Plücker conditions on plane rational curves

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with appendix by Stein Arild Strømme

CONTENTS

§1 Introduction

What are the possible numbers of nodes and ordinary cusps that plane, projective, irreducible and reduced curves of given degrees can have? A bound is given by the Plücker formulas ([1], p.120,154) relating the so called Plücker characters, but it is known that this bound is not the best possible. A great deal of work concerning this problem has been done, for instance by Veronese [2], Lefschetz [3], B.Segre [4] and Zariski [5] (p.219), [6] (p.176,186), but no final result has been found.

If one works over a field of characteristic 0 one may examine the problem by choosing either to study the possible number of singularities on a curve or on its dual curve, but this requires that we have only a certain type of tangentsingularities.

Definition: A reduced, irreducible curve, C, in $\mathbb{P}^2_{\mathbb{C}}$ is said to be a Plücker curve if C and its dual curve have only ordinary cusps and simple nodes as singularities. (An equivalent definition: A curve which Plücker characters do not have to be counted with multiplicity.)

On the other hand, it is clearly an advantage to work with a class of curves that is stable under generalization and dualization. But it is easily seen that the Plücker curves do not satisfy this condition: There exist Plücker curves being the specialization of nonplücker curves in the Hilbert scheme of plane curves of degree d, for some d (we will give examples.)

We will in this paper restrict our study to reduced, irreducible, rational plane curves over $\mathbb{C}$. By means of a correspondence between projections to planes of a fixed normal, rational curve of degree d in $\mathbb{P}^d$ and plane rational curves of degree d, we will define a new set of geometrical conditions stronger than those in the definition of a Plücker curve: This class of rational curves will be stable under generalization (in the set of rational curves in the Hilbert schemes) and dualization. Furthermore this class is in some sense maximal.

Our method will be simply to examine the set of nonplücker curves among the reduced, irreducible plane rational curves of degree d for every d. Taking the closure we will obtain new curves and conditions, and wanting equivalent conditions for a curve and its dual curve we have to add the dual of the new conditions for every possible d. These have to be treated the same way; by taking closure and examining the dual situation we get even more conditions. The result is that this process stops at this step and we get a finite list of pointconfigurations involving nodes, cusps, flexes, tangents, flextangents and cusptangents that are not allowed for the curves in our class. *)

I do not know whether these conditions also are the right ones for curves of genus ≥ 1.

As an application we prove the known fact that given any set of Plücker characters with genus 0, then there is a plane rational curve possessing these characters. The proof will be a variation over Veronese's outline of proof in [2].

In the appendix S.A Strømme describes the connection between the scheme of parametrizations of rational plane curves of given degree having only nodes and ordinary cusps as singularities and the corresponding locally closed subscheme of the Hilbert scheme. His results are deeply needed in our discussion.

*) The demand on having simple nodes for a Plücker curve may seem to be unnatural. But if we allow nonsimple nodes in the definition, they would all the same occur in the list.

§2. The results

Theorem:

Let $H_d = \mathbb{P}_{\mathbb{C}}^{\binom{d+2}{2}-1}$ be the Hilbert scheme of curves of degree d in $\mathbb{P}^2_{\mathbb{C}}$, and let $R_d \subset H_d$ be the locally closed subset consisting of reduced, irreducible rational curves having only ordinary cusps and nodes as singularities.

Then there exists a family $\{\mathcal{O}_d\}_{d\geq 3}$ of open sets $\mathcal{O}_d \subset R_d$ such that

1) If C is a plane rational curve of degree d, then $C \in \mathcal{O}_d$ if and only if $\check{C} \in \mathcal{O}_{\check{d}}$, where $\check{C}$ is the dual curve of C and $\check{d}$ its degree.

2) The family $\{\mathcal{O}_d\}$ is maximal with the property 1) and such that $\mathcal{O}_d$ does not contain any nonplücker curve and $\mathcal{O}_d$ is open in R_d, $d \geq 3$.

