THE HAUSDORFF DUAL PROBLEM: ALMOST CONNECTED GROUPS

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ABSTRACT. It will be shown that an almost connected locally compact group G has a Hausdorff unitary dual space if and only if G is a compact extension of an abelian group.

1.0. Let G be a locally compact group, G_0 its connected component of the identity. The unitary dual space $\hat{G}$ of G consists of the set of all equivalence classes of irreducible continuous unitary representations of G. $\hat{G}$ is endowed with the Fell-topology which is the inverse image of the hull-kernel topology, on the set of primitive ideals of the group C^*-algebra $C^*(G)$, under the natural map which assigns to each element π of $\hat{G}$ its kernel, when π is regarded as a representation of $C^*(G)$, [3], [4]. It seems to be a reasonable conjecture that $\hat{G}$ is a Hausdorff space if and only if G is of type I and is a compact extension of an abelian group. This hypothesis has been verified for connected groups by L. Baggett and the author, [2]. The principal aim of the present article is to settle this question for almost connected groups. Our proof is organized as follows. First we treat the case when G_0 is a noncompact semisimple Lie group. Invoking the fact, proved in [2], that $\hat{G}_0$ cannot be Hausdorff, we show using the Mackey procedure that $\hat{G}$ contains a convergent sequence with at least two limits. Hence $\hat{G}$ is non-Hausdorff. Next, we assume G_0 is an arbitarry Lie group and that $\hat{G}$ is T_2. An induction argument yields the desired structural property for G. Finally, if G is almost connected, locally compact we approximate by Lie groups to reach our theorem.

Theorem 1. An almost connected locally compact group G has a Hausdorff unitary dual space if and only if G is a compact extension of an abelian group.

Proof. 1.1. We consider first the case where the group G is a finitely connected Lie group and its identity component G_0 is semisimple and noncompact. Then, by [2, Proposition 4], the dual space of G_0 fails to be Hausdorff. It will be shown below that, in the present situation, $\hat{G}$ connot be Hausdorff. To this end we shall construct a convergent sequence from $\hat{G}$ with at least two limits. We shall assume the center Z of G_0 is finite (if Z is infinite, replace G with the group G/Z).

Let $G_0 = KAN$ be an Iwasawa decomposition. Let $\{\rho_m\}$ be a sequence of irreducible unitary complementary series representations of G_0 which converges to a reducible unitary representation ρ. The existence of such a sequence was shown in the proof of [2, Proposition 4] and, in addition, the limit ρ could be written as a direct sum $\rho = \rho^1 \oplus \rho^2$ where the trivial element ι was a subrepresentation of the restriction $\rho^1|K$ and $\rho^2|K$ was disjoint from ι. By [2] again ρ splits into a direct sum of elements of $\hat{G}_0$. Hence we can find an irreducible subrepresentation of ρ^1 whose restriction to K contains ι. As a consequence we may assume that ρ^1 and ρ^2 both are irreducible since the sequence $\{\rho_m\}$ converges to every subrepresentation of ρ.

1.2 The complementary series representations ρ_m are induced from (non-unitary) characters of a minimal parabolic subgroup of G_0, [6], hence we presume they all have the same dimension N = aleph nought. In the present situation there is only a finite number of subgroups of G containing G_0, so we can find a subsequence, also denoted by $\{\rho_m\}$, for which the stabilizer of each ρ_m in G is independent of m. Let H denote this mutual stability subgroup. Next, let for each m $\tilde{\rho}_m$ be an irreducible cocycle extension of ρ_m to H in the sense of Mackey, [8]. That is, the cocycle ω_m of each $\tilde{\rho}_m$ is the inflation to H of a cocycle on H/G_0. Within similarity there is only a finite number of cocycles of the group H/G_0. Taking a subsequence again we assume all the ω_m-s are similar, hence there exists a cocycle ω and, for each m, a coboundary δ_m such that $\omega_m = \delta_m \omega$. Here each δ_m has the form $\delta_m(x,y) = \dfrac{f_m(x,y)}{f_m(x)f_m(y)}$, where f_m is a (Borel) function of H/G_0 into the circle group.

