INTRODUCTION

Let G be a Lie group, $\hat{G}$ its unitary dual space consisting of the set of all equivalence classes of continuous irreducible unitary representations of G endowed with the hull-kernel (Fell) topology. The orbit space $\mathfrak{g}^{*}/G$, under the coadjoint action of G on the real dual space $\mathfrak{g}^{*}$ of the Lie algebra $\mathfrak{g}$ of G, is given the quotinent topology from $\mathfrak{g}^{*}$. If G is an exponential Lie group, i.e. the roots of the adjoint representation of $\mathfrak{g}$ are of the form $(1+it)f$, $t\in R$, $f\in {}^{*}$; the Kirillov correspondence is a bijection of $\mathfrak{g}^{*}/G$ onto $\hat{G}$, and Pukanszky has shown that the map $\mathfrak{g}^{*}/G\to\hat{G}$ is continuous, [11; Proposition 1]. It is still an open question if the inverse map $\Omega:\hat{G}\to\mathfrak{g}^{*}/G$ is continuous. For nilpotent G this was settled in [4], and for $*$-regular exponential groups in [3]. See also [6].

In the present article we show that the restriction of Ω to the subspace $\hat{G}_{\infty}$ of $\hat{G}$, consisting of all equivalence classes of infinite dimensional representations, is in fact continuous.

NOTATIONS. Throughout the paper we shall use basic results from the theory of induced representations of Mackey, [9], which we assume known to the reader. If K is a closed subgroup of the Lie group G, and S a unitary representation of K, $\mathrm{ind}_{K}^{G}(S)$ denotes the unitary representation of G induced from S.

CONTINUITY ON $\hat{G}_\infty$.

THEOREM. Let G be an exponential Lie group, $\mathfrak{g}^*$ the dual of its Lie algebra. Then the restriction of the Kirillov map to the subspace $(\mathfrak{g}^*/G)_\infty$ of the coadjoint orbit space, consisting of the set of all orbits that correspond to elements of $\hat{G}_\infty$, is bicontinuous, where these subspaces are assumed to be equipped with the relativized quotient topology and hull-kernel topology, respectively.

PROOF. Continuity of the map $\mathfrak{g}^*/G \to \hat{G}$ was shown in [11; Proposition 2]. Let $\Omega : \hat{G} \to \mathfrak{g}^*/G$ denote the inverse map. We proceed to show $\Omega|\hat{G}_\infty$ is continuous, assuming inductively that the result is true for all exponential Lie groups of dimension smaller than $\dim(G)$.

Let $\{T_m\}_{m=1}^{\infty}$ denote a sequence of elements in $\hat{G}_\infty$ that converges to T_0, $T_0 \in \hat{G}_\infty$. We are going to prove that a subsequence of $\{\Omega(T_m)\}_{m=1}^{\infty}$ converges to the orbit $\Omega(T_0)$ in the topology of $\mathfrak{g}^*/G$ relativized to $(\mathfrak{g}^*/G)_\infty$. We fix a closed normal connected subgroup N of codimension one in G. Applying the Mackey machine to N and G, [9; Theorem 8.1], we shall partition the proof according to the following four cases, taking a subsequence of $\{T_m\}_{m=1}^{\infty}$ (also denoted by $\{T_m\}_{m=1}^{\infty}$) if necessary.

(I) The restriction $S_m = T_m|N$ is irreducible for each $m=0,1,2,\ldots$ In this case T_m is an extension of S_m for every m.