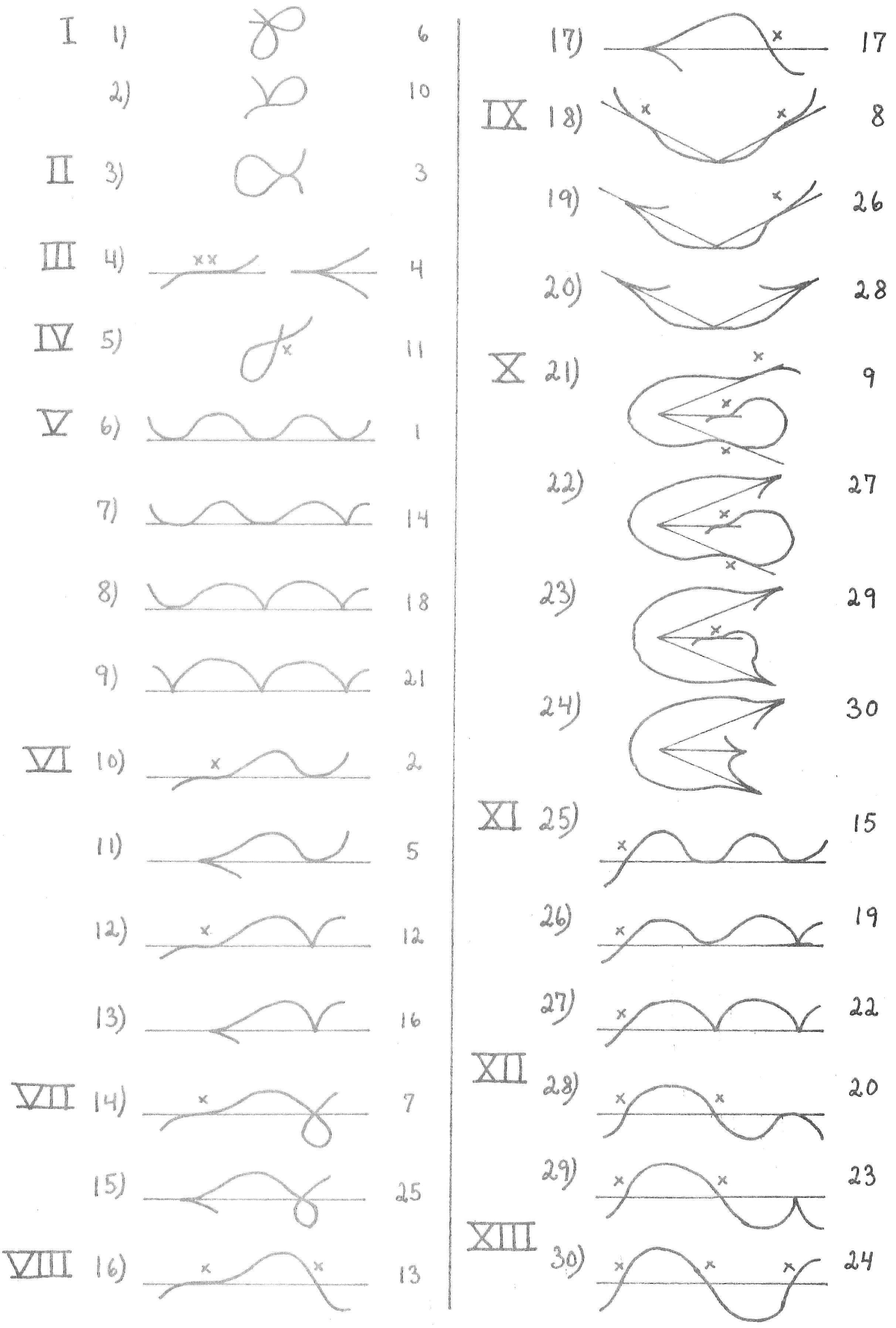
Remarks:

a) We will see that $\mathcal{O}_3 = \emptyset$.

b) There exist Plücker curves that are specializations of nonplücker curves.

We get the remark b) from the Theorem and the remark a) as follows: We have that R_d contains all the rational Plücker curves of degree d, and since every plane irreducible, reduced cubic is a Plücker curve the fact that $\mathcal{O}_3 = \emptyset$ and that the family $\{\mathcal{O}_d\}$ is maximal lead to the existence of a d for which the Plücker curves do not form an open subset of R_d.

More interesting than the pure existence of the family $\{\mathcal{O}_d\}$ are the geometrical properties of the curves in $\mathcal{O}_d$. We will decribe the reduced, irreducible rational curves not in $\mathcal{O}_d$, and the easiest way doing this is setting up the following (symbolic) list. We have 13 families of curves. The first example given in each family represents the characteristic figure of the general member in the family, the


I 1) 6
2) 10
II 3) 3
III 4) 4
IV 5) 11
V 6) 1
7) 14
8) 18
9) 21
VI 10) 2
11) 5
12) 12
13) 16
VII 14) 7
15) 25
VIII 16) 13
17) 17
IX 18) 8
19) 26
20) 28
X 21) 9
22) 27
23) 29
24) 30
XI 25) 15
26) 19
27) 22
XII 28) 20
29) 23
XIII 30) 24

following ones represent as we shall see, specializations. To the right of the drawings is the dual type numbered. A x means a flex. The family III is meant to illustrate higher order flex/cusp.

We will in fact prove the following.

Proposition:

For $d \geq 4$ a reduced, irreducible rational curve of degree d is not in $\mathcal{O}_d$ if and only if it has points and tangents as one or more of the examples in the list. For $d=3$ we have to add the cuspidal cubic to the list.

§3: Proof of the theorem and construction of the list.

3.1 We can sketch the construction in the following way: The Plücker conditions exclude the situations 1-6 and 10, 11 for every d. Taking the closure we will obtain the specializations 7, 8, 9, 12, 13. Wanting equivalent conditions for a curve and its dual curve we must add 14, 16, 18, 21 for every possible d. Taking the closure we obtain the specializations 15, 17, 19, 20, 22, 23, 24. The dual situations of these give 25-30.

We see from the list that part 1) of the theorem is satisfied (proof omitted). Remark a) follows from the fact (for instance using Noether's theorem) that 30) is generical for plane cubics. The main difficulty in proving the theorem will be to get the maximality stated.

