MATRIC MASSEY PRODUCTS AND FORMAL MODULI I

by

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Introduction. It is now folklore that the hull of a deformation functor of an algebraic geometric object, in some way is determined by the appropriate cohomology of the object and its "Massey products", see [M], [May]. The first hints in this direction occurs in Douadys exposé in [Car] (1961).

In 1975 I proved that, in fact, there is a kind of Massey product structure induced by the obstruction calculus characterizing this hull, see [La1].

Independently many authors have published results in this direction, see f.ex. [Pa1], [S & S], for references.

This, and a forthcoming paper, are concerned with the problem of actual calculation of these Massey products in two special cases, that of a k-algebra A and of an A-module E.

In §1 we recall the general machinery of [La1] which is common for all the cases we have in mind.

In §2 we prove the the usual matric Massey products, properly adjusted to our needs, for $\mathrm{Ext}_A^{\cdot}(E,E)$ determine the formal moduli of the A-module E, i.e. the hull of the deformation functor of E. As a corollary we obtain the following result

(2.10) Any complete local ring A is uniquely determined by $\mathrm{Ext}_A^i(k,k)$, $i = 1,2$ and the matric Massey products $\overset{n}{\otimes}\,\mathrm{Ext}^1 \dashrightarrow \mathrm{Ext}^2$.

CONTENTS

§1 Formal moduli and Massey products

Let X be some algebraic geometric object, say a k-algebra A or an A-module E, and consider the deformation functor

$$\mathrm{Def}_X : \underline{1} \to \underline{\mathrm{Sets}}$$

see [La1].

Let $A^i = A^i(k,X;O_X)$ be the corresponding cohomology. If X is a k-algebra A, then $A^i = H^i(k,A;A)$ is the André cohomology, and if X is an A-module E, $A^i = \mathrm{Ext}^i_A(E,E)$.

By [La1], (4.2.4), we know that <u>the formal moduli of X</u>, i.e. the hull of Def_X, is determined by a morphism of complete local k-algebras

$$o : T^2 = \mathrm{Sym}_k(A^{2*})^\wedge \to T^1 = \mathrm{Sum}_k(A^{1*})^\wedge$$

constructed using only the "obstruction calculus" of $A^\bullet$.

In fact, (4.2.4) of [La1] implies that the formal moduli H has the form

$$H \simeq T^1 \underset{T^2}{\otimes} k ,$$

provided A^1 and A^2 has countable dimensions as k-vector spaces.

We shall assume, in the what follows, that

$$\dim_k A^i < \infty \quad \text{for} \quad i = 1,2.$$

Pick a basis $\{x_1,\ldots,x_d\}$ of A^{1*} and a basis $\{y_1,\ldots,y_r\}$ of A^{2*}. Denote by $\{x_1^*,\ldots,y_d^*\}$ and $\{y_1^*,\ldots,y_r^*\}$ the corresponding dual bases of A^1 resp. A^2.

Put $f_j = o(y_j)$, $j = 1,\ldots,r$. Then by (4.2.4) of [La] the ideal $(\underline{f})$ of T^1 generated by the f_j's is contained in $\underline{m}^2_{T^1}$. Moreover $H \simeq T^1/(\underline{f})$. Now, for any surjective homomorphism of local artinian k-algebras $\pi : R \to S$, such that $\underline{m}_R \cdot \ker \pi = 0$, consider the diagram

$$\begin{array}{ccc} \mathrm{Mor}(H,R) & \to & \mathrm{Def}_E(R) \\ \downarrow & & \downarrow \\ \mathrm{Mor}(H,S) & \to & \mathrm{Def}_E(S). \end{array}$$

Suppose given a morphism $\phi: H \to S$ corresponding to the lifting $X_\phi \in \mathrm{Def}_X(S)$, then in the diagram below, we may always lift the map ϕ' to a map $\bar{\phi}$ making the resulting diagram commutative

(1)
$$\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & \ker \pi \arrow[d] \\
A^{2*} \subseteq \underline{m}_{T^2} \subseteq T^2 \arrow[r, "o"] & T^1 \arrow[r, dashed, "\bar{\phi}"] \arrow[d] \arrow[dr, "\phi'"] & R \arrow[d, "\pi"] \\
 & H \arrow[r, "\phi"] \arrow[dr, "\phi_1"'] & S \arrow[d] \\
 & & S/\underline{m}^2
\end{tikzcd}$$

The obstruction for lifting X_ϕ to R is, by construction of o, and functoriality, given by the restriction of $o \circ \bar{\phi}$ to A^{2*}. In fact $o \circ \bar{\phi}$ induces a linear map $A^{2*} \to \ker \pi$, i.e. an element $o(X_\phi, \pi) \in A^2 \otimes \ker \pi = A^2(k, X; O_X \otimes_k \ker \pi)$, which is the uniquely defined obstruction. Notice that we have the following identity

(2)
$$o(X_\phi, \pi) = \sum_k y_j^* \otimes \phi(\bar{f}_j).$$

Notice also that the image X_{ϕ_1} of X_ϕ by the map $\mathrm{Def}_X(S) \to \mathrm{Def}_X(S/\underline{m}^2)$ corresponds to the map ϕ_1. Moreover ϕ_1 is uniquely determined by the induced map on the cotangent level

$$t_\phi : A^{1*} = {}^{\underline{m}}H/\underline{m}^2{}_H \to {}^{\underline{m}}/\underline{m}^2$$

thus by an element $t_\phi \in A^1 \otimes {}^{\underline{m}}/\underline{m}^2$ which under the isomorphism $\mathrm{Def}_X(S/\underline{m}^2) \simeq A^1 \otimes {}^{\underline{m}}/\underline{m}^2$ corresponds to X_{ϕ_1}. If $t_\phi = \Sigma_{i=1}^d x_i^* \otimes t_i$, $t_i \in {}^{\underline{m}}/\underline{m}^2$ then $\phi_1(x_i) = t_i$, $i = 1, \ldots, d$.

On the other hand, having fixed a basis $\{\bar{v}_1, \ldots, \bar{v}_p\}$ for ${}^{\underline{m}}/\underline{m}^2$, we find that $t_\phi = \Sigma_{l=1}^p \alpha_l \otimes \bar{v}_l$, $\alpha_l \in A^1$.

Thus there is a one to one correspondence between maps ϕ_1 and sequences $\alpha_1, \ldots, \alpha_p$ of elements of A^1.

Pick an $\underline{n} = (n_1, \ldots, n_d) \in \underline{N}^d$ with $|\underline{n}| = \Sigma_{i=1}^d n_i = N$ and let $i_1 < i_2 < \cdots < i_p$ be the indices i for which $n_i \neq 0$.

Consider the ideal $J_{\underline{n}} \subseteq k[u_1,\dots,u_d]$ generated by the set of monomials $\{u_1^{t_1} \cdots u_d^{t_d} \mid \exists i,\ t_i > n_i\}$.

Put $R_{\underline{n}} = k[u_1,\dots,u_d]/J_{\underline{n}}$, $S_{\underline{n}} = R_{\underline{n}}/(u_1^{n_1} \cdots u_d^{n_d})$ and let $v_l = u_{i_l}$ be the image of u_{i_l} in $R_{\underline{n}}$ (resp. $S_{\underline{n}}$). Obviously $v_1,\dots,v_p$ generates $R_{\underline{n}}$ (resp. $S_{\underline{n}}$) as k-algebra, and induces a basis $\{\bar{v}_1,\dots,\bar{v}_p\}$ of $\underline{m}_{\underline{n}}/\underline{m}_{\underline{n}}^2$. Fix this basis.