Replacing each $\tilde{\rho}_m$ with $f_m^{-1}\tilde{\rho}_m$ we obtain a sequence, still denoted $\{\tilde{\rho}_m\}$, of irreducible ω-representations of H extending the ρ_m-s. By continuity of the inducing map, [4], we have

$$\mathrm{ind}_{G_0}^{H}(\rho_m) \underset{m}{\rightarrow} \mathrm{ind}_{G_0}^{H}(\rho^1 \oplus \rho^2) = \mathrm{ind}_{G_0}^{H}(\rho^1) \oplus \mathrm{ind}_{G_0}^{H}(\rho^2).$$

Let τ^i be an irreducible subrepresentation of $\mathrm{ind}_{G_0}^{H}(\rho^i)$, $i = 1,2$. Then τ^i lies over the G-orbit of ρ^i, $i = 1,2$, see e.g. [1, Theorem 2.2]. By elementary properties of the hull-kernel topology we can find a sequence $\{\tau_m\}$ of elements from $\hat{H}$ such that each τ_m is a subrepresentation of $\mathrm{ind}_{G_0}^{H}(\rho_m)$ and, in addition, $\{\tau_m\}$ converges to τ^1. Here τ_m restricts to G_0 as a multiple of ρ_m, [1, Theorem 2.2]. Whence we can find irreducible ω^{-1}-representations σ_m of H, with each σ_m equal to a multiple of the identity on G_0, and such that $\tau_m = \tilde{\rho}_m \otimes \sigma_m$, [8, Theorem 8.3]. Now since the ω^{-1}-dual of H/G_0 is finite we may assume that all the σ_m-s are equivalent (taking a subsequence), say $\sigma_m = \sigma^1$, and therefore $\tau_m = \tilde{\rho}_m \otimes \sigma^1$, for all m.

Let W be a fixed Hilbert space of dimension N, where N = aleph nought is the common dimension of all the τ_m-s. Let v be an arbitrary unit vector in the Hilbert space of τ^1. Since $\tau_m \underset{m}{\rightarrow} \tau^1$ we can find a sequence $\{v_m\}$ of vectors from W, bounded by 1, such that

$$\langle \tau_m(y)v_m, v_m \rangle \underset{m}{\rightarrow} \langle \tau^1(g)v, v \rangle \tag{1}$$

uniformly in y on each compact subset of H and, in addition,

$$\langle \tau_m(g)v_m, v_m \rangle \underset{m}{\rightarrow} \langle \tau^1(g)v, v \rangle \tag{2}$$

for all continuous complex valued functions g with compact support on H, $g \in C_c(H)$.

We show next that $\{\tau_m\}$ is a Cauchy sequence in the complete metric space $\mathrm{Irr}_N(H)$ of all irreducible representations of H on the Hilbert space W, [18, 3.7.4].

Let $h \in H$ and let $\varepsilon > 0$ be given. We pick a neighbourhood V of the identity in H such that

$$\|g-g^y\|_1 < \varepsilon/15 , \quad \text{for all } y \in V,$$

where the integration is w.r.t. a left Haar measure on H, and $g^y(x) = g(y^{-1}x)$; $x,y \in H$. Let $m(h)$ be a positive integer such that

$$|<\tau_m(g^*\ast g^h)v_m,v_m> - <\tau^1(g^*\ast g^h)v,v>| < \varepsilon/15 \tag{3}$$

whenever $m \geq m(h)$. For every y in V we obtain

$$\begin{aligned}
&|<\tau_m(yh)\tau_m(g)v_m,\tau_m(g)v_m> - <\tau^1(yh)\tau^1(g)v,\tau^1(g)v>| \leq \\
&|<\tau_m(yh)\tau_m(g)v_m,\tau_m(g)v_m> - <\tau_m(h)\tau_m(g)v_m,\tau_m(g)v_m>| \\
&\quad + |<\tau_m(h)\tau_m(g)v_m,\tau_m(g)v_m> - <\tau^1(h)\tau^1(g)v,\tau^1(g)v>| \\
&\quad + |<\tau^1(g^h)v,\tau^1(g)v> - <\tau^1(g^{yh})v,\tau^1(g)v>| \\
&\quad \leq \|(\tau_m(g^{yh})-\tau_m(g^h))v_m\| \, \|\tau_m(g)v_m\|+\varepsilon/5 \\
&\quad + \|(\tau^1(g^h)-\tau^1(g^{yh}))v\| \, \|\tau^1(g)v\| \\
&\quad \leq \|g^{yh}-g^h\|_1+\varepsilon/15+\|g^h-g^{yh}\|_1 < \varepsilon/5 ,
\end{aligned}$$

for each $m \geq m(h)$.