3.2 Let V_d be the triples of homogeneous polynomials over $\mathbb{C}$ of degree d in two variables, (t_0, t_1), and $R_d \subset H_d$ as in the theorem. We have a morphism, ψ', from the open subset $X_d' \subset \mathbb{P}(V_d)$ consisting of the triples without a common zero, to H_d in the following way:

Let the monomials of degree d in X, Y, Z be a basis for H_d. If $p=(p_0,p_1,p_2)\in X_d'$, then the resultant $R_p(X,Y,Z) = \mathrm{Res}(Xp_2-Zp_0, Yp_2-Zp_1)$ is of degree $2d$ in X,Y,Z, and by expansion of the determinant it is easily seen that Z^d is a factor of $R_p(X,Y,Z)$. So $\frac{1}{Z^d}R_p(X,Y,Z)\in H_d$, and this defines the map ψ'. Furthermore if $\psi'(p)$ is an irreducible polynomial, then it is the equation of the curve parametrized by p. (It has the right degree and its set of zeros contains the parametrized curve.)

Denote $\psi'^{-1}(R_d)=X_d$, then X_d is an open subset of $\mathbb{P}(V_d)$, and $\lambda=\psi'|_{X_d}: X_d\to R_d$ is a surjection since every rational curve over $\mathbb{C}$ can be parametrized. By the Corollary to the Theorem in the appendix we have.

3.2.1 If $U\subset R_d$ then $\lambda^{-1}(\bar{U}) = \overline{\lambda^{-1}(U)}$

We can identify X_d with an open subset of $\mathbb{P}(M_{3,d+1})$ where $M_{3,d+1}$ is the space of $3x(d+1)$ matrices, namely identifying

$$p=\Big(\sum_{j=0}^{d} a_{ij}t_0^{d-j}t_1^{\,j}\Big)_{i=0,1,2} \quad \text{and} \quad A_p=(a_{ij})_{\substack{i=0,1,2\\ j=0,1,\ldots,d}} .$$

We have $Rk(A_p)=3$ for every $p\in X_d$ so we can define a morphism $g: X_d\to Gr(d-3,d) =$ The Grassmannian of codimension 3 subspaces of $\mathbb{P}^d$, by sending a matrix to its "kernel". Let C be a fixed normal rational curve of degree d in $\mathbb{P}^d$. We can then think of A_p as a projection of C to a $\mathbb{P}^2$ with appropriate chosen coordinates, and then g is just forgetting the $\mathbb{P}^2$ and giving the center of projection.

We have an action of $PGL(3)$ on $\mathbb{P}(M_{3,d+1})$ by left multiplication, and this action restricts to X_d because the properties of the curves in R_d are independent of choice of coordinates. Furthermore the fibres of g are in this way isomorphic to $PGL(3)$.

Let $\mathcal{U}_d$ be the image of g in $Gr(d-3,d)$, then $\mathcal{U}_d$ is open by the definition of the Grassmannian.

Using 3.2.1 and the definition of the Grassmannian we have

Proposition: Let $\lambda: X_d \to R_d$ and $g: X_d \to \mathcal{U}_d$ be as above. If $U \subset \bar{R}_d$ is such that $\lambda^{-1}(U)$ is invariant under the action of $PGL(3)$ then $\bar{U} = \lambda g^{-1}\overline{(g\lambda^{-1}(U))}$

The sets we are going to study in R_d will satisfy the condition of the proposition because, as we shall see, they will be determined by geometrical properties of curves which are independent of projectiv equivalence. Hence the proposition tells us that our topological study of families of curves (as described in 3.1) can be translated to a study of centers of projections to varying $\mathbb{P}^2$'s of a fixed normal rational curve of degree d in $\mathbb{P}^d$.

3.3 Notation:

With d given, let C be a once and for all fixed normal rational curve of degree d in $\mathbb{P}^d$, and let $G(k)$ denote the Grassmannian of linear subspaces of dimension k in $\mathbb{P}^d$. For $p \in C$ and $0 \leq r \leq d-1$ let $\mathbb{P}_p^r$ denote the osculating $\mathbb{P}^r$ for C in p.

The symbols X, Y, Z will be used for resp. codimension $3,2,1$ subspaces of $\mathbb{P}^d$.

3.4 Now we will start the construction of the list. The two first families in the list (I:curves having points of multiplicity ≥ 3 and II:curves having tacnodes) are exceptional because they are outside R_d, so do not have to be treated. We will define a closed set $W_d \subset \mathcal{U}_d$ such that our wanted $\mathcal{O}_d$ will be $\lambda g^{-1}(\mathcal{U}_d \setminus W_d)$. We will examine coincidencemanifolds involving $\mathcal{U}_d$, other Grassmannians and powers of C. When two or more points in a power of C are equal, they are regarded as infinitesimally near. We will have to describe 11 coincidences, and for the moment denoting the projection of the coincidence to $\mathcal{U}_d$ in each case for $W_{d,i}$, $i=1,\ldots,11$, we automatically get $W_{d,i}$ closed, and $W_d = \bigcup_{i=1}^{11} W_{d,i}$.

For some d some of the coincidences will become empty, but then the corresponding dual situation is also nonexistent so this will not contradict the maximality stated in the theorem. So the following construction is taken for every $d \geqq 3$.