Now let $\alpha_1,\dots,\alpha_p \in A^1$ and consider the corresponding map $\phi_1 : H \to S_{\underline{n}}/\underline{m}_{\underline{n}}^2$.

<u>Definition (1.1)</u>. Any map $\phi_{\underline{n}}$ making the following diagram commutative

$$\begin{tikzcd}
T^2 \arrow[r, "o"] & T^1 \arrow[d, "\rho"'] & R_{\underline{n}} \arrow[d, "\pi_{\underline{n}}"] \\
& H \arrow[r, "\phi_{\underline{n}}"] \arrow[dr, "\phi_1"'] & S_{\underline{n}} \arrow[d] \\
& & S_{\underline{n}}/\underline{m}_{\underline{n}}^2
\end{tikzcd}$$

is called a defining system for the Massey product

$$\langle \alpha_1,\dots,\alpha_p;\underline{n}\rangle = o(X_{\phi_{\underline{n}}},\pi_{\underline{n}}) \in A^2 .$$

When $\alpha_l = x_{i_l}^{*}$ we shall write $\langle \underline{x}^{*};\underline{n}\rangle$ for the Massey product $\langle x_{i_1}^{*},\dots,x_{i_p}^{*};\underline{n}\rangle$.

Suppose now that for some $N \geq 2$ and every $j = 1,\dots,r$ we have

$$f_j = \sum_{|\underline{n}|=N} a_{j,\underline{n}}\, \underline{x}^{\underline{n}} + \text{higher terms}$$

and consider any map $\bar{\phi}_{\underline{n}} : T^1 \to R_{\underline{n}}$ such that $\bar{\phi}_{\underline{n}} \circ \pi_{\underline{n}} = \rho \circ \phi_{\underline{n}}$. Then

$\bar{\phi}_{\underline{n}}(f_j) = a_{j,\underline{n}}\, \bar{u}_{i_1}^{n_{i_1}} \cdots \bar{u}_{i_p}^{n_{i_p}} \in \ker \pi_{\underline{n}} \simeq k$. Applying the identity (2) we find

$$(3) \qquad a_{j,\underline{n}} = y_j(\langle \underline{x}^{*};\underline{n}\rangle).$$

It follows that if we let f_j^N be the degree N (leading) form of f_j, then

(4) $$f_j^N(\underline{x}) = \sum_{|\underline{n}|=N} y_j\langle \underline{x}^*;\underline{n}\rangle \cdot \underline{x}^{\underline{n}} .$$

Consider the diagram:

(5) $$\begin{tikzcd} T^2 \arrow[r, "o"'] & T^1 \arrow[r, dashed, "\bar{\phi}_{N-1}"] \arrow[d, "\rho"'] & k[u_1,\dots,u_d]/\underline{m}^{N+1} \arrow[d, "\pi"] \\ & H \arrow[r, "\phi_{N-1}"'] & k[u_1,\dots,u_d]/\underline{m}^N \end{tikzcd}$$

where $\rho\circ\phi_{N-1}(x_i) \equiv u_i (\mathrm{mod}\ \underline{m})$, $\underline{m} = (u_1,\dots,u_d)$.

Let $X_{\phi_{N-1}} \in \mathrm{Def}_X(k[u_1,\dots,u_d]/\underline{m}^N)$ correspond to ϕ_{N-1}. Notice that by assumption $X_{\phi_{N-1}}$ is a lifting of the universal lifting of X to $k[u_1,\dots,u_d]/\underline{m}^2$. Notice also that $\ker \pi = \underline{m}^N/\underline{m}^{N+1} = \bigoplus_{|\underline{n}|=N} k\cdot(\underline{u}^{\underline{n}})$. An easy argument then shows that the obstruction for lifting $X_{\phi_{N-1}}$ to $k[u_1,\dots,u_d]/\underline{m}^{N+1}$ is given by:

(6) $$\begin{aligned} o(X_{\phi_{N-1}},\pi) &= \sum_{|\underline{n}|=N} \langle \underline{x}^*;\underline{n}\rangle \otimes \underline{u}^{\underline{n}} = \sum_j y_j^* \otimes \Big(\sum_{|\underline{n}|=N} y_j\langle \underline{x}^*;\underline{n}\rangle \underline{u}^{\underline{n}}\Big) \\ &= \sum_j y_j^* \otimes f_j^N(\underline{u}) . \end{aligned}$$

Now consider the diagram

(7) $$\begin{tikzcd} T^2 \arrow[r, "o"'] & T^1 \arrow[r, dashed, "\bar{\phi}_N"] \arrow[d] & R_{N+1} = k[u_1,\dots,u_d]/(\underline{m}^{N+2}+\underline{m}(f_1^N,\dots,f_r^N)) \arrow[d, "\pi'_{N+1}"] \\ & H \arrow[r, dashed, "\phi_N"] \arrow[dr, "\phi_{N-1}"'] & S_N = k[u_1,\dots,u_d]/(\underline{m}^{N+1}+(f_1^N,\dots,f_r^N)) \arrow[d, "\pi_N"] \\ & & S_{N-1} = k[u_1,\dots,u_d]/\underline{m}^N \end{tikzcd}$$

Since S_N is $k[u_1,\dots,u_d]/\underline{m}^{N+1}$ divided by the ideal generated by the obstruction for lifting $X_{\phi_{N-1}}$, we may lift $X_{\phi_{N-1}}$ to S_N, therefore we may find maps ϕ_N and $\bar{\phi}_N$ making the diagram commutative.

Pick a monomial basis $\{u^{\underline{n}}\}_{\underline{n}\in\bar{B}_{N-1}}$ for S_{N-1} (take simply all $u^{\underline{n}}$ with $|\underline{n}| \leq N-1$) and pick a monomial basis $\{u^{\underline{n}}\}_{\underline{n}\in B_N}$ for $\ker \pi_N = \underline{m}^N/\underline{m}^{N+1}+(f_1,\dots,f_r^N)$. Put $\bar{B}_N = \bar{B}_{N-1} \cup B_N$. For every $\underline{n}$ with $|\underline{n}| \leq N$ we have a unique relation in S_N

(8) $$u^{\underline{n}} = \sum_{\underline{m}\in\bar{B}_N} \beta_{\underline{n},\underline{m}}\, u^{\underline{m}}$$

Since by construction $o(X_{\phi_{N-1}}, \pi_N) = 0$, this relation together with (6) implies that for every $\underline{m} \in B_N$ (or $\bar{B}_N$ if one insists).

