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BETTINUMBERS OF MONOID ALGEBRAS. APPLICATIONS TO 2-DIMENSIONAL TORUS IMBEDDINGS

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<u>Contents</u>

<u>Notation</u>

k any field

Λ a monoid (the unit is denoted by 1)

$A = k[\Lambda]$ the monoid k-algebra

$\underline{m}$ the maximal ideal of A generated by $\Lambda_+ = \Lambda - \{1\}$

$\underline{\underline{Z}}_+$ the non-negative integers

Introduction

The starting point of this paper is the rather elementary observation (1.2), which leads to a formula (1.3) for the Betti numbers of a monoid algebra in terms of the combinatorial properties of the monoid, see [La 2]. The rest of the paper is concerned with the application of this formula to the case of 2-dimensional torus embeddings, see [Od]. More specifically: In §1 we give a method for computing the Betti numbers $\beta_i = \dim_k \mathrm{Tor}_i^A(k,k)$ when A is the monoid algebra over k of a commutative monoid Λ with cancellation law, and no non-trivial inverses. Proposition 1.3 relates the Betti numbers to the local homology of the simplicial set associated to $\Lambda_+ = \Lambda - \{1\}$ ordered such that $\lambda \leq \lambda \cdot \mu$, when $\lambda, \mu \in \Lambda$. In §2 this is used to compute the Betti numbers of 2-dimensional torus embeddings A. In particular we prove that the Betti series

$$B(t) = \sum_{n \geq 0} \beta_n t^n$$

of A is a rational function $\frac{P(t)}{Q(t)}$. The main result of this paper is, in fact, the explicite computation of the denominator $Q(t)$, see Corollary 2.20.

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§1 Betti numbers of monoid algebras

Fix a field k and let Λ be a commutative monoid with cancellation law, i.e. such that $\lambda \cdot \mu = \lambda \cdot \mu'$ implies $\mu = \mu'$. Let $A = k[\Lambda]$ and put $\underline{m} = \Lambda_+ \cdot A$ where $\Lambda_+ = \Lambda - \{1\}$. Assume $A/\underline{m} = k$, i.e. assume Λ has no non-trivial subgroups. Put $\beta_i = \dim_k \mathrm{Tor}_i^A(k,k)$, the i-th Betti number of $k[\Lambda]$. Then the power series $B(t) = \Sigma_{n \geq 0} \beta_n t^n$ is called the Betti series of A. The purpose of this first paragraph is to give a method for computing the Betti series of A using only combinatorial properties of Λ_+.

Let Λ_+ be ordered as follows: $\lambda_1 \leq \lambda_2$ if and only if there exists a $\mu \in \Lambda$ such that $\mu \cdot \lambda_1 = \lambda_2$. There is a natural presheaf (projective system)

$$F: \Lambda_+ \to \underline{\mathrm{Ab}}$$

defined by: $F(\lambda) = A$

where $F(\lambda_1 \leq \lambda_2): F(\lambda_2) \to F(\lambda_1)$ is multiplication by

$$\mu = \frac{\lambda_2}{\lambda_1} .$$

<u>Lemma 1.1</u>

$$\lim_{\vec{\Lambda}_+} F = (\Lambda_+) \cdot A = \underline{m}.$$

<u>Proof</u>

For every $\lambda \in \Lambda_+$, consider the morphism $\eta_\lambda: F(\lambda) \to A$, the multiplication by λ. This defines a morphism $\eta: \lim\limits_{\vec{\Lambda}_+} F \to \underline{m}$.

Given an element $\alpha \in \underline{m}$, there is a unique representation $\alpha = \Sigma_{i=n} \alpha_i \cdot \lambda_i$; $\alpha_i \in k$, $\lambda_i \in \Lambda_+$. Consider α_i as an element of $F(\lambda_i)$ and let $\bar{\alpha}_i$ be the image of α_i in $\lim\limits_{\vec{\Lambda}_+} F$. Define

$\mu: \underline{m} \to \lim\limits_{\vec{\Lambda}_+} F$ by $\mu(\alpha) = \Sigma_{i=1}^{N} \bar{\alpha}_i$. Then μ is an inverse of η.

Q.E.D.

<u>Lemma 1.2</u>

$$\lim_{\vec{\Lambda}_+}{}^{(n)} F = 0 \quad \text{for} \quad n \geq 1.$$

<u>Proof</u>

By [La 1,(1,1.4)] it is enough to show that F is coflabby (coflasque). Let $\lambda \in \Lambda_+$ and suppose $\Lambda_1 \subseteq \{\lambda' \in \Lambda_+ \mid \lambda \leq \lambda'\}$ is such that if $\lambda' \in \Lambda_1$ and $\lambda' \leq \lambda''$ then $\lambda'' \in \Lambda_1$.

F is coflabby if in this situation

$$\varprojlim_{\Lambda_1} F \longrightarrow \varprojlim_{\{\lambda' \in \Lambda_+ | \lambda \leq \lambda'\}} F = F(\lambda) = A$$

is an injection.

However, the proof of Lemma 1.1 applies to show that $\varprojlim_{\Lambda_1} F = \{\frac{\lambda'}{\lambda} | \lambda' \in \Lambda_1\} \cdot A$ and that the morphism $\varprojlim_{\Lambda_1} F \longrightarrow \varprojlim_{\{\lambda' \in \Lambda_+ | \lambda < \lambda'\}} F = A$ is the obvious inclusion. Therefore we are done.

Q.E.D

Consider the resolving complex $C_\bullet(\Lambda_+;-)$ for $\varprojlim_{\Lambda_+}$, see [La1, (1.2)]. By Lemma 1.2, $C_\bullet(\Lambda_+;F)$ is an A-free resolution of the maximal ideal $\underline{m}$ of A. Therefore

$$\mathrm{Tor}_i^A(k,k) \simeq \begin{cases} k & i = 0 \\ H_{i-1}(C_\bullet(\Lambda_+;F) \otimes_A k) & i \geq 1 \end{cases}$$

Now $C_\bullet(\Lambda_+;F) \otimes_A k = C_\bullet(\Lambda_+;F \otimes_A k)$, therefore

$$H_{i-1}(C_\bullet(\Lambda_+;F) \otimes_A k) = \varprojlim_{\Lambda_+}{}^{(i-1)}(F \otimes_A k).$$