III Higher order flex/cusp

A curve has such a point if and only if there is a line intersecting a branch of the curve 4 times in a point. Thinking of the inverse image of the line for an arbitrary projection $\mathbb{P}^d \dashrightarrow \mathbb{P}^2$ of C we can look at

$$\{(X,Z,p) \in \mathcal{U}_d \times G(d-1) \times C \mid X \subset Z \supseteq \mathbb{P}^3_p\}$$

IV Nonsimple nodes

A nonsimple node is a node where one branch intersects its tangent with multiplicty $\geqq 3$.

$p \in C$ is projected to a flex if and only if the center of projection X satisfy $X \cap (\mathbb{P}^2_p \setminus \mathbb{P}^1_p) \neq \emptyset$. Wanting a closed condition we must demand $X \cap \mathbb{P}^2_p \neq \emptyset$, and this is equivalent to the existence of a $\mathbb{P}^{d-1}$ such that $X \subset \mathbb{P}^{d-1} \supset \mathbb{P}^2_p$ so we can look at

$$\{(X,Y,Z,p_1,p_2) \in \mathcal{U}_d \times G(d-2) \times G(d-1) \times C^2 \mid X \subset Y \subset Z \supseteq \mathbb{P}^2_{p_2} \ \& \ p_1,p_2 \in Y\}$$

If $X \cap \mathbb{P}^1_{p_2} \neq \emptyset$ (which gives a cusp for $p_2 \notin X$), then we would have got something treated in I, which is impossible for $X \in \mathcal{U}_d$. If $P_1 = P_2$, then $X \subset Z \supseteq \mathbb{P}^3_{p_2}$ which was treated in III:

This illustrates the specializations we will get: A demand on having a special flex specializes to a cusp. Furthermore a demand on having a certain line as tangent in a point specializes in the point being a cusp: $X \subset \mathbb{P}^{d-1} \supset \mathbb{P}^1_p \to X \cap \mathbb{P}^1_p \neq \emptyset$, and at last a node may specialize in a cusp: $X \subset \mathbb{P}^{d-2} \ni p_1, p_2 \to X \subset \mathbb{P}^{d-2} \supseteq \mathbb{P}^1_{p_1}$, when $p_1 = p_2$, so $X \cap \mathbb{P}^1_{p_1} \neq \emptyset$. We see in the list that these are the specializations we get from the first example in every family, but we have to be a little more careful when examining them because there is more than one point involved.

V Tritangent.

The tangentline corresponds to a $\mathbb{P}^{d-1}$ so we look at

$$\{(X,Z,P_1,P_2,P_3) \in \mathcal{U}_d \times G(d-1) \times C^3 \mid X \subset Z \supset \mathbb{P}^1_{p_1} \cup \mathbb{P}^1_{p_2} \cup \mathbb{P}^1_{p_3}\}$$

But this does not exclude the possibility of X intersecting one or more of the tangents, on the other hand a $\mathbb{P}^{d-3} \subset \mathbb{P}^{d-1}$ intersecting one or more of the tangents is always a specialization of a $\mathbb{P}^{d-3}$ in the $\mathbb{P}^{d-1}$ not intersecting the tangents because the codimension is 2. So we get 7),8) and 9) in the list.

<u>Remark1:</u> A same kind of argument will also work in the remaining families except IX and X, so we will just study these two families w.r.t. specialization.

Furthermore in the tritangent case, if $p_1 = p_2$ then $X \subset Z \supseteq \mathbb{P}^3_{p_1}$, which is treated in III.

VI Flextangent being tangent in another point.

$$\{(X,Z,p_1,p_2) \in \mathcal{U}_d \times G(d-1) \times C^2 \mid X \subset Z \supset \mathbb{P}^2_{p_1} \cup \mathbb{P}^1_{p_2}\}$$

As above $p_1 = p_2$ is treated in III.

We have now described the centers of projection giving non-plücker curves in U_d for every d. Taking the closure and examining the dual situation, we obtain the first examples in the families VII-X which must be studied in the same way for every possible d because of part 1) in the theorem.

Before we go on we need some lemmas:

Lemma 1: Let C be a normal rational curve of degree d in $\mathbb{P}^d$ and $p_1,\ldots,p_s \in C$. If n_i, $i=1,\ldots,s$ are nonnegative integers such that $\sum_{i=1}^{s} (n_i+1) \leq d+1$, then $\mathbb{P}_{p_1}^{n_1},\ldots,\mathbb{P}_{p_s}^{n_s}$ are in general position, i.e. they generate a $\mathbb{P}^{N-1}$ where $N=\sum_{i=1}^{s} (n_i+1)$.

Proof: This is easily seen using that a hyperplane intersects C with multiplicity d.

As a result we have for $d=4$ that two oscullating planes intersect in one and only one point.

Using the language of the previous construction we have

Lemma 2: Let $d \geq 4$, C as in lemma 1. Then two flexes/cusps that coincide give a higher order cusp or the center of projection will intersect C.

