FINELY HARMONIC FUNCTIONS WITH BOUNDED DIRICHLET INTEGRAL WITH RESPECT TO THE GREEN MEASURE

Bernt Øksendal

Abstract.

We consider finely harmonic functions h on a fine, Greenian domain $V \subset \mathbb{R}^d$ with bounded Dirichlet integral wrt. Gm, i.e.

(*)
$$\int_V |\nabla h(y)|^2 G(x,y)\,dm(y) < \infty \text{ for } x \in V,$$

where m denotes the Lebesgue measure, $G(x,y)$ the Green function. We use Brownian motion and stochastic calculus to prove that such functions h always have boundary values h^* along a.a. Brownian paths. This partially extends results by Doob, Brelot and Godefroid, who considered ordinary harmonic functions with bounded Dirichlet integral wrt. m and Green lines in stead of Brownian paths.

As a consequence og Theorem 1 we obtain several properties equivalent to (*), one of these being that h is the harmonic extension to V of a random "boundary" function h^* (of a certain type), i.e. $h(x) = E^x[h^*]$ for all $x \in V$. Another application is that the polar sets are removable singularity sets for finely harmonic functions satisfying (*). This is in contrast with the situation for finely harmonic functions with bounded Dirichlet integral wrt. m.

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§1. Introduction and statement of results

Properties of harmonic functions with bounded Dirichlet integral have been studied by several authors. In 1962 Doob [4], extending earlier works by Brelot and Godefroid, proved that a harmonic function h on a domain V in $\mathbb{R}^d$ $(d \geq 2)$ admitting a Green function and with a bounded Dirichlet integral, i.e.

$$(1.1) \qquad \int_V |\nabla h|^2 dm < \infty$$

(where m denotes Lebesgue measure in $\mathbb{R}^d$) always has a fine boundary function h^* and $h \to h^*$ along the Green lines of V. Doob (and Brelot and Godefroid) used a measure on the space of all Green lines.

In this article we use Brownian motion and stochastic calculus to prove a result of this type and establish a corresponding L^2-isometry (Theorem 1) in the more general situation when h is a <u>finely</u> harmonic function on a <u>fine</u> domain V in $\mathbb{R}^d$ with a Green function G. The assumption that h has a finite Dirichlet integral is replaced by the assumption that

$$(1.2) \qquad \int_V |\nabla h(y)|^2 G(x,y)\,dm(y) < \infty \quad \text{for all } x \in V ,$$

i.c. that h has a finite Dirichlet integral wrt. the Green measure. (It is known (Debiard and Gaveau [2]) that ∇h exists a.e. wrt. m on V.)

In the case when h is harmonic in the ordinary sense on an ordinary Greenian domain V then (1.1) is a stronger assumption than (1.2), because $G(x,y) \to 0$ as $y \to \partial V$ (the boundary of V) and the singularity of $G(x,y)$ at $y = x$ is m-integrable. In the general <u>fine</u> situation it turns out that

<u>(1.1) implies that (1.2) holds quasi-everywhere</u>,

i.e. everywhere outside some polar set.

To see this let W be a bounded subset of V and assume that (1.1) holds. Then by the Fubini theorem

$$\int_W \left(\int_V |\nabla h(y)|^2 G(x,y)\,dm(y)\right) dm(x) = \int_V |\nabla h(y)|^2 \left(\int_W G(x,y)\,dm(x)\right) dm(y)$$

$$< \infty\,, \quad \text{since} \quad \sup_y \left(\int_W G(x,y)\,dm(x)\right) < \infty\,.$$

So (1.2) holds for a.a. $x \in W$ wrt. m.

In particular, the function $H(x) = \int_V |\nabla h(y)|^2 G(x,y)\,dm(y)$ is not infinite everywhere in V. But then it follows from Theorem 2.4 in Fuglede [9] that $H(x)$ is a fine potential in V and therefore finite quasi-everywhere, as asserted.