(9) $$\sum_{|\underline{n}|=N} \beta_{\underline{n},\underline{m}} \langle \underline{x}^* ; \underline{n} \rangle = 0$$

Write $$\ker \pi'_{N+1} = (\underline{m}^{N+1}+(f_1^N,\dots,f_r^N))/(\underline{m}^{N+2}+\underline{m}(f_1^N,\dots,f_r^N))$$
$$= (f_1^N,\dots,f_r^N)/\underline{m}(f_1^N,\dots,f_r^N) \oplus I_{N+1}$$

Pick a monomial basis for $I_{N+1} = \underline{m}^{N+1}/(\underline{m}^{N+2}+\underline{m}^{N+1} \cap \underline{m}(f_1^N,\dots,f_r^N))$ of the form $\{u^{\underline{n}}\}_{\underline{n}\in B'_{N+1}}$. We may assume that for $\underline{n} \in B'_{N+1}$, $u^{\underline{n}}$ is of the form $u_k \cdot u^{\underline{m}}$ for some $\underline{m} \in B_N$. Put $\bar{B}'_{N+1} = \bar{B}_N \cup B'_{N+1}$. For every $\underline{n}$ with $|\underline{n}| \leq N+1$ we have a unique relation in R_{N+1}

(10) $$u^{\underline{n}} = \sum_{\underline{m}\in\bar{B}'_{N+1}} \beta'_{\underline{n},\underline{m}}\, u^{\underline{m}} + \sum_j \beta'_{\underline{n},j} f_j^N .$$

Let

(11) $$f_j^{N+1} = \bar{\phi}_N(f_j) = f_j^N + \sum_{\underline{n}\in\bar{B}'_{N+1}} b_{j,\underline{n}}\, u^{\underline{n}}$$

then by definition of o, the obstruction for lifting X_{ϕ_N} to R_{N+1} is

(12) $$o(X_{\phi_N}, \pi'_{N+1}) = \sum_j y_j^* \otimes f_j^{N+1}$$
$$= \sum_j y_j^* \otimes f_j^N + \sum_{\underline{m}\in\bar{B}'_{N+1}} \left(\sum_j{}^* y \otimes b_{j,\underline{n}}\, u^{\underline{n}}\right)$$

<u>Definition (1.2).</u> The map ϕ_N is called a defining system for the Massey products

$$\langle \underline{x}^*;\underline{n}\rangle = \sum_j b_{j,\underline{n}}\, y_j^* \in A^2, \quad \text{for } \underline{n} \in B'_{N+1}.$$

With these notations we have:

$$(13) \qquad f_j^{N+1} = \sum_{\underline{m}\in B'_N} y_j \langle \underline{x}^*;\underline{m}\rangle \underline{u}^{\underline{m}} + \sum_{\underline{n}\in B'_{N+1}} y_j \langle \underline{x}^*;\underline{n}\rangle \underline{u}^{\underline{n}}$$

where we have put $B'_N = \{\underline{n} \mid |\underline{n}|=N\}$.

Consider the diagram

$$(14) \qquad \begin{tikzcd}
T^2 \arrow[r, "o"] & T^1 \arrow[r, dashed, "\bar{\phi}_{N+1}"] \arrow[d] & R_{N+2} = k[u_1,\dots,u_d]/(\underline{m}^{N+3}+\underline{m}\cdot(f_1^{N+1},\dots,f_r^{N+1})) \arrow[d, "\pi'_{N+2}"] \\
& H \arrow[r, dashed, "\phi_{N+1}"] \arrow[dr, "\phi_N"'] & S_{N+1} = R_{N+1}/(f_1^{N+1},\dots,f_r^{N+1}) \arrow[d, "\pi_{N+1}"] \\
& & S_N
\end{tikzcd}$$

Since S_{N+1} is R_{N+1} divided by the ideal generated by the obstruction for lifting X_{ϕ_N} to R_{N+1}, we may lift X_{ϕ_N} to S_{N+1}, therefore we may find maps ϕ_{N+1} and $\bar{\phi}_{N+1}$ making the diagram above commutative.

Pick a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in B_{N+1}}$ for $\ker \pi_{N+1}$ such that $B_{N+1} \subseteq B'_{N+1}$. Put $\bar{B}_{N+1} = \bar{B}_N \cup B_{N+1}$. Then $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in\bar{B}_{N+1}}$ is a monomial basis for S_{N+1}. For every $\underline{n}$ with $|\underline{n}| \leq N+1$ we therefore have a unique relation in S_{N+1}

$$(15) \qquad \underline{u}^{\underline{n}} = \sum_{\underline{m}\in\bar{B}_{N+1}} \beta'_{\underline{n},\underline{m}}\, \underline{u}^{\underline{m}}$$

Since by construction $o(X_{\phi_N}, \pi_{N+1}) = 0$, this implies for every $\underline{m} \in B_{N+1}$ the following identity:

$$(16) \qquad \sum_{\underline{n}\in B'_{N+1}} \beta_{\underline{n},\underline{m}} \langle \underline{x}^*;\underline{n}\rangle = 0$$

which is analogous to (9).

Write, again, $\ker \pi'_{N+2} = (\underline{m}^{N+2} + (f_1^{N+1},\dots,f_r^{N+1})) / (\underline{m}^{N+3} + \underline{m}(f_1^{N+1},\dots,f_r^{N+1}))$
$= (f_1^{N+1},\dots,f_r^{N+1}) / \underline{m}(f_1^{N+1},\dots,f_r^{N+1}) \oplus I_{N+2}$.

Pick a monomial basis for $I_{N+2} = \underline{m}^{N+2} / (\underline{m}^{N+3} + \underline{m}^{N+2} \cap \underline{m}(f_1^{N+1},\dots,f_r^{N+1}))$ of the form $\{u^{\underline{n}}\}_{\underline{n}\in B'_{N+2}}$, where we may assume that for $\underline{n} \in B'_{N+2}$, $u^{\underline{n}}$ is of the form $u_k \cdot u^{\underline{m}}$ for some $\underline{m} \in B_{N+1}$ and some k. Put $\bar{B}'_{N+2} = \bar{B}'_{N+1} \cup B'_{N+2}$. For every $\underline{n}$ with $|\underline{n}| \leqslant N+2$ we have a unique relation in R_{N+2}

$$u^{\underline{n}} = \sum_{\underline{m}\in\bar{B}'_{N+2}} \beta'_{\underline{n},\underline{m}} u^{\underline{m}} + \sum_j \beta'_{\underline{n},j} f_j^{N+1} \tag{17}$$

of the same form as (10).

Let

$$f_j^{N+2} = \bar{\phi}_{N+1}(f_j) = f_j^{N+1} + \sum_{\underline{n}\in B'_{N+2}} c_{j,\underline{n}} u^{\underline{n}} . \tag{18}$$

Again, by definition of o, the obstruction for lifting $X_{\phi_{N+2}}$ to R_{N+2} is

$$\begin{aligned} o(X_{\phi_{N+1}}, \pi'_{N+2}) &= \sum_j y_j^* \otimes f_j^{N+2} \\ &= \sum_j y_j^* \otimes f_j^{N+1} + \sum_{\underline{n}\in B'_{N+2}} \Big(\sum_j y_j^* \otimes c_{j,\underline{n}} u^{\underline{n}}\Big) \end{aligned} \tag{19}$$

<u>Definition (1.3).</u> The map ϕ_{N+1} is called a defining system for the Massey products $\langle \underline{x}^*; \underline{n}\rangle = \Sigma_j c_{j,\underline{n}} y_j^* \in A^2$, for $\underline{n} \in B'_{N+2}$.