Let C be any compact subset of H. We cover C with a finite number of translates of V, $C \subset h_1V \cup h_2V \cup \cdots \cup h_rV$. Put $m_0 = \max\{m(h_1),\ldots,m(h_r)\}$. Then, for any y in C, we have

$$|<\tau_m(g^*\ast g^y)v_m,v_m> - <\tau^1(g^*\ast g^y)v,v>| < \varepsilon/5 \tag{4}$$

Let $w \in W$, $w \neq 0$, and let ε be given, $0 < \varepsilon < 1$. For each m, the set $\{\tau_m(g)v_m : g \in C_c(H)\}$ is dense in the Hilbert space W. Hence we can find functions $g_m \in C_c(H)$ such that

$$\|w-\tau_m(g_m)v_m\| < \varepsilon/5(\|w\|+1) \tag{5}$$

Put $u_m = \tau_m(g_m)v_m$, and let C be an arbitrary compact subset of H. By (4) we can find m_0 such that

$$|<\tau_m(y)u_m,u_m> - <\tau_n(y)u_n,u_n>| < \varepsilon/5 \tag{6}$$

for each y in C, whenever $m,n \geqslant m_0$. Using (6) we obtain for all y in C,

$$\begin{aligned}
&|<\tau_m(y)w,w>-<\tau_n(y)w,w>| \\
&\leqslant |<\tau_m(y)w,w>-<\tau_m(y)w,u_m>|+|<\tau_m(y)w,u_m>-<\tau_m(y)u_m,u_m>| \\
&+ |<\tau_m(y)u_m,u_m>-<\tau_n(y)u_n,u_n>|+|<\tau_n(y)u_n,u_n>-<\tau_n(y)u_n,w>| \\
&+ |<\tau_n(y)u_n,w>-|<\tau_n(y)w,w>| \\
&\leqslant \|w\|\|w-u_m\|+\|w-u_m\|\|u_m\|+\varepsilon/5+\|u_n\|\|u_n-w\| \\
&+ \|u_n-w\|\|w\| < 5\cdot\varepsilon/5 = \varepsilon ,
\end{aligned}$$

whenever $m,n \geqslant m_0$, since by (5)

$$\|u_m\| = \|\tau_m(g_m)v_m\| \leqslant \|w\|+\varepsilon/5 < \|w\|+1 .$$

This shows that $\{\tau_m\}$ is a Cauchy sequence in the complete metric space $\mathrm{Irr}_N(H)$, see [3, 18.1].

Now $\tau_m = \tilde{\rho}_m \otimes \sigma^1$, for all m, and it follows easily that $\{\tilde{\rho}_m\}$ is a Cauchy sequence in the complete space $\omega\text{-}\mathrm{Irr}_N(H)$ of all irreducible ω-representations of H on the Hilbert space W.

Thus let $\tilde{\rho}$ be a limit for the sequence $\{\tilde{\rho}_m\}$ in $\omega\text{-}\mathrm{Irr}_N(H)$.

Recall that $\tau_m \underset{m}{\to} \tau^1$, where the restriction $\tau^1|G_0$ contains ρ^1 as a direct summand. Assume first that ρ^1 doesn't occur as a subrepresentation of $\tilde{\rho}|G_0$. Then the sequence $\{\tau_m\}$ has two non-equivalent limits, $\tilde{\rho}\otimes\sigma^1$ and τ^1. By Mackey, [8], $\pi_m = \mathrm{ind}_H^G(\tau_m)$ is irreducible for each m, and by continuity of inducing the sequence $\{\pi_m\}$ converges to both $\mathrm{ind}_H^G(\tau^1)$ and $\mathrm{ind}_H^G(\tilde{\rho}\otimes\sigma^1)$, which are nonequivalent since, by the above, their restrictions to G_0 are concentrated on different G-orbits. Therefore it follows that $\hat{G}$ is non-Hausdorff.