By continuity of the restriction map we have $S_m \to S_0$ in $\hat{N}_\infty$; and by our inductive hypothesis the sequence of orbits $\{\Omega(S_m)\}_{m=1}^{\infty}$ conververges to $\Omega(S_0)$ in $(\mathfrak{n}^*/N)_\infty$. Hence there exist functionals

f_m in $\Omega(S_m)$, $m=0,1,2,\ldots$, such that $f_m \underset{m}{\to} f_0$. Let $V=\mathbb{R}e$ be a complementary subspace to $\mathfrak{n}$ in $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{n}+V$, and let e^* in $\mathfrak{g}^*$ be the element dual to e. We shall identify $\mathfrak{n}^*$ with a subspace of $\mathfrak{g}^*$ by means of the relation $\mathfrak{g}^*=\mathbb{R}e^*+\mathfrak{n}^*$. Thus, regarding f_m as an element of $\mathfrak{g}^*$, let $\tilde{S}_m$ be the class in $\hat{G}$ corresponding to the orbit of f_m, $m=0,1,2,\ldots$. $\tilde{S}_m$ is known to be an extension of S_m hence, by Mackey, there exists a character χ_m of G, with $\chi_m|_N=1$, such that $T_m=\tilde{S}_m\cdot\chi_m$. Now $\{f_m\}_{m=1}^{\infty}$ converges to f_0 in $\mathfrak{g}^*$, and therefore $\tilde{S}_m \underset{m}{\to} \tilde{S}_0$ by continuity of the Kirillov map, [11; Proposition 1]. We fix an infinite dimensional Hilbert space H_∞ with a countable basis. The space of all irreducible unitary representations of G on H_∞, denoted by $\mathrm{Irr}_\infty(G)$, is topologized as in [5;18.1.9], then the canonical map $\mathrm{Irr}_\infty(G)\to\hat{G}$ which assigns to each representation its unitary equivalence class, is continuous and open, [5;3.5.8]. Therefore we can find representations $t_m, \tilde{s}_m$ in $\mathrm{Irr}_\infty(G)$ of classes $T_m, \tilde{S}_m$ respectively, so that $t_m \underset{m}{\to} t_0$ and $\tilde{s}_m\to\tilde{s}_0$. This means

$$\|t_m(g)v-t_0(g)v\| \underset{m}{\to} 0$$

and

$$\|\tilde{s}_m(g)v-\tilde{s}_0(g)v\| \underset{m}{\to} 0\ ,$$

uniformly on compacta in G, for all v in H_∞, where we let $\|\cdot\|$ denote the norm in H_∞, [5; Proposition 18.1.9]. Let $v\in H_\infty$, $g\in G$. Then

$$\begin{aligned}
|\chi_0(g)-\chi_m(g)|\cdot\|v\| &= \|\chi_0(g)\tilde{s}_0(g)v-\chi_m(g)\tilde{s}_0(g)v\| \\
&< \|\chi_0(g)\tilde{s}_0(g)v-\chi_m(g)\tilde{s}_m(g)v\| + \|\chi_m(g)\tilde{s}_m(g)v-\chi_m(g)\tilde{s}_0(g)v\| \\
&= \|\chi_0(g)\tilde{s}_0(g)v-\chi_m(g)\tilde{s}_m(g)v\| + \|\tilde{s}_m(g)v-\tilde{s}_0(g)v\| \underset{m}{\to} 0\ ,
\end{aligned}$$

uniformly on compacta in G. It follows that the sequence of characters $\{\chi_m\}_{m=1}^{\infty}$ converges to χ_0. Let h_m be the functional in $\mathfrak{g}^*$ corresponding to χ_m; m=0,1,2,... By the above f_m+h_m lies in the orbit $\Omega(T_m)$ and $f_m+h_m \underset{m}{\to} f_0+h_0$. Therefore we have shown that $\Omega(T_m)\underset{m}{\to}\Omega(T_0)$.

(II) Each T_m, m=0,1,2,..., is induced from a representation S'_m of N, $T_m=\mathrm{ind}_N^G(S'_m)$.

By the Mackey machine the restrictions $T_m|_N$ are supported on the orbits $G\cdot S'_m$ in $\hat{N}$ under the action of G by conjugation. Now $T_m|_N \underset{m}{\to} T_0|_N$, hence we can find a sequence $\{x_m\}_{m=0}^{\infty}$ of elements in G such that $x_m\cdot S'_m \underset{m}{\to} x_0\cdot S'_0$. We put $S_m=x_m\cdot S'_m$; then $T_m=\mathrm{ind}_N^G(S_m)$, m=0,1,2,... (the stability groups under G are all equal to N). Applying our inductive hypothesis we obtain $\Omega(S_m)\underset{m}{\to}\Omega(S_0)$ in $\mathfrak{n}^*/N$; thus we can find functionals f_m in $\Omega(S_m)$ with $f_m \underset{m}{\to} f_0$. Using the fact that $f_m+\mathfrak{n}^{\perp}\subset\Omega(T_m)$ we conclude that $\Omega(T_m)\underset{m}{\to}\Omega(T_0)$.