Observe that the projective system $F \otimes_A k$ is isomorphic to $\coprod_{\lambda \in \Lambda_+} k(\lambda)$, where $k(\lambda)$ is the projective system defined by:

$$k(\lambda)(\lambda') = \begin{cases} 0 & \text{if } \lambda' \neq \lambda \\ k & \text{if } \lambda' = \lambda \end{cases}$$

Put for any $\lambda \in \Lambda_+$,

$$\hat{\lambda} = \{\lambda' \in \Lambda_+ | \lambda' \leq \lambda\}$$

$$L(\lambda) = \{\lambda' \in \Lambda_+ | \lambda' \leq \lambda,\ \lambda' \neq \lambda\} = \hat{\lambda} - \{\lambda\}$$

It is easy to see that there are isomorphisms:

$$\varprojlim_{\Lambda_+}{}^{(n)} k(\lambda) \simeq \varprojlim_{\hat{\lambda}}{}^{(n)} k(\lambda) \quad \text{for } n \geq 0$$

In fact this follows from the existence of a $\coprod$-projective resolution of $k(\lambda)$, trivial outside of $\hat{\lambda}$, see [La 1, (1.2)].

Let $\underline{k}_\lambda$ be the constant projective system on $\hat{\lambda}$ defined by $\underline{k}_\lambda(\lambda') = k$, and let $\underline{k}'_\lambda$ be the subprojective system of $\underline{k}_\lambda$ defined by $\underline{k}'_\lambda(\lambda') = 0$ if $\lambda' = \lambda$ and $\underline{k}'_\lambda(\lambda') = k$ if $\lambda' \neq \lambda$. Then there is an exact sequence of projective systems on $\hat{\lambda}$

$$0 \to \underline{k}'_\lambda \to \underline{k}_\lambda \to k(\lambda) \to 0$$

As

$$\varinjlim_{\hat{\lambda}}{}^{(n)} \underline{k}_\lambda = \begin{cases} k & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases}$$

and since

$$\varinjlim_{\hat{\lambda}}{}^{(n)} \underline{k}'_\lambda \simeq \varinjlim_{L(\hat{\lambda})}{}^{(n)} \underline{k} \simeq H_n(E(\lambda);k) \qquad n \geq 0$$

where $\underline{k}$ is the constant projective system k on $L(\lambda)$, and where we denote by $E(\lambda)$ the simplicial set defined by the ordered set $L(\lambda)$, see [La 1, (1.1)], we obtain an exact sequence

$$0 \to \varinjlim_{\hat{\lambda}}{}^{(1)} k(\lambda) \to \varinjlim_{\hat{\lambda}} \underline{k}'_\lambda \to k \to \varinjlim_{\hat{\lambda}} k(\lambda) \to 0$$

and isomorphisms:

$$\varinjlim_{\hat{\lambda}}{}^{(n)} k(\lambda) \simeq H_{n-1}(E(\lambda);k) \qquad n \geq 2$$

Notice that $\varinjlim_{\hat{\lambda}} k(\lambda) = 0$ unless λ is minimal in Λ_+, in which case $\varinjlim_{\hat{\lambda}} k(\lambda) \simeq k$, and $\varinjlim_{\hat{\lambda}}{}^{(1)} k(\lambda) = 0$.

If λ is not minimal, then

$$\varinjlim_{\hat{\lambda}}{}^{(1)} k(\lambda) \simeq \tilde{H}_o(E(\lambda);k)$$

where $\tilde{H}_\bullet$ is the augmented homology.

Summing up we have proved the following

<u>Proposition 1.3</u>

$$\mathrm{Tor}_n^\Lambda(k,k) \simeq \begin{cases} k & n = 0 \\ k^\rho & n = 1 \\ \coprod_{\lambda \in \Lambda_+} \tilde{H}_{n-2}(E(\lambda);k) & n \geq 2 \end{cases}$$

where ρ is the number of minimal elements of Λ_+.

§2 Application to 2-dimensional Torus embeddings

Let $\Lambda' \subseteq \mathbb{Z}_+^2$ be the saturated rational monoid generated by (m_1,n_1) and (m_2,m_2) satisfying the two conditions

i) $(m_i,n_i) = 1 \quad i = 1,2$

ii) The system $\{(m_1,n_1),(m_2,n_2)\}$ is right-oriented i.e. satisfies

$$m_1 \cdot n_2 - m_2 \cdot n_1 = p > 0$$

Whenever needed we shall consider $\mathbb{Z}_+^2$ as embedded in $\mathbb{Q}^2$ or $\mathbb{R}^2$ by the obvious inclusions.

Consider the linear transformation

$$T\colon \mathbb{Q}\times\mathbb{Q} \to \mathbb{Q}\times\mathbb{Q}$$

given by

$$T(m_1,n_1) = (p,0)$$

$$T(m_2,n_2) = (0,p)$$

We may represent T by the 2×2 matrix

$$T = \begin{pmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{pmatrix}$$

We are interested in the image of Λ' under the transformation T, denoted by $\Lambda = T(\Lambda')$. In particular we are interested in the subset $\Lambda \cap [0,p]^2$. If $p = 1$, then $\Lambda = \mathbb{Z}_+^2$, and therefore $A = k[\Lambda] = k[x_1,x_2]$. This case presents no problem, so we assume $p > 1$. Consider the intersection $\Lambda_1 = \Lambda\cap\{(1,n)\in\mathbb{Z}_+^2\}$. The following lemma holds:

Lemma 2.1

There exists $\xi \in \mathbb{Z}_+^2$ with $0 < \xi < p$ such that

$$\Lambda_1 = \{(1,\xi+\eta\cdot p) \mid \eta\in\mathbb{Z}_+\}.$$

Moreover, for this ξ we have

$$\begin{aligned} \Lambda_n &= \Lambda\cap(\{n\}\times\mathbb{Z}) \\ &= \{(n,n\cdot\xi+\eta\cdot p) \mid \eta\in\mathbb{Z} \text{ and } n\cdot\xi+\eta\cdot p\geq 0\}. \end{aligned}$$

Proof

Since $(m_2,n_2) = 1$, there exists an integer pair $(x_o,y_o) \in Z^2$ such that

$$T(x_o,y_o) = (n_2x_o-m_2y_o,-n_1x_o+m_1y_o) \in (\{1\}\times\underline{\underline{Z}}).$$