Proof: We have for $i=1,2$ $q_i \in X \cap \mathbb{P}_{p_i}^2$, $p_i \to p_0$ and $X \to X_0$ where X is the center of projection. There are two cases:

If the q_i's are generical distinct, then we have a line in X intersecting the two oscullating planes. Going to the limit we get X_0 intersects $\mathbb{P}_{p_0}^2$ in a line, therefore $p_0 \in X_0$ or X_0 gives a higher order cusp.

If $q_1=q_2$, then by lemma 1, $d=4$, and $\{q_1\} = \mathbb{P}_{p_1}^2 \cap \mathbb{P}_{p_2}^2$.

Looking at the dual curve $\check{C}$ in $\check{\mathbb{P}}^4$ given by the oscullating $\mathbb{P}$'s, we see that q_1 corresponds to a $\mathbb{P}^3$ in $\check{\mathbb{P}}^4$, containing the tangents corresponding to $\mathbb{P}^2_{p_1}$ and $\mathbb{P}^2_{p_2}$. Going to the limit q_0 corresponds to an oscullating $\mathbb{P}^3$ for $\check{C}$, but that means $q_0=p_0$.

Eventually by using lemma 2 it is easily seen for the following families in the case $d\geq 4$, that a $X\in U_d$ will give something treated in III or VI if two of the points on C coincide.

Remark 2: For $d=3$ we must examine this situation in the cases X and XIII (which are the only possible nonempty coincidences for $d=3$).

VII Flextangent through a node

$$\{(X,Y,Z,p_1,p_2,p_3)\in \mathcal{U}_d \times G(d-2)\times G(d-1)\times C^3 \mid X\subset Y\subset Z\supseteq \mathbb{P}^2_{p_3} \ \&\ p_1,p_2\in Y\}$$

F.S (Further specializations): 12), 13)

VIII Flextangent through another flex.

$$\{(X,Z,p_1,p_2)\in \mathcal{U}_d \times G(d-1) \times C^2 \mid X\subset Z\supseteq \mathbb{P}^2_{p_1}, p_2 \ \&\ X\cap \mathbb{P}^2_{p_2} \neq \emptyset\}$$

F.S: 12), 13)

IX Two flextangents intersecting in a point on the curve.

$$\{(X,Y,p_1,p_2,p_3)\in \mathcal{U}_d \times G(d-2) \times C^3 \mid X\subset Y\ni p_3 \ \&\ \dim (Y\cap \mathbb{P}_{p_i})\geq 1 \ i=1,2\}$$

Ad remark 1: Fix distinct p_1,p_2,p_3 and a $\mathbb{P}^{d-2}$ as in the definition. If $\mathbb{P}^1_{p_i}\subset \mathbb{P}^{d-2}\cap \mathbb{P}^2_{p_i}$ for $i=1$ or 2 then we get something treated in IV, otherwise let $L_i=\mathbb{P}^2_{p_i}\cap \mathbb{P}^{d-2}$ $i=1,2$. Then a $\mathbb{P}^{d-3}$ intersecting one or both $\mathbb{P}^2_{p_i}$ in the point $L_i\cap \mathbb{P}^1_{p_i}$ is a specialization of a $\mathbb{P}^{d-3}\subset \mathbb{P}^{d-2}$ intersecting $\mathbb{P}^2_{p_i}$ in a point in $L_i\setminus \mathbb{P}^1_{p_i}$.

Hence 14 and 15 are specializations of 13.

X. Three concurrent flextangents

$$\{(X,Y,p_1,p_2,p_3) \in \mathcal{U}_d \times G(d-2) \times C^3 \mid X\subset Y \ \ \& \ \dim(Y\cap \mathbb{P}^2_{p_i}) \geq 1 \ \ i=1,2,3\}$$

Ad remark 2: When $d=3$ the set is empty, because the dual curve would have been of degree 3 or 4 with 3 cusps or 2 cusps and a flex on a line.

Ad remark 1: When $d=4$, fix distinct p_1,p_2,p_3. We shall have a $\mathbb{P}^1$ contained in a $\mathbb{P}^2$ and $\dim(\mathbb{P}^2\cap \mathbb{P}^2_{p_i}) \geq 1$. The $\mathbb{P}^2$ is determined by the three points of intersection between the osculating planes (lemma 1), because if they did not generate a $\mathbb{P}^2$ they would have to be a common point of three osculating planes: Looking at the dual situation in $\check{\mathbb{P}}^4$ this corresponds to a $\mathbb{P}^3$ containing 3 tangents of $\check{C}$, but that is impossible by lemma 1, since $\check{C}$ is a rational curve of degree 4.

Let $\mathcal{U}\subset C^3$ be the triples of distinct points. Then the coincidence over $\mathcal{U}$ is isomorphic to $\mathcal{U} \times \check{\mathbb{P}}^2$, where the 3 points of intersection determine the coordinates in $\mathbb{P}^2$. Therefore the coincidence is irreducible and projecting to $\mathcal{U}_d$ we get an irreducible set. Since the centers of projection giving one or more cusp instead of flex form a proper closed subset of this set, the irreducibility gives us 22, 23, 24, as specializations of 21.

When $d\geq 5$, we may always find a $\mathbb{P}^{d-3}$ through 3 points, and using the same method as in IX we find the remark 1 satisfied.