As a consequence of Theorem 1 we obtain several properties equivalent to (1.2), one of these being that h is the harmonic extension to V of a random function h^* (of a certain type), i.e. $h(x) = E^x[h^*]$ for all $x \in V$ (Theorem 2). Another application is that the polar sets are removable singularity sets for a finely harmonic function h satisfying (1.2) (Theorem 3). This result is in contrast with the situation for finely harmonic functions h satisfying (1.1). In this case it is known that polar sets need not be removable singularity sets (see Fuglede [8], Théorème 12 and p. 153). Thus the condition (1.1) does not imply (1.2) in general.

§2. Boundary behaviour and removable singularity sets

In the following $B_t(\omega)$, $\omega \in \Omega$, $t \geq 0$ will denote Brownian motion in $\mathbb{R}^d$ $(d \geq 2)$. The probability law of B_t starting at $x \in \mathbb{R}^d$ is denoted by P^x and E^x is the expectation operator wrt. P^x.

For a finely open set $V \subset \mathbb{R}^d$ we will let $\tau_V = \inf\{t > 0 \,;\, B_t \notin V\}$ be the first exit time from V ($\tau_V = \infty$ if $B_t \in V$ for all $t > 0$). If $\tau_V < \infty$ a.s. <u>the harmonic measure</u> λ_x^V at x wrt. V is defined by

$$(2.1) \qquad \int_{\partial V} f d\lambda_x^V = E^x[f(B_{\tau_V})] \,,$$

if f is a bounded, continuous real function on ∂V, the boundary of V.

The <u>Green function</u> $G(x,y)$ of a fine domain $V \subset \mathbb{R}^d$ is defined by

$$G(x,y)dm(y) = \int_0^\infty P^x[B_s \in dy,\ s < \tau_V]ds,$$

provided the integral converges.
Intuitively, $G(x,y)dm(y)$ is the expected length of time Brownian motion starting at x stays in $dm(y)$ before it exits from V. See Chung [1] for more information.

<u>LEMMA 1</u>. Let h be a finely harmonic function in a finely open set $V \subset \mathbb{R}^d$ with a Green function G. Let τ_V be the first exit time from V. Then

$$(2.2) \qquad E^x[\int_0^{\tau_V} |\nabla h(B_s)|^2 ds] = \int_V |\nabla h(y)|^2 G(x,y)dm(y)$$

for all $x \in V$.

Proof. By the Fubini theorem we have (χ denotes the indicator function)

$$E^x\left[\int_0^{\tau_V} |\nabla h(B_s)|^2 ds\right] = E^x\left[\int_0^{\infty} |\nabla h(B_s)|^2 \chi_{[0,\tau_V)}(s)ds\right]$$

$$= \int_0^{\infty}\left(\int_V |\nabla h(y)|^2 \cdot P^x[B_s \in dy\ ,\ s < \tau_V]\right)ds$$

$$= \int_V |\nabla h(y)|^2 \left(\int_0^{\infty} P^x[B_s \in dy\ ,\ s < \tau_V]ds\right) = \int_V |\nabla h(y)|^2 G(x,y)dm(y),$$

which proves Lemma 1.

LEMMA 2. Let f be a real, finely continuous function on $\mathbb{R}^d$. Then

$$t \to f(B_t(\omega))$$

is continuous on $[0,\infty)$, for a.a. $\omega \in \Omega$.

Proof. By Theorem 3.5.1 in Chung [1] the function $t \to f(B_t(\omega))$ is right continuous on $[0,\infty)$, a.s. Left continuity follows by the same argument as in the proof of Theorem 4.5.9 in the same book: Choose $c > 0$ and define the reverse process

$$\tilde{B}_t = \begin{cases} B_{c-t} & \text{for } 0 \leq t \leq c \\ B_0 + B_t - B_c & \text{for } c < t \end{cases}$$

Then $\tilde{B}_t$ is again a Brownian motion, so $t \to f(\tilde{B}_t)$ is right continuous, a.s. Since this holds for all $c > 0$ the function $t \to f(B_t)$ is left continuous, a.s.

