1. Introduction. There are three problems which have been studied concerning antiautomorphisms of von Neumann algebras; the existence problem, the conjugacy problem, and their description. The latter problem includes whether they are spatial of a particular form, i.e. of the form $x + w^*xw$ with $w$ a conjugate linear isometry of a prescribed type. In the present paper we shall study the spatial problem, with main emphasis on antiautomorphisms $\alpha$ leaving the center elementwise on fixed, called central in the sequel, and with $\alpha$ an involution, i.e. $\alpha^2 = 1$. This problem with variations has previously been studied in [2,6]. E.g. it was shown in [6] that a central involution $\alpha$ is automatically spatial with $w^2$ a selfadjoint unitary operator in the center of the von Neumann algebra.

It turns out that the general problem of whether a central antiautomorphism is spatial has a solution similar to that of automorphisms, with proof also quite similar. We include these results for the sake of completeness. The main new ingredient in the paper is that if $\alpha$ is a central involution of the von Neumann algebra $M$ then $\alpha$ is necessarily on the form $\alpha(x) = Jx^*J$ with $J$ a conjugation, unless the commutant $M'$ of $M$ has a direct summand of type $I_n$ with $n$ odd. In the latter case it may happen that $\alpha$ can only be written in the form $\alpha(x) = -jx^*j$ with $j^2 = -1$. 
2. The results. Recall that two projections $e$ and $f$ in a von Neumann algebra $M$ acting on a Hilbert space $H$ are said to be equivalent, written $e \sim f \pmod M$, or just $e \sim f$, if there is a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$. $e$ is said to be cyclic, written $e = [M' \xi]$ if there is a vector $\xi \in H$ such that $e$ is the projection onto the subspace spanned by vectors of the form $x' \xi$, $x' \in M'$. If $w$ is a conjugate linear operator we denote by $w^*$ its adjoint, viz, $(w^* \xi, \eta) = (w \eta, \xi)$. We denote by $\omega_\xi$ the positive functional $\omega_\xi(x) = (x \xi, \xi)$ on $M$.

**Lemma.** Let $M$ be a von Neumann algebra acting on a Hilbert space $H$. Suppose $\alpha$ is a central antiautomorphism of $M$. Let $\xi$ be a unit vector in $H$, and suppose $[M' \xi] \sim \alpha([M' \xi]) \pmod M$. Then we have:

(i) There exists a unit vector $\eta \in H$ such that $\omega = \omega_\eta \circ \alpha$ on $M$.

(ii) $[M \xi] \sim [M \eta] \pmod {M'}$.

(iii) There exists a conjugate linear partial isometry $w$ on $H$ such that $w^*w = [M \xi]$, $ww^* = [M \eta]$, and $w^*xw[M \xi] = \alpha(x)[M \xi]$, $x \in M$.

(iv) If $\eta = \xi$ is cyclic and $\alpha^{2n} = 1$, the identity map, then $w$ can be chosen so that $w^{2n} = 1$.

**Proof.** Let $e = [M' \xi]$ be the support of the vector state $\omega_\xi$. Let $f = \alpha^{-1}(e)$. By assumption $e \sim f$, so there exists a partial isometry $v \in M$ such that $v^*v = e$, $vv^* = f$. Then $v \xi$ is a unit vector such that $\omega_{v \xi}(f) = (vv^* v \xi, v \xi) = (e \xi, \xi) = 1$, whence $v \xi \in f(H)$. Since
\[ w(x) = \omega(v^*xv), \] the support of \[ \omega \] is \( f = vev^* \), hence \( \omega \) is separating for \( f \). Since \( \omega \alpha \) is a normal state with support \( f \) there exists by [1, Ch. III, §1, Thm. 4] a unit vector \( \eta \in f(H) \) such that \( \omega \alpha = \omega \eta \). This proves (i).

Note that \( f = vev^* = v[M'\xi]v^* = [M'\xi] \). Suppose \( 0 + x \in f \) is positive. Then \( x = a^{-1}(y) \) with \( y \in M \) positive, so that \( \omega(y) = \omega(x) \neq 0 \). In particular, \( \omega \) is faithful on \( f \), so that its support is \([M'\eta] = [fM'\eta] = f \). Thus \([M'\eta] = [M'\xi] \), and (ii) follows from [1, Ch. III, §3 Cor.]