With these notations, we have the following

$$f_j^{N+2} = \sum_{\underline{l}\in B'_N} \langle \underline{x}^*;\underline{l}\rangle u^{\underline{l}} + \sum_{\underline{m}\in B'_{N+1}} y_j \langle \underline{x}^*;\underline{m}\rangle u^{\underline{m}} + \sum_{\underline{n}\in B'_{N+2}} y_j \langle \underline{x}^*;\underline{m}\rangle u^{\underline{n}} \tag{20}$$

Clearly this process may be continued indefinitely. For every $k \geqslant 0$ we obtain a diagram

$$
(21) \qquad
\begin{tikzcd}
T^2 \arrow[r, "o"] & T^1 \arrow[r, "\bar{\phi}_{N+k}"] \arrow[d] & R_{N+k+1} \arrow[d, "\pi'_{N+k+1}"] \\
 & H \arrow[r, "\phi_{N+k}"] & S_{N+k} \arrow[d, "\pi_{N+k}"] \\
 & & S_{N+k-1}
\end{tikzcd}
$$

a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n}\in\bar{B}_{N+k}}$ for S_{N+k}, such that for every $\underline{n}$ with $|\underline{n}| \leqslant N+k$ there is a unique relation in S_{N+k}

$$(22) \qquad \underline{u}^{\underline{n}} = \sum_{\underline{m}\in\bar{B}_{N+k}} \beta_{\underline{n},\underline{m}}\, \underline{u}^{\underline{m}} ,$$

inducing the identity

$$(23) \qquad \sum_{\underline{n}\in B'_{N+k}} \beta_{\underline{n},\underline{m}} \langle \underline{x}^{*};\underline{n}\rangle = 0 , \qquad \underline{m} \in B_{N+k}.$$

And there is a corresponding basis $\{\underline{u}^{\underline{n}}\}_{\underline{m}\in B'_{N+k+1}}$ for the component I_{N+k+1} of $\ker \pi'_{N+k+1} = (f_1^{N+k},\ldots,f_r^{N+k})/\underline{m}(f_1^{N+k},\ldots,f_r^{N+k}) \oplus I_{N+k+1}$ such that in R_{N+k+1} we have for every $\underline{n}$ with $|\underline{n}| \leqslant N+k+1$

$$(24) \qquad \underline{u}^{\underline{n}} = \sum_{\underline{m}\in\bar{B}'_{N+k+1}} \beta'_{\underline{n},\underline{m}}\, \underline{u}^{\underline{n}} + \sum_{j} \beta'_{\underline{n},j}\, f_j^{N+k}$$

where we have put $\bar{B}_{N+k+1} = \bar{B}_{N+k} \cup B'_{N+k+1}$.

Moreover,

$$(25) \qquad f_j^{N+k+1} = \bar{\phi}_{N+k}(f_j) = f_j^{N+k} + \sum_{\underline{n}\in B'_{N+k+1}} \underline{u}^{\underline{n}}.$$

The obstruction for lifting $X_{\phi_{N+k}}$ to R_{N+k+1} is

$$(26) \qquad o(X_{\phi_{N+k}}, \pi'_{N+k+1}) = \sum_j y_j^{*} \otimes f_j^{N+k+1}$$

$$= \sum_j y_j^{*} \otimes f_j^{N+k} + \sum_{\underline{n}\in B'_{N+k+1}} \left(\textstyle\sum_j y_j^{*} \otimes \omega_{j,\underline{n}}\, \underline{u}^{\underline{n}}\right)$$

<u>Definition (1.4)</u>. The map ϕ_{N+k} is called a defining system for the Massey products $\langle \underline{x}^{*};\underline{n}\rangle = \Sigma_j\, \omega_{j,\underline{n}}\, y_j^{*} \in A^2$ for $\underline{n} \in B'_{N+k+1}$.

In particular, we find for every k,

(27) $$f_j^{N+k+1} = \sum_{l=o}^{k+1} \sum_{\underline{n}\in B'_{N+l}} y_j \langle \underline{x}^*;\underline{n}\rangle \underline{u}^{\underline{n}}$$

Notice that by (4.2.4) of [La1] we have

(28) $$H \simeq \lim_{\overleftarrow{k}} S_{N+k}$$

therefore

(29) $$H \simeq k[[u_1,\dots,u_d]]/(\bar{f}_1,\dots,\bar{f}_d)$$

where

(30) $$\bar{f}_j = \lim_{k\to\infty} f_j^{N+k}.$$

Formally we may therefore write

(31) $$\bar{f}_j = \sum_{l=o}^{\infty} \sum_{\underline{n}\in B'_{N+l}} y_j \langle \underline{x}^*;\underline{n}_l\rangle \underline{u}^{\underline{n}}.$$

§2 Massey products for $\mathrm{Ext}_A^{\bullet}(E,E)$

In this paragraph we shall let A be any k-algebra and we shall let X, in §1, be some A-module E. We shall thus be concerned with the deformation functor of E as an A-module

$$\mathrm{Def}_E\colon \underline{l} \to \underline{\mathrm{Sets}}$$

defined as follows,

$$\mathrm{Def}_E(S) = \left\{ \begin{array}{ccc|c} S \otimes_k A & \to & \mathrm{End}(E_S) & E_S \text{ is } S\text{-flat} \\ \downarrow & & \downarrow & \downarrow \\ A & \to & \mathrm{End}(E) & E_S \otimes_S k = E \end{array} \right\} \Big/ \text{iso.}$$

As is well known, the corresponding cohomology is

$$A^i = \mathrm{Ext}_A^i(E,E)$$

The deformation theory for modules, as hinted at on page 150 of [La1], parallels the corresponding theory for algebras. There is a global theory and a relative theory, and the main theorem (4.2.4) of [La1] holds. There are no surprices, and we shall therefore leave the details to the reader.

Pick any free resolution $L.$ of E as an A-module, and consider the associated single complex $\mathrm{Hom}^{\bullet}_A(L.,L.)$ of the double complex $\mathrm{Hom}_A(L.,L.)$. By definition we have

$$\mathrm{Hom}^p_A(L.,L) = \prod_{m\geq o} \mathrm{Hom}(L_m, L_{m-p})$$

Let $d_i : L_i \to L_{i-1}$ be the differential of $L.$, then

$$d^p : \mathrm{Hom}^p_A(L.,L.) \to \mathrm{Hom}^{p+1}(L.,L.)$$

is defined by

$$d^p(\{\alpha^p_i\}_{i\geq o}) = d_i o \alpha^p_{i-1} - (-1)^p \alpha^p_i o d_{i-p}$$

Clearly $\mathrm{Hom}^{\bullet}_A(L.,L.)$ is a graded differential associative A-algebra, multiplication being the composition of $\mathrm{Hom}^{\bullet}(L.,L.)$.

<u>Lemma (2.1)</u>. There is a natural isomorphism

$$\mathrm{Ext}^i_A(E,E) \simeq H^i(\mathrm{Hom}^{\bullet}_A(L.,L.)), \quad i \geq 0.$$

Consider any surjective morphism $\pi : R \to S$ in $\underline{l}$, such that $\underline{m}_R \cdot \ker \pi = 0$.

Assume there exists a lifting $\{L.\otimes_k S, d_i(S)\}$ of the complex $\{L., d_i\}$, i.e. of the free resolution $L.$ of E.

This means that there exists a commutative diagram of the form

$$\begin{array}{ccccccccccccc}
0 & \leftarrow & H_o(L.\otimes_k S) & \leftarrow & L_o\otimes_k S & \xleftarrow{d_1(S)} & L_1\otimes_k S & \xleftarrow{d_2(S)} & L_2\otimes_k S & \xleftarrow{d_3(S)} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & H_o(L.) & \leftarrow & L_o & \xleftarrow{d_1} & L_1 & \xleftarrow{d_2} & L_2 & \xleftarrow{d_3} & \cdots
\end{array}$$

where for every i, the composition

$$d_{i+1}(S) \circ d_i(S) = 0.$$

We shall see that any such lifting is, in fact, an $A\otimes_k S$-free resolution of $H_o(L.\otimes_k S) = E_S$, and that E_S is a lifting of $E = H_o(L.)$ to S.