The remaining possibility is that ρ^1 occurs as a direct summand of $\tilde{\rho}|G_0$. In this case we replace τ^1 with τ^2 in the above construction. Recall that τ^2 was assumed to be an irreducible subrepresentation of $\mathrm{ind}_{G_0}^H(\rho^1 \oplus \rho^2)$ whose restriction to G_0 contains ρ^2 as a direct summand. Thus we can find a sequence $\{\tilde{\rho}_m \otimes \sigma^2\}$ such that $\tilde{\rho}_m \otimes \sigma^2$ is an irreducible subrepresentation of $\mathrm{ind}_{G_0}^H(\rho_m)$ for each m, and the sequence converges to both τ^2 and $\tilde{\rho} \otimes \sigma^2$.

Next, we show that τ^2 and $\tilde{\rho} \otimes \sigma^2$ are in fact nonequivalent. Let $N_G(K)$ be the normalizer of K in G. It is known that $N_G(K)$ meets every connected component of G since K is maximal compact and G_0 is semisimple.

In particular, we can find coset representatives $x_1, x_2, \ldots, x_r$ of G/G_0 such that $x_i K x_i^{-1} = K$, $1 \leqslant i \leqslant r$. Hence each K-fixed vector of ρ^1 is also a K-fixed vector of $x_i \rho^1$, $(x_i \rho^1(y) = \rho^1(x_i^{-1} y x_i))$, $1 \leqslant i \leqslant r$. It follows that $x\rho^1|K$ contains the trivial representation for each x in G, and as a consequence, the G-orbits of ρ^1 and ρ^2 in $\hat{G}_0$ are distinct. Accordingly τ^2 and $\tilde{\rho} \otimes \sigma^2$ are nonequivalent. Inducing the sequence $\{\tilde{\rho}_m \otimes \sigma^2\}$ and its limits to G we see as before that $\hat{G}$ fails to be a Hausdorff space. This completes our argument in case G_0 is a semisimple Lie group.

1.3. Remark. The above construction led to a sequence $\{\pi_m\}$ of $\hat{G}$, each π_m of dimension aleph nought, that converges to the nonequivalent representations π^1 and π^2, where the restriction of π^1 to a maximal compact subgroup K of G_0 contains the identity representation ι, whereas $\pi^2|K$ is disjoint from ι.

2. We are now ready to complete the proof of Theorem 1 in the Lie case. Assume the identity component G_0 is an arbitrary Lie group

of finite index in G. Our argument will go by induction on the dimension $\dim(G)$. The theorem is obviously correct for finite groups $(\dim(G) = 0)$.

Assume that the result is true for all Lie groups of dimension less than n, where n is a positive integer, and let G be a group of dimension equal to n possessing Hausdorff dual space. First, if G contains a compact normal subgroup K of positive dimension then we apply the inductive hypothesis to the factor group G/K, and obtain easily that G is a compact extension of an abelian group. Therefore we presume that G contains no compact normal subgroups of positive dimension. We have already seen that the connected component of the identity in G can not be semisimple. Thus let A be of maximal dimension among the abelian, connected, normal subgroups of G which are contained in G_0. The group A is nontrivial since the center of the nilradical of G_0 is invariant under all automorphisms of G_0. Put $r = \dim(A)$. Then A is isomorphic to a direct product $\underline{\underline{R}}^{r-j} \times \underline{\underline{T}}^{j}$ where the torus part $\underline{\underline{T}}^{j}$ is G-invariant since the vector group $\underline{\underline{R}}^{r-j}$ has no nontrivial compact subgroups. It follows that $j = 0$ and A is isomorphic to $\underline{\underline{R}}^{r}$. Then, since by the inductive hypothesis G/A is amenable, we can mimic the arguments of [2, p.66]. Hence we deduce first that A is central in G (since the stabilizer of each element in $\hat{A}$ is seen to equal all of G). Secondly, let K be a compact normal subgroup of G/A for which the factor group in G/A is abelian (the inductive hypothesis applied to G/A).

Let $p\colon G \to G/A$ denote the canonical map. Since the second cohomology group $H^2(K,A)$ is known to be trivial, the group $p^{-1}(K)$ must be isomorphic to the direct product $K \times A$. In particular, K is normal in G. Let G_1 denote the factor group G/K, and let A_1 be the image of A in G_1 under the canonical map $G \to G_1$.