With \( \eta \) as above define a conjugate linear operator \( w: M\xi + M\eta \) by

\[ w(x) = \alpha^{-1}(x^*) \eta. \]

Then

\[ \|w(x)\|^2 = \|\alpha^{-1}(x^*)\eta\|^2 = (\alpha^{-1}(x^*)\eta, \eta) = (x^*x, \xi) = \|x\|^2, \]

so that \( w \) extends to a conjugate linear isometry of \([M\xi](H) \) onto \([M\eta](H). \) Extend \( w \) to all of \( H \) by

\[ w = [M]\eta \]

defining it to be 0 on \([M\xi](H) \). Since \( w^*w = [M\xi] \) we have for \( x, y \in M, \)

\[ w^*wy = w^*\alpha^{-1}(y^*)\eta = w^*\alpha^{-1}(y^*\alpha(x^*))\eta = w^*(y^*\alpha(x^*))\eta = \alpha(y)x^*\eta. \]

Thus (iii) follows.

Finally, if \( \eta = \xi \) is cyclic then \( w \) is a conjugate linear isometry such that \( w^*w = \alpha(x), x \in M. \) By definition of \( w, w^{2k}x^* = \alpha^{-2k}(x^*)x, \)

\( k \in \mathbb{N}; \) hence in particular, \( w^{2n}x^* = x^* \) for all \( x, \) so that \( w^{2n} = 1. \)

QED.

**Theorem 1.** Let \( M \) be a von Neumann algebra and \( \alpha \) an antiautomorphism such that \( \alpha(e) = e \) for all projections \( e \in M. \) Then \( \alpha \) is spatial.
**Proof.** We first note that if \( e' \) is a projection in \( M' \) then the map \( \alpha_{e'}: Me' \to Me' e' \) defined by

\[
\alpha_{e'}(xe') = \alpha(x)e'
\]

is an antiautomorphism. Indeed, if \( x \in M \) let \( c_x \) denote the central projection which is the intersection of all central projections \( q \) in \( M \) with \( qx = x \). Since the assumption on \( \alpha \) implies \( \alpha \) is central, \( c_x = c(\alpha(x)) \). By [5, Lem. 3.1.1] \( xe' = 0 \) if and only if \( 0 = c_x e' = c(\alpha(x)) e' \). Thus \( \alpha_{e'} \) is well defined and injective. Since it is clearly surjective, the assertion follows.

To prove the theorem let by Zorn's lemma \( p' \) be a projection in \( M' \) maximal with respect to the property that \( \alpha_p' \) is spatial on \( Mp' \). Suppose \( p' \neq 1 \) and let \( q' = 1 - p' \). Let \( \xi \) be a unit vector in \( q'(H) \) and let by Lemma (i) \( \eta \) be a unit vector in \( q'(H) \) such that \( \omega_\eta = \omega_\xi \alpha \) on \( Mq' \). Let \( w: [M\xi](H) \to [M\eta](H) \) be as in Lemma (iii). By Lemma (ii) \( [M\xi] \sim [M\eta] (\mod M') \) so there is \( u \in M' \) such that \( u^* u = [M\eta] \), \( uu^* = [M\xi] \). Then \( uw \) is a conjugate linear partial isometry which is 0 on \( [M\xi](H) \perp \) and isometric on \( [M\xi](H) \) onto itself, such that if \( x \in M[M\xi] \) then

\[
(uw)^* x^*(uw) = w^* u^* xu w = w^* x w = \alpha[M\xi](x),
\]

using that \( u \in M' \) and \( [M\xi] u = u \). Thus \( \alpha_{p'} + [M\xi] = \alpha_{p'} + [M\xi] \) is spatial, contradicting the maximality of \( p' \). Thus \( p' = 1 \), completing the proof.

**Theorem 2.** Let \( M \) be a von Neumann algebra with no direct summand of type II\(_\infty\) with finite commutant. Then each central antiautomorphism of \( M \) is spatial.
Proof. Let \( \alpha \) be a central antiautomorphism of \( M \). We may consider the different types separately. The type I portion is taken care of by [6, Lem. 4.3]. Suppose \( M \) is finite. Let \( \Phi \) be the centervalued trace on \( M \) which is the identity on the center. By uniqueness of \( \Phi \), \( \Phi \circ \alpha = \Phi \), hence \( \Phi(\alpha(e)) = \Phi(e) \) for all projections \( e \). It follows that \( e \sim \alpha(e) \) for all projections, hence \( \alpha \) is spatial by Theorem 1.

Assume \( M \) is of type II\(_\infty\) with \( II\_\infty \) commutant. Since the identity is the sum of central projections which are countably decomposable with respect to the center, we may assume the center is countably decomposable. By [5, Lem. 3.3.6] there is a cyclic projection \( e = [M'\xi], \xi \) a unit vector, in \( M \) with central support 1 such that \( eq \) is infinite for all central projections \( q \neq 0 \) in \( M \). Since \( \alpha \) maps infinite projections onto infinite projections, \( f = \alpha^{-1}(e) \) is infinite and is the support of \( \omega_\xi \alpha \). Since \( M' \) is infinite there is a unit vector \( \eta \) such that \( \omega_\xi \alpha = \omega_\eta \) [1, Ch. III, §8, Cor. 10]. Thus \( f = [M'\eta] \) is countably decomposable, and \( fq \) is infinite for all central projections \( q \neq 0 \), and the central support of \( f \) equals that of \( e \) since \( \alpha \) is central. By [1, Ch. III, §8, Cor. 5] \( f \sim e \). By Lemma (iii) and the maximality argument employed in the proof of Theorem 1, \( \alpha \) is spatial.