Both contentions are obviously true for $S = k$, so by induction we may assume they hold for S. If we then are able to prove the corresponding statements for R, we are through.

But first we have an existence problem. Given a lifting E_S of E to S it is easy to see that there is a corresponding lifting $\{L.\otimes_k S,\ d_i(S)\}$ of $\{L.,d_i\}$ to S. By assumption we have conversally that any such lifting $\{L.\otimes_k S,\ d_i(S)\}$ of $\{L.,d_i\}$ to S determines a lifting $E_S = H_o(L.\otimes_k S)$ and is, itself, an $A \otimes_k S$-free resolution of E_S.

Pick one such lifting $\{L.\otimes_k S,\ d_i(S)\}$, and let us compute the obstruction for lifting $\{L.\otimes_k S,\ d_i(S)\}$ to R. This obstruction is then, clearly, an obstruction for lifting E_S to R.

For every i, pick a lifting $d_i'(R)\colon L_i\otimes_k R \to L_{i-1}\otimes_k R$ of $d_i(S)\colon$ $L_i\otimes_k S \to L_{i-1}\otimes_k S$, to R. This is obviously possible, since all L_i are A-free.

Since $d_i(S)\circ d_{i-1}(S) = 0$ and since $I = \ker \pi$ is killed by the maximal ideal $\underline{m}_R$ of R, the composition $d_i'(R)\circ d_{i-1}'(R)\colon L_i\otimes_k R \to$ $L_{i-2}\otimes_k R$ is induced by a unique map

$$O_i\colon L_i \to L_{i-2} \otimes_k I$$

The family $\{O_i\}_{i\geq 0}$ defines an element

$$O \in \mathrm{Hom}^2(L.,L.) \otimes_k I$$

One checks that $d^2 O = 0$, so that O is a 2-cocycle of $\mathrm{Hom}_A^{\bullet}(L.,L.)$, defining an element

$$o(E_S,\pi) \in \mathrm{Ext}_A^2(E,E) \otimes_k I.$$

It is easily seen that $o(E_S,\pi)$ is independent of the choice of the $d_i'(R)$'s lifting the $d_i(S)$'s.

Moreover, if $o(E_S,\pi) = 0$, there exists an element $\xi \in \mathrm{Hom}_A^1(L.,L.) \otimes_k I$ such that $d\xi = -O$. Put

$$d_i(R) = d'_i(R)+\varepsilon_i ,$$

then one finds

$$d_i(R) \circ d_{i-1}(R) = 0$$

and $\{L.\otimes_k R,\ d_i(R)\}$ is a lifting of $\{L.\otimes_k S,\ d_i(S)\}$ to R.
Now let $\{L.\otimes_k R; d_i(R)\}$ be any lifting of $\{L.\otimes_k S,\ d_i(S)\}$ to R, then there is an exact sequence of complexes

$$0 \to \{L.\otimes_k I,\ d_i \otimes 1_I\} \to \{L.\otimes_k R,\ d_i(R)\} \to \{L.\otimes_k S,\ d_i(S)\} \to 0$$

inducing a long exact sequence

$$\begin{array}{llll} \to H_n(L.\otimes_k I) & \to H_n(L.\otimes_k R) & \to H_n(L.\otimes_k S) & \\ \to H_{n-1}(L.\otimes_k I) & \to \cdots & \to H_1(L.\otimes_k S) & \\ \to H_o(L.\otimes I) & \to H_o(L.\otimes R) & \to H_o(L.\otimes_k S) & \to 0 \end{array}$$

from which it follows that

$$H_n(L.\otimes_k R) = 0 \quad \text{for} \quad n \geq 1, \quad \text{and}$$

$$0 \to E \otimes_k I \to H_o(L.\otimes R) \to E_S \to 0$$

is exact.
Therefore $H_o(L.\otimes R) = E$ is a lifting of E_R to R.
Moreover, given two liftings $\{L.\otimes_k R,\ d_i(R)_l\}$, $l = 1,2$ of $\{L.\otimes_k S,\ d_i(S)\}$, corresponding to two liftings E_R^1 and E_R^2 of E_S, the differences $d_i(R)_1 - d_i(R)_2$ induce maps

$$\eta_i : L_i \to L_{i-1} \otimes_k I.$$

The family $\{\eta_i\}_{i \geq o}$ is a 1-cocycle of $\mathrm{Hom}^{\bullet}(L.,L.)$ defining an element $\bar{\eta} \in \mathrm{Ext}^1_A(E,E)$.

In this way we obtain a surjective map

$$\{\text{liftings of } E_S \text{ to } R\} \times \mathrm{Ext}^1_A(E.E) \to \{\text{liftings of } E_S \text{ to } R\}$$

making the set of liftings of E_S to R a principal homogenous space (torsor) over $\mathrm{Ext}^1_A(E,E)$.

We have established the following,

Proposition (2.2). Let $E_S \in \mathrm{Def}_E(S)$ correspond to the lifting $\{L.\otimes_k S,\ d_i(S)\}$ of $L.$ to S. Then there is a uniquely defined obstruction

$$o(E_S,\pi) \in \mathrm{Ext}^2_A(E,E) \otimes_k I$$

given in terms of the 2-cocycle O of $\mathrm{Hom}^{\bullet}_A(L.,L.) \otimes I$ defined above, such that $o(E_S,\pi) = 0$ iff E_S may be lifted to R.

Moreover, if $o(E_S,\pi) = 0$ then the set of liftings of E_S to R is a principal homogeneous space (torsor) over $\mathrm{Ext}^1_A(E,E)$.

Thus we have at hand a nice obstruction calculus for Def_E given entirely in terms of the complex $L.$ and its liftings.

Using this we shall apply the constructions of §1 and compute the Massey products, $\langle \underline{x}^*, \underline{n}\rangle$ for $\underline{n} \in B'_{N+k}$. In fact, the $\langle \underline{x}^*, \underline{n}\rangle$ of §1 will turn out to be some generalized "ordinary" Massey products of the differential graded k-algebra $\mathrm{Hom}^{\bullet}_A(L.,L.)$.

Pick a basis $\{x_1,\dots,x_d\}$ of $\mathrm{Ext}^1_A(E,E)^*$ and a basis $\{y_1,\dots,y_r\}$ of $\mathrm{Ext}^2_A(E,E)^*$. Denote by $\{x_1^*,\dots,x_d^*\}$ and $\{y_1^*,\dots,y_r^*\}$ the corresponding dual bases of Ext^1 and Ext^2.

Let for $i = 1,\dots,d$, $X_i \in \mathrm{Hom}^1_A(L.,L.)$ be a cocycle representing x_i^* and let for $j = 1,\dots,r$, $Y_j \in \mathrm{Hom}^2_A(L.,L.)$ be a cocycle representing y_j^*.

Pick an $\underline{n} = (n_1,\dots,n_d)$ with $|\underline{n}| = \Sigma_{i=1}^d n_i = N$ and consider as in §1 the k-algebras $S_{\underline{n}}$ and $R_{\underline{n}}$. Fix the basis $\{\bar{v}_1 \cdots \bar{v}_p\}$ of $\underline{m}_{\underline{n}}/\underline{m}_{\underline{n}}^2$. Recall that we have in $R_{\underline{n}}$ the following slightly confusing identities

$$(1) \qquad \begin{aligned} v_l &= u_{i_l}, \quad l = 1,\dots,p. \\ u_i &= 0 \quad \text{if } i \notin \{i_1,\dots,i_p\} \end{aligned}$$

insisted upon because it makes the notations more streamlined later on.