The set $\{(m_1,n_1),(m_2,n_2)\}$ forms a basis for $\underline{\underline{Q}}^2$, and there exist $\alpha, \beta_o \in \underline{\underline{Q}}$ such that

$$(x_o,y_o) = \alpha(m_1,n_1)+\beta_o(m_2,n_2) \tag{$*$}$$

But T is a linear map so we have

$$T(x_o,y_o) = \alpha\cdot T(m_1,n_1)+\beta_o\cdot T(m_2,n_2)$$

$$= \alpha\cdot(p,0)+\beta_o\cdot(0,p) \in (\{1\}\times\underline{\underline{Z}}).$$

This implies $\alpha = \frac{1}{p}$ and from equation $(*)$ and the fact $(m_1,n_1) = 1$ we deduce that $\beta_o \notin \underline{\underline{Z}}$. So there exists an integer $\mu \in \underline{\underline{Z}}$ such that $0 < \beta_o+\mu < 1$ and

$$T((x_o,y_o)+\mu(m_2,n_2)) = \alpha\cdot(p,o)+(\beta_o+\mu)(0,p) \in (\{1\}\times[0,p]).$$

Put $\beta = \beta_o+\mu$ and $(x,y) = (x_o,y_o)+\mu(m_2,n_2) \in \underline{\underline{Z}}^2_+$.

$\alpha, \beta \in \underline{\underline{Q}}$ are rational numbers, and γ the product of their denominators. The numbers $\gamma\cdot\alpha$, $\gamma\cdot\beta$ are integers, and

$$\gamma\cdot(x,y) \in \Lambda'.$$

Since the monoid Λ' is saturated, it follows that $(x,y) \in \Lambda'$. Let $\xi = \beta\cdot p$. Then $T(n\cdot(x,y)) = (n,n\cdot\xi)$. Now consider the equivalence

$$\begin{aligned} n\cdot\xi + \eta\cdot p &= n\cdot\beta\cdot p + \eta\cdot p \\ &= (n\cdot\beta+\eta)\cdot p \geqslant 0 \\ \iff \quad & n\cdot\beta+\eta \geqslant 0 . \end{aligned}$$

If $n\cdot\xi + \eta\cdot p \geqslant 0$ then we have

$$(n,n\cdot\xi+\eta\cdot p) = T(n(x,y)+\eta(m_2,n_2))$$

$$= T(n\cdot\alpha(m_1,n_1)+(n\cdot\beta+\eta)(m_2,n_2))$$

and $(n,n\cdot\xi+\eta\cdot p) \in \Lambda$. This follows from the fact that an integer pair, positively generated by (m_1,n_1) and (m_2,n_2) is element of Λ'.

Suppose $(x,y),(x',y') \in \Lambda'$ satisfy $T(x,y) \in \Lambda_a$, $T(x',y') \in \Lambda_a$ for some $a \in \underline{\underline{Z}}_+$. Then we have

$$n_2 \cdot x - m_2 \cdot y = n_2 \cdot x' - m_2 \cdot y'$$

or equivalently

$$n_2(x-x') = m_2(y-y')$$

Since $(m_2,n_2) = 1$ this is equivalent to

$$x-x' = c \cdot m_2 \qquad y-y' = c \cdot n_2$$

for some $c \in \underline{\underline{Z}}$. But then we have

$$\begin{aligned}-n_1 \cdot x + m_1 \cdot y &= -n_1(c \cdot m_2 + x') + m_1(y' + c \cdot n_2)\\ &= -n_1 \cdot x' + m_1 \cdot y' - c(n_1 \cdot m_2 - m_1 \cdot n_2)\\ &= -n_1 \cdot x' + m_1 \cdot y' + c \cdot p\end{aligned}$$

It is easy to see that this proves the lemma.

Q.E.D.

Thus we have a complete description of Λ given by

$$\Lambda = \{(a,b) \in \underline{\underline{Z}}_+^2 \mid a \cdot \xi \equiv b \pmod p\}.$$

If we interchange (m_1,n_1) and (m_2,n_2) and apply the proof of Lemma 2.1 we get a number $\eta \in \underline{\underline{Z}}_+$ satisfying

i) $0 < \eta < p$

ii) $\eta \cdot \xi \equiv 1 \pmod p$

The use of this will appear later.

<u>Remark 2.2</u>

One of the advantages with this description of Λ is the following property of Λ: If $\lambda = (a,b)$, $\lambda' = (a',b') \in \Lambda$ and if $\lambda' - \lambda = (a'-a, b'-b) \in \underline{\underline{Z}}_+^2$, then $\lambda' - \lambda \in \Lambda$.
In fact since for $(a,b),(a',b') \in \Lambda$; $b \equiv a \cdot \xi \pmod p$, $b' \equiv a' \cdot \xi \pmod p$ and $a'-a \geq 0$, $b'-b \geq 0$ we find $b'-b = (a'-a) \cdot \xi \pmod p$ therefore $(a'-a, b'-b) \in \Lambda$. Notice that this implies that the order relation on Λ (see §1) induced by the order relation on Λ' is the restriction of the ordinary order relation on $\underline{\underline{Z}}_+^2$.

Definition 2.3

Let $P \in \underline{\underline{Z}}_+^2$. Define the ordered set $\hat{P}$ associated with P by $\hat{P} = \{\lambda\in\Lambda \mid \lambda\leqslant P\} \subseteq \Lambda$. The associated simplicial set will also be denoted by $\hat{P}$.

Correspondingly we shall let $L(P) = \{\lambda\in\Lambda \mid \underset{\neq}{\leqslant} P\}$ also denote the associated simplicial set. (When $P \in \Lambda$, this is preciely the set $E(P)$ of paragraph 1.)

Remark 2.4

Notice that for $P \in \underline{\underline{Z}}_+^2 - \Lambda$ we have $L(P) = \hat{P}$.

Lemma 2.5

Let ξ and η be defined as above. Let $U \subseteq \underline{\underline{Z}}_+^2$ be the set defined by

$$U = \{(a,b)\in\underline{\underline{Z}}_+^2 \mid b>p+a\cdot\xi \text{ or } a>p+b\cdot\eta\}$$

Then for any $P \in U$

$$\tilde{H}_n(L(P)) = 0 \qquad n \geqslant 0.$$

Proof

It is obviously sufficient to prove the lemma in the case where $P = (a,b)$ satisfies the condition $b > p+a\cdot\xi$. Given $P = (a,b)\in\underline{\underline{Z}}_+^2$, and suppose $b > p+a\cdot\xi$. Then there exist integers $\alpha, \beta \in \underline{\underline{Z}}$ such that

$$b - a\cdot\xi = \alpha\cdot p + \beta$$

with $0 < \beta \leqslant p$ and $\alpha \geqslant 1$. We shall prove the lemma by induction on the integer a.