Finally, assume \( M \) is of type III. Then each normal state is a vector state [1, Ch. III, §8, Cor. 10] so the conclusion of Lemma (i) holds. Since any two countably decomposable projections with the same central supports are equivalent in \( M \), the argument from the II\(_\infty\) case applies to conclude that \( \alpha \) is spatial. Q.E.D.
Remark 1. The above theorem reflects the situation for automorphisms of von Neumann algebras. For a factor $M$ of type $\text{II}_1$ with finite commutant it was shown by Kadison [4] that an automorphism is spatial if and only if it preserves the trace, or equivalently the dimension of projections. By Theorem 1 the latter condition is sufficient for an antiautomorphism $\alpha$ to be spatial. Conversely, if $\alpha$ is spatial the argument of Kadison on [4, p.324] can be repeated word by word to conclude that $\alpha$ preserves the dimension of projections.

The difficulty in the above situation can be avoided if $\alpha$ is periodic.

Theorem 3. Let $M$ be a von Neumann algebra and $\alpha$ a periodic central antiautomorphism. Then $\alpha$ is spatial. Furthermore, if each normal state on $M$ is a vector state (e.g. if $M$ has a separating vector, or $M'$ is properly infinite) then there exists a conjugate linear isometry $w$ that $\alpha(x) = w^* x w$ with $w^{2n} = 1$, where $2n$ is the period of $\alpha$.

Proof. Let $e$ be a projection in $M$. In order to show $\alpha(e) - e$ we may, since $\alpha$ is central, assume by the Comparison Theorem that $\alpha(e) \leq e$. Iterating we have $e = \alpha^{2n}(e) \langle \alpha^{2n-1}(e) \langle \cdots \langle \alpha(e) \langle e$. Thus $\alpha(e) - e$, and $\alpha$ is spatial by Theorem 1.

Now assume each normal state is a vector state. Let $\psi$ be a unit vector. Then the state

$$
\omega = \frac{1}{2n} \sum_{k=1}^{2n} \omega_{\alpha^k} \psi
$$

is a normal state.
is a normal \( \alpha \)-invariant state. Thus \( \omega = \omega_\zeta \) for a unit vector \( \zeta \), and \( \omega_\zeta \circ \alpha = \omega_\zeta \). By the proof of Lemma (iv) there exists a conjugate linear partial isometry \( w \) with support and range \( [M\zeta] \) such that \( w^{2n} = [M\zeta] \), and \( w^* x w [M\zeta] = \alpha(x)[M\zeta] \). A maximality argument now completes the proof.

The above theorem states that for a periodic \( \alpha \) with \( M' \) large then \( w \) can be chosen with \( w^{2n} = 1 \). Our last result gives a sharper statement if \( \alpha \) an involution, Special cases of this result appeared in [6]. Recall that a conjugation is a conjugate linear isometry \( J \) such that \( J^2 = 1 \).

**Theorem 4.** Let \( M \) be a von Neumann algebra whose commutant has no direct summand of type \( I_n \) with \( n \) an odd integer. If \( \alpha \) is a central involution on \( M \) then there exists a conjugation \( J \) such that \( \alpha(x) = Jx^* J \), \( x \in M \).