LEMMA 3. Let $U \subset \mathbb{R}^d$ be finely open and let τ be a stopping time. Then for a.a. ω we have:

If $B_\tau(\omega) \in U$ then there exists $\varepsilon > 0$ such that $B_t(\omega) \in U$ for all $t \in (\tau(\omega) - \varepsilon\ ,\ \tau(\omega))$.

Proof. Since the fine topology is completely regular we can for each $x \in U$ find a finely continuous function $y \to f_x(y)$ on $\mathbb{R}^d$ such that $0 \leq f_x \leq 1$, $f_x \equiv 1$ on $\mathbb{R}^d \setminus U$ and $f_x(x) = 0$. Let $D_x \subset U$ be a fine neighbourhood of x such that $f_x < \frac{1}{2}$ on D_x. The family $\{D_x\}_{x \in U}$ covers U, so by Doob's quasi-Lindelöf principle ([3]) we can find a countable subfamily $\{D_{x_k}\}_{k=1}^{\infty}$ such that

$$K = U \setminus \bigcup_{k=1}^{\infty} D_{x_k}$$

is polar. Put

$$f = \sum_{k=1}^{\infty} 2^{-k} f_{x_k} .$$

Then f is finely continuous, $f \equiv 1$ on $\mathbb{R}^d \setminus U$ and $f < 1$ on $U \setminus K$. Assume $B_\tau \in U$. Since K is polar $B_\tau \notin K$ and therefore $f(B_\tau) < 1$, a.s. By Lemma 2 $t \to f(B_t)$ is continuous a.s. So for a.a. ω there exists $\varepsilon > 0$ such that $f(B_t) < 1$ for $\tau - \varepsilon < t < \tau$. This implies that $B_t \in U$ for $\tau - \varepsilon < t < \tau$ and Lemma 3 is proved.

LEMMA 4. Let h be a finely harmonic function in a fine domain $V \subset \mathbb{R}^d$. Then there exists an increasing sequence of fine bounded domains $V_n \subset V$ such that with $\tau_n = \tau_{V_n}$ we have

$$\tau_n \uparrow \tau_V \quad \text{a.s. as } n \to \infty \tag{2.3}$$

and

$$E^x[h^2(B_{\tau_n})] = h^2(x) + E^x[\int_0^{\tau_n} |\nabla h(B_s)|^2 ds] < \infty \tag{2.4}$$

for all n and all $x \in V$.

Proof. Choose $x \in V$. Then there exists a fine bounded neighbourhood $U_x \ni x$ with compact closure $\bar{U}_x \subset V$ and a sequence of functions h_n harmonic (in the ordinary sense) in a neighbourhood of $\bar{U}_x$ such that $h_n \to h$ uniformly on $\bar{U}_x$.

(Fuglede [7], Theorem 4.1.)

Put $\tau = \tau_{U_x}$. Then by Ito's formula

$$h_n(B_\tau) - h_n(x) = \int_0^\tau \nabla h_n(B_s)dB_s \quad \text{for all} \quad n .$$

So by the basic isometry for Ito integrals

$$E^x[(h_n(B_\tau)-h_n(x))^2] = E^x[\int_0^\tau |\nabla h_n(B_s)|^2 ds] ,$$

i.e.

$$E^x[h_n^2(B_\tau)] = h_n^2(x) + E^x[\int_0^\tau |\nabla h_n(B_s)|^2 ds] ,$$

since $\quad E^x[h_n(B_\tau)] = h_n(x) , \quad$ for all n .