Now the remaining families come from the dual situations of the specializations above.

XI Bitangent through a flex

$$\{(X,Z,p_1,p_2,p_3)\in \mathcal{U}_d \times G(d-1) \times C^3 \mid X\subset Z\supset p_1, \mathbb{P}^1_{p_2}, \mathbb{P}^1_{p_3} \ \ \& \ X\cap \mathbb{P}^2_{p_1} \neq \emptyset\}$$

F.S: For $d\geq 6$: 7), 8), 9)

XII Tangent through two flexes.

$\{(X,Z,p_1,p_2,p_3) \in \mathcal{U}_d \times G(d-1) \times C^3 \mid X \subset Z \supset \mathbb{P}^1_{p_3}, p_1, p_2 \ \& \ X \cap \mathbb{P}^2_{p_i} \neq \emptyset \ i=1,2\}$

F.S: For $d \geq 5$: 26), 27). For $d \geq 6$: 8), 9)

XIII Three colinear flexes

$\{(X,Z,p_1,p_2,p_3) \in \mathcal{U}_d \times G(d-1) \times C^3 \mid X \subset Z \ni p_i \ \& \ X \cap \mathbb{P}^2_{p_i} \neq \emptyset, \ i=1,2,3\}$

Ad remark 2: When $d=3$ we get an exceptional case since this situation is generical, the specialization being the cuspidal cubic. The dual curve of a cuspidal cubic is itself a cuspidal cubic, so we do not get any new conditions,

F.S: For $d \geq 4$: 29), $d \geq 5$: 27), $d \geq 6$: 9)

The theorem now follows from the construction.

We have not proved independency between these 13 families, but looking at the construction the fact $\mathcal{U}_3 = \emptyset$ leads to the existence of Plücker curves of degree 6 with 3 colinear cusps. Such a curve may for instance be constructed as the dual curve of a Plücker curve of degree 4 with 3 concurrent flextangents and no cusps. An example is the curve parametrized by $p(t) = (t^3(t-4), (t-1)^3(2t+1), t(t-1))$ with flexes for $t=0,1,-2$ giving $(0,1,0), (1,0,0), (16,27,2)$ and flextangents $X=0, Y=0$ and $27X-16Y=0$ all through $(0,0,1)$. By calculation one can prove this is a Plücker curve. Denoting $p=(p_0,p_1,p_2)$, then the dual curve is parametrized by $(p_1p_2' - p_2p_1', \ p_2p_0' - p_0p_2', \ p_0p_1' - p_1p_0')$, and using the resultant-map in 3.2 we may find the equation of the sextic.

§4 The Plücker characters of a rational curve.

4.1 The Plücker formulas for a rational Plücker curve of degree d, class $\check{d}$, with i flexes and κ cusps are

1) $\check{d}=2(d-1)-\kappa$ 2) $i=3(d-2)-2\kappa$

The other Plücker characters are then given by the genusformulas.

The possible Plücker characters are, by 2), given by those (d,κ) where $d>1$ and $0\leq\kappa\leq[\frac{3}{2}(d-2)]$. From the construction of $\{\mathcal{O}_d\}$ we will get.

Proposition:

For every pair of integers (d,κ) with $d\geq 4$, $0\leq\kappa\leq[\frac{3}{2}(d-2)]$ and $(d,\kappa)\neq(4,3)$ there exists a curve in $\mathcal{O}_d$ with κ cusps.

The reason why we have to exclude (4,3) is the fact that the three cusptangents intersect, since the dual curve is of type (3,0) having 3 flexes on a line. Since every curve of type (3,0),(3,1) or (4,3) are Plücker curves, the proposition gives us the existence of rational Plücker curves to every set of Plücker characters with genus 0.

4.2 Proof of the proposition

1) and 2) give, using $i=\check{d}-2+d-2-\kappa$:

4.2.1 $d-2<\kappa\leq\frac{3}{2}(d-2)\Leftrightarrow 0\leq i<\check{d}-2$ and then $\check{d}<d$,

$\kappa=d-2 \Leftrightarrow i=\check{d}-2$ and then $\check{d}=d$.

4.2.1 tells it is sufficient to consider the case, for every d, $\kappa\leq d-2$, because either the curve or the dual curve is of this type. We will not use this directly, but 4.2.1 is crucial since wanting to have κ cusps means, in the sense of §3, choosing a center of projection that intersects C's developable of tangents in κ points, and when $\kappa\leq d-2$ it is always possible to find a dimension $d-3$ subspace containing κ points.

Notation:

We will use the notation of §3: C is a fixed normal rational curve of degree $d\geq 4$ in $\mathbb{P}^d$, and let D be its developable of tangents.

Let $\mathcal{U}_d^1 \subset Gr(d-3)$ be the centers of projection that give rise to birational projections of C, so $\mathcal{U}_d \subset \mathcal{U}_d^1$. Eventually by using the same method as in §3 for the families I and II, one shows that the centers of projection giving curves of type I-XIII form a closed subset of $\mathcal{U}_d^1$, denote this by W_d^1.