**Proof.** Let \( M \) act on a Hilbert space \( H \) and assume first \( M \) has no direct summand of type \( I \). By [6, Thm. 3.7] there exist central projections \( p \) and \( q \) in \( M \) such that \( \alpha|pM \) is implemented by a conjugation on \( p(H) \) and \( \alpha|qM \) by a conjugate isometry \( j \) with \( j^2 = -q \). To prove the theorem it suffices to modify \( j \) so that \( \alpha|qM \) is implemented by a conjugation. We therefore assume \( \alpha(x) = -jx^* j \) for \( x \in M \), where \( j^2 = -1 \). In particular \( \alpha \) extends to an involution \( \alpha \) of \( B(H) \) implemented by \( j \), which leaves \( M' \) globally invariant. Since \( M' \) has no direct summand of type \( I \), neither does the fixed point algebra \( A \) of \( \alpha \) in \( M' \) [3, 7.4.3], hence the Halving Lemma for Jordan algebras [3, 5.2.14] yields the existence of projections \( e, f \in A \) with sum 1 and a symmetry \( s \in A \).
such that \( ses = f \). Let \( e_{11} = e, e_{12} = es, e_{21} = se = fs, e_{22} = f \). Then 
\[ \{e_{ij} : i, j = 1, 2\} \]
is a set of matrix units which generates a \( I_2 \)-factor \( M_2 \). Since \( \alpha(e_{12}) = e_{21} \), \( \alpha(e_{ii}) = e_{ii}' \), \( \alpha \) leaves \( M_2 \) globally invariant. Thus \( B(H) = B(H_0) \otimes M_2 \), and \( \alpha = \alpha_1 \otimes \alpha_2 \) with \( \alpha_1 \) an involution of \( B(H_0) \), and \( \alpha_2 = \alpha|M_2 \) an involution of \( M_2 \). For simplicity of notation we identify \( M \) with \( M \otimes 1 \), and consider \( M \) as a subalgebra of \( B(H_0) \). Since an involution of a factor is implemented by a conjugate linear isometry \( v \) with \( v^2 = 1 \) or \(-1\), \([6, \text{Thm. 3.7}]\), it follows that \( j = j_1 \otimes j_2 \) with \( j_1^2 = 1 \), and \( \alpha|M = \alpha_1|M \) is implemented by \( j_1 \). If \( j_1^2 = -1 \) replace \( j_2 \) by a conjugate linear isometry \( v \) with square \(-1\), and if \( j_1^2 = +1 \) by \( v \) with square \(+1\). In either case \( J = j_1 \otimes v \) is a conjugation implementing \( \alpha_1 \), and hence \( \alpha \) on \( M \).

It remains to consider the case when \( M \) is of type I. Since \( \alpha \) is central we may consider the different direct summands separately, hence we may assume \( M \) is homogeneous of type \( I_n' \), \( n \in \mathbb{N} \cup \{\infty\} \), with \( M' \) homogeneous of type \( I_r' \), \( r \in \mathbb{N} \cup \{\infty\} \), see e.g. \([1, \text{Ch. III, \S 3, Prop. 2}]\) applied to \( M \) and \( M' \). For a Hilbert space \( K \) let \( t \) denote the transpose on \( B(K) \) with respect to some orthonormal basis, and let \( q \) be the involution

\[
q(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]
on the complex \( 2 \times 2 \) matrices. By \([7, \text{Thm. 2.6}]\) \( M \) is a direct sum \( M = M_1 \oplus M_2 \) such that \( \alpha \) leaves each \( M_i \) invariant; \( M_1 = B(H_1) \otimes Z_1 \), \( M_2 = B(H_2) \otimes B(\mathbb{C}^2) \otimes Z_2 \), where in both cases \( Z_i \) is an abelian von Neumann algebra with \( Z_i' \) of type \( I_r \). In the first case \( \alpha|M_1 = t \otimes 1 \), hence \( \alpha|M_1 \) is implemented by a conjugation, see e.g. \([3, \text{Section}\]
In the second case \( a|_{M^2} = t \otimes q \otimes i \). Now \( q \) is implemented by a conjugate linear isometry \( j \) such that \( j^2 = -1 \), while \( t \) is implemented by a conjugation \( J \). Since by assumption \( M' \) is of type \( I_r \) with \( r \) even or \( r = \infty \) there exists a conjugate linear isometry \( j \) with \( j^2 = -1 \), which implements a central involution on \( \mathbb{Z}_2^2 \), see [3, Section 7.5]. Thus \( J \otimes j \otimes j \) is a conjugation which implements \( a \) on \( M_2 \). This completes the proof of the theorem.

**Remark 2.** The conclusion of Theorem 4 is false if \( M' \) is of type \( I_{2n} \) with \( n \in \mathbb{N} \) odd. Let for example \( M = M \otimes \mathbb{C} \otimes \mathbb{C}^1 \), so that \( M' = \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \) with \( m \) even and \( n \) odd. Then there exists \( j \) on \( \mathbb{C}^m \) such that \( j^2 = -1 \), while each involution on \( M' \) is conjugate to the transpose map. Let \( a(x \otimes 1) = (-j^* x) \otimes 1 \) on \( M \). Then \( a \) is not implemented by a conjugation. Indeed, if \( J \) is a conjugation on \( \mathbb{C}^m \otimes \mathbb{C}^n \) implementing \( a \), then \( J \) also implements an involution on \( M' = \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \), hence there would exist a conjugation \( J' \) on \( \mathbb{C}^n \) such that \( J x J' = (j \otimes j') x (j \otimes j') \) for all \( x \in \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \). Since \( J^2 = 1 \) and \( (j \otimes j')^2 = -1 \), this is impossible by [7, Lem. 3.9], hence \( a \) is not implemented by a conjugation. This example also shows that the assumption on the normal states being vector states is necessary in Theorem 3.
References