Letting $n \to \infty$ we obtain, using Lemma 1,

$$E^x[h^2(B_\tau)] = h^2(x) + E^x[\int_0^\tau |\nabla h(B_s)|^2 ds] < \infty .$$

The family $\{U_x\}_{x \in V}$ covers V , so by Doob's quasi-Lindelöf principle [3] we can find a countable subfamily denoted by $\{W_n\}$ such that

$$\bigcup_{n=1}^{\infty} W_n = V \setminus K ,$$

where K is a polar set. Now define

$$V_n = \bigcup_{k=1}^{n} W_k \quad ; \quad n = 1, 2, \ldots .$$

Since K is polar (2.3) holds.

We prove (2.4) by induction: The argument above proves that (2.4) holds for $n = 1$. To prove the induction step assume that it holds for $n = k$. Put $S_0 = \tau_k$, $T = \tau_{k+1}$ $(= \tau_{V_k \cup W_{k+1}})$. Define

$$S_1 = \inf\{t > S_0 \; ; \; B_t \notin W_{k+1}\}$$
$$S_2 = \inf\{t > S_1 \; ; \; B_t \notin V_k\}$$

and inductively

$$S_{2j+1} = \inf\{t > S_{2j} \; ; \; B_t \notin W_{k+1}\}$$
$$S_{2j+2} = \inf\{t > S_{2j+1} \; ; \; B_t \notin V_k\} \quad ; \quad j = 0, 1, 2, \ldots .$$

Then $\{S_j\}$ is an increasing sequence of stopping times. Since $S_j \leq T < \infty$ a.s. the limit

$$S = \lim_{j\to\infty} S_j$$

exists a.s. and $S \leq T$.
Since $B_{S_{2j+1}} \in \partial_f W_{k+1}$ for all j (∂_f denotes fine boundary) we must have $B_S \notin W_{k+1}$ a.s., by Lemma 3.
Similarly $B_S \notin V_k$ a.s. Thus $S \geq T$ and therefore $S = T$.

Therefore it suffices to prove that

$$\text{(2.5)} \qquad E^x[h^2(B_{S_j})] = h^2(x) + E^x[\int_0^{S_j} |\nabla h(B_s)|^2 ds] \quad \text{for all } j .$$

For if (2.5) is established then the induction step of (2.4) follows by bounded convergence if we let $j \to \infty$. (Recall that h is bounded on V_{k+1}) .

We establish (2.5) by induction on j . The strong Markov property states that if τ is a stopping time and η is measurable wrt. $\{B_s \; ; \; s \geq 0\}$, then

$$\text{(2.6)} \qquad E^x[\theta_\tau \eta | B_\tau] = E^{B_\tau}[\eta] ,$$

where θ_t is the shift operator:

$$\theta_t(g_1(B_{t_1})\ldots g_i(B_{t_i})) = g_1(B_{t_1+t})\ldots g_i(B_{t_i+t}) .$$

(See Dynkin [5], Theorem 3.11, p. 100 or Øksendal [10], (7.15).)

Assume (2.5) holds for a given j. For simplicity put $a = S_j$, $b = S_{j+1}$. Then, using (7.16) in [8]

$$E^x[h^2(B_b)] = E^x[E^x[h^2(B_b)|B_a]] = E^x[E^{B_a}[h^2(B_b)]]$$

$$= E^x[h^2(B_a) + E^{B_a}[\int_0^b |\nabla h(B_s)|^2 ds]]$$

$$(2.7) \qquad = h^2(x) + E^x[\int_0^a |\nabla h(B_s)|^2 ds] + E^x[E^{B_a}[\psi]],$$

where $\psi = \int_0^b |\nabla h(B_s)|^2 ds = \int_0^\infty |\nabla h(B_s)|^2 \chi_{[s,\infty)}(b) ds$.