Suppose $a = 0$. Then $L(P)$ has a final object and the homology vanishes.

Suppose $a > 0$. Let $P = (a,b) \in U$, and suppose the formula is valid for all $(m,c) \in U$ with $m < a$. Notice that Lemma 2.1

implies $(a,b-\beta) = (a,a\cdot\xi+\alpha\cdot p) \in \Lambda$.

Now it easy to see that

i) $L(P) = (a-1,b)^{\wedge} \cup (a,b-\beta)^{\wedge}$

ii) $(a-1,b-\beta)^{\wedge} = (a-1,b)^{\wedge} \cap (a,b-\beta)^{\wedge}$

Apply the Mayer-Vietoris sequence and obtain the long exact sequence

$$\cdots \to \tilde{H}_n(a-1,b-\beta) \to \tilde{H}_n(a,b-\beta) \oplus \tilde{H}_n(a-1,b)$$
$$\to \tilde{H}_n(L(P)) \to \tilde{H}_{n-1}(a-1,b-\beta) \to \cdots\cdots$$

where $\tilde{H}_\bullet(P)$ is the homology of the ordered set associated with P. But now we have $b > p+a\cdot\xi > p+(a-1)\cdot\xi$ and $b-\beta = \alpha\cdot p+a\cdot\xi \geq p+a\cdot\xi > p+(a-1)\cdot\xi$, so $(a-1,b) \in U$ and $(a-1,b-\beta) \in U$. The induction hypothesis implies

$$\tilde{H}_n(a-1,b-\beta) = \tilde{H}_n(a-1,b) = 0 \qquad \forall\ n \geq 0.$$

$(a,b-\beta) \in \Lambda$ and $(a,b-\beta)^{\wedge}$ has a final object; therefore

$$\tilde{H}_n(a,b-\beta) = 0 \qquad \forall\ n \geq 0$$

Thus, using the exactness of the above sequence, we get

$$\tilde{H}_n(P) = 0 \qquad \forall\ n \geq 0$$

which proves the lemma.

Q.E.D.

Definition 2.6

Let $P \in \underline{\underline{Z}}_+^2$. The maximal polygon associated with P, M(P) is the set of maximal elements of the convex hull $C(L(P)) \subseteq \underline{\underline{R}}_+^2$ of L(P). $\underline{\underline{R}}_+^2$ is regarded as an ordered set with the obvious ordering relation.

Put $M_o(P) = L(P) \cap M(P)$. Then the following lemma holds.

Lemma 2.7

$M_o(P)$ is the set of maximal elements of $L(P)$.

Proof

Let $\max L(P)$ be the set of maximal elements of $L(P)$. Obviously $M_o(P) \subseteq \max L(P)$. Assume $\lambda \in \max L(P)$ and $\lambda \notin M_o(P)$. $M(P)$ is a convex polygon and λ has to sit strictly below some edge e. Pick vertices of e; $\mu, \mu' \in M_o(P)$, $\mu \neq \mu'$, and consider the element $\eta = \mu+\mu'-\lambda$. Since $\eta \in \underline{\underline{Z}}_+^2$ we have seen (Remark 2.2) that $\eta \in \Lambda$. An easy argument then shows that $\eta \in L(P)$ and that η is above the edge e, a contradiction.

Q.E.D.

It is easily seen that $M(P)$ must lie inside a square, $p\times p$, with P as the maximal point.

Lemma 2.8

For every $P \in \underline{\underline{Z}}_+^2$ with $P \geq (p,p)$, and every $\lambda \in \Lambda$:

$$M(P+\lambda) = M(P)+\lambda$$

Proof

It is enough to show the equality $M_o(P+\lambda) = M_o(P)+\lambda$. So let $\mu \in M_o(P)$. Then $\lambda \leq \mu+\lambda < P+\lambda$. Now choose $\eta \in M_o(P+\lambda)$ such that $\lambda \leq \mu+\lambda \leq \eta < P+\lambda$. Then we have $\mu \leq \eta-\lambda < P$. Since $\mu, \eta, \lambda \in \Lambda$, the remark (2.2) implies $\eta-\lambda \in \Lambda$, thus we get $\mu = \eta-\lambda$ or $\eta = \mu+\lambda$. Consequently $\mu+\lambda \in M_o(P+\lambda)$ and $M_o(P)+\lambda \subseteq M_o(P+\lambda)$. To prove the inverse inclusion, we first notice that if $\mu \in M_o(P+\lambda)$ then $\mu \geq \lambda$. This follows from the fact that $P \geq (p,p)$ and that $M_o(P+\lambda)$ sits inside a square $p\times p$ with $P+\lambda$ as the maximal point.

So let $\mu \in M_o(P+\lambda)$. Then $\mu < P+\lambda$ or $\mu-\lambda < P$. Choose $\eta \in M_o(P)$ such that $\mu-\lambda \leq \eta < P$. This implies $\mu \leq \eta+\lambda < P+\lambda$. But $\mu \in M_o(P+\lambda)$ so the last equation implies $\mu = \eta+\lambda$ which proves the lemma.

Q.E.D.

Definition 2.9

Let $P \in \underline{\underline{Z}}_+^2$ and denote by:

$\{V_{i,j}(P)\}_{\substack{i=1,2,\dots,n \\ j=1,2,\dots,m_i}}$ the lattice point on $M(P)$ where

i is the number of the edge counted from right, and

j is the number of the lattice point on the edge, also counted from right.

Put $V_i = V_{i,1}$ for $i = 1,2,\dots,n$ and $V_{n+1} = V_{n,m_n}$. Notice that for $i = 1,2,\dots,n$ we have $m_i \geqslant 2$ and $V_{i,m_i} = V_{i+1}$.

Denote by

$\{e_{i,j}(P)\}_{\substack{i=1,\dots,n \\ j=1,\dots,m_i}}$ the edges between $V_{i,j}(P)$ and $V_{i,j+1}(P)$

For $i = 1,\dots,n$ $e_i(P) = \bigcup_{j=1}^{m_i-1} e_{i,j}(P)$ are then the edges of $M(P)$.