We will prove the proposition by induction on d, using the fact that W_d^1 is closed and the conditions are equivalent for a curve and its dual curve. This will reduce the problem to examining the nodal cubic.

Let $Y_k \subset D^k$, $0 \leq k \leq d-2$, be the k-tuples of linear independent points in D, then Y_k is open and dense in D^k.
Let $K_k = \{(X, p_1, \ldots, p_k) \in \mathcal{U}_d^1 \times Y_k \mid p_i \in X, i=1,\ldots,k\}$. We have K_k irreducible because of the linear independence, and letting $\pi : K_k \to \mathcal{U}_d^1$ be the projection, then $\pi(K_k)$ is irreducible. Let $Z_{d,k}$ denote $\overline{\pi(K_k)} \subset \mathcal{U}_d^1$, then $Z_{d,k}$ is irreducible and by counting conditions we get $\dim Z_{d,k} = 3d-6-k$. So we have $Z_{d,0} \supset Z_{d,1} \supset \ldots \supset Z_{d,d-2}$.

Let $Z_{d,d-1}$ denote the projection to $\mathcal{U}_d^1$ of $\{(X, p_1, \ldots, p_{d-1}) \in \mathcal{U}_d^1 \times C^{d-1} \mid X \cap \mathbb{P}^1_{p_i} \neq \emptyset \ i=1,\ldots,k\}$. Then, for instance by looking at explicit parametrizations of plane rational curves, (using that we are in $\mathcal{U}_d^1$ and not necessarily in $\mathcal{U}_d$) we see $Z_{d,d-2} \setminus Z_{d,d-1} \neq \emptyset$.

Suppose we have proved the existence as in the proposition for curves of degree $<d$. Assume there does not exist a curve of degree d with κ cusps having a center of projection outside W_d^1. By induction $\kappa \leq d-2$, otherwise we may dualize and get a contradiction since then $\check{d} < d$ by 4.2.1. So $Z_{d,\kappa} \setminus Z_{d,\kappa+1} \subseteq W_d^1$, and because of the irreducibility we get

4.2.2. $Z_{d,d-2} \subseteq Z_{d,\kappa} = \overline{Z_{d,\kappa} \setminus Z_{d,\kappa+1}} \subseteq W_d^1$.

A curve with κ cusps $0 \leqq \kappa \leqq d-2$ does not have to correspond to a $X \in Z_{d,\kappa}$ since we do not know whether the intersections of a X, that gives the curve, with the developable are in general position or not. But a curve of degree d with the maximal number of cusps, $[\frac{3}{2}(d-2)]$, and of the required type exists by induction since $\check{d}<d$, and for such a curve $d-2$ of the intersections with the developable have to be linear independent: Otherwise we could force one more intersection but that is impossible by the maximality (even if the center would not give a birational projection). This contradicts 4.2.2. that $Z_{d,d-2} \subseteq W_d^1$.

To start the induction for $d=4$, we may use the same method as above to prove the existence for $\kappa=0,1,2$, by looking at the properties of the curves of type $(d,\kappa)=(3,0)$ being the dual of the type $(4,3)$. Using the irreducibility of the $Z_{4,k}$'s, the only thing that does not work as earlier is if all the curves with centers of projection in $Z_{d,d-2}=Z_{4,2}$ are in the family X. But <u>every</u> curve of type $(4,2)$ has a center of projection in $Z_{4,2}$ (only 2 intersections with D), looking at the dual situation, which is also of type $(4,2)$, we get every curve of type $(4,2)$ is the dual type of the family X. A curve of type $(4,2)$ has two flexes so we get the possibilities 27 and 29 in the list. The first one is impossible since the degree is 4, the second one leads to all curves of type $(4,2)$ are in the family XII. But then we may use the same argument as in the inductionstep to get a contradiction.

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Appendix: Families of rational plane curves.

This note is concerned with the relationship between families of parametrized plane rational curves and families of unparametrized plane curves. Let d be a fixed positive integer, and denote by $\underline{X}$ the following contravariant functor on the category of k-schemes:

$\underline{X}(T)$ = set of finite T-morphisms $\phi: P^1_T \to P^2_T$ such that for all geometric points $t \to T$, the fiber ϕ_t of ϕ is birational onto its image, which is a curve of degree d with only ordinary nodes and cusps as singularities.

Clearly $\underline{X}$ is represented by an open subscheme X of projective $(3d+2)$-space. Denote by $\tilde{\phi}: P^1_X \to P^2_X$ the universal family.

Let A_1 be the Hilbert scheme of plane curves of degree d (i.e. projective $\frac{1}{2}d(d+3)$-space), and let $A \subseteq A_1$ be the open subscheme corresponding to irreducible curves with only nodes and cusps as singularities. Since $\tilde{\phi}$ is finite, the formation of $\tilde{\phi}_*(O_{P^1_X})$ commutes with arbitrary base change on X. Hence the closed subscheme $\tilde{C} \subseteq P^2_X$ defined by the zero-th Fitting ideal of $\tilde{\phi}_*(O_{P^1_X})$ defines a morphism $\psi: X \to A$.