Since $E^x[E^{B_a}[\psi]] = E^x[E^x[\theta_a\psi|B_a]] = E^x[\theta_a\psi]$

and

$$\theta_a\psi = \int_0^\infty |\nabla h(B_{a+s})|^2 \cdot \chi_{[a+s,\infty)}(b) ds$$

$$= \int_a^\infty |\nabla h(B_u)|^2 \chi_{[u,\infty)}(b) du = \int_a^b |\nabla h(B_s)|^2 ds ,$$

we obtain from (2.7) that

$$E^x[h^2(B_b)] = h^2(x) + E^x[\int_0^b |\nabla h(B_s)|^2 ds] ,$$

which establishes the induction step of (2.5) and thus completes the proof of Lemma 4.

Let V_n, τ_n be as in Lemma 4. Then we let $\mathcal{B}_n$ denote the σ-algebra of subsets of Ω generated by the random variables $\{B_{\tau_k} ; k \geq n\}$ and we define

$$\mathcal{B} = \bigcap_{n=1}^\infty \mathcal{B}_n ,$$

i.e. $\mathcal{B}$ is the tail field of the sequence $\{B_{\tau_n}\}$.

<u>THEOREM 1</u>. Let h be a finely harmonic function in a fine domain $V \subset \mathbb{R}^d$ with a Green function G, and assume that

$$\int_V |\nabla h(y)|^2 G(x,y)\,dm(y) < \infty \quad \text{for all } x \in V .$$

Then there exists a function $h^* \in L^2(\Omega, P^x)$ for all x such that

$$\lim_{t \uparrow \tau_V} h(B_t) = h^* \quad \text{a.s. } P^x \tag{2.8}$$

and

$$E^x[(h(B_{t\wedge\tau_V}) - h^*)^2] \to 0 \quad \text{as } t \uparrow \infty , \quad \text{for all } x \in V . \tag{2.9}$$

We may regard h^* as a generalized (random) boundary value function of h, in the sense that h^* is measurable wrt. the tail field $\mathcal{B}$ and h is the "harmonic extension" of h^* to V, i.e.

$$h(x) = E^x[h^*] \quad \text{for all } x \in V . \tag{2.10}$$

Moreover, we have the isometry

$$E^x[(h^*)^2] = h^2(x) + \int_V |\nabla h(y)|^2 G(x,y)\,dm(y) \quad \text{for all } x \in V . \tag{2.11}$$

<u>Proof</u>. Let V_n, τ_n be as in Lemma 4. Choose $n > m$ and $x \in V$. Then

$$E^x[h(B_{\tau_n})h(B_{\tau_m})] = E^x[E^x[h(B_{\tau_n})h(B_{\tau_m}) | B_{\tau_m}]]$$

$$= E^x[h(B_{\tau_m})E^x[h(B_{\tau_n}) | B_{\tau_m}]]$$

$$= E^x[h^2(B_{\tau_m})] .$$

Therefore

$$E^x[(h(B_{\tau_n}) - h(B_{\tau_m}))^2] = E^x[h^2(B_{\tau_n})] - 2E^x[h(B_{\tau_n})h(B_{\tau_m})] + E^x[h^2(B_{\tau_m})]$$

$$= E^x[h^2(B_{\tau_n})] - E^x[h^2(B_{\tau_m})] = E^x[\int_{\tau_m}^{\tau_n} |\nabla h(B_s)|^2 ds]$$

$$\leq \int_{V_n \setminus V_m} |\nabla h(y)|^2 G(x,y)\,dm(y) \to 0 \quad \text{as } m, n \to \infty .$$

So the sequence of functions

$$h_n = h(B_{\tau_n})$$

converges in $L^2(\Omega,P^x)$ to a function $h^* \in L^2(\Omega,P^x)$.

In particular,

$$h(x) = \lim E^x[h(B_{\tau_n})] = E^x[h^*]$$

and

$$E^x[(h^*)^2] = \lim_{n\to\infty} E^x[h^2(B_{\tau_n})] = h^2(x) + \int_V |\nabla h(y)|^2 G(x,y)\,dm(y)$$

by Lemma 1 and Lemma 4.