Let

$\{s_i(P)\}_{i=1,\dots,n}$ be the absolute values of the slopes of the $e_i(P)$'s

and let finally

$\{X_i(P)\}_{i=1,\dots,n}$, $X_i = X(V_{i,2}) - X(V_{i,1})$

and $\{Y_i(P)\}_{i=1,\dots,n}$, $Y_i = Y(V_{i,2}) - Y(V_{i,1})$ be the differences in the values of the coordinates of $V_{i,1}(P)$ and $V_{i,2}(P)$.

It is clear that $M(P)$ is determined by these families of numbers. Moreover, we deduce the following

$$Y_i(P) = s_i(P) \cdot X_i(P) \qquad i = 1,\dots,n$$

Put, as a shorthand, $\alpha_i(P) = X(P) - X(V_i(P))$ and $\beta_i(P) = Y(P) - Y(V_i(P))$, and notice that $\alpha_{i+1}(P) > \alpha_i(P)$, $\beta_{i+1}(P) < \beta_i(P)$.

For every pair (i,j), $i = 1,\dots,n$, $j = 1,\dots,m_i$ the proof of Lemma 2.7 gives the existence of unique points $Q_{i,j}(P) = (X(V_{i,j}(P)), Y(V_{i,j+1}(P)))$ and $P_{i,j}(P) = (X(V_{i,j+1}(P)), Y(V_{i,j}(P)))$

with the properties

$$L(Q_{i,j}(P)) = V_{i,j}(P)^\wedge \cup V_{i,j+1}(P)^\wedge$$

$$P_{i,j}(P)^\wedge = V_{i,j}(P)^\wedge \cap V_{i,j+1}(P)^\wedge$$

Definition 2.10

Denote by P_i the unique element of $\mathbb{Z}_+^2$ such that $P_i^\wedge = \bigcap_{j=1}^{m_i} P_{i,j}{}^\wedge$.

Let $\lambda \in \Lambda$ and let n be the number of edges of $M(\lambda)$. The next lemma will show that $M(P_i(\lambda))$ is congruent to the polygon $M(\lambda)$ with the i-th edge removed. We shall therefore index the vertices and the edges etc. of $M(P_i(\lambda))$ by restricting the corresponding indexing of $M(\lambda)$. Thus $e_i(P_i(\lambda))$ does not exist and, modulo translation, $e_j(P_i(\lambda))$ is congruent to $e_j(\lambda)$ whenever $i \neq j$. Likewise $V_i(P_i(\lambda))$ does not exist and

$V_{i-1,m_{i-1}}(P_i(\lambda)) = V_{i+1}(P_i(\lambda))$. Notice that the intersection points $P_j(P_i(\lambda))$ and $P_i(P_j(\lambda))$ are, in general, different when $i \neq j$. Let $P_{\{i,j\}}(\lambda)$ denote their intersection, i.e. the unique element of $\mathbb{Z}_+^2$ such that

$$P_{\{i,j\}}(\lambda)^\wedge = P_i(P_j(\lambda))^\wedge \cap P_j(P_i(\lambda))^\wedge.$$

In general we make the following definition, ($\lambda >> 0$ means $X(\lambda), Y(\lambda) >> 0$).

Definition 2.11

Let $\lambda \in \Lambda$ and $M(\lambda)$ as above, $\lambda >> 0$. Let $I \subseteq \{1,2,\dots,n\}$ be a set of integers different from the empty set. Define $P_I(\lambda)$ recursively via the intersection property

$$P_I(\lambda)^\wedge = \bigcap_{i \in I} P_i(P_{I-\{i\}}(\lambda))^\wedge$$

where $P_\emptyset(\lambda) = \lambda$.

Lemma 2.12 will show that $M(P_{\{i,j\}}(\lambda))$ is congruent to $M(\lambda)$ with the i-th and the j-th edge removed, and that in general $M(P_I(\lambda))$ is congruent to $M(\lambda)$ with the i-th edge removed for every $i \in I \subseteq \{1,2,\ldots,n\}$.

Lemma 2.12

Let $\lambda, M(\lambda)$ be as above and let $I \subseteq \{1,2,\ldots,n\}$ be a set of integers, the empty set included.

i) The maximal polygon $M(P_I(\lambda))$ of the set $P_I(\lambda)^\wedge$ is congruent to the maximal polygon $M(\lambda)$ of $\lambda^\wedge$ with the i-th edge removed for every $i \in I$.

ii) Let for $i = 1,2,\ldots,n$ $r_i = (\alpha_i, \beta_i)$. Then for every $j \notin I$

$$P_j(P_I(\lambda)) = \lambda - \sum_{i\in I} r_i - r_{n+1} - \sum_{\substack{h\notin I\\ h\geqslant j}} e_h - (\alpha_{j+1} - \alpha_j, 0)$$

where e_h is the vector $\overrightarrow{V_h V_{h+1}}$ associated to the edge $e_h(\lambda)$, and $\alpha_i = \alpha_i(\lambda)$, $\beta_i = \beta_i(\lambda)$.

Proof

We shall prove the lemma by induction on the number of elements of I, $\#I = k$.

The case $k = 0$ is vacous; just notice that $e_h = r_h - r_{h+1}$ so $\lambda - r_{n+1} - \sum_{\substack{h\notin I\\ h\geqslant j}} e_h = \lambda - r_j$.

Suppose the lemma holds for $\#I = k-1$, $0 < k \leqslant n$, and let $I \subseteq \{1,.,\ldots,n\}$ with $I = k$. To simplify notation, write for every $i \in I$; $P_{I,i}(\lambda) = P_i(P_{I-\{i\}}(\lambda))$. Obviously

$$P_I(\lambda)^\wedge = \bigcap_{i\in I} P_{I,i}(\lambda)^\wedge = (\min_{i\in I} X(P_{I,i}(\lambda)),\ \min_{i\in I} Y(P_{I,i}(\lambda)))^\wedge$$

so we have to study the relation between the intersection points $P_{I,i}(\lambda)$. The induction hypothesis gives

$$P_{I,j}(\lambda) = \lambda - \sum_{i\in I-\{j\}} r_i - r_{n+1} - \sum_{\substack{h\notin I-\{j\}\\ h\geq j}} e_h - (\alpha_{j+1}-\alpha_j, 0)$$

$$= \lambda - \sum_{i\in I} r_i + \sum_{\substack{h\in I\\ h>j}} e_h - (\alpha_{j+1}-\alpha_j, 0) \qquad (**)$$