We shall pick a monomial basis for the k-vectorspace $S_{\underline{n}}$ of the form

$$\{u_1^{m_1}\cdots u_d^{m_d} \mid 0 \leq m_i \leq n_i,\ \underline{m} \neq \underline{n}\}$$

written as

$$\{\underline{u}^{\underline{m}}\}_{\underline{m}\in\bar{B}_{\underline{n}}} \,.$$

With this done, let $\alpha_1,\ldots,\alpha_p \in \mathrm{Ext}^1_A(E,E)$ and consider the element $\Sigma_{l=1}^{p}\alpha_l \otimes \bar{v}_l \in \mathrm{Ext}^1_A \otimes \underline{m}_{\underline{n}}/\underline{m}^2_{\underline{n}}$.

Let $\phi_1: H \to S_{\underline{n}}/\underline{m}^2_{\underline{n}}$ be the corresponding map and let $E_{\phi_1} \in \mathrm{Def}_E(S_{\underline{n}}/\underline{m}^2_{\underline{n}})$ be the induced deformation of E.

Assume there is given a defining system $\phi_{\underline{n}}: H \to S_{\underline{n}}$ for the Massey product $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$ (see (1.1)), corresponding to a lifting $E_{\phi_{\underline{n}}} \in \mathrm{Def}_E(S_{\underline{n}})$ of E_{ϕ_1}.

Then E_{ϕ_1} is represented by a lifting $\{L.\otimes S_{\underline{n}}/\underline{m}^2_{\underline{n}};d_i(S_{\underline{n}}/\underline{m}^2_{\underline{n}})\}$ of $L.$ and $E_{\phi_{\underline{n}}}$ is represented by a lifting $\{L.\otimes S_{\underline{n}};d_i(S_{\underline{n}})\}$ of $\{L.\otimes S_{\underline{n}}/\underline{m}^2_{\underline{n}};d_i(S_{\underline{n}}/\underline{m}^2_{\underline{n}})\}$.

The family of $A\otimes_k S_{\underline{n}}$-linear maps

$$d_i(S_{\underline{n}}): L_i \otimes_k S_{\underline{n}} \to L_{i-1} \otimes_k S_{\underline{n}}$$

is uniquely determined by the restriction to $L_i \otimes 1$, thus by the family of A-linear maps

$$\alpha_{i,\underline{m}}: L_i \to L_{i-1}\,,\quad \underline{m} \in \bar{B}_{\underline{n}}$$

defined by

$$d_i(S_{\underline{n}})\,|\,L_i\otimes 1 = \sum_{\underline{m}\in\bar{B}_{\underline{n}}} \alpha_{i,\underline{m}} \otimes \underline{u}^{\underline{m}}.$$

With this notation, we may assume

$$d_i(S_{\underline{n}}/\underline{m}^2_{\underline{n}})\,|\,L_i\otimes 1 = \sum_{l=1}^{p} \alpha_{i,\underline{\varepsilon}_l} \otimes \underline{u}^{\underline{\varepsilon}_l}$$

where $\underline{\varepsilon}_l = (\underbrace{0,\ldots,1}_{i_l},0,\ldots,0) \in \bar{B}_{\underline{n}}$.

According to (1) we may also write

$$d_i(S_{\underline{n}}/\underline{m}_{\underline{n}}^2) \mid L_i \otimes 1 = \Sigma\ \alpha_{i,\underline{\varepsilon}_1} \otimes \bar{v}_1 .$$

For every $\underline{m} \in \bar{B}_{\underline{n}}$ the family $\{\alpha_{i,\underline{m}}\}_i$ is a cochain

$$\alpha_{\underline{m}} \in \mathrm{Hom}_A^1(L.,L.)$$

such that $\alpha_{\underline{\varepsilon}_1}$ is a cocycle representing the cohomology class α_1, $1 = 1,\ldots,p$, and $\alpha_{i,\underline{o}} = d_i$, $i > 0$.

Since $d_i(S_{\underline{n}}) \circ d_{i-1}(S_{\underline{n}}) = 0$ for all $i > 0$ we find that the family $\{\alpha_{\underline{m}}\}_{\underline{m}\in\bar{B}_{\underline{n}}}$ satisfies the following identities

$$\sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m} \\ \underline{m}_i \in \bar{B}_n}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2} = 0 \quad \text{for all } \underline{m} \in \bar{B}_{\underline{n}} .$$

Moreover the obstruction for lifting $E_{\phi_{\underline{n}}}$ to $R_{\underline{n}}$, i.e. the obstruction $o(E_{\phi_{\underline{n}}}, \pi_{\underline{n}})$ for lifting $\{L_i \otimes S_{\underline{n}}, d_i(S_{\underline{n}})\}$ to $R_{\underline{n}}$, is easily seen to be represented by the, (à priori), cocycle

$$\sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{n} \\ \underline{m}_i \in \bar{B}_{\underline{n}}}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2} \in \mathrm{Hom}_A^2(L.,L.).$$

<u>Proposition (2.3)</u>. Given a sequence of p cohomology classes $\alpha_1 \in \mathrm{Ext}_A^1(E,E)$, then a defining system for the Massey product $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$ corresponds to a family $\{\alpha_{\underline{m}}\}_{\underline{m}\in\bar{B}_{\underline{n}}}$ of 1-cochains of $\mathrm{Hom}_A^1(L.,L.)$, such tht for every $\underline{m} \in \bar{B}_{\underline{n}}$

$$\star \qquad \sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m} \\ \underline{m}_i \in \bar{B}_{\underline{n}}}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2} = 0$$

and such that $\alpha_{i,\underline{o}} = d_i$ for $i > 0$ and $\alpha_{\underline{\varepsilon}_1}$ represents α_1, $1 = 1,\ldots,p$.

Conversally, any such family $\{\alpha_{\underline{m}}\}_{\underline{m}\in\bar{B}_{\underline{n}}}$ give rise to a defining system for the Massey product $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$.

Moreover, given such a defining system, the Massey product $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$ is represented by the 2-cocycle

$$\sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m} \\ \underline{m}_i\in\bar{B}_{\underline{n}}}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2} \in \mathrm{Hom}_A^2(L.,L.).$$

Proof. This is just the observation that a lifting $E_{\phi_{\underline{n}}}$ of E_{ϕ_1} corresponds to a lifting $\{L.\otimes_k S_{\underline{n}};d_i(S_{\underline{n}})\}$ of $\{L.\otimes_k S_{\underline{n}}/\underline{m}_{\underline{n}}^2;d_i(S_{\underline{n}}/\underline{m}_{\underline{n}}^2)\}$, thus to families $\{\alpha_{i,\underline{m}}\}_{\underline{m}\in\bar{B}_{\underline{n}}}$ such that

$$** \qquad d_i(S_n)\mid L_i\otimes 1 = \sum_{\underline{m}\in\bar{B}_{\underline{n}}} \alpha_{i,\underline{m}}\otimes \underline{u}^{\underline{m}}.$$

The relation $d_i(S_n)\circ d_{i-1}(S_n)=0$ translates into $*$. Conversally $*$ proves that $d_i(S_n)$ defined by $**$ defines a lifting $\{L.\otimes S_{\underline{n}};d_i(S_{\underline{n}}/\underline{m}_{\underline{n}}^2)\}$, thus also a lifting $E_{\phi_{\underline{n}}}$ of E_{ϕ_1}. Finally any such $E_{\phi_{\underline{n}}}$ corresponds to a map $\phi_{\underline{n}}: H\to S_{\underline{n}}$, i.e. to a defining system. Q.E.D.

