioned in the introduction:

**THEOREM 4.** Let $U$ be as in Theorem 3 and let $h$ be a finely harmonic function on $U \setminus F$, where $F$ is a polar set. Suppose

\[
(2.15) \quad \int_U |\nabla h(y)|^2 G(x_0, y) dm(y) < \infty
\]

for some point $x_0 \in F$.

Then $h$ extends to a finely harmonic function in $U \setminus (F \setminus \{x_0\})$.

**COROLLARY.** Let $U$ be as in Theorem 3 and let $h$ be a finely harmonic function in $U \setminus \{x_0\}$, where $x_0$ is some point in $U$.

Suppose (2.15) holds. Then $h$ extends to a finely harmonic function in $U$.

**Remarks.** 1) Note that Theorem 3 contains Theorem 9.15 in Fuglede [6], because if $h$ is bounded in $V$ then (2.14) holds, by Lemma 1 and Lemma 4.

2) Consider the special case of an ordinary harmonic function $h$ on a domain (in the ordinary topology) $V$ in $\mathbb{R}^d$. Then the conclusions of Theorems 1, 2 and 3 hold in particular
if we replace the condition (1.2) by (1.1), since - as noted in
the introduction - (1.1) implies (1.2) in that case.
In Theorem 3 we must add the assumption that \( F \) is relatively
closed (a polar set is always finely closed).

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