Consider the last part of the above sum, $\Sigma_{h\in I, h>j} e_h + (X(e_j), 0)$. The fact that $\alpha_{j+1} > \alpha_j$ and $\beta_{j+1} < \beta_j$ shows that the X-value of this vector increases and the Y-value decreases with increasing $j \in I$. So it follows that

$$P_I(\lambda)^{\wedge} = P_{I,i_1}(\lambda)^{\wedge} \cap P_{I,i_k}(\lambda)^{\wedge}$$

$$= (X(P_{I,i_1}(\lambda)), Y(P_{I,i_k}(\lambda)))^{\wedge}$$

where $I = \{i_1 < i_2 < \cdots < i_k\}$. From $(**)$ we deduce that $X(P_I(\lambda)) = X(P_{I,i_1}(\lambda)) = X(\lambda - \Sigma_{i\in I} r_{i+1})$ and $Y(P_I(\lambda)) = Y(P_{I,i_k}(\lambda)) = Y(\lambda - \Sigma_{i\in I} r_i)$. In addition we get the two inequalities

$$P_{I,i_1}(\lambda) < \lambda - \sum_{i\in I} r_{i+1}$$

$$P_{I,i_k}(\lambda) < \lambda - \sum_{i\in I} r_i$$

Obviously $\lambda - \Sigma_{i\in I} r_i \geq \lambda - \Sigma_{i\in I} r_{i+1} - r_1$ and $\lambda - \Sigma_{i\in I} r_{i+1} \geq \lambda - \Sigma_{i\in I} r_i - r_{n+1}$ and therefore $\lambda - \Sigma_{i\in I} r_{i+1} - r_1 < P_I(\lambda)$ and $\lambda - \Sigma_{i\in I} r_i - r_{n+1} < P_I(\lambda)$. Thus $\lambda - \Sigma_{i\in I} r_{i+1} - r_1$ and $\lambda - \Sigma_{i\in I} r_i - r_{n+1}$ are the "endpoints" of the maximal polygon of $P_I(\lambda)$.

Using the fact that $\Sigma_{h=1}^{n} e_h = r_1 - r_{n+1}$ we have the equalities

$$\lambda - \sum_{i\in I} r_{i+1} - r_1 = \lambda - \sum_{i\in I} r_{i+1} - r_{n+1} - \sum_{h=1}^{n} e_h$$

$$= \lambda - \sum_{i\in I} (r_{i+1} - r_i) - \sum_{i\in I} r_i - r_{n+1} - \sum_{h=1}^{n} e_h$$

$$= \lambda - \sum_{i\in I} r_i - r_{n+1} - \sum_{h\notin I} e_h$$

This proves part i)

To prove ii) observe that i) implies

$$X(P_j(P_I(\lambda))) = X(\lambda - \sum_{i\in I} r_{i+1} - r_1 + \sum_{\substack{h\notin I \\ h\leq j}} e_h)$$

$$= X(\lambda - \sum_{i\in I} r_i - r_{n+1} - \sum_{\substack{h\notin I \\ h\geq j}} e_h + e_j)$$

We already know

$$Y(P_j(P_I(\lambda))) = Y(\lambda - \sum_{i\in I} r_i - r_{n+1} - \sum_{\substack{h\notin I \\ h\geq j}} e_h)$$

and therefore

$$P_j(P_I(\lambda)) = \lambda - \sum_{i\in I} r_i - r_{n+1} - \sum_{\substack{h\notin I \\ h\geq j}} e_h + (X(e_j),0)$$

which is the claimed equation for $P_j(P_I(\lambda))$, $\#I = k$.

Q.E.D.

<u>Corollary 2.13</u>

$P_I(\lambda) \in \Lambda$ if and only if $I = \{1,2,\ldots,n\}$ or $I = \emptyset$.

<u>Proof</u>

$0 \leq \sum_{i\in I} \alpha_{i+1} - \alpha_i \leq p$ with equality on the left or right if and only if $I = \emptyset$ respectively $I = \{1,2,\ldots,n\}$.

Q.E.D.

In the next few lemmas we shall relate the homology of $L(P)$ to the homology of ordered sets connected with $M(P)$. Let $P \in \underline{\underline{Z}}_+^2$ and assume $P >> 0$. Put $M = M(P)$, $V_i = V_i(P)$ etc.

<u>Lemma 2.14</u>

In the situation above we have an isomorphism for every $r \geq 0$

$$\bigoplus_{j=1}^{m_i-1} \tilde{H}_r(P_{i,j}) \simeq \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r(P_i)$$

<u>Proof</u>

Define $V = V_i - r_{i+1} \in \Lambda$. Then for $j = 1,2,\ldots,m_i$

$$P_{i,j}^{\wedge} = (X(P_{i,j}),Y(V))^{\wedge} \cup (X(V),Y(P_{i,j}))^{\wedge}$$

$$V^{\wedge} = (X(P_{i,j}),Y(V))^{\wedge} \cap (X(V),Y(P_{i,j}))^{\wedge}$$

The proof of this is left to the reader; an argument analogue to the proof of Lemma 2.7 will give the result.

Applying the reduced Mayer-Vietoris sequence, and using the fact that $V^\wedge$ has a final object, we get an isomorphism for $j = 1,2,\dots,m_i-1$ and $r \geq 0$

$$\tilde{H}_r(P_{i,j}) \simeq \tilde{H}_r(X(P_{i,j}),Y(V)) \oplus \tilde{H}_r(X(V),Y(P_{i,j})) \qquad (***)$$

But we also have for $j = 2,3,\dots,m_i-1$

$$L(V_{i,j}) = (X(P_{i,j-1}),Y(V))^\wedge \cup (X(V),Y(P_{i,j}))^\wedge$$

$$V^\wedge = (X(P_{i,j-1}),Y(V))^\wedge \cap (X(V),Y(P_{i,j}))^\wedge$$

So for every $r \geq 0$

$$\tilde{H}_r(L(V_{i,j})) \simeq \tilde{H}_r(X(P_{i,j-1}),Y(V))) \oplus \tilde{H}_r(X(V),Y(P_{i,j})) \qquad (****)$$

Summing over $j = 1,2,\dots,m_i-1$ the isomorphisms (***), changing paranthesises, and using (****) we get

$$\bigoplus_{j=1}^{m_i-1} \tilde{H}_r(P_{i,j}) \simeq \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r(X(V),Y(P_{j,1})) \oplus \tilde{H}_r(X(P_{i,m_i-1}),Y(V))$$

$$\simeq \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r(P_i) \qquad \forall\, r \geq 0$$

Q.E.D.