Remark (2.4). In the light of (2.3) we shall let the notion of a defining system for the Massey product $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$ refer to either the map $\phi_{\underline{n}}$ or the family $\{\alpha_{\underline{m}}\}$ depending on the situation.

Remark (2.5). Let $\underline{n}=(n_1,\ldots,n_d)$ be given such that $n_i=0$ for $i\notin\{i_1,\ldots,i_p\}$, then the Massey product $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$, if defined, depends only upon $\alpha_1,\ldots,\alpha_p$ and the p-uple $(n_{i_1},n_{i_2},\ldots,n_{i_p})$. Given $\alpha_1,\ldots,\alpha_p\in \mathrm{Ext}_A^1(E,E)$ and any p-uple $\underline{m}=(m_1,\ldots,m_p)$ there is no confusion in writing

$$\langle\alpha_1,\ldots,\alpha_p;\underline{m}\rangle.$$

Suppose $p=1$ and $\underline{n}=(n)$, $\alpha_1=\alpha\in \mathrm{Ext}_A^1(E,E)$, then a defining system for $\langle\alpha;\underline{n}\rangle$ is a family $\{\alpha_{\underline{m}}\}_{0<\underline{m}<\underline{n}-1}$ of 1-cochains

$$\alpha_{\underline{m}} \in \mathrm{Hom}^1_A(L.,L.)$$

such that, $\alpha_{i,o} = d_i$, $i \geq 0$ and $\alpha_{\underline{1}}$ represents α, with the property that for every $0 \leq \underline{m} < \underline{n}-1$, $\sum_{\underline{m}_1+\underline{m}_2=\underline{m}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2} = 0$.

If a defining system exists, then

$$\langle \alpha;\underline{n}\rangle = \mathrm{cl}(\sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{n} \\ o \leq \underline{m}_i \leq \underline{m}-1}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2})$$

In particular for $\underline{n} = (2)$ the Massey product $\langle\alpha,(2)\rangle$ is always defined and is represented by the 2-cocycle $\alpha\circ\alpha$. These are the "Bocksteins".

If $p = 2$ and $\underline{n} = (1,1)$, $\alpha_1,\alpha_2 \in \mathrm{Ext}^1_A(E,E)$ then the family $\{\alpha_{\underline{m}}\}_{\underline{m}\in\{(0,0),(1,0),(0,1)\}}$ where $\alpha_{(0,0)} = \{d_i\}_{i\geq 0}$, $\alpha_{(1,0)}$ represents α, and $\alpha_{(0,1)}$ represents α_2, is a defining system for $\langle\alpha_1,\alpha_2;(1,1)\rangle$ which is represented by $\alpha_{(1,0)}\circ\alpha_{(0,1)} + \alpha_{(0,1)}\circ\alpha_{(1,0)}$. Thus $\langle\alpha_1,\alpha_2;(1,1)\rangle$ is the symmetrized cup-product.

Now, having a purely cohomological expression for the (defined) Massey products $\langle\alpha_1,\ldots,\alpha_p;\underline{n}\rangle$, we shall procede as in §1, computing step by step a set of generators for the ideal of T^1 defining the formal moduli of E.

Assume, as in §1 that the formal power-series $f_j = o(y_j) \in T^1 = k[[x_1,\ldots,x_d]]$ may be written as

$$f_j = \sum_{|\underline{n}|=N} a_{j,\underline{n}}\, \underline{x}^{\underline{n}} + \text{higher terms} \quad j = 1,\ldots,r$$

for some $N \geq 2$.

Then by §1 (3), $\alpha_{j,\underline{n}} = y_j\langle x^*;n\rangle$ where, by assumption $\langle\underline{x}^*;n\rangle = \langle x^*_{i_1},x^*_{i_2},\ldots,x^*_{i_p};\underline{n}\rangle$ is (uniquely) defined.

Put $f^N_j = \Sigma_{|\underline{n}|=N}\, a_{j.\underline{n}}\, \underline{u}^{\underline{n}}$ and consider the diagram §1 (7). The map ϕ_{N-1} induces defining systems for all Massey products $\langle\underline{x}^*;\underline{n}\rangle$ for $|\underline{n}| < N$, and corresponds therefore to a family

(2) $$\{\alpha_{\underline{m}}\}_{\underline{m}\in\overline{B}_{N-1}}$$

of 1-cochains of $\mathrm{Hom}^{\bullet}_A(L.,L.)$ such that for every $i \geq 0$, $\alpha_{i,\underline{o}} = d_i$, and $\alpha_{\underline{e}_i}$ is a cocycle representing $x^*_{i,\underline{e}_i} = (\underbrace{0,\ldots,1}_{i},0,\ldots 0) \in \underline{\underline{N}}$. Moreover, for every $\underline{m} \in \overline{B}_{N-1}$

(3) $$\sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m}\\ \underline{m}_i\in\overline{B}_{N-1}}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2} = 0$$

Let $d_i(S_{N-1}): L_i \otimes_k S_{N-1} \to L_{i-1} \otimes_k S_{N-1}$ be the $A \otimes S_{N-1}$-linear map defined by

$$d_i(S_{N-1})\mid L_i \otimes 1 = \sum_{\underline{m}\in\overline{B}_{N-1}} \alpha_{i,\underline{m}}\, \underline{u}^{\underline{m}} .$$

Then (3) implies that $\{L.\otimes_k S_{N-1};\ d_i(S_{N-1})\}$ is a lifting of the universal deformation of $L.$ to S_2 defined by the map

$$\phi_1 : H \to k[u_1,\ldots,u_d]/\underline{m}^2 .$$

Recall that ϕ_1 corresponds to the deformation of $L.$, or of E if one wishes, to S_2 defined by the element

$$\sum_{i=1}^{d} x^*_i \otimes \overline{u}_i \in \mathrm{Ext}^1_A(E,E) \otimes \underline{m}/\underline{m}^2 = \mathrm{Def}_E(S_2).$$

By construction $\{L.\otimes_k S_{N-1}\ ;\ d_i(S_{N-1})\}$ induces the deformation

$E_{\phi_{N-1}} \in \mathrm{Def}_E(S_{N-1})$.

Sticking to the notations of §1, and noticing that for every $\underline{n}\in B'_N = \{\underline{m}\in\underline{\underline{N}}^d \mid |\underline{m}|=N\}$ the Massey product $<\underline{x}^*;\underline{n}>$ is represented by the 2-cycle

$$Y(\underline{n}) = \sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{n}\\ \underline{m}_i\in\overline{B}_{N-1}}} \alpha_{\underline{m}_1} \circ \alpha_{\underline{m}_2}$$

§1 (8) and (9) translates into the following. For every $\underline{m} \in B_N$, $\Sigma_{\underline{n}\in B'_N}\ \beta_{\underline{n},\underline{m}}\ Y(\underline{n})$ is a coboundary.