(III) $T_0|_N=S_0$ is irreducible and each T_m, m=1,2,3,..., is induced from a representation S'_m of N, $T_m=\mathrm{ind}_N^G(S'_m)$.

Arguing as in (II) we can find representations S_m in the orbit of $T_m|_N$, m=1,2,3,..., such that $S_m \underset{m}{\to} S_0$, $T_m=\mathrm{ind}_N^G(S_m)$, and by our inductive hypothesis, functionals f_m in $\Omega(S_m)$, m=0,1,2,..., with $f_m \underset{m}{\to} f_0$. As in (I) $T_0=\tilde{S}_0\cdot\chi_0$ where $\tilde{S}_0$ denotes the element in $\hat{G}_\infty$ corresponding to f_0; and the character χ_0 is identically one on N, and is given by a functional h_0. Using that T_m is induced from S_m, m>0, we have $\Omega(T_m)\supset f_m+\mathfrak{n}^{\perp}$, hence $f_m+h_0\in\Omega(T_m)$, and clearly $f_m+h_0 \underset{m}{\to} f_0+h_0$. We have shown that $\Omega(T_m)\underset{m}{\to}\Omega(T_0)$.

(IV) $T_0=\mathrm{ind}_N^G(S_0)$ and $T_m|_N=S_m$ is irreducible, $m=1,2,3,\ldots$ From the description of the hull-kernel topology it follows that the sequence $\{S_m\}_{m=1}^{\infty}$ converges to each element in the G-orbit of $T_0|_N$, in particular $S_m \underset{m}{\rightarrow} S_0$; and by virtue of the inductive hypothesis we can find functionals f_m in $\Omega(S_m)$, $m=0,1,2,\ldots$, with $f_m \rightarrow_m f_0$. For each $m=0,1,2,\ldots$, we let t_m be an element of $\mathrm{Irr}_\infty(G)$ in the class of T_m, so that $t_m \underset{m}{\rightarrow} t_0$ in the topology of $\mathrm{Irr}_\infty(G)$ (the canonical map $\mathrm{Irr}_\infty(G)\rightarrow\hat{G}_\infty$ is open). In particular $\{t_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathrm{Irr}_\infty(G)$ and, writing $t_m=\tilde{s}_m\cdot\chi_m$, we see as in case (I) that for $v\in H_\infty$, $g\in G$,

$$|\chi_m(g)-\chi_n(g)|\cdot\|v\| \leq \|\chi_m(g)\tilde{s}_m(g)v-\chi_n(g)\tilde{s}_n(g)v\| + \|\tilde{s}_m(g)v-\tilde{s}_n(g)v\| .$$

Using this together with the fact that $\{\tilde{s}_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathrm{Irr}_\infty(G)$ (recall that $\tilde{s}_m$ corresponds to the functional f_m of $\mathfrak{g}$), we have $\{\chi_m\}_{m=1}^{\infty}$ is a Cauchy sequence of characters of G. Let χ_0 be its limit character and denote by h_m the functional of χ_m, $m>0$. Thus $h_m \underset{m}{\rightarrow} h_0$, and therefore

$$f_m + h_m \underset{m}{\rightarrow} f_0 + h_0 .$$

We have proved $\Omega(T_m)\underset{m}{\rightarrow}\Omega(T_0)$. QED

For nilpotent Lie groups bicontinuity of the Kirillov map was first proved by Brown, [4]. This result is also a consequence of the above theorem.