The next lemma gives the relation between the homology of $L(P)$ and the homology of the intersesction points P_i.

Lemma 2.15

Let the symbols $P,M,V_{i,j}$ be as above; n is the number of edges of M. There is an isomorphism for every $r > 0$

$$\tilde{H}_r(L(P)) \simeq \left[\bigoplus_{\substack{i=1,2,\dots,n \\ j=2,3,\dots,m_i-1}} \tilde{H}_{r-1}(L(V_{i,j})) \right] \oplus \left[\bigoplus_{i=1}^{n} \tilde{H}_{r-1}(P_i) \right]$$

Proof

As a consequence of Lemma 2.7 we have

$$L(P) = \bigcup_{\substack{i=1,\dots,n \\ j=1,\dots,m_i-1}} Q_{i,j}^{\wedge}$$

where $Q_{i,j} = Q_{i,j}(P)$ and the intersections $Q_{i,j}^{\wedge} \cap Q_{i,j+1}^{\wedge}$ and $Q_{i,m_i-1}^{\wedge} \cap Q_{i+1,1}^{\wedge}$ always are ordered sets with $V_{i,j+1}$, respectively $V_{i+1,1}$, as final elements. Using the Mayer-Vietoris sequence repeatedly we find

$$\tilde{H}_r(L(P)) \simeq \bigoplus_{\substack{i=1,\dots,n \\ j=1,\dots,m_i-1}} \tilde{H}_r(Q_{i,j})$$

Apply the Mayer-Vietoris sequence once more to the system $(Q_{i,j}^{\wedge}, V^{\wedge}_{i,j}, V^{\wedge}_{i,j+1}, P^{\wedge}_{i,j})$. Since $V^{\wedge}_{i,j}$ has a final element we obtain an isomorphism for every $r > 0$

$$\tilde{H}_r(Q_{i,j}) \simeq \tilde{H}_{r-1}(P_{i,j})$$

where $i = 1,\dots,n$, $j = 1,\dots,m_i-1$. Using Lemma 2.14 the lemma follows immediately.

Q.E.D.

Lemma 2.16

Let $\lambda \in \Lambda$ and let $I \subseteq \{1,2,\dots,n\}$. Suppose $2 \leq \#I=k \leq n$. Let $P_I = P_I(\lambda)$ and $P_{I,i} = P_i(P_{I-\{i\}}(\lambda))$. Then for every $r \geq 0$ we have an isomorphism

$$\bigoplus_{i\in I} \tilde{H}_r(P_{I,i}) \simeq \bigoplus_{\substack{i\in I \\ i\neq i_k}} \tilde{H}_r(L(V_i(P_{I-\{i\}}))) \oplus \tilde{H}_r(P_I)$$

where $I = \{i_1 < \dots < i_k\}$.

Proof

Define $P_{I,i,j}$ via the intersection property

$$P^{\wedge}_{I,i,j} = P^{\wedge}_{I,i} \cap P^{\wedge}_{I,j}$$

for every pair $i,j \in I$. From the proof of Lemma 2.12 we deduce

$P^{\wedge}_{I,i_1,i_j} = P^{\wedge}_{I,i_1,i_{j-1}} \cap P_{I,i_j}$ for every $j = 2,\ldots,k$.

For $j = 1,\ldots,k-1$ we have the inequalities

$$P^{\wedge}_{I,i_1,i_j} < P_{I,i_j} < V_{i_j}(P_{I-\{i_j\}})$$

and from Lemma 2.12 the equality

$$P_{I,i_{j+1}} = V_{ij}(P_{I-\{i_j\}}) - (0,\beta_{i_{j+1}} - \beta_{i_{j+1}+1}) \qquad (*****)$$

Thus $P_{I,i_{j+1}} < V_{i_j}(P_{I-\{i_j\}})$. In addition we have the inequality $V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} < P_{I,i_1,i_{j+1}}$. The last statement is an immediate consequence of the two relations

$$V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} < P_{I,i_{j+1}}$$

$$V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} < P_{I,i_1}$$

The first follows from equation (*****), the other is easily deduced from Lemma 2.12 using the analytic formula for P_{I,i_1}.

Thus we have

i) $V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} \leqslant P_{I,i_1,i_j} < V_{i_j}(P_{I-\{i_j\}})$

ii) $V_{i_j}(P_{I-\{i_j\}}) - r_{i_j+1} \leqslant P_{I,i_{j+1}} < V_{i_j}(P_{I-\{i_j\}})$

iii) $X(P_{I,i_{j+1}}) = X(V_{i_j}(P_{I-\{i_j\}}))$

iv) $Y(P_{I,i_1,i_j}) = Y(V_{i_j}(P_{I-\{i_j\}}))$

Applying the Mayer-Vietoris sequence three times we obtain for every $r \geqslant 0$ an isomorphism

$$\tilde{H}_r(P_{I,i_1,i_j}) \oplus \tilde{H}_r(P_{I,i_{j+1}}) \simeq \tilde{H}_r(L(V_{i_j}(P_{I-\{i_j\}}))) \oplus \tilde{H}_r(P_{I,i_1,i_{j+1}})$$

But $P_{I,i_1,i_k} = P_I$ so an iterated use of the described process will give the lemma.

Q.E.D.

We are now in position to state and prove the main result of this paragraph.

Theorem 2.17

Let $\lambda \in \Lambda$, $\lambda >> 0$ and $P_I = P_I(\lambda)$, as above. Let n be the number of edges of $M(\lambda)$. Then for every integer $r \geq n$ there is an isomorphism

$$\tilde{H}_r(L(\lambda)) = \left[\bigoplus_{k=1}^{n} \bigoplus_{\#I=k-1} \bigoplus_{i\notin I} \bigoplus_{j=2}^{m_i-1} \tilde{H}_{r-k}(V_{i,j}(P_I))\right]$$

$$\oplus \left[\bigoplus_{k=2}^{n} \bigoplus_{\#I=k} \bigoplus_{\substack{i\in I\\ i\neq i_k}} \tilde{H}_{r-k}(L(V_i(P_{I-\{i\}})))\right]$$

where $P_\emptyset = \lambda$ and $I = \{i_1 < \cdots < i_k\}$.