Moreover,

(2.12) $\quad h_n \to h^*$ a.s. wrt. P^x .

Choose $y \in V$. Then by the Harnack inequalities $P^y|\mathcal{B}_n$ is boundedly (uniformly in n) absolutely continuous wrt. $P^x|\mathcal{B}_n$, if n is large enough.

So

$$h_n \to h^* \text{ in } L^2(\Omega,P^y) \text{ as well,}$$

and we have proved (2.10) and (2.11).

It remains to establish (2.8) and (2.9):

For all $t \geq 0$ and $n \in N$ we get, as before

(2.13) $$E^x[h^2(B_{t\wedge\tau_n})] = h^2(x) + E^x[\int_0^{t\wedge\tau_n} |\nabla h(B_s)|^2 ds] .$$

The same procedure as above gives, for $n > m$,

$$E^x[h(B_{t\wedge\tau_n}) - h(B_{t\wedge\tau_m}))^2] = E^x[\int_{t\wedge\tau_m}^{t\wedge\tau_n} |\nabla h(B_s)|^2 ds] \to 0 .$$

So letting $n \to \infty$ in (2.13) we obtain, using (2.12)

$$E^x[h^2(B_{t\wedge\tau})] = h^2(x) + E^x[\int_0^{t\wedge\tau} |\nabla h(B_s)|^2 ds] ,$$

where $h(B_{t\wedge\tau})$ is interpreted as h^* if $t = \tau$.

Again the same procedure as above gives that, for $t > s$,

$$E^x[(h(B_{t\wedge\tau}) - h(B_{s\wedge\tau}))^2] = E^x[\int_{s\wedge\tau}^{t\wedge\tau} |\nabla h(B_s)|^2 ds] \to 0 \quad \text{as} \quad s, t \to \tau .$$

So $\{h(B_{t\wedge\tau})\}_t$ converges in $L^2(\Omega, P^x)$ as $t \to \tau$.
The limit is necessarily equal to h^* and (2.8) and (2.9) follow.

<u>Remark</u>. Theorem 1 raises the following question: When is h^* a genuine boundary function? In other words, when is h^* B_{τ_V}-measurable, i.e. of the form $g(B_{\tau_V})$ for some function $g \in L^2(\partial V, \lambda_x)$?
Any function of the form $g(B_{\tau_V})$ is $\mathcal{B}$-measurable (since $B_{\tau_V} = \lim_{n\to\infty} B_{\tau_n}$ a.s.), but in general the family of $\mathcal{B}$-measurable functions may also contain functions which are not of this type.
For example, if

$$V = \{(x_1,x_2) ; x_1^2 + x_2^2 < 1\} \setminus \{(x_1,0) ; x_1 \leq 0\} \subset \mathbb{R}^2$$

and

$$h(x_1,x_2) = \mathrm{Arg}(x_1+ix_2) = \mathrm{Im}(\log(x_1+ix_2)) \quad ; \quad (x_1,x_2) \in V$$

then h has different boundary values as B_t approach a point (x_1,x_2) on the negative real axis from above or below. So h^* is not B_{τ_V}-measurable in this case.

THEOREM 2. Let h be a finely harmonic function on a fine domain $V \subset \mathbb{R}^d$ with a Green function G.

Then the following are equivalent:

(i) $\int_V |\nabla h(y)|^2 G(x,y)dm(y) < \infty$ for all $x \in V$

(ii) There exists a $\mathcal{B}$-measurable function $h^* \in L^2(\Omega, P^x)$ for all x such that

$$h(x) = E^x[h^*] \quad \text{for all} \quad x \in V$$

(iii) There exists a number $M < \infty$ such that

$$E^x[h^2(B_\tau)] < M$$

for all stopping times $\tau < \tau_V$.

Proof.