<u>Theorem:</u> The morphism ψ factors as follows:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\phi"'] & A \\ Y \arrow[r, "n"'] & R \arrow[u, "i"'] \end{tikzcd}$$

where $R \subseteq A$ is the closed subscheme corresponding to rational curves (with reduced subscheme structure), $n:Y \to R$ is the normalization morphism, and $\phi:X \to Y$ is a principal PGL(2)-bundle. Furthermore, Y is nonsingular and n is a homeomorphism.

Corollary: Put $\lambda = n \circ \phi : X \to R$. For any subset $U \subseteq R$, we have $\lambda^{-1}(\bar{U}) = \overline{\lambda^{-1}(U)}$.

Remark: One may show that n is an isomorphism precisely over the open subset $R_0 \subseteq R$ corresponding to curves without cusps. More precisely, if $r \in R$ corresponds to a curve with γ cusps, the germ of R at r is analytically isomorphic to a product of γ ordinary (1-dimensional) cusps and a smooth part (of dimension $3d-1-\gamma$).

Proof of the theorem: We shall define Y via its functor of points, and later show that it coincides with the normalization of R.

For any A-scheme T, let $C_T \subseteq P^2_T$ be the pullback of the universal family $C_A \subseteq P^2_A$. Consider the following functor $\underline{Y}$ on the category of A-schemes:

$\underline{Y}(T)$ = set of subschemes $S \subseteq C_T$ with the following properties:

(i) S is etale and finite over T of rank $p = \binom{d-1}{2}$.

(ii) $S^{(2)} \subseteq C_T$, where $S^{(2)}$ is the first infinitesimal neighborhood of S in P^2_T (defined by the square of the ideal of S in P^2_T).

Note that condition (ii) is equivalent to the condition (ii)': S is contained in the singular locus of the morphism $C_T \to T$ (defined, for example, by the first Fitting ideal of $\Omega^1_{C_T/T}$).

Clearly $\underline{Y}$ is represented by a locally closed subscheme Y of $\mathrm{Hilb}^p_{P^2} \times A$. I claim that the natural morphism $\nu : Y \to A$ is proper. Indeed, by the valuative criterion for properness, if suffices to complete the following commutative diagram

$$\begin{array}{ccc} T_0 & \longrightarrow & Y \\ \cap| & \nearrow & \downarrow \nu \\ T & \longrightarrow & A \end{array}$$

where T is the spectrum of a discrete valuation ring, and $T_0 = T - \{t\}$, $t \in T$ the closed point. So we are given C_T and $S_{T_0} \subseteq \mathrm{Sing}(C_{T_0}/T_0)$. Put S_T = closure of S_{T_0} in P^2_T. Then S_T is flat and finite over T and condition (ii) holds. It remains only to show that the closed fiber S_t is nonsingular. If not, there are local parameters (u,v) of P^2_t such that $I_{S_t} \subseteq (u^2, v)$. But then $I_{C_t} \subseteq (u^4, u^2 v, v^2)$, contrary the assumption that C_t has only ordinary nodes and cusps.

Now let L_Y be the blowing up of C_Y along S_Y. I claim that L_Y is flat over Y, and that for any base change $Y' \to Y$, the pullback $L_{Y'}$ of L_Y concides with the blowing up of $C_{Y'}$ along $S_{Y'}$. Indeed, the question is local on C_Y (for the etale topology) hence the claim follows from [Wahl, 1.3 and 1.6]. In particular, all the geometric fibers of L_Y are projective lines, and $L_Y \to Y$ is a P^1-bundle. Let $\phi' : X' \to Y$ be the associated principal PGL(2)-bundle [Serre]. Its functor of points on the category of Y-schemes is $X'(T)$ = set of T-isomorphisms $P^1_T \to L_T$. Let $\alpha: P^1_{X'} \to L_{X'}$ be the universal isomorphism. Then the composed map

$$P^1_{X'} \xrightarrow{\alpha} L_{X'} \longrightarrow C_{X'} \longrightarrow P^2_{X'}$$

defines a morphism $\beta: X' \to X$.

I claim that β is an isomorphism. Indeed, define a closed subscheme $\tilde{S} \subseteq P^2_X$ by the first Fitting ideal of $\tilde{\phi}_*(O_{P^1_X})$. Then $\tilde{S}$ is etale and finite over X of rank p, and defines a morphism $\phi: X \to Y$. Clearly, the map $\tilde{\phi}: P^1_X \to \tilde{C} \subseteq P^2_X$ coincides with the blowing up of $\tilde{C}$ along $\tilde{S}$. Therefore ϕ can be lifted to an inverse of β.

Summing up our result so far, we have defined the following part of the diagram of the theorem:

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[dr, "\phi"'] & A \\
 & Y \arrow[u, "\nu = i \circ n"']
\end{tikzcd}$$

Furthermore, we have shown that ϕ is a principal PGL(2)-bundle, hence Y is nonsingular. Since any rational plane curve of degree d with ordinary nodes and cusps has a total of p of these singularities, ν is injective on geometric points. Since ν is proper, it is a birational homeomorphism onto its image R. This also shows that Y is the normalization of R.

Remark: There is a natural action of PGL(2) on X. One may, starting in the other end, check that this action is free, and construct (Y, ϕ) as a geometric quotient of this action.

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