Proof This is just an iterated use of Lemma 2.15 and Lemma 2.16, where we for each step increase the order of I. Remember that if $I \neq \emptyset$, $P_I \in \Lambda$ if and only if $I = \{1,\ldots,n\}$. Therefore the process stops when $\#I = n$. Moreover, for $\#I < n$ we have $L(P_I) = \hat{P_I}$.

Q.E.D.

Now go back to the calculation of the right-hand side of the equation in Proposition 1.3. In Theorem 2.17 we made the assumption $\lambda >> 0$. In fact it suffices to know that $\lambda > \Sigma_{i=1}^{n+1} r_i$. This is to ensure that all the points needed in Lemma 2.16 really are elements of Λ.

Put

$$Z = \{\lambda\in\Lambda \mid \lambda > \sum_{i=1}^{n+1} r_i\}$$

and recall the definition of

$$U = \{(a,b)\in \underline{\underline{Z}}_+^2 \mid b>p+a\cdot\xi \text{ or } a>p+b\cdot\xi\}$$

see(2.5).

Put

$$W = (\Lambda - Z) \cap (\Lambda - U)$$

W is a finite set containing all $\lambda \in \Lambda - Z$ with the property $\tilde{H}_{\bullet}(\lambda) \neq 0$. Since for each $\lambda \in \Lambda$, $L(\lambda)$ is a finite ordered set, there exist N' such that $\tilde{H}_m(L(\lambda)) = 0$ for all $m \geq N'$. Since W is finite we may choose N' such that $\tilde{H}_m(L(\lambda)) = 0$ for all $m \geq N'$ and all $\lambda \in W$. Putting $h_m(L(\lambda)) = \dim_k \tilde{H}(L(\lambda))$ we have thus proved

$$\sum_{\lambda \in Z} h_m(L(\lambda)) = \sum_{\lambda \in \Lambda} h_m(L(\lambda))$$

for every $m \geq N'$. Going back to Theorem 2.17 we see that the problem is to calculate the number $\Sigma_{\lambda \in Z} h_{m-k}(L(V_{i,j}(P_I(\lambda))))$. So we need a lemma.

Lemma 2.18

Let $Z \subseteq \Lambda$ and N' be defined as above. Let $N = N'+n$. Pick $m \geq N$ and let (k,I,i,j) be a quadruple which occurs in Theorem 2.17. Then we have the equality

$$\sum_{\lambda \in Z} h_{m-k}(L(V_{i,j}(P_I(\lambda)))) = \sum_{\lambda \in Z} h_{m-k}(L(\lambda))$$

Proof

The map $\lambda \to V_{i,j}(P_I(\lambda))$ from Z into Λ, is obviously a rigid translation. Of course we have $\lambda \geq V_{i,j}(P_I(\lambda))$ so

$$Z \subseteq \{\lambda \in \Lambda \mid \exists \lambda' \in Z \text{ with } \lambda = V_{i,j}(P_I(\lambda'))\}.$$

Let $\lambda' \in Z$ with $V_{i,j}(P_I(\lambda')) \notin Z$. We have $m-k \geq N-k \geq N'$ and by definition of N'; $h_{m-k}(L(V_{i,j}(P_I(\lambda')))) = 0$. Since

$$\sum_{\lambda \in Z} h_{m-k}(L(V_{i,j}(P_I(\lambda)))) = \sum_{\lambda \in Z} h_{m-k}(L(\lambda)) + \sum_{\lambda' \in Z'} h_{m-k}(L(V_{i,j}(P_I(\lambda'))))$$

where $Z' \equiv \{\lambda' \in Z \mid V_{i,j}(P_I(\lambda')) \notin Z\}$ we have proved the lemma.

Q.E.D.

Theorem 2.19

Let the number N be as above. Let for every $m \geqslant N$ $\gamma_m = \Sigma_{\lambda\in\Lambda} h_m L(\lambda)$. Then there exists a recursion in the γ's : $\gamma_m = \Sigma_{k=1}^{h} R_k \cdot \gamma_{m-k}$ given by

$$R_k = \binom{n-1}{k-1} \cdot S + \binom{n}{k}(k-1) \qquad k = 1,2,\ldots,n$$

where n is the number of edges of the maximal polygon $M(\lambda)$ of λ, $\lambda >> 0$, and $S = \Sigma_{i=1}^{n} (m_i - 1)$, where m_i is the number of lattice points on the i-th edge of $M(\lambda)$.

Proof

Due to Lemma 2.18 and Theorem 2.17 the only problem is to calculate the sums $(I = \{i_1 < \cdots < i_k\})$

$$S_1 = \sum_{I=k-1} \sum_{i\notin I} \sum_{j=1}^{m_i - 1} \gamma_{m-k}$$

$$S_2 = \sum_{I=k} \sum_{\substack{i\in I \\ i\neq i_k}} \gamma_{m-k}$$

This is a purely combinatorial problem and it is easy to show that

$$S_1 = \binom{n-1}{k-1} \cdot S \cdot \gamma_{m-k}$$

$$S_2 = \binom{n}{k} \cdot (k-1) \cdot \gamma_{m-k}$$

which proves the theorem.

Q.E.D.

Corollary 2.20

Let $\Lambda' \subseteq \mathbb{Z}_+^2$ be a saturated rational monoid, and let $k[\Lambda']$ be the associated monoid algebra. Consider the corresponding isolated singularity of the affine scheme $X = \operatorname{Spec} k[\Lambda']$. The Betti serie $B(t) = \Sigma_{n\geqslant o} \beta_m t^m$ of the local ring of this singularity is rational with denomiator

$$-1 + \sum_{k=1}^{n} \left[\binom{n-1}{k-1}\cdot S+\binom{n}{k}(k-1)\right]t^k$$

Proof

Follows immediately from Theorem 2.17 and the formula of Proposition 1.3 implying $\beta_m = \gamma_{m-2}$ for $m >> 0$.

Q.E.D.

BIBLIOGRAPHY

[La 1] LAUDAL, O.A., Sur la théorie des limites projectives et inductives. Théorie homologique des ensembles ordonnés.
Annals Sci. de l'Ecole Normale Supérreure. 3^{e} série t.82 (1965), pp. 241-296.

[La 2] LAUDAL, O.A., Groups and monoids and their algebras. A cohomological study I.
Preprint Series, Institute of Mathematics, University of Oslo. No 12, (1982).

[Od] ODA, T., Torus Embeddings and Application, Tata Institute of Fundamental Research. Bombay (1978).