(i) => (ii) by Theorem 1

(ii) => (iii): Suppose (ii) holds. Let $\tau < \tau_V$ be a stopping time. First assume that $\tau < \tau_n$ for some n. Then since h^* is $\mathcal{B}$-measurable

$$E^x[h^2(B_\tau)] = E^x[(E^{B_\tau}[h^*])^2]$$

$$= E^x[(E^x[\theta_\tau h^* | B_\tau])^2]$$

$$= E^x[(E^x[h^* | B_\tau])^2]$$

$$\leq E^x[(h^*)^2] = M$$

In the general case we apply the above argument to $\tau \wedge \tau_n$ and obtain $E^x[h^2(B_{\tau\wedge\tau_n})] \leq M$. Letting $n \to \infty$ we get (iii).

(iii) => (i): If we choose $\tau = \tau_n$ as in Lemma 2 we get by Lemma 1

$$M \geq E^x[h^2(B_{\tau_n})] = h^2(x) + \int_{V_n} |\nabla h(y)|^2 G(x,y)\,dm(y) ,$$

and (i) follows.

This completes the proof of Theorem 2.

THEOREM 3. Let $U \subset \mathbb{R}^d$ be a fine domain with a Green function G and let h be a finely harmonic function on $V = U \setminus F$, where F is a polar set. Suppose

$$(2.14) \qquad \int_U |\nabla h(y)|^2 G(x,y)\,dm(y) < \infty \quad \text{for all } x \in V .$$

Then h extends to a finely harmonic function in U.

Proof. Choose finely open sets V_n as in Lemma 2 such that

$$\bigcup_{n=1}^{\infty} V_n = U \setminus F \setminus K ,$$

where K is a polar set. Then by Theorem 1 there exists a $\mathcal{B}$-measurable function $h^* \in L^2(\Omega, P^x)$ for all x such that

$$h(x) = E^x[h^*] \quad \text{for all } x \in V .$$

Define

$$\tilde{h}(x) = E^x[h^*] \; ; \; x \in U .$$

We claim that $\tilde{h}$ is finely harmonic in U.

To see this choose $x \in U$ and a fine neighbourhood D of x such that $\bar{D} \subset U$. Let T be the first exit time from D. Since $K \cup F$ is polar we must have $T < \tau_n$ for some n. Hence since h^* is $\mathcal{B}$-measurable we get by the strong Markov property

$$\tilde{h}(x) = E^x[h^*] = E^x[E^x[h^*|B_T]]$$
$$= E^x[E^{B_T}[h^*]] = \int_{\partial D} \tilde{h}(z)\,d\lambda_x^D(x) \quad ,$$

so that $\tilde{h}$ satisfies the required mean value property.

As pointed out to me by B. Fuglede it is possible to give a stronger, pointwise version of Theorem 3 by combining Theorem 3 with Theorem 2.4 in [9], mentioned in the introduction:

THEOREM 4. Let U be as in Theorem 3 and let h be a finely harmonic function on $U \setminus F$, where F is a polar set. Suppose

(2.15) $$\int_U |\nabla h(y)|^2 G(x_0,y)\,dm(y) < \infty$$

for some point $x_0 \in F$.

Then h extends to a finely harmonic function in $U \setminus (F \setminus \{x_0\})$.

COROLLARY. Let U be as in Theorem 3 and let h be a finely harmonic function in $U \setminus \{x_0\}$, where x_0 is some point in U. Suppose (2.15) holds. Then h extends to a finely harmonic function in U.

Remarks. 1) Note that Theorem 3 contains Theorem 9.15 in Fuglede [6], because if h is bounded in V then (2.14) holds, by Lemma 1 and Lemma 4.

2) Consider the special case of an ordinary harmonic function h on a domain (in the ordinary topology) V in $\mathbb{R}^d$. Then the conclusions of Theorems 1, 2 and 3 hold in particular

if we replace the condition (1.2) by (1.1), since - as noted in the introduction - (1.1) implies (1.2) in that case.

In Theorem 3 we must add the assumption that F is relatively closed (a polar set is always finely closed).

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