COROLLARY. Let G be a simply connected and connected nilpotent Lie group. Then the Kirillov map is a homeomorphism of the coadjoint orbit space $\mathfrak{g}^*/G$ onto the unitary dual space $\hat{G}$.

PROOF. The map is clearly a homeomorphism when restricted to the closed subspace of $\mathfrak{g}^*/G$ consisting of those orbits which correspond to characters of G. Combining this with the above theorem, all that remains to be proved is continuity of the inverse map $\hat{G} \to \mathfrak{g}^*/G$ at the identity representation, and this was shown in [10, Theorem 1]. We give a short argument. Let $\varepsilon>0$ be given. For $\delta>0$ and K a closed subgroup of G, we denote by $V(\delta,K)$ the neighbourhood of 1 in $\hat{K}$ consisting of all S such that for $C \subset K$, C compact, there exists a vector v in the Hilbert space H(S) of S with $|\langle S(x)v,v\rangle-1|<\delta$, for all x in C. Now let $T\in\hat{G}$, and suppose $T\in V(\delta,G)$. Then since G is nilpotent, there exists a closed normal connected subgroup N of codimension one in G such that T is induced from an irreducible S of N. By Mackey, $T=\mathrm{ind}_N^G(xS)$ for every x in G. And by continuity of the restriction map and the fact that $T|_N$ is supported on the G-orbit $G\cdot S$, we have $x\cdot S\in V(\delta,N)$ for some x in G. Assuming inductively that $\hat{N} \to \mathfrak{n}^*/N$ is continuous at the identity representation, we may conclude by proper choice of δ that the distance from the orbit $\Omega(x\cdot S)$ to 0 in $\mathfrak{n}^*$ is less than ε. Now this distance is greater than the distance from $\Omega(T)$ to 0 in $\mathfrak{g}^*$, and it follows that $\hat{G} \to \mathfrak{g}^*/G$ is continuous at 1, completing our proof. QED

AN EXAMPLE

Let $\mathfrak{g}=\mathfrak{g}_{4,9}(0)$ denote the exponential Lie algebra given by the nonzero basis relations $[e_1,e_2]=e_2$, $[e_1,e_3]=-e_3$, $[e_2,e_3]=e_4$, and let $\{e_i^*\}_{i=1}^4$ be a basis for $\mathfrak{g}^*$ dual to $\{e_i\}_{i=1}^4$. The nilradical $N=\exp(Re_2+Re_3+Re_4)$ of the group $G=\exp\mathfrak{g}$ is isomorphic with the three dimensional Heisenberg group. For $\nu\neq 0$, let S_ν be the element of $\hat{N}$ associated to the functional νe_4^* of $\mathfrak{n}$. The isotropy group of $\tilde{S}_\nu$ is all of G, and S_ν extends to G. We denote by $\tilde{S}_\nu$ the extension of S_ν that corresponds to νe_4^* in $\mathfrak{g}^*$. All the extensions of S_ν are of the form $T_{\alpha,\nu}=\tilde{S}_\nu\cdot\chi_\alpha$ where χ_α is the character of G, equal to 1 on N, with functional αe_1^*, $\alpha\in R$. The orbit $\Omega_{\alpha,\nu}$ in $\mathfrak{g}^*$ of $T_{\alpha,\nu}$ is seen to be the hyperboloid

$$x_1\nu - x_2x_3 = \alpha\cdot\nu \ , \quad x_4 = \nu \ ,$$

with a distance $\sqrt{2\alpha\nu}$ from zero, whenever $0<\nu<\alpha$. Hence a sequence $\{\Omega_{\alpha_n,\nu_n}\}$ of such orbits may converge to 0 even if $\nu_n\to 0$ and $\alpha_n\to+\infty$, let e.g. $\nu_n=1/n^2$, $\alpha_n=n$. Then the two sequences of representations, $\{T_{\alpha_n,\nu_n}\}$ and $\{\tilde{S}_{\nu_n}\}$, both converge to 1_G by continuity of the Kirillov correspondence. Still the sequence $\{\chi_{\alpha_n}\}$ of characters is unbounded. This answers in the negative a question raised at the end of [6].

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