By assumption K is finite. Then $G_1 = G/K$ must be finitely connected. As in [2, p.66] the groups G_1 and A_1 are seen to have the same Lie algebra, since A_1 is maximal abelian. We conclude that $F = G_1/A_1$ is discrete, hence is finite, and since $H^2(F,A_1) = (0)$ it follows that G_1 is isomorphic to the direct product $F \times A_1$. Thus G is a central extension of the abelian group $F \times A_1$ by the finite group K. This completes our proof of the Lie case.

<u>3</u>. Finally, let G be an almost connected locally compact group, i.e., the identity component is cocompact in G. Then the group is a projective limit, $G = \text{proj lim}(G_\alpha)$, of finitely connected Lie groups G_α, [9, §4.6]. Now, if at least one G_α is noncompact and semisimple then $\hat{G}_\alpha$ must be non-Hausdorff by §1.1 above. Since $\hat{G}_\alpha$ is naturally embedded as a closed subspace of $\hat{G}$, it is clear that $\hat{G}$ cannot be a T_2-space. Therefore assume no G_α is semisimple. If $\hat{G}$ is Hausdorff then each $\hat{G}_\alpha$ has the same property, hence G_α is a compact extension of an abelian group by the Lie case. Then G is also a compact extension of an abelian group by definition of projective limit. Conversely, if G is almost connected and, in addition, has a precompact commutator subgroup, then G is in fact a compact extension of a vector group. In particular G is of type I. In this case $\hat{G}$ is known to be Hausdorff, see e.g. [10]. Our proof of Theorem 1 is complete.

<u>4</u>. We give below some applications of Theorem 1. An element x of a topological space M is said to be a <u>Hausdorff point</u> of M if for every element y of M, not in the closure of x, we can find

two neighbourhoods in M separating x and y. An inspection of the proof of [2, Lemma 5] leads at once to the following improvement of that lemma.

Lemma 3. Let N be a closed normal subgroup of a locally compact group G, and assume that N is of type I and also that G/N is amenable. If the trivial representation is a Hausdorff point of $\hat{G}$ then the map $\gamma \to G(\gamma)$ of $\hat{N}$ into the space of all closed subgroups of G is continuous at the trivial element of $\hat{N}$. Here $G(\gamma)$ denotes the stability group in G of γ.

Our next result should be related to [10, Theorem 1].

Proposition 4. Let G be an almost connected, amenable locally compact group. Then the following statements are equivalent.

(1) The trivial representation is a Hausdorff point of $\hat{G}$.

(2) $\hat{G}$ is a Hausdorff space.

Proof. The implication (2) => (1) is clear. Assume (1) is true. Since G is amenable, its identity component must be either non-semisimple or compact. If G_0 is compact then G is compact and $\hat{G}$ is a Hausdorff space. Therefore, let A be of maximal dimension among the connected normal abelian subgroups of G. Using the above lemma with $N = A$, we see as in [2, p.66] that A is central in G. Then we argue as in the proof of Theorem 1, §2. It follows that G is a compact extension of an abelian group, proving that $\hat{G}$ is Hausdorff.

Remark. If G is a semisimple Lie group then the trivial representation may well be a Hausdorff point of $\hat{G}$. In fact, it is possible for the trivial representation to be an isolated point of the dual, [6, Remark 10].

Corollary 5. Let G be an almost connected, amenable locally compact group. Then $\hat{G}$ is a Hausdorff space if and only if for each element x of $C^*(G)$ the map $\pi \to \|\pi(x)\|$, $C^*(G)^\wedge \to \underline{\underline{R}}$, is continuous at the trivial representation.

Proof. This is a consequence of Proposition 4 and [3, 3.9.4].

Corollary 6. Let G be an almost connected locally compact group. Then $C^*(G)$ is a C^*-algebra with a continuous trace if and only if $\hat{G}$ is a Hausdorff space.

Proof. If $\hat{G}$ is Hausdorff then $C^*(G)$ has a continuous trace by Theorem 1 and the Corollary in [5]. The converse follows from [3, 4.5.3].

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