Now, pick for every $\underline{m} \in B_N$ a 1-cochain $\alpha_{\underline{m}} \in \mathrm{Hom}^1_\Lambda(L.,L.)$ such that

(4) $$d\,\alpha_{\underline{m}} = \sum_{\underline{n}\in B'_N} \beta_{\underline{n},\underline{m}}\, Y(\underline{n})$$

and consider the family

(5) $$\{\alpha_{\underline{m}}\}_{\underline{m}\in \bar{B}_N}\,.$$

Let, for every $i \geq 0$, $d_i(S_N): L_i \otimes S_N \to L_{i-1} \otimes S_N$ be defined by: $d_i(S_N) \mid L_i \otimes 1 = \Sigma_{\underline{m}\in B_N} \alpha_{i,\underline{m}} \otimes \underline{u}^{\underline{m}}$. Then (4) translates into

$$d_i(S_N) \circ d_{i-1}(S_N) = 0.$$

Consequently $\{L.\otimes_k S_N;\ d_i(S_N)\}$ is a lifting of $\{L.\otimes_k S_{N-1};\ d_i(S_{N-1})\}$ to S_N, and induces therefore a lifting $E_{\phi_N} \in \mathrm{Def}_E(S_N)$ of $E_{\phi_{N-1}}$. E_{ϕ_N}, again, corresponds to a map ϕ_N : $H \to S_N$ which we now fix.

According to (1.2) ϕ_N is a defining system for the Massey products $<\underline{x}^*;\underline{n}>$ for $\underline{n} \in B'_{N+1}$. Since ϕ_N is induced by, and induces, a family (5), we shall refer to any such family as a defining system for the Massey products $<\underline{x}^*;\underline{n}>$, $\underline{n} \in B'_{N+1}$.

By definition, see (1.2), these Massey products are given in terms of the obstruction, see §1 (11), $o(E_{\phi_N}, \pi'_{N+1})$.

By (2.2) this obstruction is defined by the 2-cocycle $o = \{o_i\}$ where

$$o_i = d'_i(R_{N+1}) \circ d'_{i-1}(R_{N+1}),$$

$d'_i(R_{N+1}): L_i \otimes R_{N+1} \to L_{i-1} \otimes R_{N+1}$ being any lifting of $d_i(S_N)$. Pick $d'_i(R_{N+1}$ such that

$$d'_i(R_{N+1}) \mid L_i \otimes 1 = \sum_{\underline{m}\in \bar{B}_N} \alpha_{i,\underline{m}} \otimes \underline{u}^{\underline{m}}$$

then streight forward calculation, using §1 (10), shows that

$$o_i = \sum_{\underline{n}\in B'_{N+1}} \Big(\sum_{|\underline{m}|\leq N+1} \sum_{\underline{m}_1+\underline{m}_2=\underline{m}} \beta'_{\underline{m},\underline{n}} \cdot \alpha_{i,\underline{m}_1} o\alpha_{i-1,\underline{m}_2} \Big) u^{\underline{n}}$$

$$+ \sum_{j=1}^{r} \Big(\sum_{|\underline{m}|\leq N+1} \sum_{\underline{m}_1+\underline{m}_2=\underline{m}} \beta'_{\underline{m},j} \cdot \alpha_{i,\underline{m}_1} o\alpha_{i-1,\underline{m}_2} \Big) f_j^N .$$

Remember that $d_i(S_N) o d_{i-1}/S_N) = 0$.

Comparing this with (1.2) and §1 (11), we have proved the following

Proposition (2.6). Given a defining system $\{\alpha_{\underline{m}}\}_{\underline{m}\in\bar{B}_{N+1}}$ for the Massey products $\langle \underline{x}^{*};\underline{n}\rangle$, $\underline{n} \in B'_{N+1}$, $\langle \underline{x}^{*};\underline{n}\rangle$ is represented by the 2-cocycle

$$Y(n) = \sum_{|\underline{m}|\leq N+1} \sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m} \\ \underline{m}_i\in\bar{B}_n}} \beta'_{\underline{m},\underline{n}} \, \alpha_{\underline{m}_1} o \, \alpha_{\underline{m}_2}$$

By §1 (16) we know that for every $\underline{m} \in B_{N+1}$ the 2-cochain

$$\beta_{\underline{m}} = \sum_{\underline{n}\in B'_{N+1}} \beta_{\underline{n},\underline{m}} \, Y(\underline{n}) \in \mathrm{Hom}_A^2(L.,L.)$$

is a coboundary. Pick one $\alpha_{\underline{m}} \in \mathrm{Hom}_A^2(L.,L.)$ such that $d\,\alpha_{\underline{m}} = \beta_{\underline{m}}$, and consider the family

(6) $$\{\alpha_{\underline{m}}\}_{\underline{m}\in\bar{B}_{N+1}} .$$

Just as above, (6) is seen to correspond to a defining system, ϕ_{N+1}, for the Massey products $\langle \underline{x}^{*};\underline{n}\rangle$, $\underline{n} \in B'_{N+2}$.

There are relations §1, (17), (18), (19), and we may copy the procedure above.

We end up with the following,

Proposition (2.7). Given a defining system $\{\alpha_{\underline{m}}\}_{\underline{m}\in\bar{B}_{N+k-1}}$ for the7 Massey products $\langle \underline{x}^{*};\underline{n}\rangle$, $\underline{n} \in B'_{N+k}$, $\langle \underline{x}^{*};\underline{n}\rangle$, is represented by the 2-cocycle

$$Y(\underline{n}) = \sum_{|m|\leq N+k} \sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m} \\ \underline{m}_i\in\bar{B}_{N+k-1}}} \beta'_{\underline{m},\underline{n}} \, \alpha_{\underline{m}_1} o \, \alpha_{\underline{m}_2}$$

Moreover, the polynomials

$$f_j^{N+k} = \sum_{l=o}^{k} \sum_{\underline{n}\in B'_{N+1}} y_j \langle \underline{x}^*; \underline{n}\rangle \underline{u}^{\underline{n}} \qquad j = 1,\ldots,r$$

induces identities §1 (22) and (23), such that if we for every $\underline{m} \in B_{N+k}$ pick a cochain $\alpha_{\underline{m}} \in \mathrm{Hom}_A^1(L.,L.)$ with

$$d\,\alpha_{\underline{m}} = \sum_{\underline{n}\in B'_{N+k}} \beta_{\underline{n},\underline{m}}\, Y(\underline{n})$$

then the family

$$\{\alpha_{\underline{m}}\}_{\underline{m}\in \bar{B}_{N+k}}$$

is a difining system for the Massey products $\langle \underline{x}^*; \underline{n}\rangle$, $\underline{n} \in B'_{N+k+1}$.

We may, refering to §1 (28), (29), (30), sum up the content of this §2 as follows

<u>Theorem (2.8)</u>. Given an A-module E, the formal moduli H of E is determined by the Massey products of $\mathrm{Ext}_A^{\bullet}(E,E)$. In fact

$$H \simeq k[[x_1,\ldots,x_d]]/(f_1,\ldots,f_r)$$

where

$$f_j = \sum_{l=2}^{\infty} \sum_{\underline{n}\in B'_l} y_j \langle \underline{x}^*; \underline{n}\rangle\, \underline{x}^{\underline{n}}.$$

<u>Corollary (2.9)</u>. Any complete local k-algebra A with residue field k is determined by $\mathrm{Ext}_A^i(k,k)$, $i = 1,2$ and its Massey-products.

<u>Proof</u>. Obviously A is the formal moduli of k as an A-module.

Q.E.D.

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